A Robust Translation Synchronization Algorithm

Supplementary Material

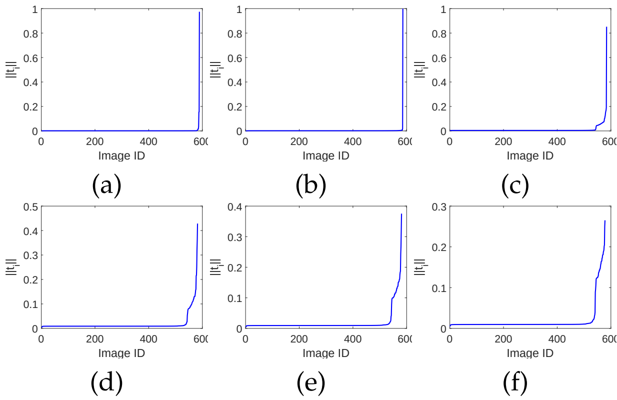

Figure 5. This figure shows $\|\boldsymbol{t}_i\|$ sorted by their magnitudes at each iteration on the Alamo dataset. (a) Iteration 1. (b) Iteration 2. (c) Iteration 3. (d) Iteration 4. (e) Iteration 5. (f) Iteration 6.

A. Details on Graph Pruning on Real Data

In this section, we provide more details on the pruning procedure which obtains a sub-graph to perform translation synchronization. Specifically, given the current sub-graph, which is initialized as the original graph, we perform translation synchronization with edge weights $w_{ij} = 1$. Let $\boldsymbol{t}_i$ be the resulting translations. If $\max \boldsymbol{t}_i \leq \delta$, we use the corresponding sub-graph to perform the alternating procedure of updating the edge weights and optimizing the image translations. Otherwise, we remove the image with the largest value in $\|\boldsymbol{t}_i\|$ and detect the maximum connected component within the remaining vertices to iterate this procedure. We set $\delta = \frac{10}{\sqrt{n}}$ in our experiments.

Figure 5 shows the iterative procedure on the Alamo dataset in 1DSFM, which removes x images. The numbers of images removed from all the datasets in 1DSFM are less than 8 images.

B. Spectral Translation Synchronization Properties

B.1. Proof of Proposition 1

Denote

$$E_{ij} = \boldsymbol{e}_{ij} \otimes I_3. \quad (15)$$

Introduce $\boldsymbol{v}_{ij,1}$ and $\boldsymbol{v}_{ij,2}$, so that $(\boldsymbol{v}_{ij}, \boldsymbol{v}_{ij,1}, \boldsymbol{v}_{ij,2})$ forms an orthonormal basis. Then we have

$$\begin{aligned} L &= \sum_{(i,j)\in\mathcal{E}} w_{ij} E_{ij}(I_3 - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)E_{ij}^T \\ &= \sum_{(i,j)\in\mathcal{E}} w_{ij} E_{ij}(\boldsymbol{v}_{ij,1}\boldsymbol{v}_{ij,1}^T + \boldsymbol{v}_{ij,2}\boldsymbol{v}_{ij,2}^T)E_{ij}^T \succeq 0. \end{aligned}$$

Consider any vector $\boldsymbol{c} \in \mathbb{R}^3$. Introduce $\boldsymbol{u} = \mathbf{1} \otimes \boldsymbol{c}$. The i-th block of $L\boldsymbol{u}$ is

$$\begin{aligned} &(L\boldsymbol{u})_i \\ =& \sum_{j\in\mathcal{N}_i} w_{ij}(I_3 - v_{ij}v_{ij}^T)\boldsymbol{u}_i + \sum_{j\in\mathcal{N}_i} -w_{ij}(I_3 - v_{ij}v_{ij}^T)\boldsymbol{u}_j \\ =& \sum_{j\in\mathcal{N}_i} w_{ij}(I_3 - v_{ij}v_{ij}^T)\boldsymbol{c} - \sum_{j\in\mathcal{N}_i} w_{ij}(I_3 - v_{ij}v_{ij}^T)\boldsymbol{c} = 0. \end{aligned}$$

Therefore, the first three eigenvalues of L are zero, and the eigenvectors are $\mathbf{1} \otimes I_3$. It is easy to check that this also applies to generalized eigen-values and eigen-vectors. □

B.2. Proof of Proposition 2

The i-th block of $L\boldsymbol{t}^{gt}$ is

$$\begin{aligned} &(L\boldsymbol{t}^{gt})_i \\ =& \sum_{j\in\mathcal{N}_i} w_{ij}(I_3 - v_{ij}^{gt}{v_{ij}^{gt}}^T)(\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}) \\ =& \sum_{j\in\mathcal{N}_i} w_{ij}\|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|(I_3 - v_{ij}^{gt}{v_{ij}^{gt}}^T)\boldsymbol{v}_{ij}^{gt} = 0. \end{aligned}$$

Therefore, $\boldsymbol{t}^{gt}$ is a fourth eigenvector L. If $\lambda_5 > 0$. Then $\boldsymbol{t}^{gt}$ is the unique fourth eigenvector.

If $\lambda_5 = 0$. Then there exists a different vector $\boldsymbol{u}_5 \neq \boldsymbol{t}^{gt}$ that is a slight pertubration of $\boldsymbol{t}^{gt}$, so that $L\boldsymbol{u}_5 = 0$. Note that

$$\begin{aligned} 0 &= \boldsymbol{u}_5^T L\boldsymbol{u}_5 \\ &= \sum_{(i,j)\in\mathcal{E}} \boldsymbol{u}_5^T E_{ij}(I_3 - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T)E_{ij}\boldsymbol{u}_5 \\ &= \sum_{(i,j)\in\mathcal{E}} \Big(\|\boldsymbol{u}_{5i} - \boldsymbol{u}_{5j}\|^2 - \big((\boldsymbol{u}_{5i} - \boldsymbol{u}_{5j})^T\boldsymbol{v}_{ij}^{gt}\big)^2\Big). \end{aligned}$$

Therefore, $\forall (i,j) \in \mathcal{E}$,

$$\|\boldsymbol{u}_{5i} - \boldsymbol{u}_{5j}\|^2 = \big((\boldsymbol{u}_{5i} - \boldsymbol{u}_{5j})^T\boldsymbol{v}_{ij}^{gt}\big)^2.$$

This means

$$\frac{\boldsymbol{u}_{5i}-\boldsymbol{u}_{5j}}{\|\boldsymbol{u}_{5i}-\boldsymbol{u}_{5j}\|}=s_{ij}\boldsymbol{v}_{ij}^{gt},$$

where $s_{ij}\in\{-1,1\}$. As $\boldsymbol{u}_5$ is a slight perturbation of $\boldsymbol{t}^{gt}$, we have $s_{ij}=1,\forall(i,j)\in\mathcal{E}$. This ends the proof. $\square$

B.3. Proof of Proposition 3

As $\lambda_4^{gt,2D}>0$, we can determine the $\boldsymbol{t}_i^{gt,2D}$ on the plane with normal $\boldsymbol{n}$, up to a global scale s and translation. In addition, we have $\boldsymbol{t}_i^{gt,2D}\neq\boldsymbol{t}_j^{gt,2D},\forall(i,j)\in\mathcal{E}$. Consider a spanning tree of $\mathcal{G}$. Without losing generality, we assume for the root r of this tree, we have $\boldsymbol{n}^T\boldsymbol{t}_r^{gt}=0$. It is easy to see that starting the root r, we can determine $\boldsymbol{n}^T\boldsymbol{t}_i^{gt}$ iteratively. This is because, $\boldsymbol{t}_i^{gt,2D}-\boldsymbol{t}_j^{gt,2D}$ is given, and $\boldsymbol{n}^T(\boldsymbol{t}_i^{gt}-\boldsymbol{t}_j^{gt})$ can be recovered from $\boldsymbol{v}_{ij}^{gt}$. $\square$

C. Topological Uniqueness Condition

In the following, we present a necessary uniqueness condition on the topological structure of $\mathcal{E}$. It is also a sufficient condition in a probabilistic sense.

Definition 1. *Introduce a rigidity matrix $A=(\mathbf{1}\otimes I_2;\boldsymbol{v};B)\in\mathbb{R}^{(|\mathcal{E}|+3)\times 2n}$ of a graph $\mathcal{G}$ with edge set $\mathcal{E}$. The elements of $\boldsymbol{v}$ are $v_{2i-1}=\cos(i\theta)$ and $v_{2i}=\sin(i\theta)$ where $\theta=\frac{2\pi}{n}$. The elements of B are zero except $B_{(i,j),2i-1}=\cos(\frac{(i+j)\theta}{2}$, $B_{(i,j),2i}=\sin(\frac{(i+j)\theta}{2}$,$B_{(i,j),2j-1}=-\cos(\frac{(i+j)\theta}{2}$, and $B_{(i,j),2j}=-\sin(\frac{(i+j)\theta}{2}$. We say $\mathcal{G}$ is rigid if $\mathrm{rank}(B)=2n$.*

Theorem 4. *$\mathcal{G}$ is rigid is a necessary condition for 2D uniqueness of translation synchronization. It is a sufficient condition in the sense that 2D uniqueness holds with probability 1 if we randomly sample $\boldsymbol{t}_i^{gt,2D}$.*

Proof: Consider an arbitrary set of n vertices $\boldsymbol{p}_i=(x_i,y_i)^T,1\le i\le n$ with edge set $\mathcal{E}$.

$$\boldsymbol{v}_{ij}=\frac{(x_i-x_j,y_i-y_j)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}},$$
$$\boldsymbol{v}_{ij}^{\perp}=\frac{(-(y_i-y_j),x_i-x_j)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}.$$

Define $J\in\mathbb{R}^{\mathcal{E}\times 2n}$, where $J((i,j),(2i-1):(2i))=\boldsymbol{v}_{ij}^{\perp}$ and $J((i,j),(2j-1):(2j))=-\boldsymbol{v}_{ij}^{\perp}$. When $x_i=\cos(\frac{\pi i}{n})$ and $y_i=\sin(\frac{\pi i}{n})$, it is clear that $\mathrm{rank}(J)=2n-3$ if and only if $\mathrm{rank}(B)=2n$. When $\mathrm{rank}(A)<2n$. Then $\mathrm{rank}(J)<2n-3$. This means the rank of $L^{gt}=J'*J$ is smaller than $2n-4$. In this case, $\lambda_i^{4,gt}=0$, and 2D uniqueness does not hold.

Suppose $\mathrm{rank}(B)=2n$. We show that $\mathrm{rank}(J)=2n-3$ when x_i,y_i are random samples. It is sufficient to show that for an edge subset $\mathcal{E}'\subset\mathcal{E}$ where $|\mathcal{E}'|=2n-3$ and $\mathrm{rank}(B')=2n$, the corresponding Jacobian matrix J has rank $\mathrm{rank}(J')=2n-3$. Denote

$$J'=\mathrm{diag}(\frac{1}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}})\overline{J}'$$

where

$$\overline{J}'((i,j),(2i-1):(2i))=(-(y_i-y_j),(x_i-x_j)),$$
$$\overline{J}'((i,j),(2j-1):(2j))=((y_i-y_j),-(x_i-x_j)).$$

Then it is clear that $\mathrm{rank}(J')=2n-3$ when $x_i=\cos(\frac{i\pi}{n})$ and $y_i=\sin(\frac{i\pi}{n})$. We show that $\mathrm{rank}(J')=2n-3$ when x_i and y_i are arbitrary. Suppose this is not true, there exists non-zero coefficients $c_i,1\le i\le 2n-3$, so that $\sum\limits_{i=1}^{2n-3}c_i\overline{J}'(i,:)=0$ for any x_i and y_i almost surely. As $\overline{J}'(i,:)$ are linear in x_i and y_i, it means that $\sum\limits_{i=1}^{2n-3}c_i\overline{J}'(i,:)=0$ holds for all x_i and y_i, including $x_i=\cos(\frac{i\pi}{n})$ and $y_i=\sin(\frac{i\pi}{n})$. $\square$

D. Proofs of Theorems 1 and Theorems 2

We begin with an analytic expression of the objective function $f(\{w_{ij}\},\{s_i\},\{\sigma_{ij}\})$ in Section D.1. Section D.2 and Section D.3 complete the proofs of Theorem 1 and Theorem 2. Section D.4 presents proofs of the propositions in Section D.1.

D.1. Analytical Expression of $f(\{w_{ij}\},\{s_i\},\{\sigma_{ij}\})$

Denote $A_{ij}=I_3-\boldsymbol{v}_{ij}^{\mathrm{inp}}\boldsymbol{v}_{ij}^{\mathrm{inp}^T}$ and $A_{ij}^{gt}=I_3-\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T}$. Introduce

$$dA_{ij}=A_{ij}-A_{ij}^{gt}$$

The following proposition characterizes an important property regarding dA_{ij}.

Proposition 4. *Consider a symmetric matrix $F\in\mathbb{R}^{3\times3}$. Then by dropping third-and-higher order terms, we have*

$$\underset{\boldsymbol{v}_{ij}^{\mathrm{inp}}}{\mathbb{E}}dA_{ij}=\frac{\sigma_{ij}^2}{2}(3\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T}-I_3),$$

and

$$\underset{\{\epsilon_{ij,k}\}}{\mathbb{E}}dA_{ij}FdA_{ij}=\frac{\sigma_{ij}^2}{2}\Big(\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T}(\mathrm{Tr}(F)-4\boldsymbol{v}_{ij}^{gt^T}F\boldsymbol{v}_{ij}^{gt})+\boldsymbol{v}_{ij}^{gt^T}F\boldsymbol{v}_{ij}^{gt}I_3+\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T}F+F\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T}\Big)\quad(16)$$

Proof. See Section D.4.1. □

The blocks of the perturbation connection Laplacian matrix $dL = L - L^{gt}$ is given by

$$dL_{ij} = \begin{cases} \sum\limits_{k \in \mathcal{N}_i} w_{ik} dA_{ik} & i = j \\ -w_{ij} dA_{ij} & (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

The following proposition characterize an expectation of dL when dA_{ij} follows the distribution described above.

Proposition 5. *Consider a positive semidefinite matrix $B \in \mathbb{R}^{3n \times 3n}$. Let $\boldsymbol{t}^{gt} \in \mathbb{R}^{3n}$ collect $\boldsymbol{t}_i^{gt}$ in its blocks. Then*

$$\begin{aligned} &\mathop{\mathbb{E}}_{\{\boldsymbol{v}_{ij}^{\text{inp}}\}} {\boldsymbol{t}^{gt}}^T dLBdL\boldsymbol{t}^{gt} = \sum_{(i,j)\in\mathcal{E}} \frac{\sigma_{ij}^2}{2} w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \\ &\Big(\mathrm{Tr}(E_{ij} B E_{ij}^T) - {\boldsymbol{v}_{ij}^{gt}}^T E_{ij} B E_{ij}^T \boldsymbol{v}_{ij}^{gt}\Big) \\ &\quad + \Big(\sum_{(i,j)\in\mathcal{E}} w_{ij}\sigma_{ij}^2 \|t_i^{gt} - t_j^{gt}\| E_{ij}^T v_{ij}^{gt}\Big)^T B \\ &\Big(\sum_{(i,j)\in\mathcal{E}} w_{ij}\sigma_{ij}^2 \|t_i^{gt} - t_j^{gt}\| E_{ij}^T v_{ij}^{gt}\Big). \end{aligned} \tag{17}$$

where E_{ij} is defined in Eq. (15).

Proof. See Section D.4.2. □

We proceed to analyze the stability of $\boldsymbol{u}_4$. First of all, all $3n$ generalized eigenvectors $\boldsymbol{u}_i, 1 \leq i \leq 3n$ satisfy that

$$\boldsymbol{u}_i^T W \boldsymbol{u}_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Note that $\sum\limits_{i=1}^{n} \boldsymbol{t}_i^{gt} = 0$. Without losing generalization, we assume

$$\sum_{i=1}^{n} s_i = 1, \qquad \sum_{i=1}^{n} s_i \boldsymbol{t}_i^{gt} = 0. \tag{18}$$

The first constraint in (18) normalizes the scale of s_i. The second equality in (18) places an additional constraint on s_i. The following proposition characterizes the top four generalized eigenvectors of L.

Proposition 6. *Under the assumptions in (18), we have*

$$(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \mathbf{1} \otimes I_3, \qquad \boldsymbol{u}_4 = \frac{\boldsymbol{t}^{gt}}{\sqrt{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2}} \tag{19}$$

The next proposition describes the perturbation in $\boldsymbol{u}_4$ with respect to the perturbation dL in L.

Proposition 7. *Suppose $\lambda_5(L^{gt}) > 0$. Then*

$$d\boldsymbol{u}_4 = -\Big(I_{3n} - U_4 U_4^T S\Big) L^{\dagger} \big(I_{3n} - S\boldsymbol{u}_4\boldsymbol{u}_4^T\big) dL\boldsymbol{u}_4. \tag{20}$$

where $U_4 = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4)$.

Proof: See Section D.5. □

Denote $\boldsymbol{s} = (s_i) \in \mathbb{R}^n$. When σ_{ij} and w_{ij} are fixed and applying (20), we can rewrite the objective function f as

$$f(\boldsymbol{s}) = \mathop{\mathbb{E}}_{\{\epsilon_{ij,k}\}} {\boldsymbol{t}^{gt}}^T dL B(\boldsymbol{s}) dL \boldsymbol{t}^{gt} \tag{21}$$

where

$$B(\boldsymbol{s}) = \Big(I_{3n} - \frac{\boldsymbol{t}^{gt}(S\boldsymbol{t}^{gt})^T}{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2}\Big) L^{\dagger} C(\boldsymbol{s}) L^{\dagger} \Big(I_{3n} - \frac{S\boldsymbol{t}^{gt}{\boldsymbol{t}^{gt}}^T}{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2}\Big)$$

where

$$\begin{aligned} C(\boldsymbol{s}) &= I_{3n} - (\mathbf{1}\boldsymbol{s}^T + \boldsymbol{s}\mathbf{1}^T) \otimes I_3 + n\boldsymbol{s}\boldsymbol{s}^T \otimes I_3 \\ &\quad - \frac{\boldsymbol{t}^{gt}(S\boldsymbol{t}^{gt})^T + S\boldsymbol{t}^{gt}{\boldsymbol{t}^{gt}}^T}{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2} + \frac{\|\boldsymbol{t}^{gt}\|^2 S\boldsymbol{t}^{gt}(S\boldsymbol{t}^{gt})^T}{(\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2)^2} \end{aligned}$$

Note that $L(\mathbf{1} \otimes I_3) = 0$ and $L\boldsymbol{t}^{gt} = 0$. We can simplify

$$\begin{aligned} L^{\dagger} C(\boldsymbol{s}) L^{\dagger} &= {L^{\dagger}}^2 + nL^{\dagger}(\boldsymbol{s}\boldsymbol{s}^T \otimes I_3) L^{\dagger} \\ &\quad + \frac{\|\boldsymbol{t}^{gt}\|^2}{(\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2)^2} (L^{\dagger} S\boldsymbol{t}^{gt})(L^{\dagger} S\boldsymbol{t}^{gt})^T. \end{aligned} \tag{22}$$

To apply Prop. 5 to derive a closed-form expression of $f(\boldsymbol{w})$, we compute

$$\begin{aligned} E_{ij} B(\boldsymbol{s}) E_{ij}^T &= \Big(E_{ij} - \frac{(\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})(S\boldsymbol{t}^{gt})^T}{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2}\Big) \\ & L^{\dagger} C(\boldsymbol{w}) L^{\dagger} \Big(E_{ij}^T - \frac{S\boldsymbol{t}^{gt}(\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})^T}{\sum\limits_{i=1}^{n} s_i \|\boldsymbol{t}_i^{gt}\|^2}\Big) \end{aligned} \tag{23}$$

To simplify $f(\boldsymbol{s})$, we note that for any vector $\boldsymbol{h} \in \mathbb{R}^3$,

$$\begin{aligned} &\mathrm{Tr}((\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})\boldsymbol{h}^T) - {\boldsymbol{v}_{ij}^{gt}}^T (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})\boldsymbol{h}^T)\boldsymbol{v}_{ij}^{gt} \\ =&\mathrm{Tr}((\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})\boldsymbol{h}^T) - \mathrm{Tr}((\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})\boldsymbol{h}^T) \\ =&0. \end{aligned}$$

It follows that we can simplify the objective function as

$$\begin{aligned} f(\boldsymbol{s}) = \sum_{(i,j)\in\mathcal{E}} &\frac{\sigma_{ij}^2}{2} w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \Big(\mathrm{Tr}(E_{ij} L^{\dagger} C(\boldsymbol{s}) L^{\dagger} E_{ij}^T) \\ &- {\boldsymbol{v}_{ij}^{gt}}^T E_{ij} L^{\dagger} C(\boldsymbol{w}) L^{\dagger} E_{ij}^T \boldsymbol{v}_{ij}^{gt}\Big). \end{aligned} \tag{24}$$

D.2. Proof of Theorem 1

According Eq. (22), it is clear that

$$L^{\dagger}C(\boldsymbol{s})L^{\dagger} \succeq {L^{\dagger}}^2,$$

and equality holds if and only if $\boldsymbol{s} = \mathbf{1}$. Therefore,

$$\begin{aligned} f(\boldsymbol{s}) \geq & \sum_{(i,j)\in\mathcal{E}} \frac{\sigma_{ij}^2}{2} w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \Big(\mathrm{Tr}(E_{ij}{L^{\dagger}}^2 E_{ij}^T) \\ & - {\boldsymbol{v}_{ij}^{gt}}^T E_{ij}{L^{\dagger}}^2 E_{ij}^T \boldsymbol{v}_{ij}^{gt}\Big) \end{aligned}$$

and equality holds when $\boldsymbol{s} = \mathbf{1}$. This ends the proof of Theorem 1. □

D.3. Proof of Theorem 2

Suppose $s_i = 1$. In this case, we optimize w_{ij} to minimize $f(\{w_{ij}\}, \{s_i\}, \{\sigma_{ij}\})$. L becomes a function of $\{w_{ij}\}$. Define

$$\overline{L} = \sum_{(i,j)\in\mathcal{E}} \sigma_{ij}^2 w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 E_{ij}^T (I_3 - \boldsymbol{v}^{gt}{\boldsymbol{v}^{gt}}^T) E_{ij},$$

and

$$\overline{\boldsymbol{g}} = \sum_{(i,j)\in\mathcal{E}} \sigma_{ij}^2 w_{ij} \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\| E_{ij}^T \boldsymbol{v}_{ij}^{gt}.$$

The objective function to be minimized is given by

$$\begin{aligned} f(\{w_{ij}^{\star}\}) = & \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \sigma_{ij}^2 w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \Big(\mathrm{Tr}(E_{ij}{L^{\dagger}}^2 E_{ij}^T) \\ & - {\boldsymbol{v}_{ij}^{gt}}^T E_{ij}{L^{\dagger}}^2 E_{ij}^T \boldsymbol{v}_{ij}^{gt}\Big) \\ = & \frac{1}{2}\mathrm{Tr}\Big(L^{\dagger}\overline{L}L^{\dagger}\Big). \end{aligned}$$

Apply the chain rule, we have

$$\begin{aligned} \frac{\partial f}{\partial w_{ij}} = & \frac{1}{2}\mathrm{Tr}\Big(L^{\dagger}\frac{\partial \overline{L}}{\partial w_{ij}}L^{\dagger} - L^{\dagger}\overline{L}L^{\dagger}\frac{\partial L}{\partial w_{ij}}L^{\dagger} \\ & - L^{\dagger}\frac{\partial L}{\partial w_{ij}}L^{\dagger}\overline{L}L^{\dagger}\Big). \end{aligned} \tag{25}$$

To simplify (25), we introduce $G = (G_1; G_2) \in \mathbb{R}^{2|\mathcal{E}|\times 3n}$ where

$$G_1 = (\sqrt{w_{ij}}{\boldsymbol{v}_{ij,1}^{gt}}^T E_{ij}), \quad G_2 = (\sqrt{w_{ij}}{\boldsymbol{v}_{ij,2}^{gt}}^T E_{ij}).$$

It is easy to check that

$$L = G^T G, \quad L^{\dagger} = G^{\dagger}{G^{\dagger}}^T,$$

and

$$\overline{L} = G^T D_4 G, \quad D_4 = I_2 \otimes \mathrm{diag}(w_{ij}\sigma_{ij}^2\|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2)$$

Moreover,

$$\frac{\partial L}{\partial w_{ij}} = \frac{1}{w_{ij}} G^T (I_2 \otimes \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T) G \tag{26}$$

$$\frac{\partial \overline{L}}{\partial w_{ij}} = \frac{2}{w_{ij}} G^T D_4 (I_2 \otimes \boldsymbol{e}_{ij}\boldsymbol{e}_{ij}^T) G \tag{27}$$

Substituting Eq. (26) and Eq. (27) into Eq. (25), we have

$$\begin{aligned} \frac{\partial f}{\partial w_{ij}} = & \frac{2}{w_{ij}}\mathrm{Tr}\Big((I_2 \otimes \boldsymbol{e}_{ij})^T {G^{\dagger}}^T G^{\dagger} D_4 (I - G^{\dagger}G^T) \\ & (I_2 \otimes \boldsymbol{e}_{ij})\Big). \end{aligned} \tag{28}$$

When $w_{ij}^{\star} = \frac{1}{\sigma_{ij}^2\|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2}$, we have $D_4 = I_{2|\mathcal{E}|}$. As

$${G^{\dagger}}^T G^{\dagger} = {G^{\dagger}}^T G^{\dagger} G^{\dagger} G^T,$$

we have $\frac{\partial f}{\partial w_{ij}} = 0$. This means $\{w_{ij}^{\star}\}$ is a critical point. Next, we show that $\{w_{ij}^{\star}\}$ is a local minimum.

Proposition 8. *Consider any vector $\boldsymbol{x} \neq \mathbf{0} \in \mathbb{R}^{|\mathcal{E}|}$, where $\sum\limits_{(i,j)\in\mathcal{E}} x_{ij} w_{ij}^{\star} = 0$. We have*

$$\sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} x_{ij} x_{i'j'} \frac{\partial^2 f}{\partial w_{ij}\partial w_{i'j'}} > 0.$$

Proof. See Section D.5.1. □

Next, we show that $\{w_{ij}^{\star}\}$ is the only critical point of f.

Proposition 9. *$\{s w_{ij}^{\star}\}$ is the only solution to $\forall (i,j) \in \mathcal{E}$,*

$$\mathrm{Tr}\Big((I_2 \otimes \boldsymbol{e}_{ij})^T {G^{\dagger}}^T G^{\dagger} D_4 (I - G^{\dagger}G^T)(I_2 \otimes \boldsymbol{e}_{ij})\Big) = 0. \tag{29}$$

Proof. See Section D.5.2. □

This ends the proof of Theorem 2. □

D.4. Proofs of Propositions

D.4.1 Proof of Prop. 4

Note that We can decompose $\boldsymbol{v}_{ij}^{\mathrm{inp}}$ to the orthogonal vectors $\boldsymbol{v}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\perp}$. $\boldsymbol{v}_{ij}^{\mathrm{inp}} = \cos\theta_{ij}\boldsymbol{v}_{ij}^{gt} + \sin\theta_{ij}\boldsymbol{v}_{ij}^{\perp}$

$$\mathbb{E}_{\boldsymbol{v}_{ij}^{\perp}} \boldsymbol{v}_{ij}^{\perp T} F \boldsymbol{v}_{ij}^{\perp} = \frac{1}{2}\Big(\mathrm{Tr}(F) - \boldsymbol{v}_{ij}^{gt\,T} F \boldsymbol{v}_{ij}^{gt}\Big),$$
$$\mathbb{E}_{\boldsymbol{v}_{ij}^{\perp}} \boldsymbol{v}_{ij}^{\perp} \boldsymbol{v}_{ij}^{\perp T} = \frac{1}{2}\Big(I_3 - \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T}\Big).$$
$$\mathbb{E}_{\boldsymbol{v}_{ij}^{\perp}} \boldsymbol{v}_{ij}^{\perp} F \boldsymbol{v}_{ij}^{gt\,T} = 0.$$
$$\mathbb{E}_{d\boldsymbol{v}_{ij}} \boldsymbol{v}_{ij}^{\mathrm{inp}} \boldsymbol{v}_{ij}^{\mathrm{inp}\,T} = \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} (\cos^2\theta_{ij} \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} + \sin^2\theta_{ij} \boldsymbol{v}_{ij}^{\perp} \boldsymbol{v}_{ij}^{\perp T}).$$
$$\begin{aligned}\mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} dA_{ij} &= \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} - \boldsymbol{v}_{ij}^{\mathrm{inp}} \boldsymbol{v}_{ij}^{\mathrm{inp}\,T} \\ &= -\mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} (\sin^2\theta_{ij}) \boldsymbol{v}_{ij}^{\perp} \boldsymbol{v}_{ij}^{\perp T} + \mathbb{E}_{\theta_{ij}} (\sin^2\theta_{ij}) \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} \\ &= \frac{\sigma_{ij}^2}{2} (3\boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} - I_3).\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{E}_{d\boldsymbol{v}_{ij}} dA_{ij} F dA_{ij} \\ =&\mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \big(\boldsymbol{v}_{ij}^{gt} d\boldsymbol{v}_{ij}^T + d\boldsymbol{v}_{ij} \boldsymbol{v}_{ij}^{gt\,T} + d\boldsymbol{v}_{ij} d\boldsymbol{v}_{ij}^T - \|d\boldsymbol{v}_{ij}\|^2 \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T}\big) \\ =&\boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \sin^2\theta_{ij} \cos^2\theta_{ij} \boldsymbol{v}_{ij}^{\perp T} F \boldsymbol{v}_{ij}^{\perp} \\ &+ \boldsymbol{v}_{ij}^{gt\,T} F \boldsymbol{v}_{ij}^{gt} \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \sin^2\theta_{ij} \cos^2\theta_{ij} \boldsymbol{v}_{ij}^{\perp T} \boldsymbol{v}_{ij}^{\perp} \\ &+ \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} F \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \sin^2\theta_{ij} \cos^2\theta_{ij} \boldsymbol{v}_{ij}^{\perp} \boldsymbol{v}_{ij}^{\perp T} \\ &+ \mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} \sin^2\theta_{ij} \cos^2\theta_{ij} \boldsymbol{v}_{ij}^{\perp} \boldsymbol{v}_{ij}^{\perp T} F \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} \\ =&\frac{\sigma_{ij}^2}{2}\Big(\boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} (\mathrm{Tr}(F) - 4\boldsymbol{v}_{ij}^{gt\,T} F \boldsymbol{v}_{ij}^{gt}) + \boldsymbol{v}_{ij}^{gt\,T} F \boldsymbol{v}_{ij}^{gt} I_3 \\ &\quad + \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T} F + F \boldsymbol{v}_{ij}^{gt} \boldsymbol{v}_{ij}^{gt\,T}\Big).\end{aligned}$$

□

D.4.2 Proof of Prop. 5

First of all, we have

$$\boldsymbol{t}^{gt\,T} dLBdL\boldsymbol{t}^{gt} = \sum_{1\le i,j\le n} \sum_{k\in\mathcal{N}_i} \sum_{l\in\mathcal{N}_j} w_{ik} w_{jl} (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_k^{gt})^T dA_{ik} B_{ij} dA_{jl} (\boldsymbol{t}_j^{gt} - \boldsymbol{t}_l^{gt}).$$

Note that $dA_{ij}, (i,j) \in \mathcal{E}$ are independent. Therefore

$$\begin{aligned}&\mathbb{E}_{\{\boldsymbol{v}_{ij}^{\mathrm{inp}}\}} \boldsymbol{t}^{gt\,T} dLBdL\boldsymbol{t}^{gt} = \\ &\mathbb{E}_{\{\boldsymbol{v}_{ij}^{\mathrm{inp}}\}} \Big(\sum_{1\le i\le n} \sum_{k\in\mathcal{N}_i} (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_k^{gt})^T w_{ik}^2 dA_{ik} B_{ii} dA_{ik} (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_k^{gt}) \\ &- \sum_{1\le i\le n} \sum_{k\in\mathcal{N}_i} w_{ik}^2 (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_k^{gt})^T dA_{ik} B_{ik} dA_{ik} (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_k^{gt})\Big) \\ &+ \mathbb{E}_{\{\boldsymbol{v}_{ij}^{\mathrm{inp}}\}} \sum_{1\le i,j\le n} \sum_{k\in\mathcal{N}_i} \sum_{l\in\mathcal{N}_j} 1(i\ne j \vee k\ne l)(w_{ik} w_{jl}) \\ &(t_i^{gt} - t_k^{gt})^T dA_{ik} B_{ij} dA_{jl} (t_j^{gt} - t_l^{gt})\Big).\end{aligned}$$

We have

$$\begin{aligned}&\mathbb{E}_{\{\boldsymbol{v}_{ij}^{\mathrm{inp}}\}} \boldsymbol{t}^{gt\,T} dLBdL\boldsymbol{t}^{gt} = \sum_{(i,j)\in\mathcal{E}} w_{ij}^2 (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt})^T \\ &\mathbb{E}_{\boldsymbol{v}_{ij}^{\mathrm{inp}}} dA_{ij} (B_{ii} + B_{jj} - B_{ij} - B_{ij}^T) dA_{ij} (\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}) \\ &+ \sum_{(i,k),(j,l)\in\mathcal{E}} 1(i\ne j \vee k\ne l) w_{ik} w_{jl} (t_i^{gt} - t_k^{gt})^T \\ &\mathbb{E}_{\boldsymbol{v}_{ik}^{\mathrm{inp}}} dA_{ik} (B_{ij} + B_{kl} - B_{il} - B_{jk}) \mathbb{E}_{\boldsymbol{v}_{jl}^{\mathrm{inp}}} dA_{jl} (t_j^{gt} - t_l^{gt}).\end{aligned} \tag{30}$$

As $\boldsymbol{v}_{ij}^{gt} = \frac{\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}}{\|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|}$. Applying Prop. 4, we have

$$\begin{aligned}&\mathbb{E}_{\{\epsilon_{ij,k}\}} \boldsymbol{t}^T dLBdL\boldsymbol{t} = \sum_{(i,j)\in\mathcal{E}} \frac{\sigma_{ij}^2}{2} w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \Big(\mathrm{Tr}(B_{ii} + B_{jj} \\ &\quad - B_{ij} - B_{ij}^T) - \boldsymbol{v}_{ij}^{gt\,T} (B_{ii} + B_{jj} - B_{ij} - B_{ij}^T) \boldsymbol{v}_{ij}^{gt}\Big) \\ &+ \sum_{(i,k),(j,l)\in\mathcal{E}} 1(i\ne j \vee k\ne l)(w_{ik} w_{jl} \sigma_{ik}^2 \sigma_{jl}^2) \\ &(t_i^{gt} - t_k^{gt})^T ((B_{ij} + B_{kl} - B_{il} - B_{jk}))(t_j^{gt} - t_l^{gt})\Big) \\ &= \sum_{(i,j)\in\mathcal{E}} \frac{\sigma_{ij}^2}{2} w_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \Big(\mathrm{Tr}(E_{ij} B E_{ij}^T) \\ &\quad - \boldsymbol{v}_{ij}^{gt\,T} E_{ij} B E_{ij}^T \boldsymbol{v}_{ij}^{gt}\Big) \\ &\quad + \sum_{(i,k),(j,l)\in\mathcal{E}} w_{ik} w_{jl} \sigma_{ik}^2 \sigma_{jl}^2 (t_i^{gt} - t_k^{gt})^T E_{ik} B E_{jl}^T (t_j^{gt} - t_l^{gt})\end{aligned} \tag{31}$$

□

D.5. Proof of Prop. 7

Note that

$$\frac{\partial \boldsymbol{u}_i^T W \boldsymbol{u}_4}{\partial \boldsymbol{v}} = 0.$$

Since $\boldsymbol{u}_i, 1 \leq i \leq 3$ do not depend on $\boldsymbol{v}$, we have

$$U_4^T W d\boldsymbol{u}_4 = 0. \tag{32}$$

Let the columns of $\overline{U}_4 \in \mathbb{R}^{3n\times(3n-4)}$ collect bases of vectors that orthogonal to U_4, i.e., $\overline{U}_4^T U_4 = 0$. Express

$$d\boldsymbol{u}_4 = \overline{U}_4 \boldsymbol{y} + U_4 \boldsymbol{x} \tag{33}$$

where $\boldsymbol{y} \in \mathbb{R}^{3n-4}$ and $\boldsymbol{x} \in \mathbb{R}^4$. Combining Eq. (33) and Eq. (32), we have

$$\boldsymbol{x} = -(U_4^T W U_4)^{-1} U_4^T W \overline{U}_4 \boldsymbol{y} = -U_4^T W \overline{U}_4 \boldsymbol{y}. \tag{34}$$

This means

$$d\boldsymbol{u}_4 = (I_{3n} - U_4 U_4^T W)\overline{U}_4 \boldsymbol{y}. \tag{35}$$

Consider the equality

$$L\boldsymbol{u}_4 = \lambda_4 W \boldsymbol{u}_4.$$

Compute the derivatives of both sides with respect to $\boldsymbol{v}$, we have

$$L d\boldsymbol{u}_4 + dL\boldsymbol{u}_4 = d\lambda_4 W \boldsymbol{u}_4. \tag{36}$$

The derivative of the eigen-value is given by

$$d\lambda_4 = \boldsymbol{u}_4^T dL \boldsymbol{u}_4. \tag{37}$$

Substituting Eq. (37) into Eq. (36), we have

$$L d\boldsymbol{u}_4 + dL\boldsymbol{u}_4 = W \boldsymbol{u}_4 \boldsymbol{u}_4^T dL \boldsymbol{u}_4 \tag{38}$$

Multiply both sides of Eq. (38) by $\overline{U}_4^T$ and combine Eq. (33), we arrive at

$$\overline{U}_4^T L \overline{U}_4 \boldsymbol{y} + \overline{U}_4^T dL \boldsymbol{u}_4 = 0. \tag{39}$$

Substituting Eq. (39) into Eq. (35), we have

$$d\boldsymbol{u}_4 = -(I_{3n} - U_4 U_4^T W)\overline{U}_4 \big(\overline{U}_4^T L \overline{U}_4\big)^{-1} \overline{U}_4^T dL \boldsymbol{u}_4. \tag{40}$$

Note that

$$L^{\dagger} = \overline{U}_4 \big(\overline{U}_4^T L \overline{U}_4\big)^{-1} \overline{U}_4^T.$$

$\square$

D.5.1 Proof of Prop. 8

We can expand

$$\begin{aligned}&\sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} x_{ij} x_{i'j'} \frac{\partial^2 f}{\partial w_{ij} \partial w_{i'j'}} \\ =& H_1 - H_2 + H_3 - H_4 + H_5 + H_6\end{aligned} \tag{41}$$

where

$$H_1 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} x_{ij}^2 \mathrm{Tr}\big(L^{\dagger} \frac{\partial^2 \overline{L}}{\partial^2 w_{ij}} L^{\dagger}\big),$$

$$\begin{aligned}H_2 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} & x_{ij} x_{i'j'} \mathrm{Tr}\big(L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \frac{\partial \overline{L}}{\partial w_{ij}} L^{\dagger} \\ & + L^{\dagger} \frac{\partial \overline{L}}{\partial w_{ij}} L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger}\big),\end{aligned}$$

$$\begin{aligned}H_3 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} & x_{ij} x_{i'j'} \mathrm{Tr}\big(L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \overline{L} L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \\ & + L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \overline{L} L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger}\big),\end{aligned}$$

$$\begin{aligned}H_4 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} & x_{ij} x_{i'j'} \mathrm{Tr}\big(L^{\dagger} \frac{\partial \overline{L}}{\partial w_{i'j'}} L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \\ & + L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \frac{\partial \overline{L}}{\partial w_{i'j'}} L^{\dagger}\big),\end{aligned}$$

$$\begin{aligned}H_5 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} & x_{ij} x_{i'j'} \mathrm{Tr}\Big(L^{\dagger} \overline{L} L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \\ & + L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \overline{L} L^{\dagger}\big),\end{aligned}$$

$$\begin{aligned}H_6 = \sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} & x_{ij} x_{i'j'} \mathrm{Tr}\big(L^{\dagger} \overline{L} L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \\ & + L^{\dagger} \frac{\partial L}{\partial w_{i'j'}} L^{\dagger} \frac{\partial L}{\partial w_{ij}} L^{\dagger} \overline{L} L^{\dagger}\big).\end{aligned}$$

Introduce

$$D_2 = I_2 \otimes \mathrm{diag}\big(\frac{x_{ij}}{w_{ij}}\big).$$

We have

$$\begin{aligned}H_1 =& 2 \sum_{(i,j)\in\mathcal{E}} \frac{x_{ij}^2}{w_{ij}} \sigma_{ij}^2 \|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2 \\ & \mathrm{Tr}\big(L^{\dagger} E_{ij} w_{ij} (\boldsymbol{v}_{ij,1}^{gt} {\boldsymbol{v}_{ij,1}^{gt}}^T + \boldsymbol{v}_{ij,2}^{gt} {\boldsymbol{v}_{ij,2}^{gt}}^T) E_{ij}^T L^{\dagger}\big) \\ =& 2\mathrm{Tr}\big(L^{\dagger} G^T D_2^2 D_4 G L^{\dagger}\big) = 2\mathrm{Tr}\big(G^{\dagger} D_2^2 D_4 {G^{\dagger}}^T\big).\end{aligned} \tag{42}$$

Through similar calculations, we have

$$H_2 = H_4 = 4\mathrm{Tr}\big(G^{\dagger} D_2 G G^{\dagger} D_2 D_4 {G^{\dagger}}^T\big), \tag{43}$$

and

$$H_3 = 2\mathrm{Tr}\big(G^\dagger D_2 {G^\dagger}^T G^T D_4 = GG^\dagger D_2 {G^\dagger}^T\big), \qquad (44)$$

and

$$H_5 = H_6 = 2\mathrm{Tr}\big(G^\dagger D_4 GG^\dagger D_2 {G^\dagger}^T G^T D_2 {G^\dagger}^T\big). \qquad (45)$$

Introduce orthonormal matrix $\overline{U} \in \mathbb{R}^{2|\mathcal{E}|\times(2|\mathcal{E}|-3n+4)}$ where

$$I - \overline{U}\overline{U}^T = GG^\dagger.$$

Substituting Eq. (42), Eq. (43), Eq. (44), and Eq. (45) into Eq. (41), we have

$$\sum_{(i,j)\in\mathcal{E}} \sum_{(i',j')\in\mathcal{E}} x_{ij}x_{i'j'} \frac{\partial^2 f}{\partial w_{ij}\partial w_{i'j'}} = 2\mathrm{Tr}\big(G^\dagger F {G^\dagger}^T\big) \qquad (46)$$

where

$$\begin{aligned} F = {} & D_2^2 D_4 - 2D_2(I - \overline{U}\overline{U}^T)D_2D_4 - 2D_2D_4(I - \overline{U}\overline{U}^T)D_2 \\ & + D_2(I - \overline{U}\overline{U}^T)D_4(I - \overline{U}\overline{U}^T)D_2 \\ & + D_4(I - \overline{U}\overline{U}^T)D_2(I - \overline{U}\overline{U}^T)D_2 \\ & + D_2(I - \overline{U}\overline{U}^T)D_2(I - \overline{U}\overline{U}^T)D_4 \\ = {} & D_2\overline{U}\overline{U}^T D_4 \overline{U}\overline{U}^T D_2 - D_2(I - \overline{U}\overline{U}^T)D_2\overline{U}\overline{U}^T D_4 \\ & - D_4\overline{U}\overline{U}^T D_2(I - \overline{U}\overline{U}^T)D_2. \end{aligned}$$

When $w_{ij} = \frac{1}{\sigma_{ij}^2\|\boldsymbol{t}_i^{gt} - \boldsymbol{t}_j^{gt}\|^2}$, we have D_4. As $\overline{U}^T G^\dagger = 0$, we have

$$\mathrm{Tr}\big(G^\dagger F {G^\dagger}^T\big) = \|G^\dagger D_2 \overline{U}\overline{U}^T\|_{\mathcal{F}}^2 = \|G^\dagger D_2 \overline{U}\|_{\mathcal{F}}^2$$

It is clear that $\mathrm{Tr}\big(G^\dagger F {G^\dagger}^T\big) \geq 0$ and equality holds if and only if $D_2 = sI_{2|\mathcal{E}|}$ $\square$

D.5.2 Proof Prop. 9

We prove a stronger result.

Lemma 1. *Consider a unitary matrix $U \in \mathbb{R}^{2m\times n}$ where $2m > n$. Let $\Sigma = \mathrm{diag}(\sigma_i) \in R^{n\times n}$ be a diagonal matrix with all positive values. Consider a vector $\boldsymbol{x} \in \mathbb{R}^m$. Suppose $\forall 1 \leq i \leq m$,*

$$Tr\Big((I_2 \otimes \boldsymbol{e}_i)^T U\Sigma U^T (I_2 \otimes \mathrm{diag}(\boldsymbol{x}))(I - UU^T)(I_2 \otimes \boldsymbol{e}_i\Big) = 0, \qquad (47)$$

then $\boldsymbol{x} = s\mathbf{1}$.

Proof. Denote $U = (U_1; U_2)$. Then Eq. (47) is equivalent to

$$\begin{aligned} & \Big((U_1\Sigma U_1^T).(I_m - U_1U_1^T) + (U_2\Sigma U_2^T).(I_m - U_2U_2^T) \\ & - (U_1\Sigma U_2^T).(U_1U_2^T) - (U_2\Sigma U_1^T).(U_2U_1^T)\Big)\boldsymbol{x} = 0. \end{aligned} \qquad (48)$$

where $A.B$ is the element-wise matrix multiplication operation.

Denote

$$\begin{aligned} A = {} & (U_1\Sigma U_1^T).(I_m - U_1U_1^T) + (U_2\Sigma U_2^T).(I_m - U_2U_2^T) \\ - {} & (U_1\Sigma U_2^T).(U_1U_2^T) - (U_2\Sigma U_1^T).(U_2U_1^T). \end{aligned}$$

We show that $\boldsymbol{x}^T A\boldsymbol{x} \geq 0$ and equality holds if and only if $\boldsymbol{x} = s\mathbf{1}$. Let i-th row of U_1 and U_2 as $\boldsymbol{u}_{1i}$ and $\boldsymbol{u}_{2i}$. It follows that

$$\begin{aligned} \boldsymbol{x}^T A\boldsymbol{x} = {} & \sum_{i=1}^m x_i^2\big(\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2i}^T\big) - \sum_{i=1}^m\sum_{j=1}^m \\ & x_ix_j\Big((\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1j}^T)(\boldsymbol{u}_{1i}\boldsymbol{u}_{1j}^T) + (\boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2j}^T)(\boldsymbol{u}_{2i}\boldsymbol{u}_{2j}^T) \\ & + (\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{2j}^T)(\boldsymbol{u}_{1i}\boldsymbol{u}_{2j}^T) + (\boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{1j}^T)(\boldsymbol{u}_{2i}\boldsymbol{u}_{1j}^T)\Big) \\ \geq {} & \sum_{i=1}^m x_i^2\big(\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2i}^T\big) - \sum_{i=1}^m\sum_{j=1}^m \\ & \frac{x_i^2 + x_j^2}{2}\Big((\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1j}^T)(\boldsymbol{u}_{1i}\boldsymbol{u}_{1j}^T) + (\boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2j}^T)(\boldsymbol{u}_{2i}\boldsymbol{u}_{2j}^T) \\ & + (\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{2j}^T)(\boldsymbol{u}_{1i}\boldsymbol{u}_{2j}^T) + (\boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{1j}^T)(\boldsymbol{u}_{2i}\boldsymbol{u}_{1j}^T)\Big) \end{aligned}$$

$$\begin{aligned} = {} & \sum_{i=1}^m x_i^2\big(\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2i}^T\big) - \sum_{i=1}^m x_i^2\Big(\boldsymbol{u}_{1i}\Sigma \\ & (\sum_{j=1}^m \boldsymbol{u}_{1j}^T\boldsymbol{u}_{1j})\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma(\sum_{j=1}^m \boldsymbol{u}_{2j}^T\boldsymbol{u}_{2j})\boldsymbol{u}_{2i}^T \\ & + \boldsymbol{u}_{1i}\Sigma(\sum_{j=1}^m \boldsymbol{u}_{2j}^T\boldsymbol{u}_{2j})\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma(\sum_{j=1}^m \boldsymbol{u}_{1j}^T\boldsymbol{u}_{jj})\boldsymbol{u}_{2i}^T\Big) \\ = {} & \sum_{i=1}^m x_i^2\big(\boldsymbol{u}_{1i}\Sigma\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma\boldsymbol{u}_{2i}^T\big) - \sum_{i=1}^m x_i^2\big(\boldsymbol{u}_{1i}\Sigma \\ & (U_1^TU_1 + U_2^TU_2)\boldsymbol{u}_{1i}^T + \boldsymbol{u}_{2i}\Sigma(U_1^TU_1 + U_2^TU_2)\boldsymbol{u}_{2i}^T\big) = 0. \end{aligned}$$

and equality holds if and only if $x_i = x_j, \forall i \neq j$. $\square$

E. Proof of Theorem 3

We begin with key lemmas regarding general-purpose stability results of eigen-values and eigen-vectors and matrix-norms in Section E.1. Section E.2 complete the proof of Theorem 3. Section E.3 presents proofs of the lemmas in Section E.1.

E.1. Key Lemmas

We first present two lemmas regarding the stability of eigen-values and eigen-vectors. Suppose that the measurement with edge $(i,j) \in \mathcal{E}$ is $\boldsymbol{v}_{ij}^{\mathrm{inp}}$, and the underlying ground truth is $\boldsymbol{v}_{ij}^{gt}$. The edge weight is $w_{ij} \in [0,1]$. Let $\boldsymbol{w} = (w_{ij})$. We define

$$L(\boldsymbol{w}) = \sum_{(i,j)\in\mathcal{E}} w_{ij} E_{ij}(I_3 - \boldsymbol{v}_{ij}^{\text{inp}} {\boldsymbol{v}_{ij}^{\text{inp}}}^T) E_{ij}^T,$$

$$L^{gt}(\boldsymbol{w}) = \sum_{(i,j)\in\mathcal{E}} w_{ij} E_{ij}(I_3 - \boldsymbol{v}_{ij}^{gt} {\boldsymbol{v}_{ij}^{gt}}^T) E_{ij}^T,$$

$$dL(\boldsymbol{w}) = \sum_{(i,j)\in\mathcal{E}} w_{ij} E_{ij}(\boldsymbol{v}_{ij}^{gt} {\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}^{\text{inp}} {\boldsymbol{v}_{ij}^{\text{inp}}}^T) E_{ij}^T.$$

It is clear that $L(\boldsymbol{w}) = L^{gt}(\boldsymbol{w}) + dL(\boldsymbol{w})$. Consider

$$L(\boldsymbol{w})\boldsymbol{u}_4 = \lambda_4 \boldsymbol{u}_4, \qquad L^{gt}(\boldsymbol{w})\boldsymbol{u}_4^{gt} = \lambda_4^{gt}\boldsymbol{u}^{gt}.$$

The first lemma characterizes an upper bound of λ_4 using $dL(\boldsymbol{w})$

Lemma 2. *We have,*

$$\lambda_4 \leq {\boldsymbol{u}^{gt}}^T dL(\boldsymbol{w})\boldsymbol{u}^{gt}. \tag{49}$$

Proof: See Section E.3.1. □

Denote $\boldsymbol{t}_{ij} = \boldsymbol{u}_{4i} - \boldsymbol{u}_{4j}$ and $\boldsymbol{t}_{ij}^{gt} = \boldsymbol{u}_{4i}^{gt} - \boldsymbol{u}_{4j}^{gt}$. The next lemma provides an upper bound on $\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\|$.

Lemma 3. *Suppose $\|dL(\boldsymbol{w})\| \leq \frac{\lambda_5^{gt}}{3}$ and*

$$\|L^{gt}(\boldsymbol{w})^{\dagger} dL(\boldsymbol{w})\|_1 + \|L^{gt}(\boldsymbol{w})^{\dagger}\|_1 {\boldsymbol{t}^{gt}}^T dL(\boldsymbol{w})\boldsymbol{t}^{gt} < 1.$$

Then

$$\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq (1-\beta)\frac{\|E_{ij}^T L^{gt}(\boldsymbol{w})^{\dagger}\|_1 \|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_\infty}{\alpha} + \beta\|\boldsymbol{t}_{ij}^{gt}\|, \tag{50}$$

and

$$\|\boldsymbol{t} - \boldsymbol{t}^{gt}\|_\infty \leq (1-\beta)\frac{\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1 \|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_\infty}{\alpha} + \beta\|\boldsymbol{t}^{gt}\|_\infty, \tag{51}$$

where

$$\beta \leq \frac{\|dL(\boldsymbol{w})\|^2}{2(\lambda_5^{gt} - {\boldsymbol{u}_4^{gt}}^T dL(\boldsymbol{w})\boldsymbol{u}_4^{gt} - \|dL(\boldsymbol{w})\|)^2},$$

$$\alpha = 1 - \big({\boldsymbol{u}_4^{gt}}^T dL(\boldsymbol{w})\boldsymbol{u}_4^{gt} + \|dL(\boldsymbol{w})\|_1\big)\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1.$$

Proof: See Section E.3.2. □

We proceed to bound $\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1$ and $\|E_{ij}^T L^{gt}(\boldsymbol{w})^{\dagger}\|_1$ with respect to references $\|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1$ and $\|E_{ij}^T L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1$ where $\overline{\boldsymbol{w}}$ is some reference edge vector.

Lemma 4. *Suppose $\lambda_5(L^{gt}(\boldsymbol{w})) > 0$ and $\lambda_5(L^{gt}(\overline{\boldsymbol{w}})) > 0$. Then*

$$\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1 \leq \frac{\|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1}{1 - \|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1 c(d\boldsymbol{w})} \tag{52}$$

$$\|E_{ij}^T L^{gt}(\boldsymbol{w})^{\dagger}\|_1 \leq \frac{\|E_{ij}^T L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1}{1 - \|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1 c(d\boldsymbol{w})} \tag{53}$$

where

$$c(d\boldsymbol{w}) = \max_{1\leq i\leq n} \sum_{j\in\mathcal{N}_i} |w_{ij} - \overline{w}_{ij}|.$$

Proof: See Section E.3.3. □

We then provide two L^∞ bounds on $dL(\boldsymbol{w})$.

Lemma 5.

$$\|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_\infty \leq \max_{1\leq i\leq n} \big(\epsilon\delta_i^{\text{in}}(\boldsymbol{w}) + \delta_i^{\text{out}}(\boldsymbol{w})\big) \tag{54}$$

$$\|dL(\boldsymbol{w})\|_1 \leq \max_{1\leq i\leq n} \Big(\epsilon \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{ij} + \sum_{j\in\mathcal{N}_i^{\text{out}}} w_{ij}\|\boldsymbol{v}_{ij}^{\text{inp}}{\boldsymbol{v}_{ij}^{\text{inp}}}^T - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T\|\Big) \tag{55}$$

where

$$\delta_i^{\text{in}}(\boldsymbol{w}) = \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{ij}\|\boldsymbol{t}_{ij}^{gt}\|$$

$$\delta_i^{\text{out}}(\boldsymbol{w}) = \sum_{j\in\mathcal{N}_i^{\text{out}}} w_{ij}\|(\boldsymbol{v}_{ij}^{\text{inp}}{\boldsymbol{v}_{ij}^{\text{inp}}}^T - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T)\boldsymbol{t}_{ij}^{gt}\|$$

Proof: See Section E.3.4. □

Next, we provide two spectral norms of $dL(\boldsymbol{w})$.

Lemma 6.

$${\boldsymbol{t}^{gt}}^T dL(\boldsymbol{w})\boldsymbol{t}^{gt} \leq \epsilon^2 \sum_{(i,j)\in\mathcal{E}^{\text{in}}} w_{ij}\|\boldsymbol{t}_{ij}^{gt}\|^2 + \sum_{(i,j)\in\mathcal{E}^{\text{out}}} w_{ij}\|\boldsymbol{t}_{ij}^{gt}\|^2 \|\boldsymbol{v}_{ij}^{\text{inp}}{\boldsymbol{v}_{ij}^{\text{inp}}}^T - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T\|, \tag{56}$$

and

$$\|dL(\boldsymbol{w})\| \leq \max_{1\leq i\leq n} \Big(\epsilon \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{ij} + \sum_{j\in\mathcal{N}_i^{\text{out}}} w_{ij} \|\boldsymbol{v}_{ij}^{\text{inp}}{\boldsymbol{v}_{ij}^{\text{inp}}}^T - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T\|\Big) \tag{57}$$

Proof: See Section E.3.5. □

Finally, we present two lemmas which are used to control w_{ij} during the alternating procedure.

Lemma 7. *Consider a hyper-parameters $\eta < 1$. Define*

$$b_l(\eta, \boldsymbol{t}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\text{inp}}) := \min_{\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq \eta \|\boldsymbol{t}_{ij}^{gt}\|} \|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|^2 \|\boldsymbol{t}_{ij}\|^2.$$

Then $b_l(\eta, \boldsymbol{t}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\text{inp}}) = 0$ when

$$\|(I_3 - \boldsymbol{v}_{ij}^{\text{inp}} {\boldsymbol{v}_{ij}^{\text{inp}}}^T) \boldsymbol{v}_{ij}^{gt}\| \leq \eta.$$

Otherwise,

$$b_l(\eta, \boldsymbol{t}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\text{inp}}) \geq 4(1-\eta)^2 \|\boldsymbol{t}_{ij}^{gt}\|^2 \sin^2(\frac{\phi_1 - \phi_2}{2})$$

where

$$\phi_1 = \text{acos}({\boldsymbol{v}_{ij}^{\text{inp}}}^T \boldsymbol{v}_{ij}^{gt}), \quad \phi_2 = \text{asin}(\eta).$$

Proof: See Section E.4. □

Lemma 8. *Consider a hyper-parameter $\eta \leq 1$. Define*

$$b_u(\eta, \boldsymbol{t}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\text{inp}}) := \max_{\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq \eta \|\boldsymbol{t}_{ij}^{gt}\|} \|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|^2 \|\boldsymbol{t}_{ij}\|^2.$$

Then

$$\begin{aligned} b_u(\eta, \boldsymbol{t}_{ij}^{gt}, \boldsymbol{v}_{ij}^{\text{inp}}) \leq & 4\sin^2(\min(\frac{\pi}{2}, \frac{\phi_1 + \phi_2}{2})) \\ & (1+\eta)^2 \|\boldsymbol{t}_{ij}^{gt}\|)^2 \end{aligned}$$

Proof: See Section E.5. □

E.2. Complete the Proof of Theorem 3

Our proof is based on the eigen stability results in Lemma 2 and Lemma 3 and the bounds in Lemma 4 to Lemma 8.

As our goal is to show the robustness of our algorithm against outliers, our proof do not aim to provide tight values of c_1, c_2, c_3. Define

$$r = \max_{(i,j)\in\mathcal{E}} \|\boldsymbol{t}_{ij}^{gt}\| / \min_{(i,j)\in\mathcal{E}} \|\boldsymbol{t}_{ij}^{gt}\|.$$

We assume that the value of r is not super big.

We show that the iterative procedure converges to a local minimum that is sufficiently close to $\boldsymbol{t}^{gt}$. Denote

$$\begin{aligned} \delta(\epsilon, \boldsymbol{w}) := \max_{1\leq i \leq n} \Big(\epsilon \sum_{j\in\mathcal{N}_i^{\text{in}}} w_{ij} + \\ \sum_{j\in\mathcal{N}_i^{\text{out}}} w_{ij} \|\boldsymbol{v}_{ij}^{\text{inp}} {\boldsymbol{v}_{ij}^{\text{inp}}}^T - \boldsymbol{v}_{ij}^{gt} {\boldsymbol{v}_{ij}^{gt}}^T \|\Big)\|. \end{aligned}$$

Applying Lemma 5, we have $\forall (i,j) \in \mathcal{E}$,

$$\|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_{\infty} \leq r\delta(\epsilon, \boldsymbol{w})\|\boldsymbol{t}_{ij}^{gt}\|,$$

and

$$\|dL(\boldsymbol{w})\| \leq \|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_1 \leq \delta(\epsilon, \boldsymbol{w}).$$

This means α and β in Lemma 3 satisfy

$$\alpha \geq 1 - 2\delta(\epsilon, \boldsymbol{w})\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1, \tag{58}$$

$$\beta \leq \frac{\delta(\epsilon, \boldsymbol{w})^2}{2(1 - 2\delta(\epsilon, \boldsymbol{w}))^2}. \tag{59}$$

Applying Lemma 3, we have

$$\frac{\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\|}{\|\boldsymbol{t}_{ij}^{gt}\|} \leq \beta + \frac{1-\beta}{\alpha} \|E_{ij}^T L^{gt}(\boldsymbol{w})^{+}\|_1 \delta(\epsilon, \boldsymbol{w})$$

It is clear that we can choose c_1, c_2, c_3 so that the output of the first iteration of our algorithm satisfies

$$\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq \frac{1}{4}\|\boldsymbol{t}_{ij}^{gt}\|, \quad \alpha \geq \frac{7}{8}.$$

Applying Lemma 7 and Lemma 8 and after some calculation, we have that the edge weights w_{ij} converge to the neighborhood of

$$\overline{w}_{ij} = \frac{\sigma_{\min}^2}{\sigma_{\min}^2 + \|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}^{gt}\|^2 \|\boldsymbol{t}_{ij}^{gt}\|^2}.$$

Theorem 3 then follows Lemma 3 and Lemma 4. □

E.3. Proof of Lemmas in Section E.1

E.3.1 Proof of Lemma 2

Note the following variational definition of

$$\lambda_4 = \min_{\boldsymbol{x}\in\mathbb{R}^{3n}, \|\boldsymbol{x}\|=1, (\boldsymbol{1}\otimes I_3)^T \boldsymbol{x} = 0} \boldsymbol{x}^T L \boldsymbol{x}. \tag{60}$$

As $(\boldsymbol{1}\otimes I_3)^T \boldsymbol{u}_4^{gt} = 0$, we have

$$\begin{aligned} \lambda_4 \leq & {\boldsymbol{u}_4^{gt}}^T L \boldsymbol{u}_4^{gt} \\ = & {\boldsymbol{u}_4^{gt}}^T (L^{gt} + dL) \boldsymbol{u}_4^{gt} = {\boldsymbol{u}_4^{gt}}^T dL \boldsymbol{u}_4^{gt}. \end{aligned}$$

□

E.3.2 Proof of Lemma 3

Let $\overline{U}_4^{gt} \in \mathbb{R}^{3n\times(3n-4)}$ collect the 5-th to $3n$-th eigenvectors L^{gt}. As $(\boldsymbol{1}\otimes I_3)^T \boldsymbol{u}_4 = (\boldsymbol{1}\otimes I_3)^T \boldsymbol{u}_4^{gt} = \boldsymbol{0}$, we can express

$$d\boldsymbol{u} = \boldsymbol{u}_4 - \boldsymbol{u}_4^{gt} = -x\boldsymbol{u}_4^{gt} + \overline{U}_4^{gt} \boldsymbol{y}. \tag{61}$$

As $\|\boldsymbol{u}_4\| = 1$, we have

$$(1-x)^2 + \|\boldsymbol{y}\|^2 = 1. \tag{62}$$

The following proposition describes the formula for $\overline{U}_4^{gt}\boldsymbol{y}$

Proposition 10. *Suppose* $\|dL\| + {\boldsymbol{u}_4^{gt}}^T dL \boldsymbol{u}_4^{gt} < \lambda_5^{gt}$. *Then*

$$\overline{U}_4^{gt}\boldsymbol{y} = -(1-x)\big(I + L^{\dagger}(\lambda_4)dL\big)^{-1}L^{\dagger}(\lambda_4)dL\boldsymbol{u}_4^{gt}. \tag{63}$$

where

$$L^{\dagger}(\lambda_4) = \overline{U}_4^{gt}(\Lambda - \lambda_4 I)^{-1}{\overline{U}_4^{gt}}^T.$$

Proof:
First of all, from $(L^{gt} + dL)(\boldsymbol{u}_4^{gt} + d\boldsymbol{u}) = \lambda_4(\boldsymbol{u}_4^{gt} + d\boldsymbol{u})$, we have

$$(L^{gt} + dL - \lambda_4 I)d\boldsymbol{u} = (\lambda_4 I - dL)\boldsymbol{u}_4^{gt}. \tag{64}$$

Substituting Eq. (61) into Eq. (64) and multiplying both sides by ${\overline{U}_4^{gt}}^T$, we arrive at

$$\big(\Lambda - \lambda_4 I + {\overline{U}_4^{gt}}^T dL \overline{U}_4^{gt}\big)\boldsymbol{y} = -(1-x){\overline{U}_4^{gt}}^T dL\boldsymbol{u}_4^{gt} \tag{65}$$

which means

$$\boldsymbol{y} = -(1-x)\big(\Lambda - \lambda_4 I + {\overline{U}_4^{gt}}^T dL\overline{U}_4^{gt}\big)^{-1}{\overline{U}_4^{gt}}^T dL\boldsymbol{u}_4^{gt}.$$

Note that

$$\begin{aligned}
&\overline{U}_4^{gt}\big(\Lambda - \lambda_4 I + {\overline{U}_4^{gt}}^T dL\overline{U}_4^{gt}\big)^{-1}{\overline{U}_4^{gt}}^T dL\boldsymbol{u}_4^{gt} \\
=&\overline{U}_4^{gt}(\Lambda - \lambda_4 I)^{-\frac{1}{2}}\big(I + (\Lambda - \lambda_4 I)^{-\frac{1}{2}}{\overline{U}_4^{gt}}^T dL\overline{U}_4^{gt} \\
&(\Lambda - \lambda_4 I)^{-\frac{1}{2}}\big)^{-1}(\Lambda - \lambda_4 I)^{-\frac{1}{2}}{\overline{U}_4^{gt}}^T dL\boldsymbol{u}_4^{gt} \\
=&(I + L^{\dagger}(\lambda_4)dL)^{-1}L^{\dagger}(\lambda_4)dL\boldsymbol{u}_4^{gt}
\end{aligned}$$

□

The following proposition provides an upper bound on x.

Proposition 11. *Suppose* $\|dL\| \leq \frac{\lambda_5^{gt}}{3}$, *we have*

$$x \leq \frac{\|dL\|^2}{2(\lambda_5^{gt} - \lambda_4 - \|dL\|)^2}. \tag{66}$$

Proof: Denote

$$\alpha = \|\big(I + L^{\dagger}(\lambda_4)dL\big)^{-1}L^{\dagger}(\lambda_4)dL\boldsymbol{u}_4^{gt}\|,$$

then

$$(1-x)^2 = \frac{1}{1+\alpha^2}. \tag{67}$$

Note that

$$\|L^{\dagger}(\lambda_4)dL\| \leq \frac{\|dL\|}{\lambda_5^{gt} - \lambda_4}.$$

It follows that

$$\alpha \leq \frac{\|dL\|}{\lambda_5^{gt} - \lambda_4 - \|dL\|}. \tag{68}$$

As $\|dL\|\frac{1}{3}\lambda_5^{gt}$, we have

$$\alpha \leq \frac{1}{2}.$$

Substituting (68) into (67), we have

$$\begin{aligned}
x \leq& 1 - \frac{1}{\sqrt{1+\alpha^2}} \leq \frac{\alpha^2}{2} \\
\leq& \frac{\|dL\|^2}{2(\lambda_5^{gt} - \lambda_4 - \|dL\|)^2}
\end{aligned} \tag{69}$$

□

We have

$$\begin{aligned}
\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| =& \|E_{ij}^T d\boldsymbol{u}\| \leq x\|E_{ij}^T\boldsymbol{u}_4^{gt}\| + (1-x) \\
&\|E_{ij}^T L^{\dagger}(\lambda_4)dL(I + L^{\dagger}(\lambda_4)dL)^{-1}\boldsymbol{u}_4^{gt}\| \\
\leq& x\|\boldsymbol{t}_{ij}^{gt}\| + (1-x)\|E_{ij}^T L^{\dagger}(\lambda_4) \\
&\sum_{i=0}^{+\infty}(-dLL^{\dagger}(\lambda_4))^i dL\boldsymbol{u}_4^{gt}\|_1 \\
\leq& x\|\boldsymbol{t}_{ij}^{gt}\| + (1-x)\frac{\|E_{ij}^T L^{\dagger}(\lambda_4)\|_1\|dL\boldsymbol{u}_4^{gt}\|_{\infty}}{1 - \|L^{\dagger}(\lambda_4)dL\|_1}
\end{aligned} \tag{70}$$

Note that

$$\begin{aligned}
\|E_{ij}^T L^{\dagger}(\lambda_4)\|_1 =& \|\sum_{i=0}^{+\infty} E_{ij}^T\lambda_4^i(L^{\dagger})^{i+1}\|_1 \\
&\leq \|E_{ij}^T L^{\dagger}\|_1\sum_{i=0}^{+\infty}(\lambda_4\|L^{\dagger}\|_1)^i \\
&= \frac{\|E_{ij}^T L^{\dagger}\|_1}{1 - \lambda_4\|L^{\dagger}\|_1}
\end{aligned} \tag{71}$$

Similarly,

$$\begin{aligned}
\|L^{\dagger}(\lambda_4)dL\|_1 =& \|\sum_{i=0}^{+\infty}\lambda_4^i L^{\dagger})^{i+1}dL\|_1 \\
&\leq \frac{\|L^{\dagger}dL\|_1}{1 - \lambda_4\|L^{\dagger}\|_1}
\end{aligned} \tag{72}$$

Substituting (71) and (72) into (70), we obtain

$$\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq x\|\boldsymbol{t}_{ij}^{gt}\| + (1-x)\frac{\|E_{ij}^T L^{\dagger}\|_1\|dL\boldsymbol{u}_4^{gt}\|_{\infty}}{1 - \lambda_4\|L^{\dagger}\|_1 - \|L^{\dagger}dL\|_1} \tag{73}$$

Similarly, we have

$$\|\boldsymbol{t}-\boldsymbol{t}^{gt}\| \leq x\|\boldsymbol{t}^{gt}\| + (1-x)\frac{\|L^{\dagger}\|_1\|dL\boldsymbol{u}_4^{gt}\|_{\infty}}{1-\lambda_4\|L^{\dagger}\|_1 - \|L^{\dagger}dL\|_1}. \tag{74}$$

We now end the proof because

$$\lambda_4 \leq {\boldsymbol{u}_4^{gt}}^T dL \boldsymbol{u}_4^{gt}.$$

$\square$

E.3.3 Proof Lemma 4

We only prove Eq. (52) as the proof of Eq. (53) is very similar.

Note that both $L^{gt}(\boldsymbol{w})$ and $L^{gt}(\overline{\boldsymbol{w}})$ share the same non-trivial eigenvectors denoted as $\overline{U}_4^{gt}$. This means,

$$L^{gt}(\boldsymbol{w})^{\dagger} = \overline{U}_4^{gt}\Big({\overline{U}_4^{gt}}^T L^{gt}(\boldsymbol{w})\overline{U}_4^{gt}\Big)^{-1}{\overline{U}_4^{gt}}^T,$$
$$L^{gt}(\overline{\boldsymbol{w}})^{\dagger} = \overline{U}_4^{gt}\Big({\overline{U}_4^{gt}}^T L^{gt}(\overline{\boldsymbol{w}})\overline{U}_4^{gt}\Big)^{-1}{\overline{U}_4^{gt}}^T.$$

It follows that,

$$\begin{aligned}
& L^{gt}(\boldsymbol{w})^{\dagger} \\
=& \overline{U}_4^{gt}\Big({\overline{U}_4^{gt}}^T L^{gt}(\overline{\boldsymbol{w}})\overline{U}_4^{gt} + {\overline{U}_4^{gt}}^T L^{gt}(d\boldsymbol{w})\overline{U}_4^{gt}\Big)^{-1}{\overline{U}_4^{gt}}^T \\
=& \overline{U}_4^{gt}({\overline{U}_4^{gt}}^T L^{gt}(\overline{\boldsymbol{w}})\overline{U}_4^{gt})^{-1}\sum_{i=0}^{\infty}\Big(-({\overline{U}_4^{gt}}^T L^{gt}(\overline{\boldsymbol{w}})\overline{U}_4^{gt})^{-1} \\
& {\overline{U}_4^{gt}}^T L^{gt}(\overline{d\boldsymbol{w}})\overline{U}_4^{gt}\Big)^i {\overline{U}_4^{gt}}^T \\
=& L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\sum_{i=0}^{n}\Big(-L^{gt}(\overline{\boldsymbol{w}})^{\dagger}L^{gt}(d\boldsymbol{w})\Big)^i.
\end{aligned}$$

Applying triangle inequality, we arrive at

$$\begin{aligned}
\|L^{gt}(\boldsymbol{w})^{\dagger}\|_1 &\leq \|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1\sum_{i=0}^{\infty}\big(\|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1\|L^{gt}(d\boldsymbol{w})\|_1\big)^i \\
&= \frac{\|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1}{1-\|L^{gt}(\overline{\boldsymbol{w}})^{\dagger}\|_1\|L^{gt}(d\boldsymbol{w})\|_1}.
\end{aligned}$$

$\square$

E.3.4 Proof of Lemma 5

We first describe two propositions regarding the non-empty blocks of $dL(\boldsymbol{w})$.

Proposition 12. *$\forall (i,j)\in\mathcal{E}^{\text{in}}$, we have*

$$\|(\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt}\| \leq \epsilon\|\boldsymbol{t}_{ij}^{gt}\|, \tag{75}$$
$$\|\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T\| \leq \epsilon. \tag{76}$$

Moreover, $\forall (i,j)\in\mathcal{E}^{\text{out}}$, we have

$$\|(\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt}\| \leq \|\boldsymbol{t}_{ij}^{gt}\|, \tag{77}$$
$$\|\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T\| \leq 1, \tag{78}$$

Proof: Express $\boldsymbol{v}_{ij}$ as

$$\boldsymbol{v}_{ij} = \cos(\theta)\boldsymbol{v}_{ij}^{gt} + \sin(\theta){\boldsymbol{v}_{ij}^{gt}}^{\perp},$$
$${\boldsymbol{v}_{ij}^{gt}}^{\perp} = \frac{(I_3 - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T)\boldsymbol{v}_{ij}}{\|(I_3 - \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T)\boldsymbol{v}_{ij}\|}.$$

Then

$$\begin{aligned}
& \boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T \\
=& (\boldsymbol{v}_{ij}^{gt}, {\boldsymbol{v}_{ij}^{gt}}^{\perp})\sin(\theta)\begin{pmatrix}\sin(\theta) & -\cos(\theta) \\ -\cos(\theta) & -\sin(\theta)\end{pmatrix}(\boldsymbol{v}_{ij}^{gt}, {\boldsymbol{v}_{ij}^{gt}}^{\perp})^T.
\end{aligned} \tag{79}$$

This means

$$\begin{aligned}
& \|\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T\| \\
=& |\sin(\theta)|\|\begin{pmatrix}\sin(\theta) & -\cos(\theta) \\ -\cos(\theta) & -\sin(\theta)\end{pmatrix}\| \\
=& |\sin(\theta)|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \|(\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt}\| \\
=& |\sin(\theta)|\|\begin{pmatrix}\sin(\theta) & -\cos(\theta) \\ -\cos(\theta) & -\sin(\theta)\end{pmatrix}(\|\boldsymbol{t}_{ij}^{gt}\|, 0)^T\| \\
=& |\sin(\theta)|\|\boldsymbol{t}_{ij}^{gt}\|.
\end{aligned}$$

We complete the proof by noting that When $(i,j)\in\mathcal{E}^{\text{in}}$, we have $|\sin(\theta)|\leq\epsilon$. $\square$

We now complete the proof of Lemma 5. Applying Eq. (75) and Eq. (77), Eq. (54) is true because

$$\|dL(\boldsymbol{w})\boldsymbol{t}^{gt}\|_{\infty} \leq \|dL(\overline{\boldsymbol{w}})\boldsymbol{t}^{gt}\|_{\infty} + \|dL(d\boldsymbol{w})\boldsymbol{t}^{gt}\|_{\infty}$$

where

$$\begin{aligned}
& \|dL(d\boldsymbol{w})\boldsymbol{t}^{gt}\|_{\infty} \\
\leq& \max_{1\leq i\leq n}\Big(\sum_{j\in\mathcal{N}_i}|dw_{ij}|\|(\boldsymbol{v}_{ij}^{gt}{\boldsymbol{v}_{ij}^{gt}}^T - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt}\|\Big) \\
\leq& \max_{1\leq i\leq n}\Big(\sum_{j\in\mathcal{N}_i^{\text{in}}}\epsilon|dw_{ij}|\|\boldsymbol{t}_{ij}^{gt}\| + \sum_{j\in\mathcal{N}_i^{\text{out}}}|dw_{ij}|\|\boldsymbol{t}_{ij}^{gt}\|\Big) \\
=& \max_{1\leq i\leq n}\Big(\epsilon\delta_i^{\text{in}}(d\boldsymbol{w}) + \delta_i^{\text{out}}(d\boldsymbol{w})\Big).
\end{aligned}$$

Eq. (55) can be proven in a similar fashion. $\square$

E.3.5 Proof of Lemma 6

First of all, we have

$$\boldsymbol{t}^{gt^T} dL(\boldsymbol{w})\boldsymbol{t}^{gt} = \boldsymbol{t}^{gt^T} dL(\overline{\boldsymbol{w}})\boldsymbol{t}^{gt} + \boldsymbol{t}^{gt^T} dL(d\boldsymbol{w})\boldsymbol{t}^{gt}$$

where

$$\begin{aligned} & \boldsymbol{t}^{gt^T} dL(d\boldsymbol{w})\boldsymbol{t}^{gt} \\ = & \sum_{(i,j)\in\mathcal{E}} (w_{ij} - \overline{w}_{ij})\boldsymbol{t}_{ij}^{gt^T}(\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T} - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt} \end{aligned}$$

Applying Eq. (79), we have

$$\begin{aligned} & \boldsymbol{t}_{ij}^{gt^T}(\boldsymbol{v}_{ij}^{gt}\boldsymbol{v}_{ij}^{gt^T} - \boldsymbol{v}_{ij}\boldsymbol{v}_{ij}^T)\boldsymbol{t}_{ij}^{gt} \\ = & \sin^2(\theta)\|\boldsymbol{t}_{ij}^{gt}\|^2 \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{t}^{gt^T} dL(d\boldsymbol{w})\boldsymbol{t}^{gt} \leq \epsilon^2 & \sum_{(i,j)\in\mathcal{E}^{\text{in}}} |w_{ij} - \overline{w}_{ij}|\|\boldsymbol{t}_{ij}^{gt}\|^2 \\ + & \sum_{(i,j)\in\mathcal{E}^{\text{out}}} |w_{ij} - \overline{w}_{ij}|\|\boldsymbol{t}_{ij}^{gt}\|^2, \end{aligned}$$

which proves Eq. (56).

Moreover,

$$\begin{aligned} \|dL(\boldsymbol{w})\| & \leq \|dL(\overline{\boldsymbol{w}}\| + \|dL(d\boldsymbol{w})\| \\ & \leq \|dL(\overline{\boldsymbol{w}}\| + \|dL(d\boldsymbol{w})\|_1. \end{aligned}$$

The rest of the proof follows that of Lemma 5 in Section E.3.4. □

E.4. Proof of Lemma 7

It is clear that the minimum value of $\|\boldsymbol{t}_{ij}\|$ and the minimum value of $\|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|$ can be obtained in isolation. The minimum value of $\|\boldsymbol{t}_{ij}\|$ is $(1-\eta)\|\boldsymbol{t}_{ij}^{gt}\|$. When $\boldsymbol{v}_{ij}^{\text{inp}}$ is in the cone specified by $\|\boldsymbol{t}_{ij} - \boldsymbol{t}_{ij}^{gt}\| \leq \eta\|\boldsymbol{t}_{ij}^{gt}\|$, the minimum value is given by $\|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|$. Otherwise, $\|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|$ is given by the difference between the angle between $\boldsymbol{v}_{ij}^{\text{inp}}$ and $\boldsymbol{v}_{ij}^{gt}$ and half-angle of the cone. This ends the proof.

□

E.5. Proof of Lemma 8

The proof is very similar to that of Lemma 7. The only difference is that the maximum value of $\|\boldsymbol{v}_{ij}^{\text{inp}} - \boldsymbol{v}_{ij}\|$ is 2.

□