Supplementary material to the article: The medial axis of closed bounded sets is Lipschitz stable with respect to the Hausdorff distance under ambient diffeomorphisms

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A Proofs

Proof of Lemma 3.3 We first note that the statement is empty if $v = 0$. Let $v \neq 0$, and consider the nested family of balls $B(p + \lambda' v, \lambda' |v|)$, parametrized by $\lambda' > 0$.

Because $\pi_{\mathcal{S}}(p + \lambda v) \neq p$, the (closed) ball $B(p + \lambda v, \lambda |v|)$ contains a point $q \in \mathcal{S}$ other than p. Since the balls $B(p + \lambda' v, \lambda' |v|)$ are nested, the point q lies inside every ball $B(p + \lambda' v, \lambda' |v|)$ with $\lambda' \geq \lambda$. Moreover, q lies in the interior of $B(p + \lambda' v, \lambda' |v|)$ for $\lambda' > \lambda$. Hence, for every $\lambda' \geq \lambda$, we have that $\pi_{\mathcal{S}}(p + \lambda' v) \neq \{p\}$ and for $\lambda' > \lambda$, that $p \notin \pi_{\mathcal{S}}(p + \lambda' v)$. □

Proof of Lemma 3.7 Let $Q = \pi_{\mathcal{S}}(x)$ be the subset of $\mathcal{S}$ that is closest to x. Because $x \in \mathrm{ax}(\mathcal{S})$, Q contains at least two points, one of them being p. We write $\lambda = |x - p|$. Since $\mathcal{S}$ and $\mathrm{ax}(\mathcal{S})$ are disjoint, $\lambda > 0$, and thus we can define $u = \frac{x-p}{\lambda}$.

Since the interior of the ball $B(x, \lambda)$ does not intersect $\mathcal{S}$, it in particular does not intersect $\mathrm{Tan}(p, \mathcal{S})$ and thus $B(x, \lambda)$ is weakly tangent at p by Remark 3.2.Let us now consider the nested family $B(p + \lambda' u, \lambda')$ of weakly tangent balls at p. By definition, $\partial B(x, \lambda) \cap \mathcal{S} = Q$ and therefore $B(p + \lambda' u, \lambda') \cap \mathcal{S} = p$ for $\lambda' < \lambda$. At the same time, Lemma 3.3 yields that for $\lambda' > \lambda$, $p \notin \pi_{\mathcal{S}}(p + \lambda' u)$. Hence the projection range in direction u equals $d(p, u, \pi_{\mathcal{S}}) = \lambda$ and we obtain $\pi_{\mathrm{ax},\mathcal{S}}(p, u) = p + \lambda u = x$ directly from Definition 3.6 . The fact that $\pi_{\mathrm{ax},\mathcal{S}}\left(\mathrm{UBP}(\mathcal{S})\right) \subseteq \overline{\mathrm{ax}(\mathcal{S})}$ is due to Lemma 2.3 □

The next two claims are used in the proof of Theorem 3.9:

Claim A.1

$$u' = \frac{(D_p F^t)^{-1}(u)}{|(D_p F^t)^{-1}(u)|}, \tag{8}$$

where $D_p F^t$ is the transpose matrix (or the adjoint operator) of DF at the point p, defined by

$$\forall v_1, v_2, \qquad \langle v_1, D_p F(v_2) \rangle = \langle D_p F^t(v_1), v_2 \rangle.$$

Proof

$$\begin{aligned} w \in D_p F(u^{\perp}) &\iff \langle D_p F^{-1}(w), u \rangle = 0 \\ &\iff \langle w, (D_p F^{-1})^t u \rangle = 0 \\ &\iff \langle w, u' \rangle = 0 \\ &\iff w \in u'^{\perp}, \end{aligned}$$

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and thus

$$D_pF(u^\perp) = u'^\perp. \tag{9}$$

In other words, we have shown that u' is orthogonal to $D_pF(u^\perp) = D_pF(T)$.

Because

$$\langle D_pF(u), (D_pF^{-1})^t(u)\rangle = \langle D_pF^{-1}(D_pF(u)), u\rangle = \langle u, u\rangle > 0,$$

we deduce that $\langle D_pF(u), u'\rangle > 0$. This is in turn equivalent to u' pointing towards the interior of $F(B(c,\rho))$. □

Claim A.2 *Let* $\|D_pF - \mathrm{Id}\,\| \le \varepsilon < 1$. *Then the angle* $\angle u, u'$ *between the vectors* u *and* u' *satisfies*

$$\cos\angle u, u' \ge \sqrt{1-\varepsilon^2}.$$

Proof We first show that $\angle u, u' < \pi/2$. Indeed, define the vector w as

$$w = (D_pF^t)^{-1}(u),$$

that is, the vector satisfying $u = D_pF^t(w)$. Then $u' = \frac{w}{|w|}$ (see equation (8)), and

$$\begin{aligned} |w|\langle u, u'\rangle &= \langle u, w\rangle = \langle D_pF^t w, w\rangle \\ &= \langle w, D_pFw\rangle = |w|^2 + \langle w, (D_pF - \mathrm{Id})w\rangle \\ &\ge |w|^2 - |w|^2\,\|D_pF - \mathrm{Id}\,\| \\ &> 0. \qquad\qquad \text{(because, by assumption, } \|DF_p - \mathrm{Id}\,\| < 1) \end{aligned}$$

Thus, $\langle u, u'\rangle > 0$, and therefore $\angle u, u' < \pi/2$.

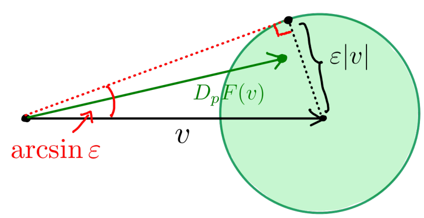

Figure 6: Since $\|v - D_pF(v)\| \le \varepsilon\,|v|$, the vector $D_pF(v)$ lies in the green ball $B(v, \varepsilon\,|v|)$. Since $\varepsilon < 1$, the angle between v and $D_pF(v)$ is upper-bounded by $\arcsin\varepsilon < \pi/2$.

Furthermore, consider a vector $v \in u^\perp$. Since $\|v - D_pF(v)\| \le \|D_pF - \mathrm{Id}\,\||v| \le \varepsilon|v|$, the angle between v and $D_pF(v)$ is upper-bounded by $\arcsin\varepsilon < \pi/2$, as illustrated in Figure 6. This yields a bound on the angle between the tangent spaces $u^\perp$ and $D_pF(u^\perp)$:

$$\sin\angle u^\perp, D_pF(u^\perp) = \sin \sup_{v\in u^\perp, w\in D_pF(u^\perp)} \angle v, w \le \varepsilon. \tag{10}$$

Using (9) and (10) we deduce that:

$$\sin\angle u, u' = \sin\angle u^\perp, u'^\perp \le \varepsilon.$$

Finally, since $\angle u, u' < \pi/2$, $\cos\angle u, u' \ge \sqrt{1-\varepsilon^2}$. This concludes the proof. □

Proof of Theorem 3.9 We first derive the bounds for the radius ρ'. As the first step, we apply Theorem 2.6to the boundary sphere $S(c,\rho)$ of the maximal empty weakly tangent ball $B(c,\rho)$. In particular, we can choose the constant s in Theorem 2.6arbitrarily large, and the constant t arbitrarily close to the reach $\mathrm{rch}(S(c,\rho)) = \rho$, to obtain:

$$\mathrm{rch}\left(F(S(c,\rho))\right) \ge \frac{1}{\left(\frac{L_F}{\rho} + L_{DF}\right)(L_F)^2} = \frac{\rho}{(L_F)^3 + \rho L_{DF}(L_F)^2} =: \rho_1.$$

This means that no open ball of radius ρ_1 tangent to the set $F(S(c,\rho))$ actually intersects $F(S(c,\rho))$. In addition, since the set $F(B(c,\rho))$ does not contain any points of $F(\mathcal{S})$ in its interior, no ball of radius ρ_1 that is tangent to $F(S(c,\rho))$ and whose centre lies inside $F(S(c,\rho))$ contains any point of $F(\mathcal{S})$.

The unit vector $u' \in DF_p(T)^\perp$ (defined in (8)) is defined such that the point $F(p) + \rho_1 u'$ lies inside the distorted ball $F(B(c,\rho))$. Due to the above observation, the ball $B(F(p) + \rho_1 u', \rho_1)$ is weakly tangent to $F(\mathcal{S})$ at $F(p)$ and contains no points of $F(\mathcal{S})$ in its interior.

Let us now consider the weakly tangent ball $B(F(p) + \rho'' u', \rho'')$, whose radius ρ'' satisfies

$$\rho'' > \frac{(L_F)^3 \rho}{1 - \rho L_{DF} (L_F)^2} =: \rho_2.$$

To shorten up the notation, we set

$$F(p) + \rho'' u' =: c''.$$

To derive a contradiction, we assume that $B(c'', \rho'')$ is maximal empty. This is equivalent to assuming that $\mathrm{int} B(c'', \rho'') \cap F(\mathcal{S}) = \emptyset$, and thus $B(c'', \rho'')$ is a maximal empty weakly tangent ball to $F(p)$. Similarly to the beginning of the proof, we now apply Theorem 2.6to the map F^{-1} and the boundary sphere $S(c'', \rho'') = \partial B(c'', \rho'')$. As a result, the reach of $F^{-1}(S(c'', \rho''))$ is at least

$$\mathrm{rch}\left(F^{-1}(S(c'', \rho''))\right) \geq \frac{\frac{(L_F)^3 \rho}{1 - \rho L_{DF} (L_F)^2}}{(L_F)^3 + \frac{(L_F)^3 \rho}{1 - \rho L_{DF} (L_F)^2} L_{DF} (L_F)^2} = \rho.$$

We conclude that there exists a ball that is tangent to the set $F^{-1}(S(c'', \rho''))$ at $F^{-1}(F(p)) = p$, whose radius is larger than ρ, and that does not contain any points of $\mathcal{S}$ in its interior. This contradicts the fact that the ball $B(c,\rho)$ is maximal empty, and completes the proof of the first part of the statement.

We now prove the bounds on the distortion of the map $\pi_{\mathrm{ax},\mathcal{S}}$. Let $\rho' \in [\rho_1, \rho_2]$ be the radius of the maximal empty weakly tangent ball at $F(p)$ in the direction u', and write $c' := F(p) + \rho' u'$ for its centre. We stress that, as a consequence of Lemma 2.3, $c' \in \overline{\mathrm{ax}(F(\mathcal{S}))}$, but it is not necessarily true that $c' \in \mathrm{ax}(F(\mathcal{S}))$.

The goal is to estimate the distance between the two centres $c = \pi_{\mathrm{ax},\mathcal{S}}(p, u)$ and $c' = \pi_{\mathrm{ax},\mathcal{S}}(F(p), u')$. Indeed, since $c - p = \rho u$ and $c' - F(p) = \rho' u'$,

$$\begin{aligned} |c - c'| &= |c - p + p - F(p) + F(p) - c'| = |\rho u + p - F(p) - \rho' c'| \\ &\leq |\rho u - \rho' u'| + |F(p) - p|. \end{aligned}$$

Due to the assumptions of the theorem,$|F(p) - p| \leq \varepsilon_1$. Furthermore, thanks to Claim A.2,

$$|\rho u - \rho' u'|^2 = \rho^2 + (\rho')^2 - 2\rho\rho' \cos\angle u, u' \leq \rho^2 + (\rho')^2 - 2\rho\rho' \sqrt{1 - (\varepsilon_2)^2}.$$

Recalling that $\rho' \in [\rho_1, \rho_2]$, we thus obtain

$$\begin{aligned} |\rho u - \rho' u'| &\leq \max\left(\sqrt{\rho^2 + (\rho_1)^2 - 2\rho\,\rho_1 \cos(\arcsin(\varepsilon_2))}, \sqrt{\rho^2 + (\rho_2)^2 - 2\rho\,\rho_2 \cos(\arcsin(\varepsilon_2))}\right) \\ &= \max\left(\sqrt{\rho^2 + (\rho_1)^2 - 2\rho\,\rho_1 \sqrt{1 - (\varepsilon_2)^2}}, \sqrt{\rho^2 + (\rho_2)^2 - 2\rho\,\rho_2 \sqrt{1 - (\varepsilon_2)^2}}\right). \end{aligned}$$

Hence,

$$|c - c'| \leq \max\left(\sqrt{\rho^2 + (\rho_1)^2 - 2\rho\,\rho_1 \sqrt{1 - (\varepsilon_2)^2}}, \sqrt{\rho^2 + (\rho_2)^2 - 2\rho\,\rho_2 \sqrt{1 - (\varepsilon_2)^2}}\right) + \varepsilon_1. \tag{11}$$

As the last step, we simplify the expression (11) (at the cost of weakening the bounds). For this, we assume that $\rho L_{DF} (L_F)^2 \leq 1/2$, so that

$$\rho_1 = \frac{\rho}{(L_F)^3 + \rho L_{DF} (L_F)^2} \geq \frac{\rho}{(L_F)^3}\left(1 - \rho \frac{L_{DF}}{L_F}\right), \tag{12}$$

$$\rho_2 = \frac{(L_F)^3 \rho}{1 - \rho L_{DF} (L_F)^2} \leq \rho (L_F)^3 \left(1 + 2\rho L_{DF} (L_F)^2\right), \tag{13}$$

where we used that, for $x \in [0, 1/2]$, $\frac{1}{1+x} \geq 1 - x$ and $\frac{1}{1-x} \leq 1 + 2x$. We note that both ρ_1 and ρ_2 tend to ρ as L_F tends to 1 and L_{DF} tends to 0. We now consider $|\rho_1 - \rho|$ and $|\rho_2 - \rho|$, and claim that

$$|\rho_1 - \rho|, |\rho_2 - \rho| \leq \rho (L_F)^3 \left(1 + 2\rho L_{DF} (L_F)^2\right) - \rho.$$

For $|\rho_2 - \rho| = \rho_2 - \rho$, the claim holds thanks to (13). To establish this for $|\rho_1 - \rho|$ requires a small calculation:

$$\begin{aligned}
|\rho_1 - \rho| = \rho - \rho_1 &\leq \rho - \frac{\rho}{(L_F)^3}\left(1 - \rho\frac{L_{DF}}{L_F}\right) && \text{(due to (12))}\\
&\leq \rho (L_F)^3 \left(1 + 2\rho L_{DF}(L_F)^2\right) - \rho && \text{(assuming the claim holds)}\\
2\rho &\leq \rho (L_F)^3 \left(1 + 2\rho L_{DF}(L_F)^2\right) + \frac{\rho}{(L_F)^3}\left(1 - \rho\frac{L_{DF}}{L_F}\right) && \\
& && \text{(reformulating the previous inequality)}\\
2 &\leq (L_F)^3 + \frac{1}{(L_F)^3} + 2\rho L_{DF}(L_F)^5 - \rho\frac{L_{DF}}{(L_F)^4},
\end{aligned}$$

where the final inequality holds because $x^3 + x^{-3} \geq 2$, and $2x^5 - x^{-4} \geq 0$, for $x \geq 1$. We now consider the function

$$\begin{aligned}
f(\delta) &= \rho^2 + \rho^2(1+\delta)^2 - 2\rho^2 (1+\delta)\sqrt{1 - (\varepsilon_2)^2}\\
&= \rho^2 \left(\delta^2 + 2\left(1 - \sqrt{1-(\varepsilon_2)^2}\right)\delta + 2\left(1 - \sqrt{1-(\varepsilon_2)^2}\right)\right).
\end{aligned}$$

The function f is a second order polynomial in δ and because all coefficients are positive, the maximum of f on the interval $[-\delta_m, \delta_m]$ is a attained at δ_m, that is,

$$\sup_{\delta \in [-\delta_m, \delta_m]} f(\delta) = f(\delta_m).$$

By combining these results, we see that

$$\begin{aligned}
&|c - c'|\\
&\leq \sqrt{f\left((L_F)^3 \left(1 + 2\rho L_{DF}(L_F)^2\right) - 1\right)} + \varepsilon_1\\
&= \sqrt{\rho^2 + \left(\rho(L_F)^3 \left(1 + 2\rho L_{DF}(L_F)^2\right)\right)^2 - 2\rho\left(\rho(L_F)^3 \left(1 + 2\rho L_{DF}(L_F)^2\right)\right)\sqrt{1-(\varepsilon_2)^2}}\\
&\quad + \varepsilon_1\\
&= \rho\sqrt{1 + (L_F)^6 \left(1 + 2\rho L_{DF}(L_F)^2\right)^2 - 2(L_F)^3\left(1 + 2\rho L_{DF}(L_F)^2\right)\sqrt{1-(\varepsilon_2)^2}} + \varepsilon_1.
\end{aligned}$$

Because both $f(\delta)$ and the bound (13) are monotone in ρ, and ρ is bounded by the radius r of the bounding sphere $S(r)$, we conclude that

$$\begin{aligned}
&|c - c'|\\
&\leq 2r\sqrt{1 + (L_F)^6 \left(1 + 4r L_{DF}(L_F)^2\right)^2 - 2(L_F)^3\left(1 + 4r L_{DF}(L_F)^2\right)\sqrt{1-(\varepsilon_2)^2}} + \varepsilon_1. \qquad (14)
\end{aligned}$$

For every point c in $\mathrm{ax}(\mathcal{S})$ we have found a point c' in $\overline{\mathrm{ax}(F(\mathcal{S}))}$ whose distance is bounded by (14), and therefore the one-sided Hausdorff distance between the two medial axes $\mathrm{ax}(\mathcal{S})$ and $\overline{\mathrm{ax}(F(\mathcal{S}))}$ is bounded by the same quantity. Because the symmetrical formulation of the statement, the same bound holds for the Hausdorff distance. ☐

Proof of Theorem 4.1 We denote $L_\varphi = \mathrm{Lip}(\phi)$. Expressions (5), (6) and (7) of the main article yield:

$$L_\varphi \leq r\varepsilon, \qquad L_{DF} = \varepsilon, \qquad L_F \leq 1 + L_\varphi \leq 1 + r\varepsilon, \qquad \varepsilon_1 \leq r^2\varepsilon, \qquad \varepsilon_2 \leq r\varepsilon. \qquad (15)$$

We deduce

$$r\varepsilon \leq 1/4 \Longrightarrow r\varepsilon(1 + r\varepsilon)^2 \leq 1/2 \Longrightarrow r L_{DF}(L_F)^2 \leq 1/2.$$

Thus, the conditions of Theorem 3.9 are satisfied. Next, we reformulate the inequality (3) of Theorem 3.9.The expression E under the square root at the right hand side of this inequality is:

$$E = 1 + (L_F)^6 \left(1 + 4rL_{DF}(L_F)^2\right)^2 - 2(L_F)^3 \left(1 + 4rL_{DF}(L_F)^2\right) \sqrt{1 - (\varepsilon_2)^2}.$$

By replacing L_F by $1 + L_\varphi$ in E, the constants, as well as the degree-one terms in L_φ, rL_{DF}, and ε_2, cancel out. More precisely,

$$E = 16r^2 L_{DF}^2 + r^2\varepsilon_2^2 + 24rL_\varphi L_{DF} + 9L_\varphi^2 + \mathcal{O}(|(rL_{DF}, L_\varphi, \varepsilon_2)|^3). \tag{16}$$

Finally, by substituting inequalities (15) into (16), we obtain

$$E \leq 50r^2\varepsilon^2 + \mathcal{O}\left(r^3\varepsilon^3\right),$$

and

$$d_H(\mathrm{ax}(\mathcal{S}), \mathrm{ax}(F(\mathcal{S}))) \leq \left(1 + \sqrt{50}\right) r^2\varepsilon + \mathcal{O}\left(r^3\varepsilon^2\right).$$

□

B Federer's tubular neighbourhood lemma

Lemma B.1 (Federer's tubular neighbourhood lemma, Theorem 4.8 (12) of [19]) *] Let* $p \in \mathcal{S}$ *and* $\mathrm{lfs}(p) > 0$*. The generalized normal space to* $\mathcal{S}$ *at* p *is characterized by the following property: For any* $\rho \in \mathbb{R}$ *satisfying* $0 < \rho < \mathrm{lfs}(p)$*,*

$$\mathrm{Nor}(p, \mathcal{S}) = \{\lambda v \in \mathbb{R}^d \mid \lambda \geq 0, |v| = \rho, \pi_{\mathcal{S}}(p + v) = \{p\}\}.$$

In particular, $\mathrm{Nor}(p, \mathcal{S})$ *is a convex cone. The generalized tangent space* $\mathrm{Tan}(p, \mathcal{S})$ *is the convex cone dual to* $\mathrm{Nor}(p, \mathcal{S})$*.*