A Efficient Implementation

In this section, we show how to make Algorithm 1 computationally tractable with negligible effects in the final regret bound. Indeed, Algorithm 1 is based on a confidence set $\mathcal{C}_t(\delta)$ (Equation 7) that requires evaluating the norm of a difference of operators g_t, which is clearly computationally infeasible. In the following, we discuss how to make both steps of the algorithm computationally tractable. The resulting algorithm, `Eff-GKB-UCB`, is provided in Algorithm 2.

Input: Decision set $\mathcal{X}$, confidence level δ, confidence sets $\mathcal{D}_t(\delta)$
for $t \in [\![T]\!]$ **do**
 `/* Step 1: Efficient Maximum Likelihood Estimate */`
 $\widehat{\boldsymbol{\alpha}}_t = \underset{\boldsymbol{\alpha} \in \mathbb{R}^{t-1}}{\arg\min}\, \mathcal{L}_t\left(\langle \boldsymbol{\alpha}, \mathbf{k}_t(\cdot)\rangle\right)$ (Equation 3)
 `/* Step 2: Efficient Optimistic Decision Selection */`
 $\mathbf{x}_t \in \underset{\mathbf{x} \in \mathcal{X}}{\arg\max} \max_{\boldsymbol{\alpha} \in \mathbb{R}^{t-1}} \langle \boldsymbol{\alpha}, \mathbf{k}_t(\widehat{\mathbf{x}})\rangle$
  s.t. $\mathcal{L}_t(\langle \boldsymbol{\alpha}, \mathbf{k}_t(\cdot)\rangle) \leq \mathcal{L}_t(\langle \widehat{\boldsymbol{\alpha}}_t, \mathbf{k}_t(\cdot)\rangle) + D_t(\delta; \mathcal{H})$ (Equation 23)
 Play $\mathbf{x}_t$ and observe y_t
end

Algorithm 2: `Eff-GKB-UCB`.

Efficient Maximum Likelihood Estimation. Since function m is convex, loss function $\mathcal{L}_t(f)$ is convex in $f = \langle \alpha, \phi \rangle \in \mathcal{H}$ and, consequently, also in the parameter vector $\alpha \in \mathbb{R}^{\mathbb{N}}$. However, optimizing over either f or α is infeasible, being both infinite-dimensional. Nevertheless, thanks to the *generalized representer theorem* [27, Theorem 1], we can restrict the optimization to the functions of the form $f(\cdot) = \sum_{s=1}^{t-1} \alpha_s k(\cdot, \mathbf{x}_s) = \langle \boldsymbol{\alpha}, \mathbf{k}_t(\cdot)\rangle$, where $\boldsymbol{\alpha} = (\alpha_s)^\top_{s \in [\![t-1]\!]}$ and $\mathbf{k}_t(\cdot) = (k(\cdot, \mathbf{x}_s))^\top_{s \in [\![t-1]\!]}$. This allows limiting the problem to the minimization of a convex function on a vector of $t-1$ real variables $\boldsymbol{\alpha} \in \mathbb{R}^{t-1}$.

Efficient Optimistic Decision Selection. To make the choice of the optimistic function, we propose a different (looser) confidence set based on the evaluation of the loss function only [2], defined for every round $t \in [\![T]\!]$ and confidence $\delta \in (0, 1)$:[9]

$$\mathcal{D}_t(\delta) := \left\{ f \in \mathcal{H} \,:\, \mathcal{L}_t(f) - \mathcal{L}_t(\widehat{f}_t) \leq D_t(\delta; \mathcal{H}) := (1 + 2R_s BK) B_t(\delta; \mathcal{H}) \right\}, \tag{22}$$

We prove in Lemma B.2 that the choice of the confidence radius ensures the inclusion property between the confidence sets $\mathcal{C}_t(\delta) \subseteq \mathcal{D}_t(\delta)$. Having fixed a decision $\widehat{\mathbf{x}} \in \mathcal{X}$,[10] the optimistic decision selection can be formulated, thanks to the generalized represented theorem [27] as the following constrained convex program:[11]

$$\begin{aligned} \min_{\boldsymbol{\alpha} \in \mathbb{R}^{t-1}} \quad & -\langle \boldsymbol{\alpha}, \mathbf{k}_t(\widehat{\mathbf{x}})\rangle \\ \text{subject to} \quad & \mathcal{L}_t(\langle \boldsymbol{\alpha}, \mathbf{k}_t(\cdot)\rangle) \leq \mathcal{L}_t(\langle \widehat{\boldsymbol{\alpha}}_t, \mathbf{k}_t(\cdot)\rangle) + (1 + 2R_s BK) B_t(\delta; \mathcal{H}), \end{aligned} \tag{23}$$

where $\widehat{\boldsymbol{\alpha}}_t$ are the parameters of the ML function computed in the previous step. Thus, the program has a linear objective function and a convex constraint, being $\mathcal{L}_t(\langle \boldsymbol{\alpha}, \mathbf{k}_t(\cdot)\rangle)$ convex in $\boldsymbol{\alpha}$.

---

[9] Computing $B_t(\delta; \mathcal{H})$ can be not straightforward for specific choices of kernel k and inverse link function μ. In such a case, we can upper bound it using Lemma 5.1 by replacing $\sup_{f \in \mathcal{H}} \log\det(\lambda^{-1} \widetilde{V}_t(\lambda; f))$ with $\max\{1, R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1}\} \log\det(\lambda^{-1} V_t(\lambda))$. This has the effect of replacing the terms $\widetilde{\gamma}_T(\mathcal{H})$ with $\max\{1, R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1}\}\gamma_T$ in the final regret bound.

[10] As customary in this literature [29, 5], we do not address the issue of optimizing over the decision space efficiently. This can be surely done efficiently when $\mathcal{X}$ is finite. When $\mathcal{X}$ is continuous, we can resort to a *discretization* based on the regularity properties of the kernel function, with a controllable effect on the final regret performances [21].

[11] Even if the represented theorem is formulated for unconstrained minimization, it admits costs functions that take $+\infty$ as value [27]. Thus, we can convert a constrained minimization into an unconstrained one by bringing the constraint into the objective function and making it take value $+\infty$ when the constraint is violated.

We now show that the choice of the new confidence set $\mathcal{D}_t(\delta)$ does not degrade the dependence on the relevant quantities compared to using $\mathcal{C}_t(\delta)$.

Theorem A.1 (Regret Bound of `Eff-GKB-UCB`)**.** *Under Assumptions 3.1, 3.2, 3.3, and 3.4,* `GKB-UCB` *with confidence radius* $(1+2R_sBK)B_t(\delta;\mathcal{H})$ *and* $\lambda > 0$*, for every* $\delta \in (0,1)$*, with probability at least* $1-\delta$*, suffers regret bounded as:* $R(\texttt{Eff-GKB-UCB}, T) = R_{\text{perm}}(T) + R_{\text{trans}}(T)$*, where:*

$$R_{\text{perm}}(T) \leq 4\sqrt{\max\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot\mu}K^2\}}(2+2R_sBK)\sqrt{\beta_T(\delta;\mathcal{H})}\left(\sqrt{\beta_T(\delta;\mathcal{H})}+2\right)\sqrt{\widetilde{\gamma}_T(f^*)}\sqrt{\frac{T}{\kappa_*}},$$

$$R_{\text{trans}}(T) \leq 8R_s(1+R_{\dot\mu}\kappa_{\mathcal{X}})\max\left\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot\mu}K^2\right\}(2+2R_sBK)^2\beta_T(\delta;\mathcal{H})\left(\sqrt{\beta_T(\delta;\mathcal{H})}+2\right)^2\widetilde{\gamma}_T(f^*).$$

The bounds of Theorem 7.2 and Theorem A.1 exhibit the same order dependence on the relevant quantities, but Theorem A.1 has a larger constant, approximately 3 times larger than Theorem 7.2 for $R_{\text{perm}}(T)$ and 9 times larger for $R_{\text{trans}}(T)$.

B Proofs

B.1 Proofs of Section 5

Lemma 5.1. *Let $\mathcal{H}$ be a RKHS induced by kernel k. Let $t \in \mathbb{N}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_t \in \mathcal{X}$ be a sequence of decisions. It holds that $\widetilde{\Gamma}_t(\mathcal{H}) \leq \max\{1, R_{\dot\mu}\mathfrak{g}(\tau)^{-1}\}\Gamma_t$.*

Proof. A direct application of Lemma C.5. □

B.2 Proofs of Section 6

Lemma B.1 (Freedman's Inequality)**.** *Let $(z_t)_{t\geq 1}$ be a real-valued martingale difference sequence adapted to the filtration $\mathcal{F}_t$ such that $z_t \leq R$ a.s. for all $t \geq 1$. Then, for every $\lambda \in (0, 3/R)$ it holds that with probability at least $1-\delta$:*

$$\forall t \geq 1: \qquad \sum_{s=1}^{t} z_s \leq \frac{\lambda}{2(1-\lambda R/3)}\sum_{s=1}^{t}\mathbb{E}[z_s^2|\mathcal{F}_{s-1}] + \frac{\log\delta^{-1}}{\lambda}. \tag{24}$$

This implies that for every $\nu > 0$, with probability at least $1-\delta$:

$$\forall t \geq 1: \qquad \sum_{s=1}^{t} z_s \leq \nu\sqrt{2\log\delta^{-1}} + \frac{R\log\delta^{-1}}{3} \quad \textit{or} \quad \sum_{s=1}^{t}\mathbb{E}[z_s^2|\mathcal{F}_{s-1}] > \nu^2. \tag{25}$$

Proof. Refer to Theorem 13.6 of [35]. □

Theorem 6.1 (A data-driven Freedman's inequality)**.** *Let $(z_t)_{t\geq 1}$ be a real-valued martingale difference sequence adapted to the filtration $\mathcal{F}_t$ such that $z_t \leq R$ a.s. for all $t \geq 1$. Let $(v_t)_{t\geq 1}$ be a process predictable by the filtration $\mathcal{F}_t$ such that for every $t \geq 1$, we have that $\sum_{s=1}^{t}\mathbb{E}[z_s^2|\mathcal{F}_{s-1}] \leq v_t$ a.s.. Then, for every $\eta > 1$ and $v_0 > 0$, with probability at least $1-\delta$, it holds that:*

$$\forall t \geq 1: \qquad \sum_{s=1}^{t} z_s \leq \sqrt{2\max\{v_0, \eta v_t\}\log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}, \tag{15}$$

where $\widehat{\ell} = \max\left\{0, \lceil\log_\eta(v_t/v_0)\rceil\right\}$.

Proof. The proof makes use of classical Freedman's inequality [11] combined with a *stitching* argument [13]. We start from the version of Freedman's inequality of Lemma B.1 taken from [35]:

$$\Pr\left(\exists t \geq 1: \sum_{s=1}^{t} z_s > \nu\sqrt{2\log\delta^{-1}} + \frac{R\log\delta^{-1}}{3}, \; \sum_{s=1}^{t}\mathbb{E}[z_s^2|\mathcal{F}_{s-1}] \leq \nu^2\right) \leq \delta. \tag{26}$$

Since $v_t \geq \sum_{s=1}^{t} \mathbb{E}[z_s^2|\mathcal{F}_{s-1}]$ a.s. for every $t \geq 1$, it immediately follows that:

$$\Pr\left(\exists t \geq 1 : \sum_{s=1}^{t} z_s > \nu\sqrt{2\log\delta^{-1}} + \frac{R\log\delta^{-1}}{3},\ v_t \leq \nu^2\right) \leq \delta. \tag{27}$$

We now proceed by performing a stitching argument with a geometric grid over the values of $\nu \geq 0$ defined as $\{\eta^\ell v_0 : \ell \in \mathbb{N}\}$ for any choice of $\eta > 1$ and $v_0 > 0$. Thus, we have:

$$\Pr\left(\exists \ell \in \mathbb{N},\ \exists t \geq 1 : \sum_{s=1}^{t} z_s > \sqrt{2\eta^\ell v_0 \log\frac{\pi^2(\ell+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(\ell+1)^2}{6\delta},\ v_t \leq \eta^\ell v_0\right) \tag{28}$$

$$\leq \sum_{\ell\in\mathbb{N}} \Pr\left(\exists t \geq 1 : \sum_{s=1}^{t} z_s > \sqrt{2\eta^\ell v_0 \log\frac{\pi^2(\ell+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(\ell+1)^2}{6\delta},\ v_t \leq \eta^\ell v_0\right) \tag{29}$$

$$\leq \sum_{\ell\in\mathbb{N}} \frac{6\delta}{\pi^2(\ell+1)^2} \leq \delta, \tag{30}$$

where line (29) follows from a union bound, line (30) is an application of Equation (27) with $\nu = \eta^\ell v_0$ and by observing that $\sum_{\ell\in\mathbb{N}} \frac{1}{(\ell+1)^2} = \frac{\pi^2}{6}$. Let us now consider the smallest value of $\widehat{\ell} \in \mathbb{N}$ such that $v_t \leq \eta^\ell v_0$:

$$\widehat{\ell} = \min\left\{\ell \in \mathbb{N} : v_t \leq \eta^\ell v_0\right\} = \max\left\{0, \left\lceil \log_\eta \frac{v_t}{v_0} \right\rceil\right\}. \tag{31}$$

For this value of $\widehat{\ell}$, we have:

$$\eta^{\widehat{\ell}} v_0 \leq \eta^{\max\left\{0, \left\lceil \log_\eta \frac{v_t}{v_0}\right\rceil\right\}} v_0 \leq \eta^{\max\left\{0, \log_\eta \frac{v_t}{v_0} + 1\right\}} v_0 \leq \max\left\{v_0, \eta v_t\right\}. \tag{32}$$

Finally, we prove the inequality:

$$\Pr\left(\exists t \geq 1 : \sum_{s=1}^{t} z_s > \sqrt{2\max\{v_0, \eta v_t\}\log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}\right) \tag{33}$$

$$\leq \Pr\left(\exists t \geq 1 : \sum_{s=1}^{t} z_s > \sqrt{2\eta^{\widehat{\ell}} v_0 \log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(\widehat{\ell}+1)^2}{6\delta}\ , v_t \leq \eta^{\widehat{\ell}} v_0\right) \tag{34}$$

$$\leq \Pr\left(\exists \ell \in \mathbb{N},\ \exists t \geq 1 : \sum_{s=1}^{t} z_s > \sqrt{2\eta^\ell v_0 \log\frac{\pi^2(l+1)^2}{6\delta}} + \frac{R}{3}\log\frac{\pi^2(l+1)^2}{6\delta}\ , v_t \leq \eta^\ell v_0\right) \tag{35}$$

$$\leq \delta,$$

where line (34) follows from Equation (32) and line (35) from line (30). $\square$

Theorem 6.2 (Bernstein-Like Dimension-Free Self-Normalized Concentration)**.** *Let $(\mathbf{x}_t)_{t\geq 1}$ be a discrete-time stochastic process predictable by the filtration $\mathcal{F}_t$ and let $(\epsilon_t)_{t\geq 1}$ be a real-valued stochastic process adapted to the $\mathcal{F}_t$ such that $\mathbb{E}[\epsilon_t|\mathcal{F}_{t-1}] = 0$, $\mathbb{V}\mathrm{ar}[\epsilon_t|\mathcal{F}_{t-1}] = \sigma_t^2 = \sigma^2(\mathbf{x}_t)$, and $|\epsilon_t| \leq R$ a.s. for every $t \geq 1$. Let $\phi : \mathcal{X} \to \mathbb{R}^{\mathbb{N}}$ be the feature mapping induced by kernel k such that $\|\phi(\mathbf{x})\|_2 \leq K$ for every $\mathbf{x} \in \mathcal{X}$. Let:*

$$S_t \coloneqq \sum_{s=1}^{t-1} \epsilon_s \phi(\mathbf{x}_s), \qquad \widetilde{V}_t(\lambda) \coloneqq \sum_{s=1}^{t-1} \sigma_s^2 \phi(\mathbf{x}_s)\phi(\mathbf{x}_s)^\top + \lambda I. \tag{16}$$

Then, for every $\delta \in (0,1)$ and $t \geq 1$, with probability at least $1-\delta$ it holds that:

$$\|S_t\|_{\widetilde{V}_t^{-1}(\lambda)} \leq \left(\sqrt{73 \log \det(\lambda^{-1}\widetilde{V}_t)} + \sqrt{3}\right) \sqrt{\log \frac{\pi^2(\rho+1)^2}{3\delta}} + \frac{3RK}{\sqrt{\lambda}} \log \frac{\pi^2(\rho+1)^2}{3\delta}, \quad (17)$$

where $\rho = \max\left\{0, \left\lceil \log\left(\frac{8R^2K^2(t-1)^3}{\lambda} \log\left(1 + \frac{K^2R^2}{\lambda}\right)\right)\right\rceil\right\}$.

Proof. The proof follows similar steps as [6, 36], using Theorem 6.1 as base inequality. For the sake of this derivation, we will suppress the dependence on λ, simply writing $\widetilde{V}_t(\lambda) = \widetilde{V}_t$.[12] Let us introduce the notation $Z_t := \|S_t\|_{\widetilde{V}_t^{-1}}$, $w_t := \|\phi(\mathbf{x}_t)\|_{\widetilde{V}_t^{-1}}$, and $\widetilde{w}_t := \sigma_t \|\phi(\mathbf{x}_t)\|_{\widetilde{V}_t^{-1}}$. We denote with $K = \sup_{\mathbf{x} \in \mathcal{X}} \|\phi(\mathbf{x})\|_2$. From the matrix inversion lemma [33], we have:

$$\widetilde{V}_t^{-1} = \widetilde{V}_{t-1}^{-1} - \frac{\widetilde{V}_{t-1}^{-1}\phi(\mathbf{x}_{t-1})\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1}\sigma_{t-1}^2}{1 + \|\phi(\mathbf{x}_{t-1})\|^2_{\widetilde{V}_{t-1}^{-1}}\sigma_{t-1}^2} \quad (36)$$

$$= \widetilde{V}_{t-1}^{-1} - \frac{\widetilde{V}_{t-1}^{-1}\phi(\mathbf{x}_{t-1})\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1}\sigma_{t-1}^2}{1 + \widetilde{w}_{t-1}^2}. \quad (37)$$

Let us decompose Z_t:

$$Z_t^2 := \|S_t\|^2_{\widetilde{V}_t^{-1}} = S_t^\top \widetilde{V}_t^{-1} S_t \quad (38)$$

$$= (S_{t-1} + \epsilon_{t-1}\phi(\mathbf{x}_{t-1}))^\top \widetilde{V}_t^{-1}(S_{t-1} + \epsilon_{t-1}\phi(\mathbf{x}_{t-1})) \quad (39)$$

$$= S_{t-1}\widetilde{V}_t^{-1}S_{t-1} + 2\epsilon_{t-1}\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_t^{-1} S_{t-1} + \epsilon_{t-1}^2 \phi(\mathbf{x}_{t-1})^\top \widetilde{V}_t^{-1}\phi(\mathbf{x}_{t-1}) \quad (40)$$

$$\leq S_{t-1}\widetilde{V}_{t-1}^{-1}S_{t-1} + \underbrace{2\epsilon_{t-1}\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_t^{-1} S_{t-1}}_{\text{(A)}} + \underbrace{\epsilon_{t-1}^2 \phi(\mathbf{x}_{t-1})^\top \widetilde{V}_t^{-1}\phi(\mathbf{x}_{t-1})}_{\text{(B)}}, \quad (41)$$

having exploited the fact that $\widetilde{V}_t \succeq \widetilde{V}_{t-1}$. We analyze terms (A) and (B) separately.

Analysis of Term (A). From the matrix inversion lemma, we have:

$$2\epsilon_{t-1}\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_t^{-1} S_{t-1} = 2\epsilon_{t-1}\Bigg(\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1} \quad (42)$$

$$- \frac{\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1}\phi(\mathbf{x}_{t-1})\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1}\sigma_{t-1}^2}{1 + \widetilde{w}_{t-1}^2}\Bigg) \quad (43)$$

$$= 2\epsilon_{t-1}\left(\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1} - \frac{\widetilde{w}_{t-1}^2}{1 + \widetilde{w}_{t-1}^2}\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1}\right) \quad (44)$$

$$= 2\epsilon_{t-1}\frac{\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1}}{1 + \widetilde{w}_{t-1}^2} =: \ell_t. \quad (45)$$

Consider now the event $\mathcal{E}_t = \mathbb{1}\{0 \leq s \leq t : Z_s \leq \beta_t\}$, being β_t a non-negative non-decreasing predictable process, whose expression will be defined later. Furthermore, let us define $\widetilde{\beta}_t = \min\left\{\beta_t, \frac{(t-1)RK}{\sqrt{\lambda}}\right\}$ which is non-decreasing as well. Under event $\mathcal{E}_t$, we know that $Z_s \leq \widetilde{\beta}_t$ thanks to Lemma C.7. Under $\mathcal{E}_t$, we bound the maximum value and the variance of ℓ_t. Let us start with the maximum value:

$$\ell_t \mathcal{E}_t \leq |\ell_t \mathcal{E}_t| \leq \left| 2\epsilon_{t-1}\frac{\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} S_{t-1}}{1 + \widetilde{w}_{t-1}^2}\mathcal{E}_t \right| \quad (46)$$

[12] With little abuse, we will ignore the fact that ϕ is an infinite-dimensional feature mapping to avoid excessive technicalities. We refer the interested reader to [32] that shows that all passages we do are indeed legal when ϕ is the feature mapping induced by an RKHS.

$$\leq \frac{2R}{1+\widetilde{w}_{t-1}^2}\|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}\|S_{t-1}\|_{\widetilde{V}_{t-1}^{-1}}\mathcal{E}_t \tag{47}$$

$$\leq \frac{2R\|\phi(\mathbf{x}_{t-1})\|_{\lambda^{-1}I}\beta_{t-1}}{1+\widetilde{w}_{t-1}^2} \tag{48}$$

$$\leq \frac{2RK}{\sqrt{\lambda}}\beta_t, \tag{49}$$

where line (47) follows from the application of Cauchy-Schwarz inequality and recalling that $|\epsilon_{t-1}| \leq R$ a.s., line (48) is obtained by observing that $\widetilde{V}_{t-1} \succeq \lambda I$ and by exploiting event $\mathcal{E}_t$, and line (49) comes from the bound on $\|\phi(\mathbf{x}_{t-1})\| \leq K$ and the monotonicity of β_t. Let us move to the variance, recalling that ℓ_t is zero mean, i.e., $\mathbb{E}[\ell_t|\mathcal{F}_{t-1}] = 0$:

$$\mathbb{E}\left[\ell_t^2|\mathcal{F}_{t-1}\right] = \mathbb{E}\left[\left(2\epsilon_{t-1}\frac{\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1}S_{t-1}}{1+\widetilde{w}_{t-1}^2}\right)^2 \mathcal{E}_t|\mathcal{F}_{t-1}\right] \tag{50}$$

$$\leq \frac{4\sigma_{t-1}^2\|\phi(\mathbf{x}_{t-1})\|^2_{\widetilde{V}_{t-1}^{-1}}\|S_{t-1}\|^2_{\widetilde{V}_{t-1}^{-1}}}{(1+\widetilde{w}_{t-1}^2)^2}\mathcal{E}_t \tag{51}$$

$$\leq \left(\frac{2\widetilde{w}_{t-1}}{1+\widetilde{w}_{t-1}^2}\right)^2 \widetilde{\beta}_{t-1}^2 \tag{52}$$

$$\leq \min\{1, 2\widetilde{w}_{t-1}\}^2\widetilde{\beta}_{t-1}^2, \tag{53}$$

where line (51) follows from Cauchy-Schwarz inequality and recalling that $\mathbb{E}[\epsilon_{t-1}|\mathcal{F}_{t-1}] = \sigma_{t-1}^2$, line (53) follows from the inequality $\frac{2x}{1+x^2} \leq \min\{1, 2x\}$ for $x \geq 0$. Summing, we obtain:

$$\sum_{s=1}^{t}\mathbb{E}\left[\ell_s^2|\mathcal{F}_{s-1}\right] \leq \sum_{s=1}^{t}\min\{1, 2\widetilde{w}_{s-1}\}^2\widetilde{\beta}_{t-1}^2 \tag{54}$$

$$\leq 4\widetilde{\beta}_t^2\sum_{s=1}^{t}\min\{1, \widetilde{w}_{s-1}\}^2, \tag{55}$$

where we bounded $\widetilde{\beta}_{t-1} \leq \widetilde{\beta}_t$ and $\min\{1, 2\widetilde{w}_{s-1}\}^2 \leq 4\min\{1, \widetilde{w}_{s-1}\}^2$. From a standard elliptical potential lemma (Lemma C.6 with $M = 1$), we obtain:

$$\sum_{s=1}^{t}\min\{1, \widetilde{w}_{s-1}\}^2 \leq 2\log\frac{\det(\widetilde{V}_t)}{\det(\widetilde{V}_0)} = 2\log\det(\lambda^{-1}\widetilde{V}_t), \tag{56}$$

where $\widetilde{V}_0 = \lambda I$. By Theorem 6.1, setting $\eta = e$, $v_0 = 1$, $v_t = 8\beta_t^2\log\det(\lambda^{-1}\widetilde{V}_t)$, we have that with probability at least $1-\delta$:

$$\forall t \geq 1: \sum_{s=1}^{t}\ell_s \leq \sqrt{2\max\left\{1, 8e\beta_t^2\log\det(\lambda^{-1}\widetilde{V}_t)\right\}\log\frac{\pi^2(\widehat{\rho}+1)^2}{6\delta}} + \frac{2RK}{3\sqrt{\lambda}}\beta_t\log\frac{\pi^2(\widehat{\rho}+1)^2}{6\delta}, \tag{57}$$

with $\widehat{\rho} = \max\left\{0, \left\lceil\log\left(8\frac{(t-1)^2R^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\right)\right\rceil\right\}$, having bounded $\widetilde{\beta}_t \leq \beta_t$ in the inequality and $\widetilde{\beta}_t \leq \frac{(t-1)RK}{\sqrt{\lambda}}$ in the expression of $\widehat{\rho}$.

Analysis of Term (B). We proceed again by using the matrix inversion lemma:

$$\epsilon_{t-1}^2\phi(\mathbf{x}_{t-1})^\top\widetilde{V}_t^{-1}\phi(\mathbf{x}_{t-1}) = \epsilon_{t-1}^2\Bigg(\phi(\mathbf{x}_{t-1})^\top\widetilde{V}_{t-1}^{-1}\phi(\mathbf{x}_{t-1}) \tag{58}$$

$$- \frac{\phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} \phi(\mathbf{x}_{t-1}) \phi(\mathbf{x}_{t-1})^\top \widetilde{V}_{t-1}^{-1} \phi(\mathbf{x}_{t-1}) \sigma_{t-1}^2}{1 + \widetilde{w}_{t-1}^2} \Bigg) \tag{59}$$

$$= \frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2}. \tag{60}$$

Let us define:

$$\ell_t := \frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} - \mathbb{E}\left[\frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} | \mathcal{F}_{t-1} \right].$$

Let us start bounding the maximum value:

$$\ell_t \le \frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} \le \frac{R^2 K^2}{\lambda}, \tag{61}$$

where we bounded $\|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2 \le \|\phi(\mathbf{x}_{t-1})\|_{\lambda^{-1} I}^2 \le \frac{K^2}{\lambda}$.

Concerning the variance, we have:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[\ell_t | \mathcal{F}_{t-1}] &= \mathbb{V}\mathrm{ar}\left[\frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} | \mathcal{F}_{t-1} \right] && (62)\\
&\le \mathbb{E}\left[\left(\frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} \right)^2 | \mathcal{F}_{t-1} \right] && (63)\\
&\le \frac{R^2 K^2}{\lambda} \mathbb{E}\left[\frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} | \mathcal{F}_{t-1} \right] && (64)\\
&= \frac{R^2 K^2}{\lambda} \frac{\sigma_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} && (65)\\
&= \frac{R^2 K^2}{\lambda} \frac{\widetilde{w}_{t-1}^2}{1 + \widetilde{w}_{t-1}^2} && (66)\\
&\le \frac{R^2 K^2}{\lambda} \min\{1, \widetilde{w}_{t-1}\}^2, && (67)
\end{aligned}$$

where line (64) derives from applying Equation (61), line (67) follows from the inequality $\frac{x}{1+x} \le \min\{1, x\}$. Summing and applying the elliptic potential lemma (Lemma C.6 with $M = 1$), we have:

$$\sum_{s=1}^{t} \mathbb{V}\mathrm{ar}[\ell_s | \mathcal{F}_{s-1}] \le \frac{R^2 K^2}{\lambda} \sum_{s=1}^{t} \min\{1, \widetilde{w}_{s-1}\}^2 \le \frac{2 R^2 K^2}{\lambda} \log\det(\lambda^{-1} \widetilde{V}_t). \tag{68}$$

Furthermore, following the same steps from Equation (64), we obtain:

$$\sum_{s=1}^{t} \mathbb{E}[\ell_s | \mathcal{F}_{s-1}] = \sum_{s=1}^{t} \mathbb{E}\left[\frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} | \mathcal{F}_{t-1} \right] \le 2 \log\det(\lambda^{-1} \widetilde{V}_t). \tag{69}$$

We now apply Theorem 6.1, setting $\eta = e$, $v_0 = 1$, $v_t = \frac{2R^2K^2}{\lambda} \log\det(\lambda^{-1} \widetilde{V}_t)$, we have that with probability at least $1 - \delta$:

$$\forall t \ge 1 : \sum_{s=1}^{t} \frac{\epsilon_{t-1}^2 \|\phi(\mathbf{x}_{t-1})\|_{\widetilde{V}_{t-1}^{-1}}^2}{1 + \widetilde{w}_{t-1}^2} \le 2 \log\det(\lambda^{-1} \widetilde{V}_t) \tag{70}$$

$$+\sqrt{2\max\left\{1,\frac{2eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\right\}\log\frac{\pi^2(\widetilde{\rho}+1)^2}{6\delta}}+\frac{R^2K^2}{3\lambda}\log\frac{\pi^2(\widetilde{\rho}+1)^2}{6\delta}, \tag{71}$$

with $\widetilde{\rho}=\max\left\{0,\left\lceil\log\left(\frac{2R^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\right)\right\rceil\right\}$.

Putting All Together. We observe that $\widehat{\rho}\geq\widetilde{\rho}$, and that $\log\det(\lambda^{-1}\widetilde{V}_t)\leq(t-1)\log\left(1+\frac{R^2K^2}{\lambda}\right)$ from Lemma C.7, we define $\rho:=\max\left\{0,\left\lceil\log\left(\frac{8R^2K^2(t-1)^3}{\lambda}\log\left(1+\frac{K^2R^2}{\lambda}\right)\right)\right\rceil\right\}$. Putting together the two bounds, we have to find β_t in order to satisfy the following condition:

$$\text{(A)}+\text{(B)}\leq\sqrt{2\max\left\{1,8e\beta_t^2\log\det(\lambda^{-1}\widetilde{V}_t)\right\}\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\frac{2RK}{3\sqrt{\lambda}}\beta_t\log\frac{\pi^2(\rho+1)^2}{6\delta} \tag{72}$$

$$+2\log\det(\lambda^{-1}\widetilde{V}_t)+\sqrt{2\max\left\{1,\frac{2eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\right\}\log\frac{\pi^2(\rho+1)^2}{6\delta}} \tag{73}$$

$$+\frac{R^2K^2}{3\lambda}\log\frac{\pi^2(\rho+1)^2}{6\delta}\leq\beta_t^2. \tag{74}$$

We proceed by bounding the maxima in the left-hand-side as $\max\{a,b\}\leq a+b$ for $a,b\geq 0$ and using the subadditivity of the square root to get a stricter condition:

$$\sqrt{2\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\sqrt{16e\beta_t^2\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\frac{2RK}{3\sqrt{\lambda}}\beta_t\log\frac{\pi^2(\rho+1)^2}{6\delta} \tag{75}$$

$$+2\log\det(\lambda^{-1}\widetilde{V}_t)+\sqrt{2\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\sqrt{\frac{4eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}} \tag{76}$$

$$+\frac{R^2K^2}{3\lambda}\log\frac{\pi^2(\rho+1)^2}{6\delta}\leq\beta_t^2. \tag{77}$$

This is a second-degree inequality in the variable β_t and, thus, we have to find the minimum value of β_t fulfilling such an inequality. Using the polynomial inequality of Proposition 7 of [2] (i.e., $x^2\leq bx+c=0\implies x\leq b+\sqrt{c}$ when $b,c\geq 0$), we have:

$$\beta_t\leq\sqrt{16e\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\frac{2RK}{3\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{6\delta} \tag{78}$$

$$+\Bigg(2\sqrt{2\log\frac{\pi^2(\rho+1)^2}{6\delta}}+2\log\det(\lambda^{-1}\widetilde{V}_t) \tag{79}$$

$$+\sqrt{\frac{4eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}}+\frac{R^2K^2}{3\lambda}\log\frac{\pi^2(\rho+1)^2}{6\delta}\Bigg)^{\frac{1}{2}} \tag{80}$$

$$\leq\left((\sqrt{16e}+\sqrt{2})\sqrt{\log\det(\lambda^{-1}\widetilde{V}_t)}+\sqrt{2\sqrt{2}}\right)\sqrt{\log\frac{\pi^2(\rho+1)^2}{6\delta}} \tag{81}$$

$$+\left(\frac{2}{3}+\frac{1}{\sqrt{3}}\right)\frac{RK}{\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{6\delta} \tag{82}$$

$$+\left(\frac{4eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}\right)^{\frac{1}{4}} \tag{83}$$

$$\leq\left(\left(\sqrt{16e}+\sqrt{2}+\frac{1}{2}\right)\sqrt{\log\det(\lambda^{-1}\widetilde{V}_t)}+\sqrt{2\sqrt{2}}\right)\sqrt{\log\frac{\pi^2(\rho+1)^2}{6\delta}} \tag{84}$$

$$+\left(\frac{2}{3}+\frac{1}{\sqrt{3}}+\sqrt{e}\right)\frac{RK}{\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{6\delta}, \tag{85}$$

where line (82) follows from the subadditivity of the square root and recalling that $\log \frac{6(\rho+1)^2}{\pi^2\delta} \geq 1$ for $t \geq 1$, to get line (85), we apply Young's inequality for products as $ab \leq a^2/2 + b^2/2$ for $a, b \geq 0$ to get:

$$\left(\frac{4eR^2K^2}{\lambda}\log\det(\lambda^{-1}\widetilde{V}_t)\log\frac{\pi^2(\rho+1)^2}{6\delta}\right)^{\frac{1}{4}} \leq \sqrt{e}\frac{RK}{\sqrt{\lambda}}\sqrt{\log\frac{\pi^2(\rho+1)^2}{6\delta}} + \frac{1}{2}\sqrt{\log\det(\lambda^{-1}\widetilde{V}_t)}. \tag{86}$$

To obtain more manageable constant, we write:

$$\beta_t \leq \left(\sqrt{73\log\det(\lambda^{-1}\widetilde{V}_t)} + \sqrt{3}\right)\sqrt{\log\frac{\pi^2(\rho+1)^2}{6\delta}} + \frac{3RK}{\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{6\delta}. \tag{87}$$

A simple inductive argument allows to conclude that, with probability at least $1-2\delta$:

$$Z_t^2 \leq \left(\sqrt{73\log\det(\lambda^{-1}\widetilde{V}_t)} + \sqrt{3}\right)\sqrt{\log\frac{\pi^2(\rho+1)^2}{6\delta}} + \frac{3RK}{\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{6\delta}. \tag{88}$$

Notice that, as requested, β_t is a non-decreasing sequence of t, since ρ is non-decreasing with t and $\det(\lambda^{-1}\widetilde{V}_t)$ is non-decreasing as well. Indeed, since $\widetilde{V}_t = \widetilde{V}_{t-1} + \sigma_{t-1}\phi(\mathbf{x}_{t-1})\phi(\mathbf{x}_{t-1})^\top$, we have that thanks to Weyl's inequality for eigenvalues $\lambda_i(\widetilde{V}_t) \geq \lambda_i(\widetilde{V}_{t-1})$ for all $i \in \mathbb{N}$, being λ_i the i-th eigenvalue [30]. It follows that $\det(\lambda^{-1}\widetilde{V}_t) \geq \det(\lambda^{-1}\widetilde{V}_{t-1})$. Rescaling $\delta \leftarrow \delta/2$, we get the result. $\square$

B.3 Proofs of Section 7

Lemma 7.1 (Good Event). *Let $t \in \mathbb{N}$, $f \in \mathcal{H}$, and $\delta \in (0,1)$, define the confidence radius as:*

$$B_t(\delta; f) \coloneqq \sqrt{\lambda}B + \frac{1}{\mathfrak{g}(\tau)}\left(\sqrt{73\log\det(\lambda^{-1}\widetilde{V}_t(\lambda; f))} + \sqrt{3}\right)\sqrt{\log\frac{\pi^2(\rho+1)^2}{3\delta}} + \frac{3RK}{\mathfrak{g}(\tau)\sqrt{\lambda}}\log\frac{\pi^2(\rho+1)^2}{3\delta},$$

where $\rho = \max\left\{0, \left\lceil\log\left(\frac{8R^2K^2(t-1)^3}{\lambda}\log\left(1+\frac{K^2R^2}{\lambda}\right)\right)\right\rceil\right\}$. Let $\mathcal{E}_\delta \coloneqq \{\forall t \geq 1 \,:\, f^* \in \mathcal{C}_t(\delta)\}$. Under Assumptions 3.1, 3.2, and 3.3, it holds that $\Pr(\mathcal{E}_\delta) \geq 1-\delta$.

Proof. First of all, we observe that $\mathcal{E}_\delta = \left\{\forall t \geq 1 \,:\, \left\|g_t(f^*) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;f)} \leq B_t(\delta; f)\right\}$. Let $t \in \mathbb{N}$ and let us define $\epsilon_t \coloneqq -y_t + \mu(f^*(\mathbf{x}_t))$. We have:

$$\begin{aligned}
&g_t(f^*) - g_t(\widehat{f}_t) && (89)\\
&\quad= \sum_{s=1}^{t-1}\mathfrak{g}(\tau)^{-1}\mu(f^*(\mathbf{x}_s))\phi(\mathbf{x}_s) + \lambda\alpha^* - \sum_{s=1}^{t-1}\mathfrak{g}(\tau)^{-1}\mu(\widehat{f}_r(\mathbf{x}_s))\phi(\mathbf{x}_s) - \lambda\widehat{\alpha}_t && (90)\\
&\quad= \sum_{s=1}^{t-1}\mathfrak{g}(\tau)^{-1}(-y_s + \mu(f^*(\mathbf{x}_s))\phi(\mathbf{x}_s)) + \lambda\alpha^* && (91)\\
&\qquad - \left(\underbrace{\sum_{s=1}^{t-1}\mathfrak{g}(\tau)^{-1}(-y_s + \mu(\widehat{f}_t(\mathbf{x}_s)))\phi(\mathbf{x}_s) + \lambda\widehat{\alpha}_t}_{\nabla\mathcal{L}_t(\widehat{f}_t)=0}\right) && (92)\\
&\quad= -\mathfrak{g}(\tau)^{-1}\sum_{s=1}^{t-1}\epsilon_s\phi(\mathbf{x}_s) + \lambda\alpha^*, && (93)
\end{aligned}$$

having exploited the first-order optimality condition for the loss evaluated in the maximum-likelihood estimate, i.e., $\nabla\mathcal{L}_t(\widehat{f}_t) = 0$ and the definition of $\epsilon_s = y_s - \mu(f^*(\mathbf{x}_s))$. Now, by computing the norm, we have:

$$\left\|g_t(f^*) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;f^*)} \leq \mathfrak{g}(\tau)^{-1} \left\|\sum_{s=1}^{t-1} \epsilon_s \phi(\mathbf{x}_s)\right\|_{\widetilde{V}_t^{-1}(\lambda;f^*)} + \lambda \left\|\alpha^*\right\|_{\widetilde{V}_t^{-1}(\lambda;f^*)}. \tag{94}$$

We can immediately bound the second term under Assumption 3.1:

$$\|\alpha^*\|^2_{\widetilde{V}_t^{-1}(\lambda;f^*)} = (\alpha^*)^\top \widetilde{V}_t^{-1}(\lambda; f^*)\alpha^* \leq \lambda^{-1}\|\alpha^*\|^2 \leq \lambda^{-1}B^2, \tag{95}$$

since $\widetilde{V}_t^{-1}(\lambda; f^*) \succeq \lambda I$. For the first term, we resort to the self-normalized concentration inequality of Theorem 6.2, recalling that the variance of the noise is $\mathbb{V}\mathrm{ar}[\epsilon_s|\mathcal{F}_{s-1}] = \dot{\mu}(f^*(\mathbf{x}_s))\mathfrak{g}(\tau)^{-1}$. □

Theorem 7.2 (Regret Bound of `GKB-UCB`)**.** *Under Assumptions 3.1, 3.2, 3.3, and 3.4,* `GKB-UCB` *with the confidence radius $B_t(\delta; f)$ as defined in Lemma 7.1 and $\lambda > 0$, for every $\delta \in (0, 1)$, with probability at least $1 - \delta$, suffers regret bounded as $R(\texttt{GKB-UCB}, T) = R_{\text{perm}}(T) + R_{\text{trans}}(T)$, where:*

$$R_{\text{perm}}(T) \leq 8(1 + 2R_sBK)\beta_T(\delta;\mathcal{H})\sqrt{\max\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot{\mu}}K^2\}\,\widetilde{\gamma}_T(f^*)}\sqrt{\frac{T}{\kappa_*}}, \tag{19}$$

$$R_{\text{trans}}(T) \leq 32R_s(1 + R_{\dot{\mu}}\kappa_{\mathcal{X}})(1 + 2R_sBK)^2\beta_T(\delta;\mathcal{H})^2 \max\left\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot{\mu}}K^2\right\}\widetilde{\gamma}_T(f^*). \tag{20}$$

Proof. We start by performing a second-order Taylor's expansion of the regret:

$$\sum_{t=1}^{T}(\mu(f^*(\mathbf{x}^*)) - \mu(f^*(\mathbf{x}_t))) = \underbrace{\sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))\,(f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t))}_{=:R_1(T)} \tag{96}$$

$$+ \underbrace{\sum_{t=1}^{T}\left(\int_{v=0}^{1}(1-v)\ddot{\mu}((1-v)f^*(\mathbf{x}_t) + vf^*(\mathbf{x}^*))\mathrm{d}v\right)(f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t))^2}_{=:R_2(T)}. \tag{97}$$

We know that $\widetilde{f}_t \in \mathcal{C}_t(\delta)$ and if the good event $\mathcal{E}_\delta$ holds, we also have $f^* \in \mathcal{C}_t(\delta)$. Using the optimism, we know that $\widetilde{f}_t(\mathbf{x}_t) \geq f^*(\mathbf{x}^*)$. We start by analyzing $R_1(T)$, recalling that $\dot{\mu}(f^*(\mathbf{x}_t)) \geq 0$:

$$\begin{aligned}
R_1(T) &= \sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))\,(f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t)) && (98)\\
&= \sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))\left(f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t) \pm \widetilde{f}_t(\mathbf{x}_t)\right) && (99)\\
&\leq \sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))(\widetilde{f}_t(\mathbf{x}_t) - f^*(\mathbf{x}_t)) && (100)\\
&= \sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))\langle\widetilde{\alpha}_t - \alpha^*, \phi(\mathbf{x}_t)\rangle && (101)\\
&\leq \sum_{t=1}^{T}\dot{\mu}(f^*(\mathbf{x}_t))\underbrace{\|\widetilde{\alpha}_t - \alpha^*\|_{\widetilde{V}_t(\lambda;f^*)}}_{\text{(a)}}\underbrace{\|\phi(\mathbf{x}_t)\|_{\widetilde{V}_t^{-1}(\lambda;f^*)}}_{\text{(b)}}, && (102)
\end{aligned}$$

where we decompose the functions as inner products and the Cauchy-Schwarz's inequality. For term (a), we apply Lemma C.4 with $f \leftarrow \widetilde{f}_t, f' \leftarrow f^*, f'' \leftarrow \widehat{f}_t$ and exploit the good event:

$$\|\widetilde{\alpha}_t - \alpha^*\|_{\widetilde{V}_t(\lambda;f^*)} \leq (1 + 2R_sBK) \tag{103}$$

$$\cdot\left(\left\|g_t(\widetilde{f}_t) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;\widetilde{f}_t)} + \left\|g_t(f^*) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;f^*)}\right) \tag{104}$$

$$\leq (1 + 2R_s BK)(B_t(\delta; \widetilde{f}_t) + B_t(\delta; f^*)) \tag{105}$$
$$\leq 2(1 + 2R_s BK)\beta_T(\delta; \mathcal{H}), \tag{106}$$

having observed that $\beta_T(\delta; \mathcal{H}) \geq \beta_t(\delta; \mathcal{H}) \geq \max\{B_t(\delta; \widetilde{f}_t), B_t(\delta; f^*)\}$. For term (b), we apply Cauchy-Schwarz's inequality:

$$\sum_{t=1}^{T} \dot{\mu}(f^*(\mathbf{x}_t)) \|\phi(\mathbf{x}_t)\|_{\widetilde{V}_t^{-1}(\lambda; f^*)} \tag{107}$$
$$\leq \sqrt{\mathfrak{g}(\tau)} \sqrt{\sum_{t=1}^{T} \dot{\mu}(f^*(\mathbf{x}_t))} \sqrt{\sum_{t=1}^{T} \mathfrak{g}(\tau)^{-1} \dot{\mu}(f^*(\mathbf{x}_t)) \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)}}. \tag{108}$$

Recalling that $\mathfrak{g}(\tau)^{-1} \dot{\mu}(f^*(\mathbf{x}_t)) \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} = \|\widetilde{\phi}(\mathbf{x}_t; f^*)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)}$, we can apply an elliptic potential lemma (Lemma C.6 with $M = \max\left\{1, \lambda^{-1}\mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\}$), where $\lambda^{-1}\mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2$ is a bound to the maximum value $\|\widetilde{\phi}(\mathbf{x}_t; f^*)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)}$ can take as:

$$\|\widetilde{\phi}(\mathbf{x}_t; f^*)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} = \mathfrak{g}(\tau)^{-1} \dot{\mu}(f^*(\mathbf{x}_t)) \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} \leq \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2 \lambda^{-1}, \tag{109}$$

as $\widetilde{V}_t^{-1}(\lambda; f^*) \succeq \lambda I$. Thus, we have:

$$\sum_{t=1}^{T} \|\widetilde{\phi}(\mathbf{x}_t; f^*)\|^2_{\widetilde{V}_t^{-1}(f^*)} \leq 2 \max\left\{1, \lambda^{-1}\mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \log\det(\lambda^{-1} \widetilde{V}_t(f^*)) \tag{110}$$
$$\leq 4 \max\left\{1, \lambda^{-1}\mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*). \tag{111}$$

The remaining term can be treated as follows, by means of a Taylor expansion:

$$\sum_{t=1}^{T} \dot{\mu}(f^*(\mathbf{x}_t)) = \sum_{t=1}^{T} \dot{\mu}(f^*(\mathbf{x}^*)) + \sum_{t=1}^{T} \left(\int_{v=0}^{1} \ddot{\mu}((1-v)f^*(\mathbf{x}^*) + v f^*(\mathbf{x}_t)) \right) (f^*(\mathbf{x}_t) - f^*(\mathbf{x}^*)) \tag{112}$$
$$\leq T\dot{\mu}(f^*(\mathbf{x}^*)) + R_s \sum_{t=1}^{T} \left(\int_{v=0}^{1} \dot{\mu}((1-v)f^*(\mathbf{x}^*) + v f^*(\mathbf{x}_t)) \right) (f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t)) \tag{113}$$
$$= \frac{T}{\kappa_*} + R_s \sum_{t=1}^{T} (\mu(f^*(\mathbf{x}^*)) - \mu(f^*(\mathbf{x}_t))) \tag{114}$$
$$= \frac{T}{\dot{\mu}(f^*(\mathbf{x}^*))} + R_s R(\texttt{GKB-UCB}, T) \tag{115}$$
$$= \frac{T}{\kappa_*} + R_s R(\texttt{GKB-UCB}, T). \tag{116}$$

where we exploited $f^*(\mathbf{x}^*) \geq f^*(\mathbf{x}_t)$, the self-concordance property (Assumption 3.4) and mean-value theorem. Putting all together, we get:

$$R_1(T) \leq 4\sqrt{\mathfrak{g}(\tau)}(1 + 2R_s BK)\beta_T(\delta; \mathcal{H}) \sqrt{\frac{T}{\kappa_*} + R_s R(\texttt{GKB-UCB}, T)} \tag{117}$$
$$\cdot \sqrt{\max\left\{1, \lambda^{-1}\mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \tag{118}$$

Let us move to the second term, using optimism and proceeding with the same rationale as before:

$$R_2(T) \leq R_{\dot{\mu}} R_s \sum_{t=1}^{T} (f^*(\mathbf{x}^*) - f^*(\mathbf{x}_t))^2 \tag{119}$$

$$
\begin{aligned}
&\leq R_{\dot{\mu}} R_s \sum_{t=1}^{T} (\widetilde{f}_t(\mathbf{x}_t) - f^*(\mathbf{x}_t))^2 && (120)\\
&\leq R_{\dot{\mu}} \sum_{t=1}^{T} \|\widetilde{\alpha}_t - \alpha^*\|^2_{\widetilde{V}_t(\lambda; f^*)} \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} && (121)\\
&\leq 4 R_{\dot{\mu}} R_s (1 + 2R_s BK)^2 \beta_T(\delta; \mathcal{H})^2 \sum_{t=1}^{T} \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} && (122)\\
&\leq 4\mathfrak{g}(\tau) R_{\dot{\mu}} R_s \kappa_{\mathcal{X}} (1 + 2R_s BK)^2 \beta_T(\delta; \mathcal{H})^2 \sum_{t=1}^{T} \mathfrak{g}(\tau)^{-1} \dot{\mu}(f^*(\mathbf{x}_t)) \|\phi(\mathbf{x}_t)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} && (123)\\
&\leq 4\mathfrak{g}(\tau) R_{\dot{\mu}} R_s \kappa_{\mathcal{X}} (1 + 2R_s BK)^2 \beta_T(\delta; \mathcal{H})^2 \sum_{t=1}^{T} \|\widetilde{\phi}(\mathbf{x}_t; f^*)\|^2_{\widetilde{V}_t^{-1}(\lambda; f^*)} && (124)\\
&\leq 16\mathfrak{g}(\tau) R_{\dot{\mu}} R_s \kappa_{\mathcal{X}} (1 + 2R_s BK)^2 \beta_T(\delta; \mathcal{H})^2 \max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*), && (125)
\end{aligned}
$$

having, in addition, exploited the fact that $\kappa_{\mathcal{X}} \geq \dot{\mu}(f^*(\mathbf{x}_t))^{-1}$. Putting all together, we have:

$$
\begin{aligned}
R(\texttt{GKB-UCB}, T) &= R_1(T) + R_2(T) && (126)\\
&\leq 4\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} && (127)\\
&\quad \cdot \left(\sqrt{\frac{T}{\kappa_*}} + \sqrt{R_s R(\texttt{GKB-UCB}, T)} \right) + R_2(T). && (128)
\end{aligned}
$$

Using the polynomial inequality of Proposition 7 of [2] (i.e., $x^2 \leq bx + c = 0 \implies x \leq b + \sqrt{c}$ when $b, c \geq 0$), we have:

$$
\begin{aligned}
\sqrt{R(\texttt{GKB-UCB}, T)} &\leq 4\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \sqrt{R_s} && (129)\\
&\quad + \sqrt{4\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \sqrt{\frac{T}{\kappa_*}} + R_2(T)}. && (130)
\end{aligned}
$$

Squaring both sides and bounding the square as $(a + b)^2 \leq 2a^2 + 2b^2$, we obtain:

$$
\begin{aligned}
&R(\texttt{GKB-UCB}, T) && (131)\\
&\leq 2 \left(4\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \sqrt{R_s} \right)^2 && (132)\\
&\quad + 2 \left(4\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \sqrt{\frac{T}{\kappa_*}} + R_2(T) \right) && (133)\\
&\leq 8\sqrt{\mathfrak{g}(\tau)} (1 + 2R_s BK) \beta_T(\delta; \mathcal{H}) \sqrt{\max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*)} \sqrt{\frac{T}{\kappa_*}} && (134)\\
&\quad + 32 R_s (1 + R_{\dot{\mu}} \kappa_{\mathcal{X}}) \mathfrak{g}(\tau) (1 + 2R_s BK)^2 \beta_T(\delta; \mathcal{H})^2 \max\left\{1, \lambda^{-1} \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} K^2\right\} \widetilde{\gamma}_T(f^*). && (135)
\end{aligned}
$$

We get the result by defining $R_{\text{perm}}(T)$ and $R_{\text{trans}}(T)$ as in the statement. □

B.4 Proofs of Appendix A

Lemma B.2 (Confidence Set). *Let $t \in \mathbb{N}$, $f \in \mathcal{H}$, and $\delta \in (0,1)$. Then, it holds that $\mathcal{C}_t(\delta) \subseteq \mathcal{D}_t(\delta)$. Furthermore, under the good event $\mathcal{E}_\delta$, for every $f = \langle \alpha, \phi \rangle \in \mathcal{D}_t(\delta)$, we have:*

$$\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda;f^*)} \le (2 + 2R_s BK)\sqrt{\beta_t(\delta;\mathcal{H})}\left(\sqrt{\beta_t(\delta;\mathcal{H})} + \sqrt{2}\right). \tag{136}$$

Proof. Following the same derivation of Lemma 2 of [2], based on Taylor's expansion and using the definitions of G_t and $\widetilde{G}_t$ in Appendix C. We have:

$$\mathcal{L}_t(f) - \mathcal{L}_t(\widehat{f}_t) \tag{137}$$

$$= (\alpha - \widehat{\alpha}_t)^\top \underbrace{\nabla \mathcal{L}_t(\widehat{f}_t)}_{=0} + (\alpha - \widehat{\alpha}_t)^\top \left(\int_{v=0}^1 (1-v)\widetilde{V}_t(\lambda; \widehat{f}_t + v(f - \widehat{f}_t))\mathrm{d}v\right)(\alpha - \widehat{\alpha}_t) \tag{138}$$

$$= \|\alpha - \widehat{\alpha}_t\|^2_{\widetilde{G}_t(\widehat{f}_t, f)} \tag{139}$$

$$\le \|\alpha - \widehat{\alpha}_t\|^2_{G_t(\widehat{f}_t, f)} \tag{140}$$

$$= \left\|g_t(f) - g_t(\widehat{f}_t)\right\|^2_{\widetilde{G}_t^{-1}(\widehat{f}_t, f)} \tag{141}$$

$$\le (1 + 2R_s BK)\left\|g_t(f) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;f)}, \tag{142}$$

where we used Equation (162) and (167). Thus, let $f \in \mathcal{C}_t(\delta)$, we have that $\left\|g_t(f) - g_t(\widehat{f}_t)\right\|_{\widetilde{V}_t^{-1}(\lambda;f)} \le B_t(\delta; f) \le B_t(\delta;\mathcal{H})$ and, consequently, $f \in \mathcal{D}_t(\delta)$.

For the second part, suppose the good event $\mathcal{E}_\delta$ holds and consider $f \in \mathcal{D}_t(\delta)$, we have via Taylor's expansion:

$$\mathcal{L}_t(f) - \mathcal{L}_t(f^*) \tag{143}$$

$$= (\alpha - \alpha^*)^\top \nabla \mathcal{L}_t(f^*) + (\alpha - \alpha^*)^\top \left(\int_{v=0}^1 (1-v)\widetilde{V}_t(f^* + v(f - f^*);\lambda)\mathrm{d}v\right)(\alpha - \alpha^*) \tag{144}$$

$$= (\alpha - \alpha^*)^\top \nabla \mathcal{L}_t(f^*) + \|\alpha - \alpha^*\|^2_{\widetilde{G}_t(f^*, f)} \tag{145}$$

$$\ge (\alpha - \alpha^*)^\top \nabla \mathcal{L}_t(f^*) + (2 + 2R_s BK)^{-1}\|\alpha - \alpha^*\|^2_{\widetilde{V}_t(\lambda;f^*)}, \tag{146}$$

where we used Equation (168). Thus, we have:

$$\|\alpha - \alpha^*\|^2_{\widetilde{V}_t(f^*;\lambda)} \tag{147}$$

$$\le (2 + 2R_s BK)(\mathcal{L}_t(f) - \mathcal{L}_t(f^*)) + (2 + 2R_s BK)(\alpha - \alpha^*)^\top \nabla \mathcal{L}_t(f^*) \tag{148}$$

$$\le (2 + 2R_s BK)(\mathcal{L}_t(f) - \mathcal{L}_t(\widehat{f}_t)) + (2 + 2R_s BK)(\mathcal{L}_t(f^*) - \mathcal{L}_t(\widehat{f}_t)) \tag{149}$$

$$+ (2 + 2R_s BK)\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda;f^*)}\|\nabla \mathcal{L}_t(f^*)\|_{\widetilde{V}_t^{-1}(\lambda;f^*)} \tag{150}$$

$$\le 2(2 + 2R_s BK)(1 + 2R_s BK)B_t(\delta;\mathcal{H}) + (2 + 2R_s BK)\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda;f^*)} B_t(\delta; f^*). \tag{151}$$

where we used the fact that $\mathcal{L}_t(f) \ge \mathcal{L}_t(\widehat{f}_t) \wedge \mathcal{L}_t(f^*) \ge \mathcal{L}_t(\widehat{f}_t)$, that $f^* \in \mathcal{C}_t(\delta) \subseteq \mathcal{D}_t(\delta)$ under the good event and $f \in \mathcal{D}_t(\delta)$, and that:

$$\|\nabla \mathcal{L}_t(f^*)\|_{\widetilde{V}_t^{-1}(\lambda;f^*)} = \|g_t(f^*) - g_t(\widehat{f}_t)\|_{\widetilde{V}_t^{-1}(\lambda;f^*)} \le B_t(\delta; f^*). \tag{152}$$

holding under the good event. By the choice of confidence radius and bounding $B_t(\delta; f^*) \leq B_t(\delta; \mathcal{H}) \leq \beta_t(\delta; \mathcal{H})$, we have the second-degree inequality:

$$\|\alpha - \alpha^*\|^2_{\widetilde{V}_t(\lambda; f^*)} \leq 2(2 + 2R_s BK)(1 + 2R_s BK)\beta_t(\delta; \mathcal{H}) \tag{153}$$

$$+ (2 + 2R_s BK)\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda; f^*)}\beta_t(\delta; \mathcal{H}). \tag{154}$$

Using the polynomial inequality of Proposition 7 of [2] (i.e., $x^2 \leq bx + c = 0 \implies x \leq b + \sqrt{c}$ when $b, c \geq 0$), we have:

$$\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda; f^*)} \leq \sqrt{2(2 + 2R_s BK)(1 + 2R_s BK)\beta_t(\delta; \mathcal{H})} + (2 + 2R_s BK)\beta_t(\delta; \mathcal{H}) \tag{155}$$

$$\leq (2 + 2R_s BK)\sqrt{\beta_t(\delta; \mathcal{H})}\left(\sqrt{\beta_t(\delta; \mathcal{H})} + \sqrt{2}\right). \tag{156}$$

having bounded $1 + 2R_s BK \leq 2 + 2R_s BK$. □

Theorem A.1 (Regret Bound of `Eff-GKB-UCB`)**.** *Under Assumptions 3.1, 3.2, 3.3, and 3.4,* `GKB-UCB` *with confidence radius $(1 + 2R_s BK)B_t(\delta; \mathcal{H})$ and $\lambda > 0$, for every $\delta \in (0, 1)$, with probability at least $1 - \delta$, suffers regret bounded as: $R(\texttt{Eff-GKB-UCB}, T) = R_{\text{perm}}(T) + R_{\text{trans}}(T)$, where:*

$$R_{\text{perm}}(T) \leq 4\sqrt{\max\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot{\mu}}K^2\}}(2 + 2R_s BK)\sqrt{\beta_T(\delta; \mathcal{H})}\left(\sqrt{\beta_T(\delta; \mathcal{H})} + 2\right)\sqrt{\widetilde{\gamma}_T(f^*)}\sqrt{\frac{T}{\kappa_*}},$$

$$R_{\text{trans}}(T) \leq 8R_s(1 + R_{\dot{\mu}}\kappa_{\mathcal{X}}) \max\left\{\mathfrak{g}(\tau), \lambda^{-1}R_{\dot{\mu}}K^2\right\}(2 + 2R_s BK)^2\beta_T(\delta; \mathcal{H})\left(\sqrt{\beta_T(\delta; \mathcal{H})} + 2\right)^2\widetilde{\gamma}_T(f^*).$$

Proof. The proof follows the same steps as Theorem 7.2, with the only difference that we exploit the bound of Lemma B.2:

$$\|\alpha - \alpha^*\|_{\widetilde{V}_t(\lambda; f^*)} \leq (2 + 2R_s BK)\sqrt{\beta_t(\delta; \mathcal{H})}\left(\sqrt{\beta_t(\delta; \mathcal{H})} + \sqrt{2}\right). \tag{157}$$

□

C Technical Lemmas

In this section, we introduce some technical concepts and lemmas to be used in the analysis. We consider $\mathbf{x} \in \mathcal{X}$ and $f = \langle \alpha, \phi \rangle, f' = \langle \alpha, \phi' \rangle \in \mathcal{H}$, we define the following quantities, analogous to those of [2]:

$$\xi(\mathbf{x}, f, f') \coloneqq \int_{v=0}^{1} \dot{\mu}((1 - v)f(\mathbf{x}) + vf'(\mathbf{x}))\mathrm{d}v, \tag{158}$$

$$\widetilde{\xi}(\mathbf{x}, f, f') \coloneqq \int_{v=0}^{1} (1 - v)\dot{\mu}((1 - v)f(\mathbf{x}) + vf'(\mathbf{x}))\mathrm{d}v, \tag{159}$$

$$G_t(f, f') \coloneqq \sum_{s=1}^{t-1} \frac{\xi(\mathbf{x}, f, f')}{\mathfrak{g}(\tau)}\phi(\mathbf{x}_s)\phi(\mathbf{x}_s)^\top + \lambda I, \tag{160}$$

$$\widetilde{G}_t(f, f') \coloneqq \sum_{s=1}^{t-1} \frac{\widetilde{\xi}(\mathbf{x}, f, f')}{\mathfrak{g}(\tau)}\phi(\mathbf{x}_s)\phi(\mathbf{x}_s)^\top + \lambda I. \tag{161}$$

We have that $\xi(\mathbf{x}, f, f') \geq \widetilde{\xi}(\mathbf{x}, f, f')$ and, consequently, we have that $G_t(f, f') \succeq \widetilde{G}_t(f, f')$. Thanks to the mean-value theorem and the definition of function $g_t(f)$, we have that:

$$g_t(f) - g_t(f') = G_t(f, f')(\alpha - \alpha'). \tag{162}$$

Using Assumption 3.4, we can easily extend Lemmas 7 and 8 of [2].

Lemma C.1 (Extension of Lemma 7 of [2])**.** *Let $\mathcal{Z} \subset \mathbb{R}$ be any bounded interval of $\mathbb{R}$ and let $f : \mathcal{Z} \to \mathbb{R}$ be a monothonically non-decreasing function such that $|\ddot{f}| \leq R_s\dot{f}$. Then, for every*

$z_1, z_2 \in \mathcal{Z}$:

$$\int_{v=0}^{1} \dot{f}(z_1 + v(z_2 - z_1))\mathrm{d}v \geq \frac{\dot{f}(z)}{1 + R_s|z_1 - z_2|}, \quad \forall z \in \{z_1, z_2\}. \tag{163}$$

Proof. Immediately follows from the same steps of [2, Lemma 7]. □

Lemma C.2 (Extension of Lemma 8 of [2])**.** *Let $\mathcal{Z} \subset \mathbb{R}$ be any bounded interval of $\mathbb{R}$ and let $f : \mathcal{Z} \to \mathbb{R}$ be a monothonically non-decreasing function such that $|\ddot{f}| \leq R_s \dot{f}$. Then, for every $z_1, z_2 \in \mathcal{Z}$:*

$$\int_{v=0}^{1} (1-v)\dot{f}(z_1 + v(z_2 - z_1))\mathrm{d}v \geq \frac{\dot{f}(z_1)}{2 + R_s|z_1 - z_2|}. \tag{164}$$

Proof. See [17, Lemma D.1]. □

From Lemma C.1 and Lemma C.2, we immediatly have:

$$\xi(\mathbf{x}, f, f') := \int_{v=0}^{1} \dot{\mu}((1-v)f(\mathbf{x}) + vf'(\mathbf{x}))\mathrm{d}v \geq \frac{\dot{\mu}(\overline{f})}{1 + R_s|f(\mathbf{x}) - f'(\mathbf{x})|}, \text{ for } \overline{f} \in \{f(\mathbf{x}), f'(\mathbf{x})\}. \tag{165}$$

$$\widetilde{\xi}(\mathbf{x}, f, f') := \int_{v=0}^{1} (1-v)\dot{\mu}((1-v)f(\mathbf{x}) + vf'(\mathbf{x}))\mathrm{d}v \geq \frac{\dot{\mu}(f(\mathbf{x}))}{2 + R_s|f(\mathbf{x}) - f'(\mathbf{x})|}. \tag{166}$$

Moreover, under Assumptions 3.2 and 3.1, we have that $|f(\mathbf{x}) - f'(\mathbf{x})| \leq 2\|f\|_\infty \leq 2BK$. This allows us to write:

$$G_t(f, f') \succeq (1 + 2R_s BK)^{-1}\widetilde{V}_t(\lambda; \overline{f}), \qquad \text{for } \overline{f} \in \{f, f'\}, \tag{167}$$
$$\widetilde{G}_t(f, f') \succeq (2 + 2R_s BK)^{-1}\widetilde{V}_t(\lambda; f). \tag{168}$$

Lemma C.3. *Let $f \in \mathcal{H}$, $\widetilde{V}_t(\lambda; f)$ and $V_t(\lambda)$ defined as in the main paper. The following semidefinite inequalities holds:*

$$\min\{1, \mathfrak{g}(\tau)R_{\dot{\mu}}^{-1}\}\widetilde{V}_t(\lambda; f) \preceq V_t(\lambda) \preceq \max\{1, \mathfrak{g}(\tau)\kappa_{\mathcal{X}}(f)\}\widetilde{V}_t(\lambda; f), \tag{169}$$

where $\kappa_{\mathcal{X}}(f) = \sup_{\mathbf{x} \in \mathcal{X}} \frac{1}{\dot{\mu}(f(\mathbf{x}))}$

Proof. For one inequality, we have:

$$\begin{aligned}
\widetilde{V}_t(\lambda; f) &= \sum_{s=1}^{t-1} \frac{\dot{\mu}(f(\mathbf{x}_s))}{\mathfrak{g}(\tau)}\phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I & (170)\\
&\succeq \mathfrak{g}(\tau)^{-1}\kappa_{\mathcal{X}}(f)^{-1}\sum_{s=1}^{t-1}\phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I & (171)\\
&\succeq \min\left\{1, \mathfrak{g}(\tau)^{-1}\kappa_{\mathcal{X}}(f)^{-1}\right\}\left(\sum_{s=1}^{t-1}\phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I\right) & (172)\\
&= \min\left\{1, \mathfrak{g}(\tau)^{-1}\kappa_{\mathcal{X}}(f)^{-1}\right\}V_t(\lambda). & (173)
\end{aligned}$$

For the other inequality, we have:

$$\widetilde{V}_t(\lambda; f) = \sum_{s=1}^{t-1} \frac{\dot{\mu}(f(\mathbf{x}_s))}{\mathfrak{g}(\tau)}\phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I \tag{174}$$

$$
\begin{aligned}
&\preceq \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}} \sum_{s=1}^{t-1} \phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I && (175)\\
&\preceq \max\left\{1, \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}}\right\} \left(\sum_{s=1}^{t-1} \phi(\mathbf{x})\phi(\mathbf{x})^\top + \lambda I\right) && (176)\\
&= \max\left\{1, \mathfrak{g}(\tau)^{-1} R_{\dot{\mu}}\right\} V_t(\lambda). && (177)
\end{aligned}
$$

□

Lemma C.4. *Let $f = \langle \alpha, \phi \rangle, f' = \langle \alpha', \phi \rangle \in \mathcal{H}$, then for every $f'' = \langle \alpha'', \phi \rangle \in \mathcal{H}$, it holds that:*

- $\|\alpha - \alpha'\|_{\widetilde{V}_t(\lambda;f')} \le (1 + 2R_s BK) \left(\|g_t(f) - g_t(f'')\|_{\widetilde{V}_t^{-1}(\lambda;f)} + \|g_t(f') - g_t(f'')\|_{\widetilde{V}_t^{-1}(\lambda;f')} \right)$;
- $\|\alpha - \alpha'\|_{V_t(\lambda)} \le (1 + 2R_s BK) \max\{1, \mathfrak{g}(\tau)\kappa_{\mathcal{X}}(f')\} \cdot$
  $\cdot \left(\|g_t(f) - g_t(f'')\|_{V_t^{-1}(\lambda)} + \|g_t(f') - g_t(f'')\|_{V_t^{-1}(\lambda)} \right)$.

Proof. From the mean-value theorem (Equation 162), we have:

$$
g_t(f) - g_t(f') = G_t(f, f')(\alpha - \alpha'). \tag{178}
$$

The first statement follows the same derivation of Proposition 4 of [2], with the only care of applying Equation (167). The second statement starts from the following intermediate passage of the proof of Proposition 4 of [2]:

$$
\begin{aligned}
\|\alpha - \alpha'\|_{\widetilde{V}_t(\lambda;f')} &\le \sqrt{1 + 2R_s BK} \left(\|g_t(f) - g_t(f'')\|_{G_t^{-1}(f,f')} + \|g_t(f') - g_t(f'')\|_{G_t^{-1}(f,f')} \right) && (179)\\
&\le (1 + 2R_s BK) \left(\|g_t(f) - g_t(f'')\|_{\widetilde{V}_t^{-1}(\lambda;f')} + \|g_t(f') - g_t(f'')\|_{\widetilde{V}_t^{-1}(\lambda;f')} \right). && (180)
\end{aligned}
$$

Then, we use the semidefinite inequality $\widetilde{V}_t(\lambda; f') \succeq \max\{1, \mathfrak{g}(\tau)\kappa_{\mathcal{X}}(f')\}^{-1} V_t(\lambda)$ (Lemma C.3):

$$
\begin{aligned}
&\|\alpha - \alpha'\|_{\widetilde{V}_t(\lambda;f')} \ge \max\{1, \mathfrak{g}(\tau)\kappa_{\mathcal{X}}(f')\}^{-1/2} V_t \|\alpha - \alpha'\|_{V_t(\lambda)} && (181)\\
&\|g_t(f) - g_t(f'')\|_{\widetilde{V}_t^{-1}(\lambda;f')} \le \max\{1, \mathfrak{g}(\tau)\kappa_{\mathcal{X}}(f')\}^{1/2} \|g_t(f) - g_t(f'')\|_{V_t^{-1}(\lambda)}. && (182)
\end{aligned}
$$

□

Lemma C.5. *Let $t \in \mathbb{N}$, $f \in \mathcal{H}$, $\mathbf{K}_t$ and $\widetilde{\mathbf{K}}_t(f)$ defined as in the main paper. It holds that:*

$$
\begin{aligned}
\log\det(\mathbf{I}_t + \lambda^{-1}\widetilde{\mathbf{K}}_t(f)) &\le \log\det(\mathbf{I}_t + \lambda^{-1} R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1} \mathbf{K}_t) && (183)\\
&\le \max\{1, R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1}\} \log\det(\mathbf{I}_t + \lambda^{-1}\mathbf{K}_t). && (184)
\end{aligned}
$$

Proof. We can look at matrix $\widetilde{\mathbf{K}}_t(f)$ as follows:

$$
\widetilde{\mathbf{K}}_t(f) = \mathfrak{g}(\tau)^{-1} \mathbf{M}(f)^{1/2} \mathbf{K}_t \mathbf{M}(f)^{1/2}, \tag{185}
$$

where $\mathbf{M}(f) = \mathrm{diag}((\dot{\mu}(f(\mathbf{x}_s))_{s \in [\![t-1]\!]})$ is a diagonal matrix. Using Horn's inequality for eigenvalues [34], we have that for every $i \in [\![t-1]\!]$:

$$
\lambda_i(\widetilde{\mathbf{K}}_t(f)) \le \lambda_i(\mathbf{K}_t) \max_{s \in [\![t-1]\!]} \dot{\mu}(f(\mathbf{x}_s)) \mathfrak{g}(\tau)^{-1} \le \lambda_i(\mathbf{K}_t) R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1}. \tag{186}
$$

Furthermore, using Weyl's inequality for eigenvalues, we have for $i \in [\![t-1]\!]$:

$$
\begin{aligned}
\lambda_i(\mathbf{I}_t + \lambda^{-1}\widetilde{\mathbf{K}}_t(f)) &\le 1 + \lambda^{-1}\lambda_i(\widetilde{\mathbf{K}}_t(f)) && (187)\\
&\le 1 + \lambda^{-1} R_{\dot{\mu}} \mathfrak{g}(\tau)^{-1} \lambda_i(\mathbf{K}_t) && (188)
\end{aligned}
$$

$$\leq 1 + \lambda^{-1} \max\{1, R_{\dot\mu}\mathfrak{g}(\tau)^{-1}\}\lambda_i(\mathbf{K}_t) \tag{189}$$

$$\leq \left(1 + \lambda^{-1}\lambda_i(\mathbf{K}_t)\right)^{\max\{1, R_{\dot\mu}\mathfrak{g}(\tau)^{-1}\}} \tag{190}$$

$$\leq \lambda_i(1 + \lambda^{-1}\mathbf{K}_t)^{\max\{1, R_{\dot\mu}\mathfrak{g}(\tau)^{-1}\}}, \tag{191}$$

where we exploited the inequality $1 + ab \leq (1+b)^a$ for $b \geq 0$ and $a \geq 1$. The statement is obtained passing to the determinant and to its logarithm. □

Lemma C.6 (Elliptic Potential Lemma (slightly extended))**.** *Let $(y_t)_{t\geq 1}$ be a sequence, let $M \geq 1$, and $V_t(\lambda) = \sum_{s=1}^{t-1} y_s y_s^\top + \lambda I$. For every $T \geq 1$, it holds that:*

$$\sum_{t=1}^{T} \min\{M, \|y_t\|_{V_t^{-1}(\lambda)}\}^2 \leq 2M \log\det(\lambda^{-1}V_t(\lambda)). \tag{192}$$

Proof. We follow the steps of Lemma 12 of [2]. Using the inequality $\min\{1, u\} \leq 2\log(1+u)$ for every $u \geq 0$, we have:

$$\sum_{t=1}^{T} \min\{M, \|y_t\|^2_{V_t^{-1}(\lambda)}\} = M \sum_{t=1}^{T} \min\{1, M^{-1}\|y_t\|^2_{V_t^{-1}(\lambda)}\} \tag{193}$$

$$\leq 2M \sum_{t=1}^{T} \log\left(1 + M^{-1}\|y_t\|^2_{V_t^{-1}(\lambda)}\right) \tag{194}$$

$$\leq 2M \sum_{t=1}^{T} \log\left(1 + \|y_t\|^2_{V_t^{-1}(\lambda)}\right), \tag{195}$$

having exploited that $M \geq 1$. Now the last equation can be bounded following the usual steps of [2], to obtain:

$$\sum_{t=1}^{T} \log\left(1 + \|y_t\|^2_{V_t^{-1}(\lambda)}\right) \leq \log\det(\lambda^{-1}V_t(\lambda)). \tag{196}$$

□

Lemma C.7. *Let S_t and $\widetilde{V}_t(\lambda)$ defined as in Theorem 6.2. The following inequalities hold:*

$$\|S_t\|^2_{\widetilde{V}_t^{-1}(\lambda)} \leq \frac{(t-1)^2K^2R^2}{\lambda}, \qquad \log\det(\lambda^{-1}\widetilde{V}_t(\lambda)) \leq (t-1)\log\left(1 + \frac{K^2R^2}{\lambda}\right). \tag{197}$$

Proof. For the first inequality, we proceed as follows:

$$\|S_t\|^2_{\widetilde{V}_t^{-1}(\lambda)} \leq \|S_t\|^2_{\lambda^{-1}I} \leq \lambda^{-1}\left(\sum_{s=1}^{t-1}|\epsilon_s|\|\phi(\mathbf{x}_s)\|\right)^2 \leq \frac{(t-1)^2K^2R^2}{\lambda}. \tag{198}$$

For the second inequality, we proceed as follows:

$$\det(\lambda^{-1}\widetilde{V}_t(\lambda)) = \lambda^{-(t-1)}\det(\widetilde{\mathbf{K}}_t(\lambda)) \tag{199}$$

$$\leq \lambda^{-(t-1)}\left(\frac{1}{t-1}\operatorname{tr}(\widetilde{\mathbf{K}}_t(\lambda))\right)^{t-1} \tag{200}$$

$$\leq \lambda^{-(t-1)}(\lambda + K^2R^2)^{(t-1)} \tag{201}$$

$$= \left(1 + \frac{K^2R^2}{\lambda}\right)^{(t-1)}. \tag{202}$$

having applied the identity of Equation (1), the determinant-trace inequality and bounded $\mathrm{tr}(\widetilde{\mathbf{K}}_t(\lambda)) \leq (t-1)(\lambda + K^2R^2)$, since the diagonal elements of $\widetilde{\mathbf{K}}_t(\lambda)$ are of the form $\lambda + \sigma(\mathbf{x})k(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}') \leq \lambda + R^2K^2$, being the variance bounded by the square of the range. $\square$