Bearing-only tracking using range-parameterized shifted Rayleigh filter

Shreya Das
Dept. of Elect. Engg.
IIT Patna
Patna, India
shreya_2121ee15@iitp.ac.in

Kundan Kumar
Dept. of Elect. Engg.
IIT Patna
Patna, India
kundan.pee16@iitp.ac.in

Rahul Radhakrishnan
Dept. of Elect. Engg.
SVNIT Surat
Surat, India
r.rahul@eed.svnit.ac.in

Shovan Bhaumik
Dept. of Elect. Engg.
IIT Patna
Patna, India
shovan.bhaumik@iitp.ac.in

OCEANS 2022 - Chennai | 978-1-6654-1821-8/22/$31.00 ©2022 IEEE | DOI: 10.1109/OCEANSChennai45887.2022.9775468

***Abstract*—In this paper, we modified the shifted Rayleigh filter (SRF) to improve its performance of tracking with bearing-only measurement. The proposed method uses multiple parallel SRF with different initial positions and it is named as range parameterized shifted Rayleigh filter (RP-SRF). The RP-SRF is applied to an underwater target motion analysis (TMA) where the target moves in a nearly straight line motion with constant velocity, and the observer maneuvers to track the target. The performance of the proposed estimator is compared with the SRF, cubature Kalman filter (CKF), and RP-CKF in terms of root mean square error (RMSE), percentage of track loss, and computational time. Simulation results indicate that the RP-SRF is more accurate than the CKF, SRF, and RP-CKF and would be ahead of its nearest competitor for underwater bearing-only tracking.**

***Index Terms*—Bearing-only tracking, target motion analysis, range-parameterization, shifted Rayleigh filter, cubature Kalman filter.**

I. Introduction

Bearing only tracking (BOT) problem finds application in different areas like aircraft surveillance, submarines, and torpedo tracking, and many more [1]–[3]. These problems are generally referred to as target motion analysis (TMA). In TMA, a moving target is tracked using noisy bearing measurements. In military applications, bearing measurements are obtained mostly from passive sensors to stay hidden from enemies. Thus it provides an efficient and effective way of tracking an enemy submarine, ship, or torpedo without getting revealed [4].

Most of the research works on BOT problems deal with the tracking of a non-maneuvering target. For such system the target becomes unobservable if the observer follows a straight line path with constant velocity. To make the system observable and track the target, the observer must maneuver [5]–[8].

As the measurement equation is non-linear, the BOT problems are solved using non-linear estimators. In such estimators, the posterior probability density of the target's state is calculated using the predicted target motion and the likelihood of the measured bearing received by the observer. Due to the fact that the measurement model is highly nonlinear and the observability issue appears, it is not a simple task to locate the target [9], [10]. Several algorithms of filtering are proposed [2], [10]– [11] to do the task in recursive way.

The extended Kalman filter (EKF) [12] and its variants [13]–[15] which are based on the linearization of a non-linear function using the first-order Taylor series approximation were the first sub-optimal nonlinear filter developed for BOT. Later various other nonlinear filters such as the cubature Kalman filter (CKF), unscented Kalman filter (UKF), Gauss-Hermite filter (GHF) *etc.* [16]–[18] were applied in it.

In another approach, sequential Monte Carlo based filtering technique is implemented for BOT which shows good performance but it is computationally very expensive [16]. Very recently, a new filtering technique named as the shifted Rayleigh filter (SRF) [19] is formulated particularly for BOT problems. It assumes the prior and posterior density functions are Gaussian and compute the first two moments for the bearing-only measurement. Its performance is comparable with PF [20] as showed in [21], [22]. The main advantage of SRF is its computation efficiency and accuracy as compared to existing filters.

As no optimal filter for BOT exists, there are continuing efforts to develop a computationally efficient nonlinear filter that can provide a better estimation accuracy for the BOT problem. The range parameterization technique has been introduced in the estimators to further improve the estimation accuracy [17], [23]. It uses a number of same filters that run parallelly with different initial ranges. Finally, these filters are weighted based on the likelihood of the measurement received to estimate the posterior mean and covariance of the relative state vector [24]. Range parameterized EKF [23] and range parameterized CKF [17] show better performance than EKF [23], [25] and CKF [17] respectively.

In this paper, we formulate a range parameterized SRF (RP-SRF) for the BOT problem in order to further improve the accuracy of the SRF. In this formulation, more than one independent SRF are implemented in parallel, each with a different initial range. The final estimate will be the weighted average of the estimates obtained from all the filters. We implement the proposed filter for an underwater target tracking scenario, where the target moves in a nearly straight-line path with a constant velocity and the own ship maneuvers smoothly in order to make the system observable. To estimate the position and velocity of the target, we implement the proposed RP-SRF along with the CKF, SRF, and RP-CKF. We compare the filtering performance in terms of root mean square error (RMSE), percentage of track loss, and computation time. We also use Cramer-Rao lower bound (CRLB) to compare the results of the estimators with that of best achievable performance. From the simulation results, it is observed that the proposed RP-SRF provides more accurate tracking than the CKF, SRF, and RP-CKF.

II. Problem Formulation

The state space equation for a two dimensional engagement scenario is developed in Cartesian coordinates. The state vector for target dynamics is $\mathcal{X}_k^{tar} = \begin{bmatrix} x_k^{tar} & y_k^{tar} & \dot{x}_k^{tar} & \dot{y}_k^{tar} \end{bmatrix}^T$. Likewise the state vector for observer dynamics is $\mathcal{X}_k^{obs} = \begin{bmatrix} x_k^{obs} & y_k^{obs} & \dot{x}_k^{obs} & \dot{y}_k^{obs} \end{bmatrix}^T$. So, the relative state vector between target and observer can be defined as $\mathcal{X}_k \triangleq \mathcal{X}_k^{tar} - \mathcal{X}_k^{obs} = \begin{bmatrix} x_k & y_k & \dot{x}_k & \dot{y}_k \end{bmatrix}^T$. The dynamics of the target assuming a near constant velocity in discrete time domain can be expressed as [16], [17]

$$\mathcal{X}_k = F\mathcal{X}_{k-1} + w_{k-1} - \mho_{k-1,k}, \tag{1}$$

where F is the state transition matrix, calculated as:

$$F = \begin{bmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

where Δ is the sampling interval. $\mho_{k-1,k}$ is a vector of inputs that accounts for observer acceleration is

$$\mho_{k-1,k} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} x_k^{obs} - x_{k-1}^{obs} - \Delta\dot{x}_{k-1}^{obs} \\ y_k^{obs} - y_{k-1}^{obs} - \Delta\dot{y}_{k-1}^{obs} \\ \dot{x}_k^{obs} - \dot{x}_{k-1}^{obs} \\ \dot{y}_k^{obs} - \dot{y}_{k-1}^{obs} \end{bmatrix} \tag{3}$$

and $w_{k-1} \sim \mathcal{N}(0, Q_{k-1})$ is the process noise with covariance

$$Q = \begin{bmatrix} \frac{\Delta^3}{3} & 0 & \frac{\Delta^2}{2} & 0 \\ 0 & \frac{\Delta^3}{3} & 0 & \frac{\Delta^2}{2} \\ \frac{\Delta^2}{2} & 0 & \Delta & 0 \\ 0 & \frac{\Delta^2}{2} & 0 & \Delta \end{bmatrix} \bar{q}, \tag{4}$$

where $\bar{q}$ is the process noise intensity.

The sensors mounted to own ship, measure the direction of the target location with respect to the true north. Such measurements can be expressed as

$$\mathcal{Y}_k = \tan^{-1}\frac{x_k}{y_k} + \nu_{\theta_k}, \tag{5}$$

where ν_{θ_k} is the noise in bearing measurement which is assumed to be zero mean, white, Gaussian with standard deviation σ_θ *i.e.* $\nu_{\theta_k} \sim \mathcal{N}(0, \sigma_\theta^2)$. For such measurements the system becomes unobservable [16], [17] and the estimation of target state is only possible when the own ship starts maneuvering.

III. Filtering Methods

A. Shifted Rayleigh filter

The shifted Rayleigh filter is a moment matching algorithm that exploits nonlinearities in bearing-only tracking, resulting in a more exact conditional mean and covariance after the current measurement update assuming the prior and posterior density to be Gaussian. The approximation used in this filter is that the posterior density is replaced by an equivalent Gaussian density at the end of each measurement update.

In the SRF, the measurement model of Eq. (5) is transformed into an augmented measurement as

$$\mathcal{Z}_k = H\mathcal{X}_k + w_k, \tag{6}$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{7}$$

and $w_k \sim \mathcal{N}(0, R_k)$ is an augmented noise vector with mean zero and covariance R_k. Following [19], [21] and [26], the expression of R_k can be written as,

$$\begin{aligned} R_k &= \sigma_\theta^2 \mathbb{E}[||H\mathcal{X}_k||^2 \,|b_1, ..., b_{k-1}] I_{2\times 2} \\ &= \sigma_\theta^2(\hat{x}_{k|k-1} + \hat{y}_{k|k-1} + \sigma^2_{\hat{x}_{k|k-1}} + \sigma^2_{\hat{y}_{k|k-1}}) I_{2\times 2}, \end{aligned} \tag{8}$$

where $\sigma^2_{\hat{x}_{k|k-1}}$ and $\sigma^2_{\hat{y}_{k|k-1}}$ are obtained from the predicted error covariance $P_{k|k-1}$.

Now, we define a transformed measurement vector for a 2-D scenario as,

$$b_k = \frac{\mathcal{Z}_k}{||\mathcal{Z}_k||} = \begin{bmatrix} sin(\mathcal{Y}_k) \\ cos(\mathcal{Y}_k) \end{bmatrix}. \tag{9}$$

Geometrically it is the projection of the augmented measurement onto the unit circle *i.e.* the direction cosine of the bearing measurement. The prior mean and error covariance for SRF are evaluated in the prediction step with Eqs. (16) and (17). The posterior mean and covariance are evaluated in the correction step with Eqs. (20) and (23) which are mentioned in the Algorithm 1.

B. Range parameterized shifted Rayleigh filter

Range parameterized method was proposed in [23] in the context of the EKF where several independent EKFs were initialized with different x positions and y positions. Each time step the filters are updated, and they are weighted for consistency with respect to the measured bearing [23]. After this process is continued for some time steps, the weights of some filters' estimate go below a threshold value, after which they are no longer processed. The number of time steps after which only one filter shall be processed depends on the observer target scenario. So, the range parameterized tracking is more computationally expensive than running a single filter.

Suppose, N_f number of filters are initialized with different initial ranges and their standard deviations, which are evaluated by dividing N_f sub-intervals of the interval (r_{min}, r_{max}). The tracking performance of a filter depends on the coefficient of variation of the estimated range, C_r. As mentioned in [23] and [27], $C_r = \sigma_{r_n}/\hat{r}_n$, where $\hat{r}_n$ is the mean of a sub-interval and σ_{r_n} is the standard deviation of that sub-interval. For each filter, the mean and the standard deviation can be calculated as

$$\hat{r}_n = r_{min}\frac{\xi^{n-1}+\xi^n}{2}, \text{ and } \sigma_{r_n} = r_{min}\frac{\xi^n+\xi^{n-1}}{\sqrt{12}}, \quad (10)$$

where $n = 1, \cdots, N_f$ and the common ratio, $\xi = (r_{max}/r_{min})^{1/N_f}$.

The estimated state of each filter is combined using weighted averaging method [23], [27]. Initially the weight of each filter is considered to be the same *i.e.* $1/N_f$ as no prior information of the truth is available at the first time step. Later the weights are updated using Bayes' theorem assuming that the predicted and the measured bearing follows Gaussian distribution. The weights are calculated as [23], [27]

$$\omega_k^n = \frac{p(\mathcal{Z}_k^n)\omega_{k-1}^n}{\Sigma_{i=1}^{N_f} p(\mathcal{Z}_k^i)\omega_{k-1}^i}, \quad (11)$$

where $p(\mathcal{Z}_k^n)$ is the likelihood of augmented measurement $\mathcal{Z}_k$ of the n^{th} filter. It can be calculated as

$$p(\mathcal{Z}_k^n) = \frac{1}{2\pi\sqrt{|V_k^n|}} exp(-\frac{1}{2}(\mathcal{Z}_k - \hat{\mathcal{Z}}_{k|k-1}^n)^T (V_k^n)^{-1}(\mathcal{Z}_k - \hat{\mathcal{Z}}_{k|k-1}^n)), \quad (12)$$

where $\hat{\mathcal{Z}}_{k|k-1}^n = H\hat{\mathcal{X}}_{k|k-1}^n$ and $\mathcal{Z}_k$ is the augmented measurement as in Eq. (6) is the augmented angle predicted by the n^{th} filter at k^{th} time step and V_k^n is the innovation covariance of the measurement for the n^{th} filter at the k^{th} time step and evaluated as

$$V_k^n = HP_{k|k-1}^n H^T + R_k^n. \quad (13)$$

The weighted average of the mean and the covariance of the N_f independent SRFs are expressed as

$$\hat{\mathcal{X}}_{k|k} = \sum_{i=1}^{N_f} \omega_k^i \hat{\mathcal{X}}_{k|k}^i \quad (14)$$

and

$$P_{k|k} = \sum_{i=1}^{N_f} \omega_k^i [P_{k|k}^i + (\hat{\mathcal{X}}_{k|k}^i - \hat{\mathcal{X}}_{k|k})(\hat{\mathcal{X}}_{k|k}^i - \hat{\mathcal{X}}_{k|k})^T] \quad (15)$$

respectively. To implement the RP-SRF, a pseudo-code is presented in Algorithm 1.

Algorithm 1: Pseudo code for RP-SRF

`Step 1: Initialization`
- Initialize each filter with $\hat{\mathcal{X}}_{0|0}^n$ and $P_{0|0}^n$.

`Step 2: Prediction step`

Prior estimate of n^{th} filter,

$$\hat{\mathcal{X}}_{k|k-1}^n = F\hat{\mathcal{X}}_{k-1|k-1}^n - \mho_{k-1,k}. \quad (16)$$

Prior error covariance,

$$P_{k|k-1}^n = FP_{k-1|k-1}^n F^T + Q_k. \quad (17)$$

Innovation covariance,

$$V_k^n = HP_{k|k-1}^n H^T + R_k^n. \quad (18)$$

`Step 3: Correction step`

The Kalman gain,

$$K_k^n = P_{k|k-1}^n H^T (V_k^n)^{-1}. \quad (19)$$

Posterior estimate of n^{th} filter,

$$\hat{\mathcal{X}}_{k|k}^n = (I - K_k^n H)\hat{\mathcal{X}}_{k|k-1}^n + \gamma_k^n K_k^n b_k, \quad (20)$$

where

$$\gamma_k^n = (b_k^T (V_k^n)^{-1} b_k)^{-\frac{1}{2}} \rho(u_k^n), \quad (21)$$

where

$$u_k^n = (b_k^T (V_k^n)^{-1} b_k)^{-\frac{1}{2}} b_k^T (V_k^n)^{-1} (H\hat{\mathcal{X}}_{k|k-1}^n). \quad (22)$$

Posterior error covariance,

$$P_{k|k}^n = (I - K_k^n H)P_{k|k-1}^n + \delta_k^n K_k^n b_k b_k^T (K_k^n)^T, \quad (23)$$

where

$$\delta_k^n = (b_k^T (V_k^n)^{-1} b_k)^{-1}(2 + u_k \rho(u_k^n) - \rho^2(u_k^n)). \quad (24)$$

`Step 4: Range parameterization`
- Calculate the likelihood of the augmented measurement with the Eq. (12).
- Evaluate weights for each SRF as shown in Eq. (11).
- Posterior estimate of RP-SRF, $\hat{\mathcal{X}}_{k|k} = \sum_{i=1}^{N_f} \omega_k^i \hat{\mathcal{X}}_{k|k}^i$.
- Posterior error covariance of RP-SRF, $P_{k|k} = \sum_{i=1}^{N_f} \omega_k^i [P_{k|k}^i + (\hat{\mathcal{X}}_{k|k}^i - \hat{\mathcal{X}}_{k|k})(\hat{\mathcal{X}}_{k|k}^i - \hat{\mathcal{X}}_{k|k})^T]$.

IV. Simulation Results

A. Target-observer scenario

Here, we consider an underwater target tracking scenario [17], where the target moves in a nearly straight line with constant velocity and the own ship maneuvers as shown in Fig. 1. To track the target, we implemented here the CKF, SRF, RP-CKF, and the proposed RP-SRF. Parameters used in the simulation are listed in Table I. The sampling time, Δ is considered here to be $1min$ with an observation period of 30 minutes.

B. Initialization of filters

The initialization of all filters is done as per [16]. The position components of the relative state vector are initialized based on the prior initial target range and the first bearing measurement. The estimated initial range of the target is considered as $\bar{r} \sim \mathcal{N}(r, \sigma_r^2)$, where r is the true initial range and σ_r is its standard deviation. The initial bearing measurement is $\bar{\theta} \sim \mathcal{N}(\theta, \sigma_\theta^2)$, where θ is the true bearing and σ_θ is its standard deviation. Likewise, using prior knowledge of the target speed, the initial speed estimate is $\bar{s} \sim \mathcal{N}(s, \sigma_s^2)$, where s is the true initial target speed. Assuming that the target is moving towards the observer, the initial course estimate is

TABLE I: Parameters of the scenario

Parameters	Values
Initial range (r)	$5km$
Target speed (s)	$4knots$
Target course	-140^o
Observer speed	$5knots$
Observer initial course	140^o
Observer final course	20^o
Observer maneuver	From 13^{th} to $17^{th}min$
Initial range standard deviation (σ_r)	$2km$
Initial target speed standard deviation (σ_s)	$2knots$
Initial bearing standard deviation (σ_θ)	1.5^o
Initial course standard deviation (σ_c)	$\pi/\sqrt{12}$
Process noise intensity ($\bar{q}$)	$1.944 \times 10^{-6} km^2/min^3$

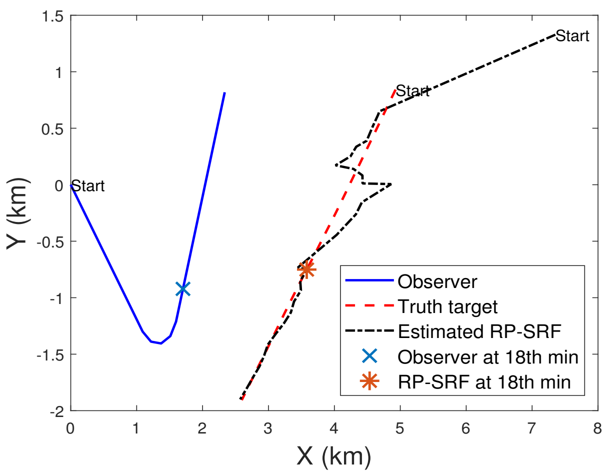


Fig. 1: Target and own ship trajectories along with the RP-SRF estimation

$\bar{c} = \bar{\theta} + \pi$.

Thereafter, the initial estimate of the relative state vector can be defined as

$$\hat{\mathcal{X}}_0 = \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{\dot{x}} \\ \bar{\dot{y}} \end{bmatrix} = \begin{bmatrix} \bar{r}\sin(\bar{\theta}) \\ \bar{r}\cos(\bar{\theta}) \\ \bar{s}\sin(\bar{c}) - \dot{x}_0^{obs} \\ \bar{s}\cos(\bar{c}) - \dot{y}_0^{obs} \end{bmatrix}. \quad (25)$$

The initial covariance is as follows [16]:

$$P_{0|0} = \begin{bmatrix} P_{xx} & P_{xy} & 0 & 0 \\ P_{yx} & P_{yy} & 0 & 0 \\ 0 & 0 & P_{\dot{x}\dot{x}} & P_{\dot{x}\dot{y}} \\ 0 & 0 & P_{\dot{y}\dot{x}} & P_{\dot{y}\dot{y}} \end{bmatrix}, \quad (26)$$

where

$$P_{xx} = \bar{r}^2\sigma_\theta^2\cos^2(\bar{\theta}) + \sigma_r^2\sin^2(\bar{\theta}) \quad (27)$$
$$P_{yy} = \bar{r}^2\sigma_\theta^2\sin^2(\bar{\theta}) + \sigma_r^2\cos^2(\bar{\theta}) \quad (28)$$
$$P_{xy} = P_{yx} = (\sigma_r^2 - \bar{r}^2\sigma_\theta^2)\sin(\bar{\theta})\cos(\bar{\theta}) \quad (29)$$
$$P_{\dot{x}\dot{x}} = \bar{s}^2\sigma_c^2\cos^2(\bar{c}) + \sigma_s^2\sin^2(\bar{c}) \quad (30)$$
$$P_{\dot{y}\dot{y}} = \bar{s}^2\sigma_c^2\sin^2(\bar{c}) + \sigma_s^2\cos^2(\bar{c}) \quad (31)$$
$$P_{\dot{x}\dot{y}} = P_{\dot{y}\dot{x}} = (\sigma_s^2 - \bar{s}^2\sigma_c^2)\sin(\bar{c})\cos(\bar{c}). \quad (32)$$

The accuracy of range parameterized filters depend on the number of parallel filters (N_f). For a large N_f value the estimation performance of the filter is better at the cost of higher computation time. Here, we choose $N_f = 10$. The initial estimated range and its standard deviation for each filter are calculated as in Eq. (10), where $r_{min} = r - 3\sigma_r$ and $r_{max} = r + 3\sigma_r$. If $r_{min} < 0$ then r_{min} is considered to be $0.1r$ [17]. All the independent N_f filters are assigned equal and normalized initial weights of $\omega_o^n = 1/N_f$, for $n = 1, ..., N_f$.

C. Performance Analysis

Performance analysis is done by comparing the RMSE plots obtained from different filters both for position and velocity, the percentage track loss, and the computation time of each filter.

The position and velocity RMSE (excluding the track loss) along with the Cramer Rao lower bound (CRLB) obtained from 500 Monte Carlo (MC) runs, are plotted in Figs. 2a–2b, respectively. From the Figs. 2a–2b, it can be observed that the RMSEs of RP-SRF is lowest followed by RP-CKF, SRF and CKF has the highest RMSEs.

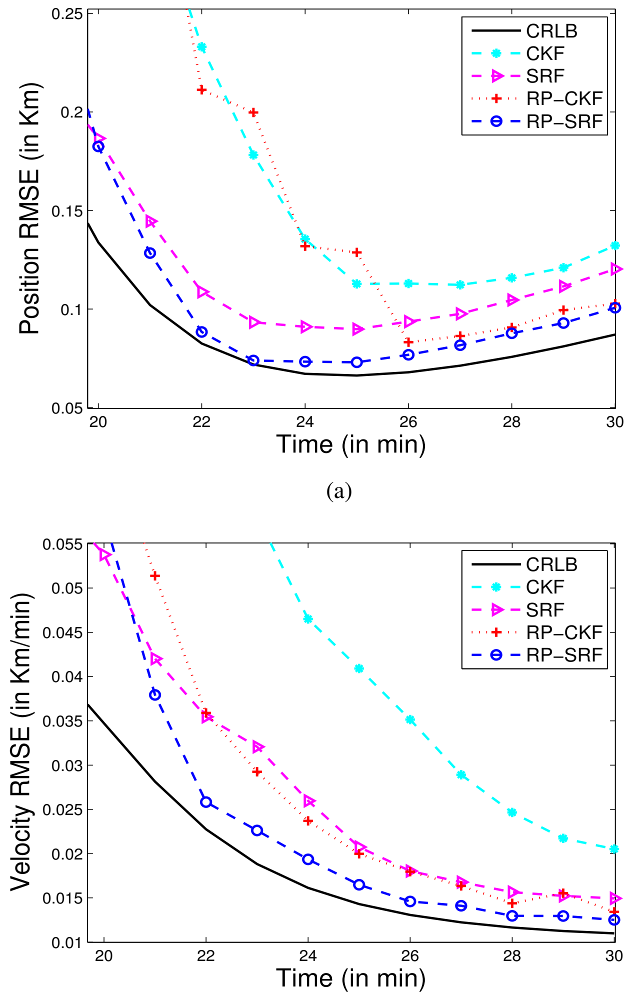
We also compare the filters in terms of percentage track loss, calculated from twenty thousand Monte-Carlo runs as shown in Table II. We define the estimator loses its track when the terminal range error goes beyond 1 Km. Table II exhibits that the RP-SRF and RP-CKF have the lowest track loss (0%) followed by SRF (0.18%), whereas the CKF has the highest track loss (2.05%). Further, we compare the computation time of the filters and presented in the same table. The RP-SRF is considered as the reference hence it is denoted by unity. From the table, we see that the computation time of the RP-SRF is about nine times more than SRF and CKF and about 1.25 times of the RP-CKF.

TABLE II: Percentage of track loss and relative computation time for all the implemented filters

Filters	Track loss (%)	Relative comp. time
CKF	2.05	0.1
SRF	0.18	0.11
RP-CKF	0	0.81
RP-SRF	0	1

V. Discussion and Conclusion

In this paper, to improve the tracking performance of SRF we implement the range parameterization technique. The proposed filter (RP-SRF) is compared with the CKF, SRF,



(b)

Fig. 2: RMSE in (a) position, and (b) velocity obtained from 500 MC runs

and RP-CKF in terms of RMSE, percentage of track loss, and relative execution time. In the simulation scenario considered here, RP-SRF shows zero track loss percentage and better RMSE than the other filters. We conclude that the proposed RP-SRF performs better than its existing counterparts in BOT problems with a caveat that comparative performance of it may not be the best in all possible engagement scenarios.

References

[1] S. S. Blackman and R. Popoli, *Design and analysis of modern tracking systems*. Artech House Publishers, 1999.

[2] J. C. Hassab, *Underwater signal and data processing*. CRC Press, 2018.

[3] S. Nardone, A. Lindgren, and K. Gong, "Fundamental properties and performance of conventional bearings-only target motion analysis," *IEEE Transactions on automatic control*, vol. 29, no. 9, pp. 775–787, 1984.

[4] R. Karlsson and F. Gustafsson, "Recursive Bayesian estimation: bearings-only applications," *IEE Proceedings-Radar, Sonar and Navigation*, vol. 152, no. 5, pp. 305–313, 2005.

[5] E. Fogel and M. Gavish, "Nth-order dynamics target observability from angle measurements," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 24, no. 3, pp. 305–308, 1988.

[6] T. L. Song, "Observability of target tracking with bearings-only measurements," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 32, no. 4, pp. 1468–1472, 1996.

[7] C. Jauffret and D. Pillon, "Observability in passive target motion analysis," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 32, no. 4, pp. 1290–1300, 1996.

[8] P. Shar and X. R. Li, "Practical approach to observability of bearings-only target tracking," in *Signal and Data Processing of Small Targets 1999*, vol. 3809, pp. 514–520, International Society for Optics and Photonics, 1999.

[9] V. J. Aidala, "Kalman filter behavior in bearings-only tracking applications," *IEEE Transactions on Aerospace and Electronic Systems*, no. 1, pp. 29–39, 1979.

[10] V. Aidala and S. Hammel, "Utilization of modified polar coordinates for bearings-only tracking," *IEEE Transactions on Automatic Control*, vol. 28, no. 3, pp. 283–294, 1983.

[11] S. C. Nardone and M. L. Graham, "A closed-form solution to bearings-only target motion analysis," *IEEE Journal of Oceanic Engineering*, vol. 22, no. 1, pp. 168–178, 1997.

[12] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with applications to tracking and navigation: theory algorithms and software*. John Wiley & Sons, 2004.

[13] T. Song and J. Speyer, "A stochastic analysis of a modified gain extended Kalman filter with applications to estimation with bearings only measurements," *IEEE Transactions on Automatic Control*, vol. 30, no. 10, pp. 940–949, 1985.

[14] A. Jawahar and S. K. Rao, "Modified polar extended Kalman filter (MP-EKF) for bearings-only target tracking," *Indian J. Sci. Technol*, vol. 9, no. 26, pp. 1–5, 2016.

[15] S. K. Rao, "Modified gain extended Kalman filter with application to bearings-only passive manoeuvring target tracking," *IEE Proceedings-Radar, Sonar and Navigation*, vol. 152, no. 4, pp. 239–244, 2005.

[16] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman filter: particle filters for tracking applications*. Artech house, 2003.

[17] P. H. Leong, S. Arulampalam, T. A. Lamahewa, and T. D. Abhayapala, "A Gaussian-sum based cubature Kalman filter for bearings-only tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 49, no. 2, pp. 1161–1176, 2013.

[18] R. Radhakrishnan, S. Bhaumik, and N. K. Tomar, "Gaussian sum shifted Rayleigh filter for underwater bearings-only target tracking problems," *IEEE Journal of Oceanic Engineering*, vol. 44, no. 2, pp. 492–501, 2018.

[19] J. Clark, R. Vinter, and M. Yaqoob, "Shifted Rayleigh filter: A new algorithm for bearings-only tracking," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 43, no. 4, pp. 1373–1384, 2007.

[20] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Transactions on signal processing*, vol. 50, no. 2, pp. 174–188, 2002.

[21] S. Arulampalam, M. Clark, and R. Vinter, "Performance of the shifted Rayleigh filter in single-sensor bearings-only tracking," in *2007 10th International Conference on Information Fusion*, pp. 1–6, IEEE, 2007.

[22] O. Straka, J. Duník, and M. Šimandl, "Performance evaluation of local state estimation methods in bearings-only tracking problems," in *14th International Conference on Information Fusion*, pp. 1–8, IEEE, 2011.

[23] N. Peach, "Bearings-only tracking using a set of range-parameterised extended Kalman filters," *IEE Proceedings-Control Theory and Applications*, vol. 142, no. 1, pp. 73–80, 1995.

[24] H. Wu, S. Chen, B. Yang, and X. Luo, "Range-parameterised orthogonal simplex cubature Kalman filter for bearings-only measurements," *IET Science, Measurement & Technology*, vol. 10, no. 4, pp. 370–374, 2016.

[25] R. Karlsson and F. Gustafsson, "Range estimation using angle-only target tracking with particle filters," in *Proceedings of the 2001 American Control Conference.(Cat. No. 01CH37148)*, vol. 5, pp. 3743–3748, IEEE, 2001.

[26] R. Radhakrishnan, S. Bhaumik, and N. K. Tomar, "Gaussian sum shifted Rayleigh filter for underwater bearings-only target tracking problems," *IEEE Journal of Oceanic Engineering*, vol. 44, no. 2, pp. 492–501, 2018.

[27] S. Arulampalam and B. Ristic, "Comparison of the particle filter with range-parameterized and modified polar EKFs for angle-only tracking," in *Signal and Data Processing of Small Targets 2000*, vol. 4048, pp. 288–299, International Society for Optics and Photonics, 2000.