A Basic Observations

In this section, we present several preliminary observations that lay the groundwork for proving our main theorems. For clarity and consistency, we fix an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ throughout this section.

Observation A.1. *For any deterministic voting rule f and an ordering σ over $\mathcal{C}$, there exists a candidate with in-degree at least $\lceil \frac{m-1}{2} \rceil$ in $\mathcal{T}(f, \mathcal{C}, \sigma)$.*

Observation A.2. *Since o_g is the optimal candidate in group g, we have $\mathsf{cost}_g(\mathsf{o}_g) \leq \mathsf{cost}_g(c)$ for any candidate c, including o. This holds for all objectives (avg-max, avg-avg, max-max and max-avg).*

Observation A.3. *Since $\mathsf{cost}(.)$ is defined as the maximum over $\mathsf{cost}_g(.)$ under the max-avg and max-max objectives, it follows that $\mathsf{cost}_g(\mathsf{o}) \leq \mathsf{cost}(\mathsf{o})$, for each group g.*

Observation A.4. *For rand-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ur})$ with output w, the expected cost of the mechanism is given by*

$$\mathbb{E}[\mathsf{cost}(\mathsf{w})] = \frac{1}{k} \sum_{g \in \mathcal{G}} \mathsf{cost}(\mathsf{w}_g).$$

Observation A.5. *For rand-rand mechanism $\Psi = (\mathsf{f}_{rd}, \mathsf{f}_{ur})$ with output w, the expected cost of the mechanism is given by*

$$\mathbb{E}[\mathsf{cost}(\mathsf{w})] = \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{n_g} \sum_{v \in g} \mathsf{cost}(\mathsf{top}(v)).$$

Proof. By the definitions of the Random Dictatorship rule (f_{rd}) and the uniform selection rule (f_{ur}), we have

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \sum_{g \in \mathcal{G}} \Pr(\mathsf{w} = \mathsf{w}_g) \cdot \mathbb{E}[\mathsf{cost}(\mathsf{w}_g)] \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \mathbb{E}[\mathsf{cost}(\mathsf{w}_g)] \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \sum_{v \in g} \Pr\left(\mathsf{w}_g = \mathsf{top}(v)\right) \cdot \mathsf{cost}(\mathsf{top}(v)) \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{n_g} \sum_{v \in g} \mathsf{cost}(\mathsf{top}(v)).
\end{aligned}$$

$\square$

Observation A.6. *For the max-avg objective and any group g, we have $\mathsf{cost}_g(\mathsf{o}) \leq \mathsf{cost}(\mathsf{o})$, as implied directly by the definition of max-avg.*

Observation A.7. *Since $\mathsf{top}(v)$ denotes the candidate closest to voter v, it follows that $\mathsf{d}(v, \mathsf{top}(v)) \leq \mathsf{d}(v, c)$ for any candidate c.*

Observation A.8. *For every voter v and every candidate c, we have $\mathsf{d}(v, c) \leq \mathsf{d}(v^{**}(c), c)$.*

Observation A.9. *For every group g, every voter $v \in g$, and every candidate c, we have $\mathsf{d}(v, c) \leq \mathsf{d}(v^*(c, g), c)$.*

Observation A.10. *Consider a distributed mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$, where f_{in} is a deterministic rule with distortion at most α. By the definition of centralized distortion, we know that:*

$$\mathsf{cost}_g(\mathsf{w}_g) \leq \alpha \cdot \mathsf{cost}_g(\mathsf{o}_g), \quad \forall g \in \mathcal{G}.$$

Observation A.11. *Consider a det-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$, where f_{ov} has distortion at most β with respect to avg objective. By the definition of centralized distortion, we know that:*

$$\frac{1}{k} \sum_{g \in \mathcal{G}} \mathsf{d}(\mathsf{w}, \mathsf{w}_g) \leq \beta \cdot \frac{1}{k} \sum_{g \in \mathcal{G}} \mathsf{d}(\mathsf{o}, \mathsf{w}_g), \quad \forall g \in \mathcal{G}.$$

B Proofs for Section 3 (Distortion Bounds of rand-det)

Theorem 3.1. *Let f_α be a deterministic voting rule with distortion at most α. Then, for the* max-avg *objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_\alpha, \mathsf{f}_{ur})) \leq \alpha + 2$.

Proof. Consider an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and rand-det mechanism $\Psi = (\mathsf{f}_\alpha, \mathsf{f}_{ur})$. Now, we obtain

$$
\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{w}_g) && \text{(Observation A.4)} \\
&= \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_{g^*(\mathsf{w}_g)}} \sum_{v \in g^*(\mathsf{w}_g)} \mathsf{d}(v, \mathsf{w}_g) && \text{(Definition of max-avg)} \\
&\leq \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_{g^*(\mathsf{w}_g)}} \sum_{v \in g^*(\mathsf{w}_g)} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) && \text{(Triangle Inequality)} \\
&= \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_{g^*(\mathsf{w}_g)}} \sum_{v \in g^*(\mathsf{w}_g)} \mathsf{d}(v, \mathsf{o}) \\
&+ \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_{g^*(\mathsf{w}_g)}} \sum_{v \in g^*(\mathsf{w}_g)} \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \\
&\leq \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_{g^*(\mathsf{w}_g)}} \sum_{v \in g^*(\mathsf{w}_g)} \mathsf{d}(\mathsf{o}, \mathsf{w}_g) && \text{(Observation A.3)} \\
&= \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \\
&\leq \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_g} \sum_{v \in g} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{w}_g) \Big) && \text{(Triangle Inequality)} \\
&= \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_g} \sum_{v \in g} \mathsf{d}(v, \mathsf{o}) \\
&+ \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \frac{1}{n_g} \sum_{v \in g} \mathsf{d}(v, \mathsf{w}_g) \\
&\leq \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}_g(\mathsf{w}_g) \\
&= 2\mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}_g(\mathsf{w}_g) \\
&\leq 2\mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \alpha \cdot \mathsf{cost}_g(\mathsf{o}_g) && \text{(Observation A.10)} \\
&\leq 2\mathsf{cost}(\mathsf{o}) + \alpha \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}_g(\mathsf{o}) && \text{(Observation A.2)} \\
&\leq 2\mathsf{cost}(\mathsf{o}) + \alpha \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{o}) && \text{(Observation A.3)} \\
&= (\alpha + 2)\mathsf{cost}(\mathsf{o}).
\end{aligned}
$$

□

Theorem 3.2. *Let f_α be a deterministic voting rule with distortion at most α. Then, for the* avg-avg *objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_\alpha, \mathsf{f}_{ur})) \leq \alpha + 2 - \frac{2}{k}$.

Proof. Consider an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and $\mathsf{rand\text{-}det}$ mechanism $\Psi = (\mathsf{f}_\alpha, \mathsf{f}_{ur})$. Now, we obtain

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}(\mathsf{w}_g) && \text{(Observation A.4)} \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}} \mathsf{cost}_{g'}(\mathsf{w}_g) \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(v, \mathsf{w}_g) && \text{(Definition of avg-avg)} \\
&\le \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g = g'} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(v, \mathsf{w}_g) && \text{(Triangle Inequality)} \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{cost}_g(\mathsf{w}_g) \\
&\le \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \alpha \cdot \mathsf{cost}_g(\mathsf{o}_g) && \text{(Observation A.10)} \\
&\le \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \alpha \cdot \mathsf{cost}_g(\mathsf{o}) && \text{(Observation A.2)} \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \Big) \\
&+ \frac{\alpha}{k} \cdot \mathsf{cost}(\mathsf{o}) && \Big(\mathsf{cost}(\mathsf{o}) = \sum_{g \in \mathcal{G}} \frac{1}{k} \mathsf{cost}_g(\mathsf{o})\Big) \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(v, \mathsf{o}) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(\mathsf{o}, \mathsf{w}_g) + \frac{\alpha}{k} \cdot \mathsf{cost}(\mathsf{o}) \\
&= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(v, \mathsf{o}) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{k-1}{k} \cdot \mathsf{d}(\mathsf{o}, \mathsf{w}_g) + \frac{\alpha}{k} \cdot \mathsf{cost}(\mathsf{o}) \\
&\le \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{k} \sum_{g' \in \mathcal{G}, g \neq g'} \frac{1}{n_{g'}} \sum_{v \in g'} \mathsf{d}(v, \mathsf{o}) \\
&+ \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{k-1}{k} \cdot \frac{1}{n_g} \sum_{v \in g} (\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{w}_g)) + \frac{\alpha}{k} \cdot \mathsf{cost}(\mathsf{o}) && \text{(Triangle Inequality)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g\neq g'}\frac{1}{n_{g'}}\sum_{v\in g'}\mathsf{d}(v,\mathsf{o})\\
&+\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{k-1}{k}\cdot(\mathsf{cost}_g(\mathsf{o})+\mathsf{cost}_g(\mathsf{w}_g))+\frac{\alpha}{k}\cdot\mathsf{cost}(\mathsf{o}) && \text{(Definition of } \mathsf{cost}_g(.))\\
&\le \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g\neq g'}\frac{1}{n_{g'}}\sum_{v\in g'}\mathsf{d}(v,\mathsf{o})\\
&+\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{k-1}{k}\cdot(\mathsf{cost}_g(\mathsf{o})+\alpha\cdot\mathsf{cost}_g(\mathsf{o}_g))+\frac{\alpha}{k}\cdot\mathsf{cost}(\mathsf{o}) && \text{(Observation A.10)}\\
&\le \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g\neq g'}\frac{1}{n_{g'}}\sum_{v\in g'}\mathsf{d}(v,\mathsf{o})\\
&+\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{k-1}{k}\cdot(\mathsf{cost}_g(\mathsf{o})+\alpha\cdot\mathsf{cost}_g(\mathsf{o}))+\frac{\alpha}{k}\cdot\mathsf{cost}(\mathsf{o}) && \text{(Observation A.2)}\\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g\neq g'}\frac{1}{n_{g'}}\sum_{v\in g'}\mathsf{d}(v,\mathsf{o})\\
&+\frac{(k-1)(\alpha+1)}{k}\cdot\mathsf{cost}(\mathsf{o})+\frac{\alpha}{k}\cdot\mathsf{cost}(\mathsf{o}) && \text{(Definition of } \mathsf{avg\text{-}avg})\\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g\neq g'}\mathsf{cost}_{g'}(\mathsf{o})+\frac{\alpha k+k-1}{k}\cdot\mathsf{cost}(\mathsf{o})\\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{cost}_{g'}(\mathsf{o})-\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\sum_{g'\in\mathcal{G},g=g'}\mathsf{cost}_{g'}(\mathsf{o})\\
&+\frac{\alpha k+k-1}{k}\cdot\mathsf{cost}(\mathsf{o})\\
&= \mathsf{cost}(\mathsf{o})-\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{k}\cdot\mathsf{cost}_g(\mathsf{o})+\frac{\alpha k+k-1}{k}\cdot\mathsf{cost}(\mathsf{o})\\
&= \mathsf{cost}(\mathsf{o})-\frac{1}{k}\cdot\mathsf{cost}(\mathsf{o})+\frac{\alpha k+k-1}{k}\cdot\mathsf{cost}(\mathsf{o})\\
&= (\alpha+2-\frac{2}{k})\cdot\mathsf{cost}(\mathsf{o}).
\end{aligned}
$$

□

Theorem 3.3. *Let* f_{un} *be a deterministic voting rule that satisfies the unanimity property. Then, for the* avg-max *and* max-max *objectives in general metric spaces,* $\mathsf{D}((\mathsf{f}_{un},\mathsf{f}_{ur}))\le 3$.

Proof. Here, we present a proof only for the avg-max objective; the proof for max-max is similar. Consider an instance $\mathcal{I}=(\mathcal{V},\mathcal{C},\mathcal{G},\pi,\mathsf{d})$ and rand-det mechanism $\Psi=(\mathsf{f}_{un},\mathsf{f}_{ur})$. By the unanimity property, for each group g, there exists a voter $v_g\in g$ who prefers w_g to o. Therefore, we have:

$$
\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \sum_{g\in\mathcal{G}}\frac{1}{k}\cdot\mathsf{cost}(\mathsf{w}_g) && \text{(Observation A.4)}\\
&= \sum_{g\in\mathcal{G}}\frac{1}{k}\cdot\frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{d}(v^*(\mathsf{w}_g,g'),\mathsf{w}_g) && \text{(Definition of } \mathsf{avg\text{-}max})\\
&\le \sum_{g\in\mathcal{G}}\frac{1}{k}\cdot\frac{1}{k}\sum_{g'\in\mathcal{G}}\Big(\mathsf{d}(v^*(\mathsf{w}_g,g'),\mathsf{o})+\mathsf{d}(\mathsf{o},\mathsf{w}_g)\Big) && \text{(Triangle Inequality)}\\
&\le \sum_{g\in\mathcal{G}}\frac{1}{k}\cdot\frac{1}{k}\sum_{g'\in\mathcal{G}}\Big(\mathsf{d}(v^*(\mathsf{o},g'),\mathsf{o})+\mathsf{d}(\mathsf{o},\mathsf{w}_g)\Big) && \text{(Observation A.9)}
\end{aligned}
$$

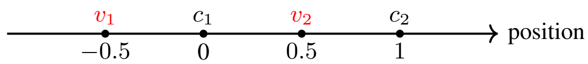

Figure 1: An example used in the proof of Theorem 3.4.

$$
\begin{aligned}
&= \mathsf{cost}(\mathsf{o}) + \sum_{g \in \mathcal{G}} \frac{1}{k} \cdot \mathsf{d}(\mathsf{o}, \mathsf{w}_g) \\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{1}{k} \sum_{g \in \mathcal{G}} \Big(\mathsf{d}(\mathsf{o}, v_g) + \mathsf{d}(v_g, \mathsf{w}_g) \Big) && \text{(Triangle Inequality)} \\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{2}{k} \sum_{g \in \mathcal{G}} \mathsf{d}(v_g, \mathsf{o}) && (\mathsf{d}(v_g, \mathsf{w}_g) \leq \mathsf{d}(v_g, \mathsf{o})) \\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{2}{k} \sum_{g \in \mathcal{G}} \mathsf{d}(v^*(\mathsf{o}, g), \mathsf{o}) && \text{(Observation A.9)} \\
&= 3\mathsf{cost}(\mathsf{o}).
\end{aligned}
$$

□

Theorem 3.4. *In general metric spaces, any* rand-det *mechanism has distortion of at least* 3 *for the* avg-max *and* max-max *objectives, even when the metric space is a line.*

Proof. We provide a proof for the avg-max objective. A similar argument can be used to prove the result for the max-max objective as well.

Consider a rand-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$. We construct an instance with candidates $\mathcal{C} = \{c_1, c_2\}$ and voters $\mathcal{V} = \{v_1, v_2\}$ in a single group. c_1 and c_2 are located at positions 0 and 1, respectively. v_1 and v_2 with preference profiles $\pi_1 = (c_1, c_2)$ and $\pi_2 = (c_2, c_1)$, are also positioned at $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. Without loss of generality, assume that Ψ selects c_2 as the representative of the group, and thus the final winner is c_2. Refer to Figure 1 for a visual illustration. By the definition of the avg-max objective, and since there is only one group, we have $\mathsf{cost}(c_1) = \frac{1}{2}$ and $\mathsf{cost}(c_2) = \frac{3}{2}$. Therefore, the distortion of Ψ is

$$\mathsf{D}(\Psi) \geq \frac{\mathsf{cost}(c_2)}{\mathsf{cost}(c_1)} = 3.$$

□

Theorem 3.5. *Any* rand-det *mechanism has a metric distortion of at least* 5 *for* max-avg*, even when the metric is a line.*

Proof. Consider a rand-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$. We construct an instance with candidates $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, and voters $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$, split into two groups $g_1 = \{v_1, v_2\}$ and $g_2 = \{v_3, v_4\}$, all within a line metric.

Let σ be an arbitrary ordering of the candidates. Without loss of generality suppose c_1 is a candidate with in-degree at least $\lceil \frac{m-1}{2} \rceil = 2$ in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$. By Observation A.1, we know such a candidate always exists. Without loss of generality, suppose c_2 and c_3 are two candidates that have a directed edge toward c_1 in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$. Again, without loss of generality, assume $c_2 \succ_\sigma c_3$. Now, consider the following construction on the line metric:

- c_2, c_1, and c_3 are located at -1, 0, and 1, respectively.
- v_2 and v_3 are located at 0, v_1 and v_4 are located at -0.5 and 0.5, respectively.
- c_4 is also positioned based on σ, ensuring that the input to $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$ remains valid. We consider three cases:
  - case 1: If $c_2 \succ_\sigma c_3 \succ_\sigma c_4$, c_4 is located at 10.

Figure 2: An example used in case 1 of Theorem 3.5. Different groups are represented using different colors.

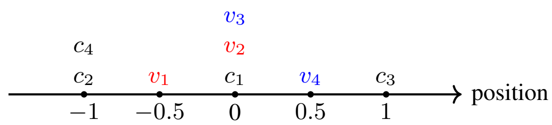

Figure 3: An example used in case 2 of Theorem 3.5. Different groups are represented using different colors.

- case 2: If $c_2 \succ_\sigma c_4 \succ_\sigma c_3$, c_4 is located at -1.
- case 3: If $c_4 \succ_\sigma c_2 \succ_\sigma c_3$, c_4 is located at 1. Refer to Figures 2 to 4.

Note that when two candidates are equidistant from a voter, multiple preference profiles can be consistent with the underlying metric space. By the definition of the Bias tournament, we know that

- A directed edge exists from c_2 to c_1. Thus, c_2 is selected as the representative of g_1.
- Likewise, there exists a directed edge from c_3 to c_1, making c_3 the representative of g_2.

Clearly, c_1 is the optimal candidate in all cases. By the definition of the max-avg objective, we have $\mathsf{cost}(c_2) = \mathsf{cost}(c_3) = \frac{5}{4}$ and $\mathsf{cost}(c_1) = \frac{1}{4}$. The mechanism must now select the final winner from the group representatives, c_2 or c_3. We calculate the distortion of Ψ as follows:

$$\begin{aligned}\mathsf{D}(\Psi) &\geq \min\left(\frac{\mathsf{cost}(c_2)}{\mathsf{cost}(\mathsf{o})}, \frac{\mathsf{cost}(c_3)}{\mathsf{cost}(\mathsf{o})}\right) \\ &\geq \frac{\mathsf{cost}(c_2)}{\mathsf{cost}(c_1)} \\ &= 5.\end{aligned}$$

$\square$

Theorem 3.6. *Any* rand-det *mechanism has a metric distortion of at least* $5 - \frac{2}{k}$ *for* avg-avg.

Proof. Consider a rand-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$. We construct an instance with candidates $\mathcal{C} = \{c_1, c_2, \ldots, c_{m=2k}\}$, voters $\mathcal{V} = \{v_1, v_2, \ldots, v_{n=2k}\}$, and k groups $g_i = \{v_{2i-1}, v_{2i}\}$ for $1 \leq i \leq k$. Let σ be an arbitrary ordering of the candidates. Without loss of generality, suppose that c_{2k} is a candidate with in-degree at least $\lceil \frac{m-1}{2} \rceil = k$ in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$. Again, without loss of generality, assume that $c_1, c_2, \ldots, c_k$ are k candidates that have directed edges toward c_{2k} in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$.

We construct a connected graph G with $2k+3$ vertices $u_1, u_2, \ldots, u_{2k+3}$ and use the shortest-path distances in G as the underlying metric d. Each voter and candidate is placed on one of the vertices (a vertex may host multiple entities). The construction of G is as follows:

- Place candidate c_{2k} at vertex u_1.
- For each $1 \leq i \leq k+1$, add an edge between u_1 and u_{2i}, and another edge between u_{2i} and u_{2i+1}. This creates $k+1$ branches extending from the central vertex u_1.
- For each $1 \leq i \leq k$, place voter v_{2i-1} at vertex u_1, voter v_{2i} at vertex u_{2i}, and candidate c_i at vertex u_{2i+1}.

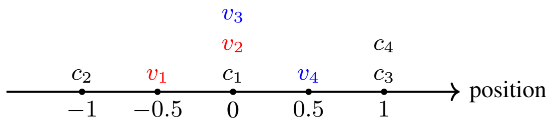

Figure 4: An example used in case 3 of Theorem 3.5. Different groups are represented using different colors.

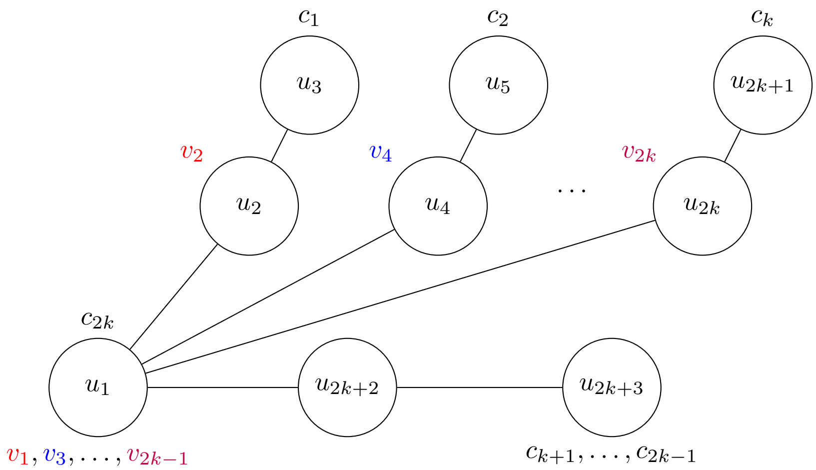

Figure 5: The graph G used in Theorem 3.6. Different groups are represented using different colors.

- For each $k+1 \leq i \leq 2k-1$, place candidate c_i at vertex u_{2k+3}.

Refer to Figure 5 for an illustration of the construction.

Pairwise distances between candidates and voters are presented in Table 2 and Table 3. One can verify that the preference profiles defined in Table 6 are consistent with the metric space d derived from distances in G. Note that there are multiple preference profiles consistent with d.

By the definition of the Bias tournament, we conclude that the representative of g_i is c_i for all $1 \leq i \leq k$, thus, one of these representatives is the final winner. Our analysis shows that $\mathsf{cost}(c_{2k}) = \frac{1}{2}$, and $\mathsf{cost}(c_i) = \frac{5-\frac{2}{k}}{2}$, for all $1 \leq i \leq k$. We can now derive the distortion of mechanism Ψ:

$$\begin{aligned}\mathsf{D}(\Psi) &\geq \frac{\min_{1\leq i\leq k}\Big(\mathsf{cost}(c_i)\Big)}{\mathsf{cost}(\mathsf{o})} \\ &\geq \frac{\frac{5-\frac{2}{k}}{2}}{\mathsf{cost}(c_{2k})} \\ &= 5 - \frac{2}{k}.\end{aligned}$$

$\square$

C Proofs for Section 4 (Distortion Bounds of rand-rand)

Theorem 4.1. *For the max-max objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_{rd}, \mathsf{f}_{ur})) \leq 3$.

$\mathsf{d}(\cdot,\cdot)$	c_{2k}	c_i
v_{2i-1}	0	2
v_{2i}	1	1
v_{2j-1}	0	2
v_{2j}	1	3

Table 2: For any $1 \leq i, j \leq k$ with $i \neq j$, the pairwise distances between the candidates $c_1, c_2, \ldots, c_k$ and the voters are as shown in Theorem 3.6.

$\mathsf{d}(\cdot,\cdot)$	c_i
v_{2j-1}	2
v_{2j}	3

Table 3: For any $k+1 \leq i \leq 2k-1$ and $1 \leq j \leq k$, the pairwise distances between the candidates $c_{k+1}, c_{k+2}, \ldots, c_{2k-1}$ and the voters are as shown in Theorem 3.6.

Proof. Consider an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and rand-rand mechanism $\Psi = (\mathsf{f}_{rd}, \mathsf{f}_{ur})$. For any voter $v \in \mathcal{V}$, we have

$$\begin{aligned}
\mathsf{cost}(\mathsf{top}(v)) &= \mathsf{d}(v^{**}(\mathsf{top}(v)), \mathsf{top}(v)) && \text{(Definition of max-max)} \\
&\leq \mathsf{d}(v^{**}(\mathsf{top}(v)), \mathsf{o}) + \mathsf{d}(\mathsf{o}, \mathsf{top}(v)) && \text{(Triangle Inequality)} \\
&\leq \mathsf{d}(v^{**}(\mathsf{top}(v)), \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{top}(v)) && \text{(Triangle Inequality)} \\
&\leq \mathsf{d}(v^{**}(\mathsf{o}), \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{top}(v)) && \text{(Observation A.8)} \\
&\leq \mathsf{d}(v^{**}(\mathsf{o}), \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) && \text{(Observation A.7)} \\
&= \mathsf{d}(v^{**}(\mathsf{o}), \mathsf{o}) + 2\mathsf{d}(v, \mathsf{o}) \\
&\leq 3\mathsf{d}(v^{**}(\mathsf{o}), \mathsf{o}) && \text{(Observation A.8)} \\
&= 3\mathsf{cost}(\mathsf{o}) && (\mathsf{cost}(\mathsf{o}) = \mathsf{d}(v^{**}(\mathsf{o}), \mathsf{o})).
\end{aligned}$$

Combining this with Observation A.5, we have

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \frac{1}{k} \sum_{g \in \mathcal{G}} \frac{1}{n_g} \sum_{v \in g} \mathsf{cost}(\mathsf{top}(v)) \\
&\leq 3\mathsf{cost}(\mathsf{o}).
\end{aligned}$$

□

Theorem 4.2. *For the avg-max objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_{rd}, \mathsf{f}_{ur})) \leq 3$.

Proof. Consider an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and rand-rand mechanism $\Psi = (\mathsf{f}_{rd}, \mathsf{f}_{ur})$. By definition of avg-max for any voter $v \in \mathcal{V}$, we have

$$\mathsf{cost}(\mathsf{top}(v)) = \frac{1}{k} \sum_{g \in \mathcal{G}} \mathsf{cost}_g(\mathsf{top}(v)).$$

Now, for any voter $v \in g'$, and any group $g \in \mathcal{G}$ we have

$$\begin{aligned}
\mathsf{cost}_g(\mathsf{top}(v)) &= \mathsf{d}(v^*(\mathsf{top}(v), g), \mathsf{top}(v)) && \text{(Definition of } \mathsf{cost}_g(.)\text{)} \\
&\leq \mathsf{d}(v^*(\mathsf{top}(v), g), v) + \mathsf{d}(v, \mathsf{top}(v)) && \text{(Triangle Inequality)} \\
&\leq \mathsf{d}(v^*(\mathsf{top}(v), g), v) + \mathsf{d}(v, \mathsf{o}) && \text{(Observation A.7)} \\
&\leq \mathsf{d}(v^*(\mathsf{top}(v), g), \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) && \text{(Triangle Inequality)} \\
&\leq \mathsf{d}(v^*(\mathsf{o}, g), \mathsf{o}) + 2\mathsf{d}(v, \mathsf{o}) && \text{(Observation A.9)} \\
&= \mathsf{cost}_g(\mathsf{o}) + 2\mathsf{d}(v, \mathsf{o}) && \text{(Definition of } \mathsf{cost}_g(.)\text{)} \\
&\leq \mathsf{cost}_g(\mathsf{o}) + 2\mathsf{d}(v^*(\mathsf{o}, g'), \mathsf{o}) && \text{(Observation A.9)} \\
&= \mathsf{cost}_g(\mathsf{o}) + 2\mathsf{cost}_{g'}(\mathsf{o}) && \text{(Definition of } \mathsf{cost}_g(.)\text{)}.
\end{aligned}$$

Voter	Preference Profile
v_{2i-1}	$\sigma\uparrow c_i\uparrow c_{2k}$
v_{2i}	$\sigma\uparrow c_{2k}\uparrow c_i$

Table 4: Preference profiles of voters v_{2i-1} and v_{2i} for any $1 \leq i \leq k$ in Theorem 3.6.

Combining this with Observation A.5, we obtain

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{cost}_{g'}(\mathsf{top}(v)) \\
&\leq \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\left(2\mathsf{cost}_g(\mathsf{o}) + \mathsf{cost}_{g'}(\mathsf{o})\right) \\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\left(2\mathsf{cost}_g(\mathsf{o}) + \frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{cost}_{g'}(\mathsf{o})\right) \\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}2\mathsf{cost}_g(\mathsf{o}) + \mathsf{cost}(\mathsf{o}) && \text{(Definition of } \mathsf{cost}(.)) \\
&= 3\mathsf{cost}(\mathsf{o}) && \text{(Definition of } \mathsf{cost}(.)).
\end{aligned}$$

□

Theorem 4.3. *For the* max-avg *objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_{rd}, \mathsf{f}_{ur})) \leq 3$.

Proof. Consider an instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and rand-rand mechanism $\Psi = (\mathsf{f}_{rd}, \mathsf{f}_{ur})$. By the definition of the max-avg objective, for any voter $v \in \mathcal{V}$, we have

$$\begin{aligned}
\mathsf{cost}(\mathsf{top}(v)) &= \max_{g\in\mathcal{G}}\mathsf{cost}_g(\mathsf{top}(v)) \\
&= \mathsf{cost}_{g^*(\mathsf{top}(v))}(\mathsf{top}(v)) \\
&= \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\mathsf{d}(v', \mathsf{top}(v)) && \text{(Definition of } \mathsf{cost}_g(.)).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mathsf{cost}_{g^*(\mathsf{top}(v))}(\mathsf{top}(v)) &= \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\mathsf{d}(v', \mathsf{top}(v)) \\
&\leq \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\Big(\mathsf{d}(v, \mathsf{top}(v)) + \mathsf{d}(v, v')\Big) && \text{(Triangle Inequality)} \\
&\leq \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, v')\Big) && \text{(Observation A.7)} \\
&\leq \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\Big(\mathsf{d}(v, \mathsf{o}) + \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, v')\Big) && \text{(Triangle Inequality)} \\
&= 2\mathsf{d}(v, \mathsf{o}) + \frac{1}{n_{g^*(\mathsf{top}(v))}}\sum_{v'\in g^*(\mathsf{top}(v))}\mathsf{d}(\mathsf{o}, v') \\
&= 2\mathsf{d}(v, \mathsf{o}) + \mathsf{cost}_{g^*(\mathsf{top}(v))}(\mathsf{o}) \\
&\leq 2\mathsf{d}(v, \mathsf{o}) + \mathsf{cost}(\mathsf{o}) && \text{(Observation A.6)}.
\end{aligned}$$

Combining this with Observation A.5, we obtain

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\mathsf{cost}(\mathsf{top}(v)) \\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\mathsf{cost}_{g^*(\mathsf{top}(v))}(\mathsf{top}(v)) && \text{(Definition of } \mathsf{cost}(.)) \\
&\le \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}(2\mathsf{d}(v,\mathsf{o})+\mathsf{cost}(\mathsf{o})) \\
&= \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}2\mathsf{d}(v,\mathsf{o}) \\
&= \mathsf{cost}(\mathsf{o}) + \frac{2}{k}\sum_{g\in\mathcal{G}}\mathsf{cost}_g(\mathsf{o}) && \text{(Definition of } \mathsf{cost}_g(.)) \\
&\le \mathsf{cost}(\mathsf{o}) + \frac{2}{k}\sum_{g\in\mathcal{G}}\mathsf{cost}(\mathsf{o}) && \text{(Observation A.6)} \\
&= 3\mathsf{cost}(\mathsf{o}).
\end{aligned}$$

□

Theorem 4.4. *For the* avg-avg *objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_{rd},\mathsf{f}_{ur})) \le 3 - 2/kn^*$ *where* n^* *represents the maximum value of* n_g *over all groups.*

Proof. Consider an instance $\mathcal{I} = (\mathcal{V},\mathcal{C},\mathcal{G},\pi,\mathsf{d})$ and rand-rand mechanism $\Psi = (\mathsf{f}_{rd},\mathsf{f}_{ur})$. By the definition of the avg-avg objective, for any voter $v \in \mathcal{V}$, we have

$$\mathsf{cost}(\mathsf{top}(v)) = \frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{cost}_{g'}(\mathsf{top}(v)).$$

For any voter $v \in \mathcal{V}$ and group g', we have

$$\begin{aligned}
\mathsf{cost}_{g'}(\mathsf{top}(v)) &= \frac{1}{n_{g'}}\sum_{v'\in g'}\mathsf{d}(v',\mathsf{top}(v)) && \text{(Definition of } \mathsf{cost}_g(.)) \\
&\le \frac{1}{n_{g'}}\sum_{v'\in g'}\Big(\mathsf{d}(v,\mathsf{top}(v))+\mathsf{d}(v',v)\Big) && \text{(Triangle Inequality)} \\
&\le \frac{1}{n_{g'}}\sum_{v'\in g'}\Big(\mathsf{d}(v,\mathsf{o})+\mathsf{d}(v',v)\Big) && \text{(Observation A.7)} \\
&= \mathsf{d}(v,\mathsf{o}) + \frac{1}{n_{g'}}\sum_{v'\in g'}\mathsf{d}(v',v).
\end{aligned}$$

Combining this with Observation A.5, we obtain

$$\begin{aligned}
\mathbb{E}[\mathsf{cost}(\mathsf{w})] &= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\mathsf{cost}_{g'}(\mathsf{top}(v)) && \text{(Definition of } \mathsf{cost}(.)) \\
&\le \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\Big(\mathsf{d}(v,\mathsf{o})+\frac{1}{n_{g'}}\sum_{v'\in g'}\mathsf{d}(v',v)\Big) \\
&= \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\mathsf{d}(v,\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\frac{1}{n_{g'}}\sum_{v'\in g'}\mathsf{d}(v',v) \\
&= \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\frac{1}{n_{g'}}\sum_{v'\in g'}\mathsf{d}(v',v) && \text{(Definition of avg-avg)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\frac{1}{n_{g'}}\sum_{v'\in g',\, v'\neq v}\Big(\mathsf{d}(v,\mathsf{o})+\mathsf{d}(\mathsf{o},v')\Big) && \text{(Triangle Inequality)}\\
&= \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\frac{1}{n_{g'}}\sum_{v'\in g'}\Big(\mathsf{d}(v,\mathsf{o})+\mathsf{d}(\mathsf{o},v')\Big)\\
&-\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G},g'=g}\frac{1}{n_{g'}}\sum_{v'\in g',v'=v}\Big(\mathsf{d}(v,\mathsf{o})+\mathsf{d}(\mathsf{o},v')\Big)\\
&= \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\sum_{g'\in\mathcal{G}}\frac{1}{n_{g'}}\sum_{v'\in g'}\Big(\mathsf{d}(v,\mathsf{o})+\mathsf{d}(\mathsf{o},v')\Big)\\
&-\frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\cdot\frac{1}{n_g}\cdot 2\mathsf{d}(v,\mathsf{o})\\
&= \mathsf{cost}(\mathsf{o}) + 2\mathsf{cost}(\mathsf{o}) - \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\cdot\frac{1}{n_g}\cdot 2\mathsf{d}(v,\mathsf{o}) && \text{(Definition of avg-avg)}\\
&= 3\mathsf{cost}(\mathsf{o}) - \frac{1}{k}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\sum_{v\in g}\frac{1}{k}\cdot\frac{1}{n_g}\cdot 2\mathsf{d}(v,\mathsf{o})\\
&= 3\mathsf{cost}(\mathsf{o}) - \frac{2}{k^2}\sum_{g\in\mathcal{G}}\frac{1}{n_g}\cdot\mathsf{cost}_g(\mathsf{o}) && \text{(Definition of } \mathsf{cost}_g(.))\\
&\leq 3\mathsf{cost}(\mathsf{o}) - \frac{2}{k^2}\sum_{g\in\mathcal{G}}\frac{1}{n^*}\cdot\mathsf{cost}_g(\mathsf{o})\\
&= (3-\frac{2}{kn^*})\mathsf{cost}(\mathsf{o}) && \text{(Definition of avg-avg)}.
\end{aligned}
$$

□

Theorem 4.5. *In the metric setting, any* rand-rand *mechanism has distortion of at least* 3 *for the* max-avg *and* max-max *objectives, even when the metric space is a line.*

Proof. We provide a proof for the max-avg objective. A similar argument can be used to prove the result for the max-max objective as well.

Consider any rand-rand mechanism Ψ. We construct an instance with candidates $\mathcal{C} = \{c_1, c_2, c_3\}$ and voters $\mathcal{V} = \{v_1, v_2\}$, where each voter belongs to a distinct group: $v_1 \in g_1$ and $v_2 \in g_2$. The instance is constructed as follows, similarly to the configuration in Figure 6:

- Place candidates c_1, c_2, and c_3 at coordinates -1, 0, and 1, respectively.
- Place voter v_1 at coordinate $-\frac{1}{2}$ and voter v_2 at coordinate $\frac{1}{2}$.
- Let the preference profiles be $\pi_1 = (c_1, c_2, c_3)$ and $\pi_2 = (c_3, c_2, c_1)$.

One can verify that the preference profiles are consistent with the distances. Candidates c_1 and c_3 are selected as representatives of g_1 and g_2, and mechanism Ψ must select one of them as the final winner.

By the definitions of the max-avg objective and given that each group consists of a single voter we have:

$$\mathsf{cost}(c_1) = \mathsf{cost}(c_3) = \frac{3}{2}, \quad \text{and} \quad \mathsf{cost}(c_2) = \frac{1}{2}.$$

Therefore, the distortion of Ψ is at least

$$
\begin{aligned}
\mathsf{D}(\Psi) &\geq \frac{\min\big(\mathsf{cost}(c_1), \mathsf{cost}(c_3)\big)}{\mathsf{cost}(c_2)}\\
&= 3.
\end{aligned}
$$

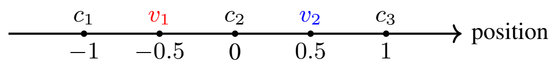

Figure 6: An instance used to prove Theorem 4.5.

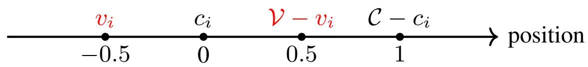

Figure 7: Configuration of candidates and voters in instance $\mathcal{I}_i$ used in the proofs of Theorems 4.6 and 4.7.

□

Theorem 4.6. *In the metric setting, any* rand-rand *mechanism has distortion of at least* $3 - \frac{2}{n}$ *for the* avg-max *objective, even when the metric space is a line.*

Proof. Suppose there are m candidates, $\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$, and $n = m$ voters, $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Each voter v_i has preference

$$\pi_i = (c_i, c_{i+1}, \ldots, c_m, c_1, c_2, \ldots, c_{i-1}).$$

We construct m instances, $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m$ on the line metric, each containing a single group. Across all instances, the sets $\mathcal{V}$, $\mathcal{C}$, and the preference profiles are the same; only the underlying metric space differs.

Now consider any rand-rand mechanism Ψ. Let p_i be the probability that Ψ selects c_i as the winner.

For each $1 \le i \le m$, construction of instance $\mathcal{I}_i$ is as follows (similar to Figure 7):

- Candidate c_i is located at 0, while all other candidates are located at 1.
- Voter v_i is located at -0.5, and all other voters are located at 0.5.

It is straightforward to verify that the constructed instances are consistent with the specified preference profiles.

For any instance $\mathcal{I}_i$ we have:

$$\left(\sum_{1 \le j \le m, j \ne i, j \in \mathbb{N}} 3p_j \right) + p_i = 3 - 2p_i.$$

We know that the total probability must sum up to 1:

$$\sum_{1 \le i \le m, i \in \mathbb{N}} p_i = 1.$$

Hence, there exists some index i such that $p_i \le \frac{1}{m}$. Substituting this into the distortion formula, we get:

$$\mathsf{D}(\Psi) \ge 3 - \frac{2}{m} = 3 - \frac{2}{n}.$$

Therefore, the distortion of any rand-rand mechanism is at least $3 - \frac{2}{n}$ (equivalently, $3 - \frac{2}{m}$).

□

Theorem 4.7. *For the* max *objective in the centralized setting, the distortion of any randomized voting rule is at least* $3 - \epsilon$*, for any constant* $\epsilon > 0$*, even when the metric is a line.*

Proof. The result directly follows from the proof of Theorem 4.6, as the same instance construction and analysis apply in the centralized setting under the max objective. As the number of voters n increases, the distortion approaches 3. Therefore, for any constant $\epsilon > 0$, we can construct an instance where the distortion of mechanism Ψ exceeds $3 - \epsilon$, provided that $n > \frac{2}{\epsilon}$. □

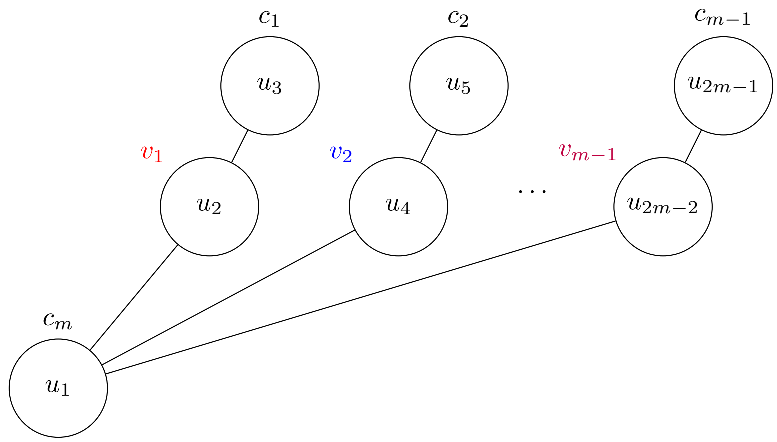

Figure 8: Tree graph used in the proof of Theorem 4.8. Different groups are distinguished with distinct colors.

$\mathsf{d}(\cdot,\cdot)$	c_i	c_j	c_m
v_i	1	3	1

Table 5: Pairwise distances between candidates and voters in Theorem 4.8 for all $1 \leq i, j \leq k$ with $i \neq j$.

Theorem 4.8. *Any* rand-rand *mechanism has metric distortion of at least* $3 - \frac{2}{n}$ *for the* avg-avg *objective.*

Proof. Consider any rand-rand mechanism Ψ, we construct an instance with k groups $g_i = \{v_i\}$ for $1 \leq i \leq k$, candidates $\mathcal{C} = \{c_1, c_2, \dots, c_{m=k+1}\}$, and voters $\mathcal{V} = \{v_1, v_2, \dots, v_{n=k}\}$.

We now construct a connected graph G with $n + m$ vertices $u_1, u_2, \dots, u_{n+m}$ and then use the shortest-path distances in G as the underlying metric space d. Each voter and candidate is placed on one of the vertices (a vertex may host multiple entities). The construction of G is as follows:

- For each $1 \leq i \leq k$, add an edge between u_1 and u_{2i}, another edge between u_{2i} and u_{2i+1}. This creates k branches extending from the central vertex u_1.
- For each $1 \leq i \leq k$ place voter v_i at vertex u_{2i}, and candidate c_i at vertex u_{2i+1}.
- Place candidate c_m at vertex u_1.

See Figure 8 for a visual representation and Table 5 for the corresponding distances.

It is easy to see that the following holds for any $1 \leq i \leq k$ and $1 \leq j \leq k$, s.t. $i \neq j$: For any $1 \leq i \leq k$, we have $\mathsf{top}(v_i) = c_i$.

Therefore representative of group g_i is candidate c_i. Consequently c_m is not the representative of any group, and thus can not be the winner decided by mechanism Ψ. We calculate the distortion using these information:

$$\mathsf{cost}(c_i) = \frac{(n-1) \cdot 3 + 1 \cdot 1}{n} = 3 - \frac{2}{n} \qquad (1 \leq i < m),$$
$$\mathsf{cost}(c_m) = 1.$$

Hence we have

$$\begin{aligned}\mathsf{D}(\Psi) &\geq \min_{1\leq i<m}\left(\frac{\mathsf{cost}(c_i)}{\mathsf{cost}(\mathsf{o})}\right)\\ &\geq \frac{\mathsf{cost}(c_1)}{\mathsf{cost}(c_m)} = 3-\frac{2}{n}.\end{aligned}$$

□

D Proofs for Section 5 (Distortion Bounds of det-det)

Theorem 5.1. *Let f_β be a deterministic voting rule with distortion at most β, and f_{un} be an unanimous deterministic rule. For the* avg-max *objective in general metric spaces,* $\mathsf{D}((\mathsf{f}_{un}, \mathsf{f}_\beta)) \leq 2\beta + 3$.

Proof. Consider an arbitrary instance $\mathcal{I} = (\mathcal{V}, \mathcal{C}, \mathcal{G}, \pi, \mathsf{d})$ and mechanism $\Psi = (\mathsf{f}_{un}, \mathsf{f}_\beta)$. For each group g, let v_g be a voter who prefers the representative w_g over the optimal candidate o. By the unanimity of f_{un}, such a voter always exists. We now derive an upper bound on the distortion of Ψ.

$$\begin{aligned}
\mathsf{cost}(\mathsf{w}) &= \frac{1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(v^*(\mathsf{w},g),\mathsf{w}) && \text{(Definition of avg-max)}\\
&\leq \frac{1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(v^*(\mathsf{w},g),\mathsf{o}) + \mathsf{d}(\mathsf{o},\mathsf{w}) && \text{(Triangle Inequality)}\\
&\leq \frac{1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(v^*(\mathsf{o},g),\mathsf{o}) + \mathsf{d}(\mathsf{o},\mathsf{w}) && \text{(Observation A.9)}\\
&= \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},\mathsf{w}) && \text{(Definition of avg-max)}\\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},\mathsf{w}_g) + \mathsf{d}(\mathsf{w},\mathsf{w}_g) && \text{(Triangle Inequality)}\\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{\beta+1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},\mathsf{w}_g) && \text{Observation A.11}\\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{\beta+1}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},v_g) + \mathsf{d}(v_g,\mathsf{w}_g) && \text{(Triangle Inequality)}\\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{2\beta+2}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},v_g) && (\mathsf{d}(v_g,\mathsf{w}_g)\leq \mathsf{d}(v_g,\mathsf{o}))\\
&\leq \mathsf{cost}(\mathsf{o}) + \frac{2\beta+2}{k}\sum_{g\in\mathcal{G}} \mathsf{d}(\mathsf{o},v^*(\mathsf{o},g)) && \text{(Observation A.9)}\\
&= (2\beta+3)\cdot\mathsf{cost}(\mathsf{o}) && \text{(Definition of avg-max).}
\end{aligned}$$

This proves the desired upper bound of $2\beta + 3$ for $\mathsf{D}(\Psi)$. □

Theorem 5.2. *For the* max-max *objective in general metric spaces,* $\mathsf{D}(\mathsf{m}_{ad}) \leq 3$.

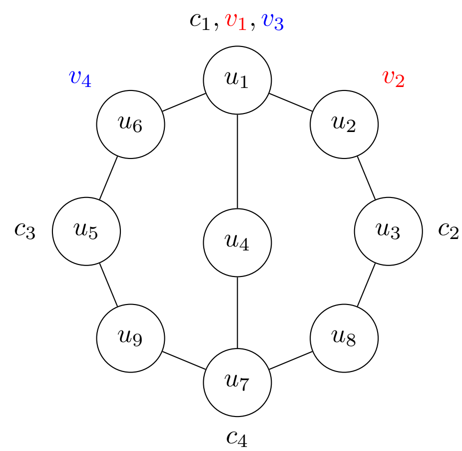

Figure 9: The figure illustrates the graph G in Theorem 5.3. Different groups are distinguished by different colors.

Proof. Suppose that the *Arbitrary Dictator* mechanism selects the top candidate of voter v as the final winner. We follow a similar proof strategy as in Theorem 4.1.

$$
\begin{aligned}
\mathsf{cost}(\mathsf{w}) &= \mathsf{d}(v^{**}(\mathsf{w}), \mathsf{w}) && \text{(Definition of max-max)}\\
&= \mathsf{d}(v^{**}(\mathsf{top}(v)), \mathsf{top}(v)) && (\mathsf{w} = \mathsf{top}(v))\\
&\leq \mathsf{d}(v, v^{**}(\mathsf{top}(v))) + \mathsf{d}(v, \mathsf{top}(v)) && \text{(Triangle Inequality)}\\
&\leq \mathsf{d}(v, v^{**}(\mathsf{top}(v))) + \mathsf{d}(v, \mathsf{o}) && \text{(Observation A.7)}\\
&\leq \mathsf{d}(v, \mathsf{o}) + \mathsf{d}(\mathsf{o}, v^{**}(\mathsf{top}(v))) + \mathsf{d}(v, \mathsf{o}) && \text{(Triangle Inequality)}\\
&\leq 3\mathsf{d}(\mathsf{o}, v^{**}(\mathsf{o})) && \text{(Observation A.8)}\\
&= 3 \cdot \mathsf{cost}(\mathsf{o}) && \text{(Definition of max-max).}
\end{aligned}
$$

Therefore, the distortion of the *Arbitrary Dictator* mechanism is at most 3 for the max-max cost objective in the metric setting. □

Theorem 5.3. *Any* det-det *mechanism has metric distortion of at least* 5 *for the* max-avg *objective.*

Proof. Consider any det-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$, we construct an instance with 2 groups $g_1 = \{v_1, v_2\}$ and $g_2 = \{v_3, v_4\}$, set of candidates $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$, and set of voters $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$.

Let σ be an arbitrary ordering of the candidates. By Observation A.1, there exists a candidate with in-degree at least $\lceil \frac{m-1}{2} \rceil = 2$ in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$. Without loss of generality, suppose c_1 is such a candidate. Also, suppose c_2 and c_3 are the two candidates that have directed edges toward c_1 in the tournament. Let c_4 denote the remaining candidate in $\mathcal{C}$.

We now construct a connected graph G with 9 vertices $u_1, u_2, \ldots u_9$, and then use the shortest-path distances in G as the underlying metric space d. We position each voter and candidate at one of the vertices (note that a single vertex may host multiple entities). Refer to Figure 9 for representation of the graph.

We define the preference profiles in a way that is consistent with the distances shown in Table 7, as well as $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$, represented in Table 6. Note that there may be other preference profiles consistent with metric space d.

Since both c_2 and c_3 defeat c_1 in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$, they must be the representatives of groups g_1 and g_2, respectively. Thus, the set of representatives is $\{c_2, c_3\}$.

In *over-group* voting, let the preference profiles of the representatives c_2 and c_3 be (c_2, c_4, c_1, c_3) and (c_3, c_4, c_1, c_2), respectively. One can verify that these preferences are consistent with the metric space. The final winner cannot be c_1 due to unanimity—both representatives prefer c_4 over c_1. Therefore, the final winner must be one of c_2, c_3, or c_4.

Voter	Preference Profile
v_1	$\sigma \uparrow c_2 \uparrow c_1$
v_2	$\sigma \uparrow c_1 \uparrow c_2$
v_3	$\sigma \uparrow c_3 \uparrow c_1$
v_4	$\sigma \uparrow c_1 \uparrow c_3$

Table 6: Preference profiles of voters in Theorem 5.3

$\mathsf{d}(\cdot,\cdot)$	c_1	c_2	c_3	c_4
v_1	0	2	2	2
v_2	1	1	3	3
v_3	0	2	2	2
v_4	1	3	1	3

Table 7: Pairwise distances between the candidates and the voters derived from the graph G in Theorem 5.3.

We now calculate the distortion:

$$\begin{aligned}
\mathsf{D}(\Psi) &\geq \min\Big(\frac{\mathsf{cost}(c_2), \mathsf{cost}(c_3), \mathsf{cost}(c_4)}{\mathsf{cost}(\mathsf{o})}\Big) \\
&\geq \frac{\frac{5}{2}}{\mathsf{cost}(c_1)} && \Big(\mathsf{cost}(c_2) = \mathsf{cost}(c_3) = \mathsf{cost}(c_4) = \frac{5}{2}\Big) \\
&= 5 && \Big(\mathsf{cost}(c_1) = \frac{1}{2}\Big).
\end{aligned}$$

□

E Proofs for Section 6 (Distortion Lower Bounds in Euclidean Space)

Theorem 6.1. *Any* rand-rand *mechanism, with respect to the* avg-avg *cost function, has a distortion of at least $\sqrt{5} - \epsilon$ for any constant $\epsilon > 0$ in Euclidean space.*

Proof. Suppose l is an integer. Consider a scenario in $(l+1)-$dimensional space with $l+2$ candidates, $c_1, c_2, \ldots, c_{l+2}$. There are $k = l + 1$ groups, each consisting of a single voter. We construct the instance as follows:

- Let q_i denote the point in $\mathbb{R}^{l+1}$ whose i-th coordinate is 1 and all other coordinates are 0, for $1 \leq i \leq l + 1$.
- Place candidate c_i at point q_i for each $1 \leq i \leq l + 1$.
- Place candidate c_{l+2} at $\left(\frac{1}{l+1}, \frac{1}{l+1}, \ldots, \frac{1}{l+1}\right)$.
- There are $l + 1$ groups. In the i-th group, there is a single voter, denoted by v_i, located at the point $\left(\frac{1}{2(l+1)}, \frac{1}{2(l+1)}, \ldots, \frac{1}{2(l+1)}, \frac{l+2}{2(l+1)}, \frac{1}{2(l+1)}, \ldots, \frac{1}{2(l+1)}\right)$ where the i-th coordinate is $\frac{l+2}{2(l+1)}$ and all other coordinates are $\frac{1}{2(l+1)}$. This point lies at the midpoint between candidates c_{l+2} and c_i. The top-ranked candidate for voter v_i is c_i.

Let us now compute the distortion in this instance. Note that for all $1 \leq i, j \leq l+1$ such that $i \neq j$ we have:

$$\begin{aligned}
\mathsf{d}(c_i, v_j) &= \sqrt{\left(\frac{1}{2(l+1)}\right)^2 (l-1) + \left(\frac{l+2}{2(l+1)}\right)^2 + \left(\frac{2l+1}{2l+2}\right)^2} \\
&= \sqrt{\frac{5l+4}{4l+4}},
\end{aligned}$$

and for all $1 \leq i \leq l+1$ we have:

$$\begin{aligned}
\mathsf{d}(c_i, v_i) &= \mathsf{d}(c_{l+2}, v_i) \\
&= \sqrt{\left(\frac{1}{2(l+1)}\right)^2 l + \left(\frac{l}{2(l+1)}\right)^2} \\
&= \sqrt{\frac{l}{4(l+1)}}.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\mathsf{cost}(c_{l+2}) &= \frac{1}{2}\sqrt{\frac{l}{l+1}}, \\
\mathsf{cost}(c_i) &= \frac{l\sqrt{\frac{5l+4}{4l+4}} + \frac{1}{2}\sqrt{\frac{l}{l+1}}}{l+1} && (1 \leq i \leq l+1).
\end{aligned}$$

We know that in group i, the representative is candidate c_i. Therefore, the final winner is one of the candidates $c_1, c_2, \ldots, c_{l+1}$. To lower bound the distortion of any rand-rand mechanism Ψ under the avg-avg cost function, we can compare the cost of each c_i, for $1 \leq i \leq l+1$, with the cost of c_{l+2}. Specifically, the ratio $\frac{\mathsf{cost}(c_i)}{\mathsf{cost}(c_{l+2})}$ provides a lower bound on the distortion:

$$\begin{aligned}
\mathsf{D}(\Psi) &\geq \frac{\mathsf{cost}(c_i)}{\mathsf{cost}(c_{l+2})} \\
&= \frac{l}{l+1}\sqrt{\frac{5l+4}{l}} + \frac{1}{l+1} && (1 \leq i \leq l+1),
\end{aligned}$$

and thus we have:

$$\lim_{l \to \infty} \frac{l}{l+1}\sqrt{\frac{5l+4}{l}} + \frac{1}{l+1} = \sqrt{5}.$$

Therefore, for any real number $\epsilon > 0$, we can construct an instance whose distortion exceeds $\sqrt{5} - \epsilon$. When $l = 2$ the instance lies within an equilateral triangle with vertices at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. A diagram of this instance is shown in Figure 10.

$\square$

Theorem 6.2. *Any* rand-det *mechanism, with respect to the* avg-avg *cost function, has a distortion of at least $2 + \sqrt{5} - \epsilon$ for any constant $\epsilon > 0$ in Euclidean space.*

Proof. Consider any rand-det mechanism $\Psi = (\mathsf{f}_{in}, \mathsf{f}_{ov})$. let $\mathcal{C} = \{c_1, c_2, \ldots, c_{2m}\}$ be the set of candidates, and let σ be an arbitrary ordering over these candidates.

By Observation A.1, we know that in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$, there always exists a candidate with in-degree at least $\lceil \frac{2m-1}{2} \rceil = m$. Without loss of generality, assume that c_{m+1} is such a candidate.

Without loss of generality, assume that $c_1, c_2, \ldots, c_m$ are m candidates that have directed edges toward c_{m+1} in the tournament.

We now construct the following instance in $(m+1)$-dimensional Euclidean space:

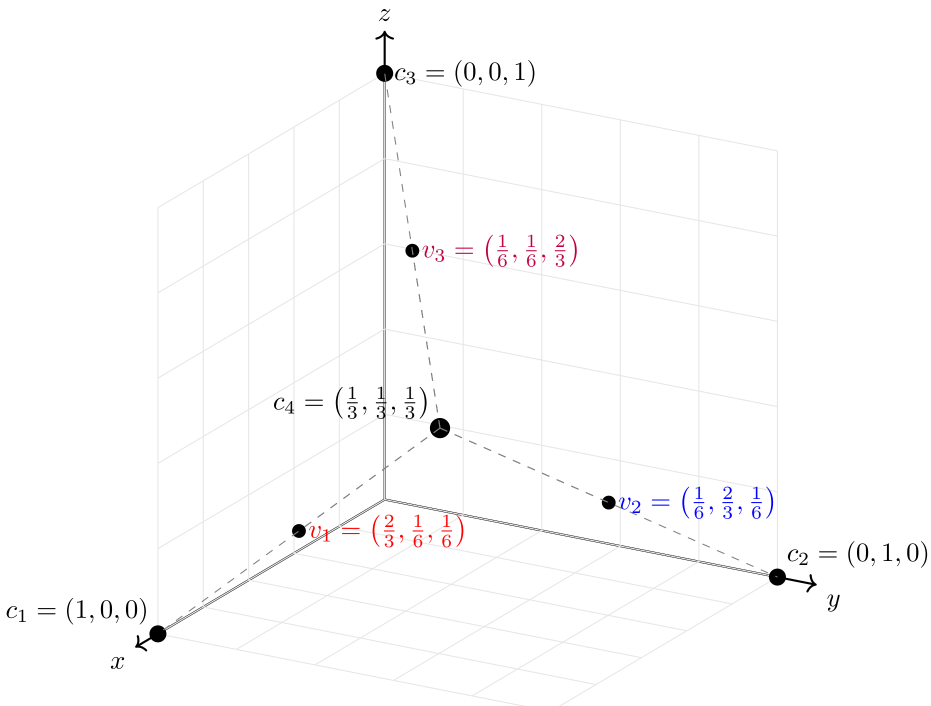

Figure 10: A 3-dimensional Euclidean instance illustrating the lower bound construction for rand-rand mechanisms under the avg-avg cost function when $l = 2$. Candidates c_1, c_2, and c_3 are placed at the unit basis vectors. The point $c_4 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ lies at the centroid of the triangle formed by these candidates. Voters v_1, v_2, and v_3 are each positioned at the midpoint between the centroid c_4 and their top-ranked candidate c_i. Different groups are indicated using distinct colors.

- Let q_i be the point in $\mathbb{R}^{m+1}$ whose i-th coordinate is 1 and all other coordinates are 0, for $1 \leq i \leq m+1$.
- Place candidate c_i at point q_i for each $1 \leq i \leq m$.
- Place candidate c_i at point q_{m+1} for each $m+2 \leq i \leq 2m$.
- Place candidate c_{m+1} at the point $\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right)$.
- We have $k = m$ groups. In the i-th group, there are two voters: v_{2i-1} and v_{2i}.
  - Voter v_{2i-1} is located at the point $\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right)$, which coincides with the position of candidate c_{m+1}.
  - Voter v_{2i} is located at the point
  $$\left(\frac{1}{2(m+1)}, \frac{1}{2(m+1)}, \ldots, \frac{m+2}{2(m+1)}, \ldots, \frac{1}{2(m+1)}\right),$$
  where the i-th coordinate is $\frac{m+2}{2(m+1)}$ and all other coordinates are $\frac{1}{2(m+1)}$. This point lies at the midpoint between candidates c_{m+1} and $c_i (1 \leq i \leq m)$.
- The ordinal preferences of the v_{2i-1} and v_{2i} are as follows:
  - $\pi_{2i-1} = \sigma \uparrow c_i \uparrow c_{m+1}$
  - $\pi_{2i} = \sigma \uparrow c_{m+1} \uparrow c_i$

  These preferences are consistent with the distances:
  - The distance from v_{2i-1} to all candidates except c_{m+2} is the same.

- The distance from v_{2i-1} to c_{m+2} is zero.
- v_{2i} is closer to c_{m+1} and c_i than to the other candidates, and equidistant from all remaining ones.
- v_{2i} has equal distance to c_{m+1} and c_i.

Now, let us calculate the distortion for this instance. Note that for all $1 \leq i, j \leq m$ we have:

$$\begin{aligned}\mathsf{d}(c_i, c_{m+1}) &= \mathsf{d}(c_i, v_{2j-1}) \\ &= \sqrt{\left(\frac{1}{m+1}\right)^2 m + \left(\frac{m}{m+1}\right)^2} \\ &= \sqrt{\frac{m}{m+1}},\end{aligned}$$

and for all $1 \leq i, j \leq m$ such that $i \neq j$ we have:

$$\begin{aligned}\mathsf{d}(c_i, v_{2j}) &= \sqrt{\left(\frac{1}{2(m+1)}\right)^2 (m-1) + \left(\frac{m+2}{2(m+1)}\right)^2 + \left(\frac{2m+1}{2m+2}\right)^2} \\ &= \sqrt{\frac{5m+4}{4m+4}},\end{aligned}$$

and for all $1 \leq i \leq m$ we have:

$$\begin{aligned}\mathsf{d}(c_i, v_{2i}) &= \sqrt{\left(\frac{1}{2(m+1)}\right)^2 m + \left(\frac{m}{2(m+1)}\right)^2} \\ &= \sqrt{\frac{m}{4(m+1)}}.\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\mathsf{cost}(c_{m+1}) &= \frac{m\left(\frac{0+\sqrt{\frac{m}{4(m+1)}}}{2}\right)}{m} \\ &= \frac{1}{4}\sqrt{\frac{m}{m+1}}, \\ \mathsf{cost}(c_i) &= \frac{\frac{\sqrt{\frac{m}{m+1}}+\sqrt{\frac{5m+4}{4m+4}}}{2}(m-1) + \frac{\sqrt{\frac{m}{m+1}}+\sqrt{\frac{m}{4(m+1)}}}{2}}{m} \qquad (\forall\, 1 \leq i \leq m).\end{aligned}$$

By the definition of $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$, the representative of group i is candidate $c_i (1 \leq i \leq m)$. Therefore, the final winner is one of the candidates $c_1, c_2, \ldots, c_m$. To obtain a lower bound on the distortion of this mechanism, we consider the ratio $\frac{\mathsf{cost}(c_i)}{\mathsf{cost}(c_{m+1})}$ for any $1 \leq i \leq m$:

$$\begin{aligned}\mathsf{D}(\Psi) &\geq \frac{\mathsf{cost}(c_i)}{\mathsf{cost}(c_{m+1})} \\ &= 2 + \frac{m-1}{m}\sqrt{\frac{5m+4}{m}} + \frac{1}{m} \qquad (1 \leq i \leq m),\end{aligned}$$

and we have:

$$\lim_{m \to \infty} 2 + \frac{m-1}{m}\sqrt{\frac{5m+4}{m}} + \frac{1}{m} = 2 + \sqrt{5}.$$

Therefore, for any real number $\epsilon > 0$, we can construct an instance with a distortion greater than $2 + \sqrt{5} - \epsilon$.

When $m = 2$ the instance lies within an equilateral triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. A diagram of this instance is shown in Figure 11.

□

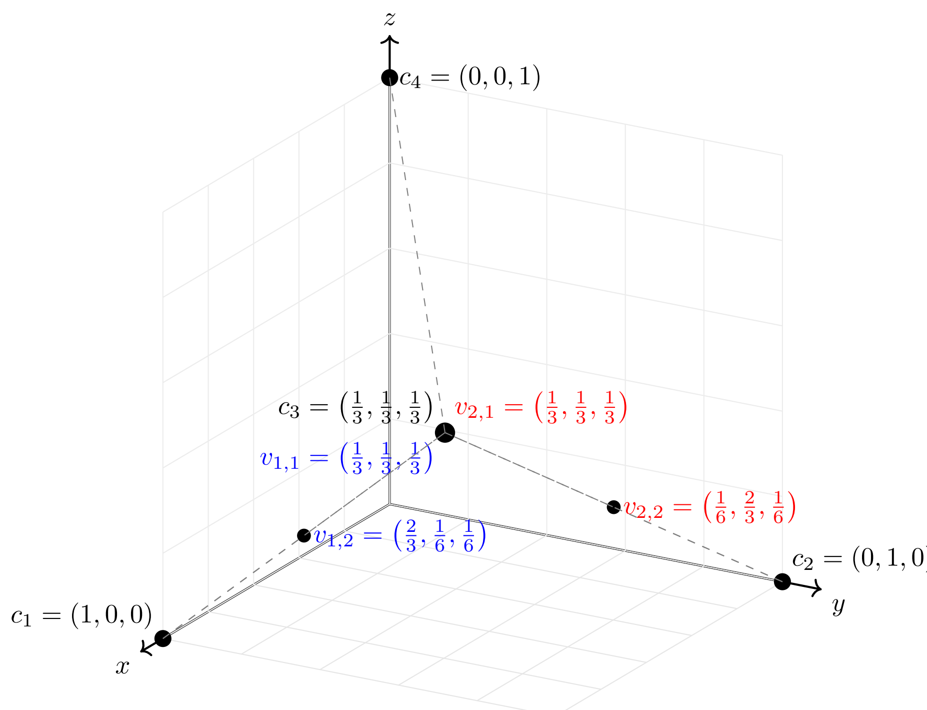

Figure 11: An illustration of the constructed instance when $m = 2$. The candidates are positioned at the corners and centroid of the 3D simplex (i.e., the equilateral triangle embedded in $\mathbb{R}^3$). Candidate c_3 is placed at the centroid, representing the candidate with high in-degree in $\mathcal{T}(\mathsf{f}_{in}, \mathcal{C}, \sigma)$. Each group contains two voters: v_{2i-1} is located at the centroid, while v_{2i} is placed at the midpoint between c_3 and c_i for $i = 1, 2$. Different groups are distinguished using different colors.