A Examples of Φ-equilibria

We now introduce two notable examples of sets Φ, namely Correlated and Coarse Correlated equilibria.

Correlated equilibria (CEs) are obtained by considering all possible deviation strategies, *i.e.*,

$$\Phi_{\text{CE}} = \left\{ \phi \in [0,1]^{\ell \times \ell} : \sum_{b \in A} \phi(a,b) = 1 \quad \forall a \in A \right\}.$$

This set models a player that can observe its own recommendation and deviate to any other action with some probability.

Another important class of equilibria is Coarse Correlated Equilibria (CCEs) that are defined by the set

$$\Phi_{\text{CCE}} = \left\{ \phi \in [0,1]^{\ell \times \ell} : \phi(a,b) = \phi(a',b) \quad \forall a, a' \in A \wedge \sum_{b \in A} \phi(a,b) = 1 \quad \forall a \in A \right\}.$$

This models a player whose deviations are forced to be equal for each recommended action $a \in A$. Intuitively, the player has to decide their own deviation strategy before seeing the recommended action a. This greatly simplifies the possible deviation since $\phi \in \Phi_{\text{CCE}}$ can simply be identified with the marginals it induces.

B Missing proof from Section 3 (hardness)

Lemma 3.2. *For each tuple $\boldsymbol{a} \in \mathcal{A}$, define the utility of the "left team" as $u_{\text{L}}(\boldsymbol{a}) = \sum_{i \in V} u_{i_{\text{L}}}(\boldsymbol{a})$ and the utility of the "right team" as $u_{\text{R}}(a) = \sum_{i \in V} u_{i_{\text{R}}}(a)$. Then $u_{\text{L}}(\boldsymbol{a}) = -u_{\text{R}}(\boldsymbol{a})$.*

Proof. The statement follows from straightforward calculations

$$\begin{aligned}
u_{\text{L}}(\boldsymbol{a}) &= \sum_{i \in V} u_{i_{\text{L}}}(\boldsymbol{a}) \\
&= \sum_{i \in V} \sum_{j:(i,j) \in E} \tilde{A}^{i_{\text{L}}, j_{\text{R}}}(a_{i_{\text{L}}}, a_{j_{\text{R}}}) \\
&= -\sum_{i \in V} \sum_{j:(i,j) \in E} (A^{j,i})^\top (a_{i_{\text{L}}}, a_{j_{\text{R}}}) && (\tilde{A}^{i_{\text{L}}, j_{\text{R}}} = -(A^{j,i})^\top) \\
&= -\sum_{i \in V} \sum_{j:(i,j) \in E} (A^{i,j})^\top (a_{j_{\text{L}}}, a_{i_{\text{R}}}) && (\text{By swapping the sum and the identity of } i \text{ and } j) \\
&= -\sum_{i \in V} \sum_{j:(i,j) \in E} A^{i,j}(a_{i_{\text{R}}}, a_{j_{\text{L}}}) \\
&= -\sum_{i \in V} \sum_{j:(i,j) \in E} \tilde{A}^{i_{\text{R}}, j_{\text{L}}}(a_{i_{\text{R}}}, a_{j_{\text{L}}}) && (\tilde{A}^{i_{\text{R}}, j_{\text{L}}} = A^{i,j}) \\
&= -\sum_{i \in V} u_{i_{\text{R}}}(\boldsymbol{a}) \\
&= -u_{\text{R}}(\boldsymbol{a}),
\end{aligned}$$

concluding the proof. □

Lemma 3.3. *Given a $z \in \mathcal{S}^\nu$, and $i \in V$, it holds that $\|m_{i_{\text{L}}}(z) - m_{i_{\text{R}}}(z)\|_\infty \leq \nu$. Moreover, given a $z \in \Delta(A^{|\mathcal{N}|})$ and an $i \in V$, the set of safe deviations of player i_{L} is $\Phi^{\mathcal{S},\nu}_{i_{\text{L}}}(z) = \{x_{i_{\text{L}}} : \|x_{i_{\text{L}}} - m_{i_{\text{R}}}(z)\|_\infty \leq \nu\}$, which guarantees $\Phi^{\mathcal{S}}_{i_{\text{L}}}(z) \neq \emptyset$.*

Proof. Consider the case $j \le k$:

$$\begin{aligned} C^j_{i_{\mathrm{L}}}(z) &= \sum_{\boldsymbol{a} \in \mathcal{A}} C^j_{i_{\mathrm{L}}}(\boldsymbol{a}) z(\boldsymbol{a}) \\ &= \sum_{\boldsymbol{a} \in A^n : a_{i_{\mathrm{L}}} = j, a_{i_{\mathrm{R}}} \neq j} z(\boldsymbol{a}) - \sum_{\boldsymbol{a} \in A^n : a_{i_{\mathrm{L}}} \neq j, a_{i_{\mathrm{R}}} = j} z(\boldsymbol{a}) \\ &= \sum_{\boldsymbol{a} \in \mathcal{A} : a_{i_{\mathrm{L}}} = j} z(\boldsymbol{a}) - \sum_{\boldsymbol{a} \in \mathcal{A} : a_{i_{\mathrm{R}}} = j} z(\boldsymbol{a}) \\ &= m_{i_{\mathrm{L}}}(z|j) - m_{i_{\mathrm{R}}}(z|j) \end{aligned}$$

and thus $C^j_{i_{\mathrm{L}}}(z) \le \nu$, with $j \le k$ implies that $m_{i_{\mathrm{L}}}(z|j) - m_{i_{\mathrm{R}}}(z|j) \le \nu$. On the other hand $C^j_{i_{\mathrm{L}}}(z) \le \nu$, with $j > k$ implies that $m_{i_{\mathrm{R}}}(z|j) - m_{i_{\mathrm{L}}}(z|j) \le \nu$ which concludes the statement. $\square$

C Missing proofs and additional details from Section 4 (membership)

C.1 Explicit definition of $A(\tilde{z})$

The correspondence $Q : [0,1]^{\ell n} \rightrightarrows [0,1]^{\ell n}$ is given by $Q(\tilde{z}) = \{z : A(\tilde{z})z \le b(\tilde{z})\}$ where

$$A(\tilde{z}) = \begin{array}{c} \overbrace{\hphantom{D_1(\tilde{z})}}^{\ell} \\ \left[\begin{array}{c:c:c:c} \begin{matrix} D_1(\tilde{z}) \\ I_{\ell \times \ell} \\ 1_\ell^\top \\ -1_\ell^\top \end{matrix} & & & \\ \hdashline & \begin{matrix} D_2(\tilde{z}) \\ I_{\ell \times \ell} \\ 1_\ell^\top \\ -1_\ell^\top \end{matrix} & & \\ \hdashline & & \ddots & \\ \hdashline & & & \begin{matrix} D_n(\tilde{z}) \\ I_{\ell \times \ell} \\ 1_\ell^\top \\ -1_\ell^\top \end{matrix} \end{array}\right] \end{array} \begin{array}{l} m \\ \ell \\ 1 \\ 1 \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \in \mathbb{R}^{n \cdot (m + \ell + 2) \times \ell n}$$

where $D_i(\tilde{z}) \in \mathbb{R}^{m \times \ell}$ is a matrix such that $[0_{m\times\ell}, \cdots, D_i(\tilde{z}), \cdots, 0_{m\times\ell}]z \le 0_m \iff C^j_i(\tilde{p})p_i \le 0 \, \forall j \in [m]$. In particular, for each i, only the components of z corresponding to the strategies of the i-th player matter and correspond to the strategy p_i. We define g the flattening function which takes a product distribution $\tilde{p} = \bigotimes_{i \in [n]} \tilde{p}_i$ and returns the corresponding "unflatted" vector $\tilde{z}$ while we call h its inverse. Thus, we can write that $D_i(g(\tilde{p}))p_i \le 0_m$ if and only if $C^j_i(p_i \otimes \tilde{p}_{-i}) \le 0$ for all $j \in [m]$. Notice that $p_i \mapsto C^j_i(p_i \otimes \tilde{p}_{-i})$ is linear and can be written as $C^j_i(p_i \otimes \tilde{p}_{-i}) = c^j_i(\tilde{p})^\top p_i$, where $c^j_i(\tilde{p}) \in \mathbb{R}^\ell$ and each component $c^j_i(\tilde{p})_{\bar{a}}$, indexed by $\bar{a}$, is given by $\sum_{\boldsymbol{a} \in \mathcal{A} : a_i = \bar{a}} C^j_i(\boldsymbol{a}) \prod_{k \neq i} \tilde{p}_k(a_k)$. Consequently, we can take

$$D_i(\tilde{z}) = \begin{bmatrix} c^1_i(h(\tilde{z})) \\ \vdots \\ c^m_i(h(\tilde{z})) \end{bmatrix} \in \mathbb{R}^{m \times \ell n}$$

C.2 Proof of Claim 4.3 and Claim 4.4 from Proposition 4.2

Claim 4.3. *The functions $\tilde{z} \mapsto A(\tilde{z})z$ and $\tilde{z} \mapsto b(\tilde{z})^\top z$ defining the correspondence are L-Lipschitz for every $z \in [0,1]^{\ell n}$ where L has a representation polynomial in the size of the instance.*

Proof. First note that $b(\tilde{z})$ does not depend on $\tilde{z}$ and thus is trivially 0-Lipschitz. Thus, we only need to prove the statement about $A(\tilde{z})$. We recall the exact definition of $A(\tilde{z})$ given in Appendix C.1.

We analyze the Jacobian of the $D_i(\tilde{z})$. Any entry of $D_i(\tilde{z})$ would correspond to a cost j of the i-th player (rows) and to an action $\hat{a} \in A$ (columns), and any component ℓ of $\tilde{z}$ would correspond to a player i' (possibly different from i) and an action $\bar{a} \in A$. Thus, by defining $\tilde{p} = h(\tilde{z})$, we can compute the following

$$\frac{\partial c_i^j(h(\tilde{z}))_{\hat{a}}}{\partial \tilde{z}_\ell} = \frac{\partial c_i^j(\tilde{p})_{\hat{a}}}{\partial \tilde{p}_{i'}(\bar{a})} \frac{\partial \tilde{p}_{i'}(\bar{a})}{\partial \tilde{z}_\ell}$$

The second term is clearly 1, as the function h just rearranges the components $\tilde{z}$, while the first term is easily bounded as follows

$$\begin{aligned}\left|\frac{\partial c_i^j(\tilde{p})_{\hat{a}}}{\partial \tilde{p}_{i'}(\bar{a})}\right| &= \left|\frac{\partial \sum_{\boldsymbol{a}\in\mathcal{A}:a_i=\hat{a}} C_i^j(\boldsymbol{a}) \prod_{k\in[n],k\neq i} \tilde{p}_k(a_k)}{\partial \tilde{p}_{i'}(\bar{a})}\right| \\ &= \left|\sum_{\boldsymbol{a}\in\mathcal{A}:a_{i'}=\bar{a},a_i=\hat{a}} C_i^j(\boldsymbol{a}) \prod_{k\in[n],k\neq i',k\neq i} \tilde{p}_k(a_k)\right| \leq \ell^n.\end{aligned}$$

The following elementary lemma lets us conclude the proof.

Lemma C.1. *Let $M : \mathbb{R}^K \to \mathbb{R}^{m\times n}$ be a matrix valued function such that $\left|\frac{\partial M_{i,j}(\tilde{z})}{\partial \tilde{z}_k}\right| \leq C$ for all $i \in [m], j \in [n], k \in [K]$ then*

$$\|(M(\tilde{z}) - M(\tilde{z}'))z\| \leq Cm\sqrt{nK}\|\tilde{z} - \tilde{z}'\|,$$

for all $z \in [0,1]^K$.

Indeed,

$$\begin{aligned}\|(A(\tilde{z}) - A(\tilde{z}'))z\| &\leq \ell^n(\ell n)\sqrt{\ell n \cdot n(m+\ell+2)}\|\tilde{z} - \tilde{z}'\| \\ &\leq 2\ell^{n+2}n^2\sqrt{m}\|\tilde{z} - \tilde{z}'\|\end{aligned}$$

and thus $L = \text{poly}(\ell^n, m, n)$ concluding the proof. ☐

Claim 4.4. *The operator $F : [0,1]^{\ell n} \to [0,1]^{\ell n}$ is G-Lipschitz where G has a representation polynomial in the size of the instance.*

Proof. We can get a simple upper bound on the Lipschitz constant of F by bounding its gradient. In particular $F(z) = (-\nabla_{p_1} u_1(p), \ldots, -\nabla_{p_n} u_n(p))$, where as usual z is the unrolling of the product distribution $p = \bigotimes_{i=1}^n p_i$. We can consider any component of F, which will correspond to some player $i \in [n]$ and action $\bar{a} \in A$, and consider some component of z which will correspond to some player $j \in [n]$ and some action $\tilde{a} \in A$. The component of F selected corresponds to $-\frac{\partial u_i(p)}{\partial p_i(\bar{a})}$ We can then consider the following:

$$-\frac{\partial^2 u_i(p)}{\partial p_j(\tilde{a})\partial p_i(\bar{a})} = -\sum_{\boldsymbol{a}\in\mathcal{A}:a_i=\bar{a},a_j=\tilde{a}} u_i(\boldsymbol{a}) \prod_{k\neq i,j} p_k(a_k)$$

and thus $\left|\frac{\partial^2 u_i(p)}{\partial p_j(\tilde{a})\partial p_i(\bar{a})}\right| \leq \ell^n$. The mean value theorem trivially concludes the proof:

$$\|F(z) - F(z')\| \leq \|J_F(\xi)\| \cdot \|z - z'\|$$

for some ξ on the segment connecting z and z'. Now for any matrix $M \in \mathbb{R}^{m\times n}$ it holds that $\|M\| \leq \sqrt{mn} \cdot \sup_{i,j} |M_{i,j}|$ and thus $\|J_F(\xi)\| \leq n\ell^{n+1} = G$, concluding the proof. ☐

Claim 4.5. *Let $\nu' = \min(\frac{\epsilon}{4}, \frac{\nu}{1+n\nu}, \frac{1}{2n})$ and $\epsilon' = \frac{\epsilon}{4(1+\nu')}$, then p is such that*

$$u_i(p) \geq u_i(\tilde{p}_i \otimes p_{-i}) - \epsilon \quad \forall \tilde{p}_i \in \Phi_i^{\mathcal{S},\nu}(p)$$

and $p \in \mathcal{S}^\nu$.

First, we claim that p is ν-safe. Indeed, $\|z^i\|_1 \in [1-\nu', 1+\nu']$ and for each $i \in [n], j \in [m]$ and we can directly compute

$$\sum_{\boldsymbol{a}\in A^n} C_i^j(\boldsymbol{a}) \prod_{j\in[n]} z^j(a_j) \le \nu'$$

then, we can divide the left and right hand side by $\prod_{j\in[n]} \|z^j\|_1 \ge (1-\nu')^n$ and, obtain that:

$$C_i^j(p) = \sum_{\boldsymbol{a}\in A^n} C_i^j(\boldsymbol{a}) \prod_{j\in[n]} p_j(a_j) \le \frac{\nu'}{(1-\nu')^n} \le \frac{\nu'}{1-n\nu'} \le \nu$$

where in the last inequality we assumed that $\nu' \le \frac{\nu}{1+n\nu}$. This shows that indeed $p \in \mathcal{S}^\nu$.

For any ν-safe deviation $\tilde{p}_i \in \Delta(A)$ (*i.e.* it holds that $C_i^j(\tilde{p}_i \otimes p_{-i}) \le \nu$), we can consider $\tilde{z}$ defined as $\tilde{z} = [z^1, \ldots, \tilde{p}^i, \ldots, z^n]$. By the definition of the correspondence (see Appendix C.1) we have that $\tilde{z} \in Q_\nu(z)$, since $\tilde{z}^i = \tilde{p}_i$, and thus, $\tilde{z}$ is a valid deviation of QUASIVI and thus

$$\sum_{\boldsymbol{a}\in A^n} C_i^j(\boldsymbol{a}) \tilde{p}_i(a_i) \prod_{j\in[n]:j\neq i} z^j(a_j) \le \nu',$$

dividing both sides by $\prod_{j\in[n]:j\neq i} \|z^j\|_1 \ge (1-\nu')^{n-1}$ we obtain $C_i^j(\tilde{p}_i \otimes p_{-i}) \le \frac{\nu'}{(1-\nu')^{n-1}} \le \nu$ which holds for every ν-safe deviation.

Now, we also claim that $p = \bigotimes_{i\in[n]} p_i$ satisfies the equilibrium constraints. Since z is a solution to QUASIVI we have that for all $\tilde{z} \in Q_\nu(z)$ the following holds:

$$F(z)^\top(\tilde{z} - z) \ge -\epsilon' \implies -\sum_{i\in[n]} \sum_{\bar{a}\in A} \sum_{\boldsymbol{a}\in\mathcal{A}:a_i=\bar{a}} u_i(\boldsymbol{a}) \prod_{j\neq i} z^j(a_j)(\tilde{z}^i(\bar{a}) - z^i(\bar{a})) \ge -\epsilon',$$

which implies that $\sum_{\bar{a}\in A} \sum_{\boldsymbol{a}\in\mathcal{A}:a_i=\bar{a}} u_i(\boldsymbol{a}) \prod_{j\neq i} z^j(a_j)(\tilde{z}^i(\bar{a}) - z^i(\bar{a})) \le \epsilon'$ once we specialize to $\tilde{z}$ such that $\tilde{z}^j = z^j$ for all $j \neq i$. Rearranging and dividing the equation above by $\gamma = \prod_{j\in[i],j\neq i} \|z^j\|_1 \|\tilde{z}^i\|_1 \|z^i\|_1$ we get

$$\frac{u_i(p)}{\|\tilde{z}^i\|_1} \ge \frac{u_i(\tilde{p}_i \otimes p_{-i})}{\|z^i\|_1} - \frac{\epsilon'}{\gamma}$$

and since $\|z^j\|_1 \in [1-\nu', 1+\nu']$ for all $j \in [n]$ and $\|\tilde{z}^i\|_1 \in [1-\nu', 1+\nu']$ this implies that $\gamma \ge (1-\nu')^{n+1}$ and thus $\frac{u_i(p)}{1-\nu'} \ge \frac{u_i(\tilde{p}_i \otimes p_{-i})}{1+\nu'} - \frac{\epsilon'}{(1-\nu')^{n+1}}$. Then, since $u_i(\boldsymbol{a}) \in [0,1]$, we get that $u_i(p) \ge u_i(\tilde{p}_i \otimes p_{-i}) - \left(2\nu' + \frac{\epsilon'}{(1-\nu')^n}(1+\nu')\right)$.[6] By taking $\nu' = \min(\frac{\epsilon}{4}, \frac{\nu}{1+n\nu}, \frac{1}{2n})$ and $\epsilon' = \frac{\epsilon}{4(1+\nu')}$ we have that $2\nu' + \frac{\epsilon'}{(1-\nu')^n}(1+\nu') \le \frac{\epsilon}{2} + \frac{\epsilon}{4}\frac{1}{1-n\nu'} \le \epsilon$ which concludes the proof.

C.3 Additional technical lemmas

Lemma C.1. *Let $M : \mathbb{R}^K \to \mathbb{R}^{m\times n}$ be a matrix valued function such that $\left|\frac{\partial M_{i,j}(\tilde{z})}{\partial \tilde{z}_k}\right| \le C$ for all $i \in [m], j \in [n], k \in [K]$ then*

$$\|(M(\tilde{z}) - M(\tilde{z}'))z\| \le Cm\sqrt{nK}\|\tilde{z} - \tilde{z}'\|,$$

for all $z \in [0,1]^K$.

Proof. Let $\{m_i : [0,1]^K \to \mathbb{R}^n\}_{i=1}^m$ be the functions defining the rows of M and $h_i(\tilde{z}|z) = m_i(\tilde{z})^\top z$. With this notation it is easy to check that $\nabla_{\tilde{z}} h_i(\tilde{z}|z) = J_{m_i}(\tilde{z})^\top z$ and thus $\|\nabla_{\tilde{z}} h_i(\tilde{z}|z)\| \le \|J_{m_i}(\tilde{z})\| \|z\| \le C\sqrt{mnK}$.

By the mean value theorem, we have that for some ξ in the segment connecting $\tilde{z}$ and $\tilde{z}'$, we have

$$\begin{aligned} |(m_i(\tilde{z}) - m_i(\tilde{z}'))^\top z| &\le \|\nabla_{\tilde{z}} h(\xi|z)\| \cdot \|\tilde{z} - \tilde{z}'\| \\ &\le C\sqrt{mnK}\|\tilde{z} - \tilde{z}'\| \end{aligned}$$

[6] Since $x\frac{1-y}{1+y} \ge x - 2y$ for each $x, y \in [0,1]$.

Finally,

$$\begin{aligned}\|(M(\tilde{z}) - M(\tilde{z}'))z\|^2 &= \sum_{i=1}^{m}((m_i(\tilde{z}) - m_i(\tilde{z}'))^\top z)^2 \\ &\leq C^2 m^2 nK \|\tilde{z} - \tilde{z}'\|^2\end{aligned}$$

concluding the proof. □