A Appendix

A.1 Proof of equivalent formulation results

A.1.1 Proof of Proposition 1

Proof. First we can express $|\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle|^2$ as a function of $\boldsymbol{x}^+, \boldsymbol{x}^-, \boldsymbol{a}_i^+$:

$$
\begin{aligned}
|\langle \boldsymbol{x}, \boldsymbol{a}_i \rangle|^2 &= \left|(\mathrm{Re}(\boldsymbol{x}) - i\mathrm{Im}(\boldsymbol{x}))^T(\mathrm{Re}(\boldsymbol{a}_i) + i\mathrm{Im}(\boldsymbol{a}_i))\right|^2 \\
&= (\langle \mathrm{Re}(\boldsymbol{x}), \mathrm{Re}(\boldsymbol{a}_i)\rangle + \langle \mathrm{Im}(\boldsymbol{x}), \mathrm{Im}(\boldsymbol{a}_i)\rangle)^2 + (\langle \mathrm{Re}(\boldsymbol{x}), \mathrm{Im}(\boldsymbol{a}_i)\rangle - \langle \mathrm{Im}(\boldsymbol{x}), \mathrm{Re}(\boldsymbol{a}_i)\rangle)^2 \\
&= \langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^2 + \langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^2. \qquad (14)
\end{aligned}
$$

We can then rewrite f as:

$$
\begin{aligned}
f(\boldsymbol{x}) &= \sum_i \left(\langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^2 + \langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^2 - \langle \boldsymbol{x}^{\natural+}, \boldsymbol{a}_i^+\rangle^2 - \langle \boldsymbol{x}^{\natural-}, \boldsymbol{a}_i^+\rangle^2\right)^2 \\
&= \sum_i \Bigg(\langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^4 + \langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^4 + \langle \boldsymbol{x}^{\natural+}, \boldsymbol{a}_i^+\rangle^4 + \langle \boldsymbol{x}^{\natural-}, \boldsymbol{a}_i^+\rangle^4 \\
&\qquad + 2\langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^2 + 2\langle \boldsymbol{x}^{\natural+}, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^{\natural-}, \boldsymbol{a}_i^+\rangle^2 \\
&\qquad - 2\langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^{\natural-}, \boldsymbol{a}_i^+\rangle^2 - 2\langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^{\natural-}, \boldsymbol{a}_i^+\rangle^2 \\
&\qquad - 2\langle \boldsymbol{x}^+, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^{\natural+}, \boldsymbol{a}_i^+\rangle^2 - 2\langle \boldsymbol{x}^-, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{x}^{\natural+}, \boldsymbol{a}_i^+\rangle^2 \Bigg). \qquad (15)
\end{aligned}
$$

From here we can notice that the scalar products can be written as scalar products with $(\boldsymbol{a}_i^+)^{\otimes 4}$, indeed for two vectors $\boldsymbol{u}, \boldsymbol{v}$, we have

$$
\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{a}_i^+\rangle^2\langle \boldsymbol{v}, \boldsymbol{a}_i^+\rangle^2 &= \left(\sum_{i_1} u_{i_1}(\boldsymbol{a}_i^+)_{i_1}\right)\left(\sum_{i_2} u_{i_2}(\boldsymbol{a}_i^+)_{i_2}\right)\left(\sum_{i_3} u_{i_3}(\boldsymbol{a}_i^+)_{i_3}\right)\left(\sum_{i_4} u_{i_4}(\boldsymbol{a}_i^+)_{i_4}\right) \\
&= \sum_{i_1,i_2,i_3,i_4} (\boldsymbol{a}_i^+)_{i_1}(\boldsymbol{a}_i^+)_{i_2}(\boldsymbol{a}_i^+)_{i_3}(\boldsymbol{a}_i^+)_{i_4} u_{i_1}u_{i_2}v_{i_3}v_{i_4} \\
&= \langle (\boldsymbol{a}_i^+)^{\otimes 4}, \boldsymbol{u} \otimes \boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{v}\rangle. \qquad (16)
\end{aligned}
$$

This allows to write f in the following form:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_i \Bigg\langle & (\boldsymbol{a}_i^+)^{\otimes 4}, (\boldsymbol{x}^+)^{\otimes 4} + (\boldsymbol{x}^-)^{\otimes 4} + (\boldsymbol{x}^{\natural+})^{\otimes 4} + (\boldsymbol{x}^{\natural-})^{\otimes 4} \\
&+ 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^-)^{\otimes 2} + 2(\boldsymbol{x}^{\natural+})^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} \\
&- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} \\
&- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2} \Bigg\rangle \\
= \Bigg\langle & \sum_i (\boldsymbol{a}_i^+)^{\otimes 4}, (\boldsymbol{x}^+)^{\otimes 4} + (\boldsymbol{x}^-)^{\otimes 4} + (\boldsymbol{x}^{\natural+})^{\otimes 4} + (\boldsymbol{x}^{\natural-})^{\otimes 4} \\
&+ 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^-)^{\otimes 2} + 2(\boldsymbol{x}^{\natural+})^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} \\
&- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} \\
&- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2} \Bigg\rangle. \qquad (17)
\end{aligned}
$$

We get the expected result by chosing:

$$\begin{aligned}\mathbf{U}(\boldsymbol{x}^+) := &(\boldsymbol{x}^+)^{\otimes 4} + (\boldsymbol{x}^-)^{\otimes 4} + (\boldsymbol{x}^{\natural+})^{\otimes 4} + (\boldsymbol{x}^{\natural-})^{\otimes 4}\\ &+ 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^-)^{\otimes 2} + 2(\boldsymbol{x}^{\natural+})^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2}\\ &- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural-})^{\otimes 2}\\ &- 2(\boldsymbol{x}^+)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2} - 2(\boldsymbol{x}^-)^{\otimes 2} \otimes (\boldsymbol{x}^{\natural+})^{\otimes 2}. \end{aligned} \tag{18}$$

□

A.2 Proof of landscape results

A.2.1 Proof of Proposition 2

Proof. In order to use tensor $\mathbf{S}$, it is be useful to notice that:

$$\begin{aligned}\langle \mathbf{S}, \boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{v}\rangle &= \sum_{i_1,i_2,i_3,i_4} \mathbf{S}_{i_1,i_2,i_3,i_4} u_{i_1}u_{i_2}v_{i_3}v_{i_4}\\ &= \sum_{i_1=i_2\neq i_3=i_4} u_{i_1}u_{i_2}v_{i_3}v_{i_4}\\ &+ \sum_{i_1=i_3\neq i_2=i_4} u_{i_1}u_{i_2}v_{i_3}v_{i_4}\\ &+ \sum_{i_1=i_4\neq i_2=i_3} u_{i_1}u_{i_2}v_{i_3}v_{i_4}\\ &+3 \sum_{i_1=i_2=i_3=i_4} u_{i_1}u_{i_2}v_{i_3}v_{i_4}\\ &= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{u},\boldsymbol{v}\rangle^2. \end{aligned} \tag{19}$$

Then we have:

$$\begin{aligned} c\langle \mathbf{S}, \mathbf{U}(\boldsymbol{x}^+)\rangle =c\Big(&3\|\boldsymbol{x}^+\|^4 + 3\|\boldsymbol{x}^-\|^4 + 3\|\boldsymbol{x}^{\natural+}\|^4 + 3\|\boldsymbol{x}^{\natural-}\|^4\\ &+ 2(2\langle \boldsymbol{x}^+,\boldsymbol{x}^-\rangle^2 + \|\boldsymbol{x}^+\|^2\|\boldsymbol{x}^-\|^2) + 2(2\langle \boldsymbol{x}^{\natural+},\boldsymbol{x}^{\natural-}\rangle^2 + \|\boldsymbol{x}^{\natural+}\|^2\|\boldsymbol{x}^{\natural-}\|^2)\\ &- 2(2\langle \boldsymbol{x}^+,\boldsymbol{x}^{\natural-}\rangle^2 + \|\boldsymbol{x}^+\|^2\|\boldsymbol{x}^{\natural-}\|^2) - 2(2\langle \boldsymbol{x}^-,\boldsymbol{x}^{\natural-}\rangle^2 + \|\boldsymbol{x}^-\|^2\|\boldsymbol{x}^{\natural-}\|^2)\\ &- 2(2\langle \boldsymbol{x}^-,\boldsymbol{x}^{\natural+}\rangle^2 + \|\boldsymbol{x}^-\|^2\|\boldsymbol{x}^{\natural+}\|^2) - 2(2\langle \boldsymbol{x}^+,\boldsymbol{x}^{\natural+}\rangle^2 + \|\boldsymbol{x}^+\|^2\|\boldsymbol{x}^{\natural+}\|^2)\Big). \end{aligned} \tag{20}$$

We can notice that $\|\boldsymbol{x}^+\|^2 = \|\boldsymbol{x}^-\|^2 = \|\boldsymbol{x}\|^2$, that $\langle \boldsymbol{x}^+,\boldsymbol{x}^-\rangle = 0$ (and same result holds for $\boldsymbol{x}^\natural$) and that:

$$\begin{aligned} &\langle \boldsymbol{x}^+,\boldsymbol{x}^{\natural+}\rangle^2 + \langle \boldsymbol{x}^+,\boldsymbol{x}^{\natural-}\rangle^2 + \langle \boldsymbol{x}^-,\boldsymbol{x}^{\natural+}\rangle^2 + \langle \boldsymbol{x}^-,\boldsymbol{x}^{\natural-}\rangle^2\\ =&2\left(\langle \mathrm{Re}(\boldsymbol{x}),\mathrm{Re}(\boldsymbol{x}^\natural)\rangle + \langle \mathrm{Im}(\boldsymbol{x}),\mathrm{Im}(\boldsymbol{x}^\natural)\rangle\right)^2 + 2\left(\langle \mathrm{Re}(\boldsymbol{x}),\mathrm{Im}(\boldsymbol{x}^\natural)\rangle - \langle \mathrm{Im}(\boldsymbol{x}),\mathrm{Re}(\boldsymbol{x}^\natural)\rangle\right)^2\\ =&2|\langle \boldsymbol{x},\boldsymbol{x}^\natural\rangle|^2. \end{aligned} \tag{21}$$

We can thus simplify the previous expression as:

$$c\langle \mathbf{S}, \mathbf{U}(\boldsymbol{x})\rangle = 8c\left(\|\boldsymbol{x}\|^4 + \|\boldsymbol{x}^\natural\|^4 - |\langle \boldsymbol{x},\boldsymbol{x}^\natural\rangle|^2 - \|\boldsymbol{x}\|^2\|\boldsymbol{x}^\natural\|^2\right) := g(\boldsymbol{x}^+).$$

□

A.2.2 Proof of Proposition 3

Proof. To simplify notation, we denote $J = \{(i_1,i_2,i_3,i_4) \mid 1 \le i_1,i_2,i_3,i_4 \le 2n\}$ and all summations over j are for $j \in J$. We also write:

$$\mathbf{U}(\boldsymbol{x}^+) = \sum_{k=1}^{K} \varepsilon_k \boldsymbol{x}_{1,k}\otimes\boldsymbol{x}_{2,k}\otimes\boldsymbol{x}_{3,k}\otimes\boldsymbol{x}_{4,k}$$

with $\boldsymbol{x}_{i,k} \in \{\boldsymbol{x}^+, \boldsymbol{x}^-, \boldsymbol{x}^{\natural+}, \boldsymbol{x}^{\natural-}\}$ and $\varepsilon_k = \pm 1$.

Recall that:

$$\begin{aligned}
f(\boldsymbol{x}^+) &= c\langle \mathbf{T}, \mathbf{U}(\boldsymbol{x}^+)\rangle \\
g(\boldsymbol{x}^+) &= c\langle \mathbf{S}, \mathbf{U}(\boldsymbol{x}^+)\rangle \\
\nabla f(\boldsymbol{x}^+) &= \sum_j c\mathbf{T}_j \nabla \mathbf{U}_j(\boldsymbol{x}^+) \\
\nabla g(\boldsymbol{x}^+) &= \sum_j c\mathbf{S}_j \nabla \mathbf{U}_j(\boldsymbol{x}^+).
\end{aligned} \tag{22}$$

Using this we can write:

$$\begin{aligned}
\langle \nabla f(\boldsymbol{x}^+), \nabla g(\boldsymbol{x}^+)\rangle &= \langle \nabla g(\boldsymbol{x}^+), \nabla f(\boldsymbol{x}^+)\rangle \\
&= \langle \nabla g(\boldsymbol{x}^+), \sum_j c\mathbf{T}_j \nabla \mathbf{U}_j\rangle \\
&= \langle \nabla g(\boldsymbol{x}^+), \sum_j c(\mathbf{T}_j - \mathbf{S}_j + \mathbf{S}_j)\nabla \mathbf{U}_j\rangle \\
&= \langle \nabla g(\boldsymbol{x}^+), \nabla g(\boldsymbol{x}^+)\rangle + \langle \nabla g(\boldsymbol{x}^+), \sum_j c(\mathbf{T}_j - \mathbf{S}_j)\nabla \mathbf{U}_j\rangle \\
&= \|\nabla g(\boldsymbol{x}^+)\|^2 + \sum_i \sum_j (\nabla g(\boldsymbol{x}^+))_i (\nabla \mathbf{U}_j)_i c(\mathbf{T} - \mathbf{S})_j \\
&= \|\nabla g(\boldsymbol{x}^+)\|^2 + \sum_i \sum_k \sum_{i_1,i_2,i_3,i_4} (\nabla g(\boldsymbol{x}^+))_i c(\mathbf{T} - \mathbf{S})_{i_1,i_2,i_3,i_4} \varepsilon_k \\
&\times \Bigg(\frac{\partial(\boldsymbol{x}_{1,k})_{i_1}}{\partial \boldsymbol{x}_i^+} (\boldsymbol{x}_{2,k})_{i_2} (\boldsymbol{x}_{3,k})_{i_3} (\boldsymbol{x}_{4,k})_{i_4} \\
&+ \frac{\partial(\boldsymbol{x}_{2,k})_{i_2}}{\partial \boldsymbol{x}_i^+} (\boldsymbol{x}_{1,k})_{i_1} (\boldsymbol{x}_{3,k})_{i_3} (\boldsymbol{x}_{4,k})_{i_4} \\
&+ \frac{\partial(\boldsymbol{x}_{3,k})_{i_3}}{\partial \boldsymbol{x}_i^+} (\boldsymbol{x}_{1,k})_{i_1} (\boldsymbol{x}_{2,k})_{i_2} (\boldsymbol{x}_{4,k})_{i_4} \\
&+ \frac{\partial(\boldsymbol{x}_{4,k})_{i_4}}{\partial \boldsymbol{x}_i^+} (\boldsymbol{x}_{1,k})_{i_1} (\boldsymbol{x}_{2,k})_{i_2} (\boldsymbol{x}_{3,k})_{i_3} \Bigg).
\end{aligned} \tag{23}$$

If $\boldsymbol{x}_{\ell,k} = \boldsymbol{x}^+$, then $\frac{\partial(\boldsymbol{x}_{\ell,k})_{i_\ell}}{\partial x_i} = \mathbf{1}_{i=i_\ell}$. If $\boldsymbol{x}_{\ell,k} = \boldsymbol{x}^-$, then $\frac{\partial(\boldsymbol{x}_{\ell,k})_{i_\ell}}{\partial x_i} = -\mathbf{1}_{i=i_\ell+n}$ if $i_\ell \leq n$ and $\mathbf{1}_{i=i_\ell-n}$ otherwise. In each case we can write:

$$\sum_i (\nabla g(\boldsymbol{x}^+))_i \frac{\partial(\boldsymbol{x}_{\ell,k})_{i_\ell}}{\partial x_i} = (\boldsymbol{M}_{\ell,k} \nabla g(\boldsymbol{x}^+))_{i_\ell}$$

for some $\boldsymbol{M}_{\ell,k} \in \left\{\boldsymbol{I}_{2n}, \boldsymbol{O}_{2n}, \begin{pmatrix} \boldsymbol{O}_n & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \boldsymbol{O}_n \end{pmatrix}\right\}$. Then we have

$$\begin{aligned}
\langle \nabla f(\boldsymbol{x}^+), \nabla g(\boldsymbol{x}^+)\rangle = \|\nabla g(\boldsymbol{x}^+)\|^2 + \sum_k \Big(& c\langle \mathbf{T} - \mathbf{S}, \varepsilon_k \boldsymbol{M}_{1,k} \nabla g(\boldsymbol{x}^+) \otimes \boldsymbol{x}_{2,k} \otimes \boldsymbol{x}_{3,k} \otimes \boldsymbol{x}_{4,k}\rangle \\
& + c\langle \mathbf{T} - \mathbf{S}, \varepsilon_k \boldsymbol{x}_{1,k} \otimes \boldsymbol{M}_{2,k} \nabla g(\boldsymbol{x}^+) \otimes \boldsymbol{x}_{3,k} \otimes \boldsymbol{x}_{4,k}\rangle \\
& + c\langle \mathbf{T} - \mathbf{S}, \varepsilon_k \boldsymbol{x}_{1,k} \otimes \boldsymbol{x}_{2,k} \otimes \boldsymbol{M}_{3,k} \nabla g(\boldsymbol{x}^+) \otimes \boldsymbol{x}_{4,k}\rangle \\
& + c\langle \mathbf{T} - \mathbf{S}, \varepsilon_k \boldsymbol{x}_{1,k} \otimes \boldsymbol{x}_{2,k} \otimes \boldsymbol{x}_{3,k} \otimes \boldsymbol{M}_{4,k} \nabla g(\boldsymbol{x}^+)\rangle \Big) \\
\geq \|\nabla g(\boldsymbol{x}^+)\|^2 - \sum_k \Big(& c\|\mathbf{T} - \mathbf{S}\|_{\mathrm{op}} \|\varepsilon_k \boldsymbol{M}_{1,k} \nabla g(\boldsymbol{x}^+)\| \|\boldsymbol{x}_{2,k}\| \|\boldsymbol{x}_{3,k}\| \boldsymbol{x}_{4,k}\| \| \\
& + c\|\mathbf{T} - \mathbf{S}\|_{\mathrm{op}} \|\varepsilon_k \boldsymbol{M}_{2,k} \nabla g(\boldsymbol{x}^+)\| \|\boldsymbol{x}_{1,k}\| \|\boldsymbol{x}_{3,k}\| \boldsymbol{x}_{4,k}\| \| \\
& + c\|\mathbf{T} - \mathbf{S}\|_{\mathrm{op}} \|\varepsilon_k \boldsymbol{M}_{3,k} \nabla g(\boldsymbol{x}^+)\| \|\boldsymbol{x}_{1,k}\| \|\boldsymbol{x}_{2,k}\| \boldsymbol{x}_{4,k}\| \| \\
& + c\|\mathbf{T} - \mathbf{S}\|_{\mathrm{op}} \|\varepsilon_k \boldsymbol{M}_{4,k} \nabla g(\boldsymbol{x}^+)\| \|\boldsymbol{x}_{1,k}\| \|\boldsymbol{x}_{2,k}\| \boldsymbol{x}_{3,k}\| \| \Big) \\
\geq \|\nabla g(\boldsymbol{x}^+)\|^2 - \sum_k & 4\delta_0 c(\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^3 \|\nabla g(\boldsymbol{x}^+)\| \qquad (24) \\
\geq \|\nabla g(\boldsymbol{x}^+)\|^2 - & C_1 \delta_0 c(\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^3 \|\nabla g(\boldsymbol{x}^+)\|. \qquad (25)
\end{aligned}$$

Assume we have $\|\nabla g(\boldsymbol{x}^+)\| \geq C_1 \delta_0 c(\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^3$, then in particular $\nabla g(\boldsymbol{x}^+) \neq 0$, and $\boldsymbol{x}^\natural \neq 0$, so inequality (24) is strict and we must have $\langle \nabla f(\boldsymbol{x}^+), \nabla g(\boldsymbol{x}^+)\rangle > 0$, and therefore $\nabla f(\boldsymbol{x}^+) \neq 0$. □

A.2.3 PROOF OF PROPOSITION 4

We denote $J = \{(i_1, i_2, i_3, i_4) \mid 1 \leq i_1, i_2, i_3, i_4 \leq 2n\}$ and all summations over j are for $j \in J$.

Proof. We have:

$$\begin{aligned}
& |\boldsymbol{u}^T \boldsymbol{H}_f(\boldsymbol{x}^+)\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{H}_g(\boldsymbol{x}^+)\boldsymbol{u}| \\
&= \left| \boldsymbol{u}^T \left(c \sum_{j \in J} (\mathbf{T} - \mathbf{S})_j \boldsymbol{H}_{\mathbf{U}_j} \right) \boldsymbol{u} \right| \\
&= \left| \boldsymbol{u}^T \left(c \sum_{j \in J} \sum_k \varepsilon_k (\mathbf{T} - \mathbf{S})_j \boldsymbol{H}_{(\boldsymbol{x}_{1,k} \otimes \boldsymbol{x}_{2,k} \otimes \boldsymbol{x}_{3,k} \otimes \boldsymbol{x}_{4,k})_j} \right) \boldsymbol{u} \right| \\
&= \Bigg| \boldsymbol{u}^T \Bigg(c \sum_{i_1, i_2, i_3, i_4} \sum_k \varepsilon_k (\mathbf{T} - \mathbf{S})_{i_1, i_2, i_3, i_4} \\
& \sum_{\sigma \in \mathfrak{S}_4} \frac{1}{2} \frac{\partial (\boldsymbol{x}_{\sigma(1),k})_{i_{\sigma(1)}}}{\boldsymbol{x}_i^+} \frac{\partial (\boldsymbol{x}_{\sigma(2),k})_{i_{\sigma(2)}}}{\boldsymbol{x}_{i'}^+} (\boldsymbol{x}_{\sigma(3),k})_{i_{\sigma(3)}} (\boldsymbol{x}_{\sigma(4),k})_{i_{\sigma(4)}} \Bigg)_{1 \leq i, i' \leq 2n} \boldsymbol{u} \Bigg| \\
&= \Bigg| c \sum_{i_1, i_2, i_3, i_4} \sum_k \varepsilon_k (\mathbf{T} - \mathbf{S})_{i_1, i_2, i_3, i_4} \\
& \sum_{i, i'} \sum_{\sigma \in \mathfrak{S}_4} \frac{1}{2} \left(u_i \frac{\partial (\boldsymbol{x}_{\sigma(1),k})_{i_{\sigma(1)}}}{\boldsymbol{x}_i^+} \right) \left(u_{i'} \frac{\partial (\boldsymbol{x}_{\sigma(2),k})_{i_{\sigma(2)}}}{\boldsymbol{x}_{i'}^+} \right) (\boldsymbol{x}_{\sigma(3),k})_{i_{\sigma(3)}} (\boldsymbol{x}_{\sigma(4),k})_{i_{\sigma(4)}} \Bigg|.
\end{aligned} \qquad (26)$$

As we saw previously in Appendix A.2.2, we can write for some $\boldsymbol{M}_{\sigma(1),k} \in \left\{\boldsymbol{I}_{2n}, \boldsymbol{O}_{2n}, \begin{pmatrix}\boldsymbol{O}_n & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \boldsymbol{O}_n\end{pmatrix}\right\}$,

$$\sum_i \left(u_i \frac{\partial (\boldsymbol{x}_{\sigma(1),k})_{i_{\sigma(1)}}}{\boldsymbol{x}_i^+} \right) = (\boldsymbol{M}_{\sigma(1),k}\boldsymbol{u})_{i_{\sigma(1)}}.$$

Then we have:

$$\begin{aligned}
&|\boldsymbol{u}^T \boldsymbol{H}_f(\boldsymbol{x}^+)\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{H}_g(\boldsymbol{x}^+)\boldsymbol{u}| \\
&= \Bigg| c \sum_{i_1,i_2,i_3,i_4} \sum_k \varepsilon_k (\mathbf{T}-\mathbf{S})_{i_1,i_2,i_3,i_4} \\
&\frac{1}{2}\Bigg((\boldsymbol{M}_{1,k}\boldsymbol{u})_{i_1} (\boldsymbol{M}_{2,k}\boldsymbol{u})_{i_2} (\boldsymbol{x}_{3,k})_{i_3} (\boldsymbol{x}_{4,k})_{i_4} + (\boldsymbol{M}_{2,k}\boldsymbol{u})_{i_2} (\boldsymbol{M}_{1,k}\boldsymbol{u})_{i_1} (\boldsymbol{x}_{3,k})_{i_3} (\boldsymbol{x}_{4,k})_{i_4} + \\
&\cdots + (\boldsymbol{M}_{4,k}\boldsymbol{u})_{i_4} (\boldsymbol{M}_{3,k}\boldsymbol{u})_{i_3} (\boldsymbol{x}_{2,k})_{i_2} (\boldsymbol{x}_{1,k})_{i_1} \Bigg) \Bigg| \\
&= \Bigg| c \sum_k \varepsilon_k \Bigg\langle \mathbf{T}-\mathbf{S}, \\
&\frac{1}{2}\Bigg((\boldsymbol{M}_{1,k}\boldsymbol{u}) \otimes (\boldsymbol{M}_{2,k}\boldsymbol{u}) \otimes (\boldsymbol{x}_{3,k}) \otimes (\boldsymbol{x}_{4,k}) + (\boldsymbol{M}_{1,k}\boldsymbol{u}) \otimes (\boldsymbol{M}_{2,k}\boldsymbol{u}) \otimes (\boldsymbol{x}_{3,k}) \otimes (\boldsymbol{x}_{4,k}) + \\
&\cdots + (\boldsymbol{x}_{1,k}) \otimes (\boldsymbol{x}_{2,k}) \otimes (\boldsymbol{M}_{3,k}\boldsymbol{u}) \otimes (\boldsymbol{M}_{4,k}\boldsymbol{u}) \Bigg) \Bigg\rangle \Bigg| \\
&\le c \sum_k \|\mathbf{T}-\mathbf{S}\|_{\mathrm{op}} \sum_{\sigma \in \mathfrak{S}_4} \frac{1}{2} \|\boldsymbol{M}_{\sigma(1),k}\boldsymbol{u}\| \|\boldsymbol{M}_{\sigma(2),k}\boldsymbol{u}\| \|\boldsymbol{x}_{\sigma(3),k}\| \|\boldsymbol{x}_{\sigma(4),k}\| \\
&\le c \sum_k \|\mathbf{T}-\mathbf{S}\|_{\mathrm{op}} \sum_{\sigma \in \mathfrak{S}_4} \frac{1}{2} \|\boldsymbol{u}\|^2 (\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^2 \\
&< c \sum_k \delta_0 12 \|\boldsymbol{u}\|^2 (\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^2 \\
&\le C_2 \delta_0 c \|\boldsymbol{u}\|^2 (\|\boldsymbol{x}\| + \|\boldsymbol{x}^\natural\|)^2.
\end{aligned} \tag{27}$$

□

A.2.4 Proof of Proposition 5

Proof. We can notice that $\boldsymbol{x}^- = \boldsymbol{M}\boldsymbol{x}^+$ where

$$\boldsymbol{M} := \begin{bmatrix}\boldsymbol{O}_n & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \boldsymbol{O}_n\end{bmatrix}.$$

It is also well known that:

$$\boldsymbol{H}_{\|\boldsymbol{x}\|^2}(x) = 2\boldsymbol{I}_n, \quad \boldsymbol{H}_{\|\boldsymbol{M}\boldsymbol{x}\|^4}(\boldsymbol{x}) = 8\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} + 4\|\boldsymbol{x}\|^2\boldsymbol{I}_n$$

$$\boldsymbol{H}_{\langle \boldsymbol{M}\boldsymbol{x},\boldsymbol{a}\rangle^2}(\boldsymbol{x}) = 2(\boldsymbol{M}\boldsymbol{a})(\boldsymbol{M}\boldsymbol{a})^{\mathrm{T}}.$$

Thus, the Hessian of g with respect to $\boldsymbol{u}^+$ is given by

$$\boldsymbol{H}_g(\boldsymbol{x}^+) = 8c(8\boldsymbol{x}^+(\boldsymbol{x}^+)^{\mathrm{T}} + 4\|\boldsymbol{x}^+\|^2\boldsymbol{I}_n - 2\boldsymbol{x}^{\natural+}(\boldsymbol{x}^{\natural+})^{\mathrm{T}} - 2\boldsymbol{x}^{\natural-}(\boldsymbol{x}^{\natural-})^{\mathrm{T}} - \|\boldsymbol{x}^{\natural+}\|^2\boldsymbol{I}_n - \|\boldsymbol{x}^{\natural-}\|^2\boldsymbol{I}_n).$$

As $\|\boldsymbol{x}^{\natural+}\| = \|\boldsymbol{x}^{\natural-}\|$, we get the desired result. □

A.2.5 Proof of Proposition 6

Proof. Using Proposition 5 and Proposition 4, we can write:

$$
\begin{aligned}
(\boldsymbol{x}^{\natural +})^T \boldsymbol{H}_f(\boldsymbol{x}^+)\boldsymbol{x}^{\natural +} &< 8c\left(8\langle \boldsymbol{x}^+, \boldsymbol{x}^{\natural +}\rangle^2 + 4\|\boldsymbol{x}^+\|^2\|\boldsymbol{x}^{\natural +}\|^2 - 2\|\boldsymbol{x}^{\natural +}\|^2 - 2\langle \boldsymbol{x}^{\natural +}\boldsymbol{x}^{\natural -}\rangle^2 - \|\boldsymbol{x}^{\natural +}\|^4 - \|\boldsymbol{x}^{\natural +}\|^2\|\boldsymbol{x}^{\natural -}\|^2\right) \\
&\quad + C_2\delta_0 c\|\boldsymbol{x}^{\natural}\|^2(\|\boldsymbol{x}\| + \|\boldsymbol{x}^{\natural}\|)^2 \\
&\leq 8c\left(8\left(\langle \mathrm{Re}(\boldsymbol{x}^{\natural}), \mathrm{Re}(\boldsymbol{x})\rangle + \langle \mathrm{Im}(\boldsymbol{x}^{\natural}), \mathrm{Im}(\boldsymbol{x})\rangle\right)^2 + 4\|\boldsymbol{x}\|^2\|\boldsymbol{x}^{\natural}\|^2 - 2\|\boldsymbol{x}^{\natural}\|^4 - \|\boldsymbol{x}^{\natural}\|^4 - \|\boldsymbol{x}^{\natural}\|^4\right) \\
&\quad + 2C_2\delta_0 c\|\boldsymbol{x}^{\natural}\|^2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{x}^{\natural}\|^2) \\
&\leq 8c(8|\langle \boldsymbol{x}, \boldsymbol{x}^{\natural}\rangle|^2 + 4\|\boldsymbol{x}\|^2\|\boldsymbol{x}^{\natural}\|^2 - 4\|\boldsymbol{x}^{\natural}\|^4) + 2C_2\delta_0 c\|\boldsymbol{x}^{\natural}\|^2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{x}^{\natural}\|^2).
\end{aligned}
\tag{28}
$$

Then as long as the following condition is satisfied, we have $(\boldsymbol{x}^{\natural})^T \boldsymbol{H}_f \boldsymbol{x}^{\natural} < 0$:

$$
\begin{aligned}
&8c(8|\langle \boldsymbol{x}, \boldsymbol{x}^{\natural}\rangle|^2 + 4\|\boldsymbol{x}\|^2\|\boldsymbol{x}^{\natural}\|^2 - 4\|\boldsymbol{x}^{\natural}\|^4) + 2\delta_0 C_2 c(\|\boldsymbol{x}\|^2\|\boldsymbol{x}^{\natural}\|^2 + \|\boldsymbol{x}^{\natural}\|^4) \leq 0 \\
\iff &8\frac{|\langle \boldsymbol{x}, \boldsymbol{x}^{\natural}\rangle|^2}{\|\boldsymbol{x}^{\natural}\|^4} + \left(4 + \frac{1}{4}\delta_0 C_2\right)\frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{x}^{\natural}\|^2} \leq \left(4 - \frac{1}{4}\delta_0 C_2\right).
\end{aligned}
\tag{29}
$$

□

A.2.6 PROOF OF PROPOSITION 7

Proof. We borrow the notations from Sun et al. (2018b) and we let:

$$
\phi(\boldsymbol{x}) = \mathrm{argmin}_{\theta \in [0, 2\pi)} \|\boldsymbol{x} - \boldsymbol{x}^{\natural} e^{i\theta}\|
$$

(And for $\boldsymbol{x} \in \mathrm{Span}\{\boldsymbol{x}^{\natural}, i\boldsymbol{x}^{\natural}\}^{\perp}$ we just define $\phi(\boldsymbol{x}) = 0$). Intuitively $\phi(\boldsymbol{x})$ is the phase of $\boldsymbol{x}$ in the real $2d$ plane spanned by $\boldsymbol{x}^{\natural}$. This means we can write in particular:

$$
\boldsymbol{x}^+ = r\left(\cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural +} + \sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural -}\right) + \boldsymbol{v}
$$

with $\boldsymbol{v}$ orthogonal to $\mathrm{Span}\{\boldsymbol{x}^{\natural +}, \boldsymbol{x}^{\natural -}\}$.

Now consider $\boldsymbol{u} = r'\left(\cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural+} + \sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural-}\right) + \boldsymbol{v}'$ such that $\|\boldsymbol{u}\| = 1$ and write $(\boldsymbol{x}^{\natural})^{\phi(\boldsymbol{x})} = \cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural+} + \sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural-}$, we have

$$
\begin{aligned}
\boldsymbol{u}^T\boldsymbol{H}_f(\boldsymbol{x}^+)\boldsymbol{u} &> \boldsymbol{u}^T\boldsymbol{H}_g(\boldsymbol{x}^+)\boldsymbol{u} - C_2\delta_0 c(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x}\|)^2 \qquad \text{(From Proposition 4)}\\
&= 8c(8\langle\boldsymbol{u},\boldsymbol{x}^+\rangle^2 + 4\|\boldsymbol{x}^+\|^2\|\boldsymbol{u}\|^2 - 2\langle\boldsymbol{x}^{\natural+},\boldsymbol{u}\rangle^2 - 2\langle\boldsymbol{x}^{\natural-},\boldsymbol{u}\rangle^2 - 2\|\boldsymbol{x}^{\natural+}\|^2\|\boldsymbol{u}\|^2\\
&\quad - C_2\delta_0 c\|\boldsymbol{u}\|^2(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x}\|)^2 \qquad \text{(From Proposition 5)}\\
&= \|\boldsymbol{u}\|^2 8c\left(\left(8\frac{\langle\boldsymbol{u},\boldsymbol{x}^+\rangle^2}{\|\boldsymbol{u}\|^2} + 4\|\boldsymbol{x}^+\|^2 - 2r'^2\|\boldsymbol{x}^\natural\|^4 - 2\|\boldsymbol{x}^\natural\|^2\right) - \frac{C_2}{8}\delta_0(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x}\|)^2\right)\\
&= 8c\left(\left(8\langle\boldsymbol{u},\boldsymbol{x}^+\rangle^2 + 4\|\boldsymbol{x}^+\|^2 - 2r'^2\|\boldsymbol{x}^\natural\|^4 - 2\|\boldsymbol{x}^\natural\|^2\right) - \frac{C_2}{8}\delta_0(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x}\|)^2\right)\\
&= 8c\Bigg(\left(8\langle\boldsymbol{u},\boldsymbol{x}^+ - (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})} + (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\rangle^2 + 4\|\boldsymbol{x}^+ - (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})} + (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\|^2 - 2r'^2\|\boldsymbol{x}^\natural\|^4 - 2\|\boldsymbol{x}^\natural\|^2\right)\\
&\quad - \frac{C_2}{8}\delta_0(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x}\|)^2\Bigg)\\
&\geq 8c\Bigg(\Big(8\langle\boldsymbol{u},(\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\rangle^2 + 16\langle\boldsymbol{u},\boldsymbol{x}^+ - (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\rangle\langle\boldsymbol{u},(\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\rangle + 4\|(\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\|^2\\
&\quad + 8\langle\boldsymbol{x}^+ - (\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})},(\boldsymbol{x}^\natural)^{\phi(\boldsymbol{x})}\rangle - 2\|\boldsymbol{x}^\natural\|^2 - 2r'^2\|\boldsymbol{x}^\natural\|^4\Big) - \frac{C_2}{8}\delta_0(\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\| + \|\boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|)^2\Bigg)\\
&\geq 8c\Bigg(8r'^2\|\boldsymbol{x}^\natural\|^4 - 16\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| + 4\|\boldsymbol{x}^\natural\|^2 - 8\|\boldsymbol{x} - e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| - 2r'^2\|\boldsymbol{x}^\natural\|^4 - 2\|\boldsymbol{x}^\natural\|^2\\
&\quad - \frac{C_2}{8}\delta_0\left(4\|\boldsymbol{x}^\natural\|^2 + 4\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|^2\right)\Bigg)\\
&= 8c\Bigg(6r'^2\|\boldsymbol{x}^\natural\|^4 - 24\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| + 2\|\boldsymbol{x}^\natural\|^2\\
&\quad - \frac{C_2}{8}\delta_0\left(4\|\boldsymbol{x}^\natural\|^2 + 4\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|^2\right)\Bigg)\\
&\geq 8c\left(2\|\boldsymbol{x}^\natural\|^2 - 24\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| - \frac{C_2}{8}\delta_0(4\|\boldsymbol{x}^\natural\|^2 + 4\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| + \|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|^2)\right).
\end{aligned}
$$

Assume $\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\| \leq \|\boldsymbol{x}^\natural\|$. If we have

$$
2\|\boldsymbol{x}^\natural\|^2 - 24\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| \geq \frac{C_2}{8}\delta_0(5\|\boldsymbol{x}^\natural\|^2 + 4\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\|),
$$

then we have $\boldsymbol{u}^T\boldsymbol{H}_f\boldsymbol{u} > 0$.

Let $\mathcal{G}$ be the set of global minimizers of f, define:

$$
\begin{aligned}
\mathcal{R}^3_{\delta_0} &= \left\{2\|\boldsymbol{x}^\natural\|^2 - 24\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\| \geq \frac{C_2}{8}\delta_0(5\|\boldsymbol{x}^\natural\|^2 + 4\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\|\|\boldsymbol{x}^\natural\|)\right\}\\
&\quad\cap\left\{\|\boldsymbol{x} - \boldsymbol{x}^\natural e^{i\phi(\boldsymbol{x})}\| \leq \|\boldsymbol{x}^\natural\|\right\}\\
&= \left\{\boldsymbol{x} \mid d(\boldsymbol{x},\mathcal{G}) \leq \min\left(1, \frac{2 - \frac{5}{8}C_2\delta_0}{24 + \frac{4}{8}C_2\delta_0}\right)\right\}\\
&= \left\{\boldsymbol{x} \mid d(\boldsymbol{x},\mathcal{G}) \leq \frac{16 - 5C_2\delta_0}{192 + 4C_2\delta_0}\right\}. \qquad (30)
\end{aligned}
$$

For δ_0 small enough, note that $R^3_{\delta_0}$ strictly contains the set of global minimizers $\boldsymbol{x}^\natural e^{i\theta}$.

Assume that there is a local minimum at $\boldsymbol{x}^+ \in \mathbb{R}^3_{\delta_0}$. Write:

$$\gamma(t) = t\boldsymbol{x}^{+} + (1-t)\cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural+} + (1-t)\sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural-}.$$

Consider $h(t) = f(\gamma(t))$, we have (with $\boldsymbol{x}^{+} = \boldsymbol{v} + r\cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural+} + r\sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural-}$):

$$\begin{aligned} h''(t) &= \left(\frac{d\gamma(t)}{dt}\right)^{T} \boldsymbol{H}_f(\gamma(t)) \left(\frac{d\gamma(t)}{dt}\right) + \nabla f(\gamma(t)) \cdot \frac{d^2(\gamma(t))}{dt^2} \\ &= \left(\frac{d\gamma(t)}{dt}\right)^{T} \boldsymbol{H}_f(\gamma(t)) \left(\frac{d\gamma(t)}{dt}\right). \end{aligned} \tag{31}$$

Notice that $\frac{d\gamma(t)}{dt}$ can be written as $r'\left(\cos(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural+} + \sin(\phi(\boldsymbol{x}))\boldsymbol{x}^{\natural-}\right) + \boldsymbol{v}'$, therefore $h''(t) > 0$, so h is strictly convex. As $h(0)$ is a global min of h, $h(1)$ cannot be a critical of h, and therefore $\boldsymbol{x}$ cannot be a critical point of f if $\boldsymbol{x} \neq \boldsymbol{x}^{\natural}e^{i\theta}$ for some θ. □

A.2.7 Proof of Proposition 8

Proof. Recall that:

$$\nabla g(\boldsymbol{x}^{+}) = 8c\left(4\boldsymbol{x}^{+}\|\boldsymbol{x}^{+}\|^2 - 2\boldsymbol{x}^{+}\|\boldsymbol{x}^{\natural+}\|^2 - 2\boldsymbol{x}^{\natural+}\langle \boldsymbol{x}^{+}, \boldsymbol{x}^{\natural+}\rangle - 2\boldsymbol{x}^{\natural-}\langle \boldsymbol{x}^{\natural-}, \boldsymbol{x}^{+}\rangle\right).$$

We have:

$$\|\nabla g(\boldsymbol{x}^{+})\|^2 \underset{\|\boldsymbol{x}\|\to+\infty}{\sim} (32c)^2\|\boldsymbol{x}\|^6.$$

Therefore from (8), if we take δ_0 such that:

$$(32c)^2 > (C_1\delta_0 c)^2 \iff \delta_0 < \frac{32}{C_1}.$$

Then $\mathcal{R}^1_{\delta_0}$ contains all $\boldsymbol{x}$ for $\|\boldsymbol{x}\|$ large enough. Then there exists some K compact such that for all $\delta_0 \leq \frac{16}{C_1}$, we have $\boldsymbol{x} \in \mathcal{R}^1_{\delta_0}$ for all $\boldsymbol{x} \notin K$. Let E be an open set containing the critical points $\mathcal{C}$ of g. $K \backslash E$ is a compact also, therefore the function $\frac{\|\nabla g(\boldsymbol{x}^{+})\|}{C_1 c(\|\boldsymbol{x}\|+\|\boldsymbol{x}^{\natural}\|)^3}$ attains its minimum on $K\backslash E$. Take δ_0 strictly below this minimum, and from (8), $\mathcal{R}^1_{\delta_0}$ contains all $K\backslash E$.
Now let's consider a critical point of $g(\boldsymbol{x}^{+})$. We can see that:

Assume $\boldsymbol{x}^{+}$ is a critical point.

• If $\|\boldsymbol{x}^{+}\|^2 = \frac{1}{2}\|\boldsymbol{x}^{\natural+}\|^2$, then we must have $\langle \boldsymbol{x}^{+}, \boldsymbol{x}^{\natural+}\rangle = 0$ and $\langle \boldsymbol{x}^{+}, \boldsymbol{x}^{\natural-}\rangle = 0$ as $\boldsymbol{x}^{\natural+}$ is ortogonal to $\boldsymbol{x}^{\natural-}$, which means that $\|\boldsymbol{x}\|^2 = \frac{1}{2}\|\boldsymbol{x}^{\natural}\|^2$ and that $|\langle \boldsymbol{x}, \boldsymbol{x}^{\natural}\rangle|^2 = \langle \boldsymbol{x}^{+}, \boldsymbol{x}^{\natural+}\rangle^2 + \langle \boldsymbol{x}^{+}, \boldsymbol{x}^{\natural-}\rangle^2 = 0$. In that case using Proposition 6, $\boldsymbol{x}$ is strictly in $\mathcal{R}^2_{\delta_0}$. for δ_0 small enough.

• If $\|\boldsymbol{x}^{+}\|^2 \neq \frac{1}{2}\|\boldsymbol{x}^{\natural+}\|^2$, then $\boldsymbol{x}^{+}$ must be in $\mathrm{Span}\{\boldsymbol{x}^{\natural+}, \boldsymbol{x}^{\natural-}\}$, lets write $\boldsymbol{x}^{+} = \mu\boldsymbol{x}^{\natural+} + \nu\boldsymbol{x}^{\natural-}$. We must have:

$$\begin{cases} 4\mu(\mu^2+\nu^2) - 4\mu = 0 \\ 4\nu(\mu^2+\nu^2) - 4\nu = 0 \end{cases}$$

which gives either $\mu^2 + \nu^2 = 1$, or $\mu = \nu = 0$. If $\mu = \nu = 0$, then $\boldsymbol{x} = 0$, which is strictly in $\mathcal{R}^2_{\delta_0}$ for δ_0 small enough. If $\mu^2 + \nu^2 = 1$, then $\boldsymbol{x} = e^{i\theta}\boldsymbol{x}^{\natural}$, which is strictly in $\mathcal{R}^3_{\delta_0}$ for δ_0 small enough.

Therefore the set of critical points $\mathcal{C}$ is in $E := \mathrm{Int}(\mathcal{R}^2_{\delta_0} \cup \mathcal{R}^3_{\delta_0})$, the interior of $\mathcal{R}^2_{\delta_0} \cup \mathcal{R}^3_{\delta_0}$ for δ_0 small enough, and from the previous argument we must have for δ_0 small enough:

$$\mathcal{R}^1_{\delta_0} \cup \mathcal{R}^2_{\delta_0} \cup \mathcal{R}^3_{\delta_0} = \mathbb{R}^{2n}.$$

□

A.3 Proof of global convergence

A.3.1 Proof of Proposition 9

Proof. Since $\boldsymbol{A}_i$ is real symmetric and positive semidefinite for all $i = 1, \ldots, m$, by orthogonal decomposition, we can write $\boldsymbol{A}_i = \sum_{j=1}^{r} \lambda_{ij} \boldsymbol{v}_{ij} \boldsymbol{v}_{ij}^{\mathrm{T}}$, where $(\boldsymbol{v}_{ij})_{j=1}^{r}$ are orthonormal, for some $1 \le r \le n$ and $\lambda_{ij} > 0$ for all $j = 1, \ldots, r$. Define

$$V := \mathrm{Span}\{\boldsymbol{v}_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, r\}.$$

Notice that

$$\nabla f(\boldsymbol{x}^+) = 4 \sum_{i=1}^{m} \left(\langle \boldsymbol{A}_i \boldsymbol{x}^+, \boldsymbol{x}^+ \rangle - b_i \right) \boldsymbol{A}_i \boldsymbol{x}^+ = 4 \sum_{i=1}^{m} \sum_{j=1}^{r} \lambda_{ij} \langle \boldsymbol{v}_{ij}, \boldsymbol{x}^+ \rangle \left(\langle \boldsymbol{A}_i \boldsymbol{x}^+, \boldsymbol{x}^+ \rangle - b_i \right) \boldsymbol{v}_{ij} \in V.$$

Therefore, $\nabla f(\boldsymbol{x}^+) \in V$ for all $\boldsymbol{x}^+ \in \mathbb{R}^{2n}$. Denote $V^\perp$ as the orthogonal complement of the subspace V, then for any given initial point $\boldsymbol{x}_0^+$, the solution $\boldsymbol{x}^+(\cdot)$ to (13) can be decomposed as $\boldsymbol{x}^+(t) = \boldsymbol{x}_V^+(t) + \boldsymbol{x}_{V^\perp}^+(t)$, where $\boldsymbol{x}_V^+(t) \in V$ and $\boldsymbol{x}_{V^\perp}^+(t) \in V^\perp$ for all $t \ge 0$. Note that $(\boldsymbol{x}^+)'(t) = -\nabla f(\boldsymbol{x}^+(t)) \in V$, thus $\boldsymbol{x}_{V^\perp}^+(t) \equiv \boldsymbol{x}_{V^\perp}^+(0)$ and we write $\boldsymbol{x}^+(t) = \boldsymbol{x}_V^+(t) + \boldsymbol{x}_{V^\perp}^+(0)$ for all $t \ge 0$. Since $f(\boldsymbol{x}^+)$ is a decreasing function over $t \ge 0$,

$$\begin{aligned}\sum_{j=1}^{r} \lambda_{ij} \langle \boldsymbol{v}_{ij}, \boldsymbol{x}^+(t) \rangle^2 = |\langle \boldsymbol{A}_i \boldsymbol{x}^+(t), \boldsymbol{x}^+(t) \rangle| &\le \sqrt{2(\langle \boldsymbol{A}_i \boldsymbol{x}^+(t), \boldsymbol{x}^+(t) \rangle - b_i)^2 + 2b_i^2} \\ &\le \sqrt{2 f(\boldsymbol{x}^+(t)) + 2b_i^2} \le \sqrt{2 f(\boldsymbol{x}_0^+) + 2b_i^2}.\end{aligned}$$

Recall that $\lambda_{ij} > 0$ for all $j = 1, \ldots, r$, hence $\langle \boldsymbol{v}_{ij}, \boldsymbol{x}^+(t) \rangle$ is bounded over $t \ge 0$ and so is $\langle \boldsymbol{v}_{ij}, \boldsymbol{x}_V^+(t) \rangle$. As $\mathrm{Span}\{\boldsymbol{v}_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, r\} = V$, we can extract a basis of vectors $\boldsymbol{v}_{ij}$ to form a basis of V, and denote this basis as $\{\boldsymbol{u}_\ell : \ell = 1, \ldots, d\}$. Then one can write $\boldsymbol{x}_V^+(t) = \sum_{\ell=1}^{d} \zeta_\ell(t) \boldsymbol{u}_\ell$. Notice that for each $\ell = 1, \ldots, d$, there must exist (i_ℓ, j_ℓ) such that $\boldsymbol{v}_{i_\ell j_\ell} = \boldsymbol{u}_\ell$. Thus,

$$\|\boldsymbol{x}_V^+(t)\|^2 = \sum_{\ell=1}^{d} \zeta_\ell(t)^2 = \sum_{\ell=1}^{d} \langle \boldsymbol{u}_\ell, \boldsymbol{x}_V^+(t) \rangle^2 = \sum_{\ell=1}^{d} \langle \boldsymbol{v}_{i_\ell j_\ell}, \boldsymbol{x}_V^+(t) \rangle^2$$

is bounded over $t \ge 0$. Finally, $\boldsymbol{x}^+(t) = \boldsymbol{x}_V^+(t) + \boldsymbol{x}_{V^\perp}^+(0)$ is bounded over $t \ge 0$. $\square$

A.4 Proof of concentration results

A.4.1 Proof of Proposition 10

Proof. We can just look at the expectation entrywise. We can notice that:

$$\begin{aligned}\mathbb{E}\left(\left(\sum_i (\boldsymbol{a}_i^+)^{\otimes 4} \right)_{i_1, i_2, i_3, i_4} \right) &= m\sigma^4 \mathbb{E}\left(\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)^{\otimes 4} \right)_{i_1, i_2, i_3, i_4} \right) \\ &= m\sigma^4 \mathbb{E}\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_1} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_2} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_3} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_4} \right). \end{aligned} \tag{32}$$

All entries of $\boldsymbol{a}_1^+$ are independent and centered, so if one index among i_1, i_2, i_3, i_4 is different from the three others, the expectation is zero. If not, we can have either $i_1 = i_2 \ne i_3 = i_4$ or $i_1 = i_3 \ne i_2 = i_4$ or $i_1 = i_4 \ne i_2 = i_3$ or $i_1 = i_2 = i_3 = i_4$, in the first three cases:

$$\mathbb{E}\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_1} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_2} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_3} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_4} \right) = \mathbb{E}\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_1^2 \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_2^2 \right) = 1$$

and in the last case:

$$\mathbb{E}\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_1} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_2} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_3} \left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_{i_4} \right) = \mathbb{E}\left(\left(\frac{\boldsymbol{a}_1^+}{\sigma} \right)_1^4 \right) = 3$$

which achieves the proof. $\square$

A.4.2 Proof of Proposition 11

Proof. As define in Even & Massoulié (2021), we denote for some tensor $\mathbf{R}$

$$\|\mathbf{R}\|_{\text{op}} = \sup_{\|\boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3 \otimes \boldsymbol{u}_4\|=1} \left| \sum_{i_1,i_2,i_3,i_4} \mathbf{R}_{i_1,i_2,i_3,i_4} (\boldsymbol{u}_1)_{i_1} (\boldsymbol{u}_2)_{i_2} (\boldsymbol{u}_3)_{i_3} (\boldsymbol{u}_4)_{i_4} \right|.$$

According to Even & Massoulié (2021), as $\frac{\boldsymbol{a}_i^+}{\sigma}$ follow $\mathcal{N}(\mathbf{0}, \boldsymbol{I}_n)$ and $\mathbb{E}(\mathbf{T}) = \mathbf{S}$, we have for some absolute constant $C > 0$,

$$\mathbb{P}\left(\|\mathbf{T} - \mathbf{S}\|_{\text{op}} \geq C \sqrt{\frac{n + \log n + \beta m}{m}} \right) \leq e^{-\beta m}.$$

For $m \geq Kn$, as $\log n \leq n$, we have,

$$C \sqrt{\frac{n + \log n + \beta m}{m}} \leq C \sqrt{\frac{2}{K} + \beta}.$$

If we take K large enough and $\beta > 0$ small enough (e.g., $\beta < \delta_0^2/(2C^2)$ and $K > 4C/\delta_0^2$), then the right handside is strictly lower than δ_0. For such $\beta, K > 0$,

$$\mathbb{P}(\|\mathbf{T} - \mathbf{S}\|_{\text{op}} < \delta_0) \geq 1 - e^{-\beta m}.$$

□