A PROOFS

In this section, we provide the proof of Theorem 1 from section 3.4.

Proof. Let us consider 10 apply simple rearrangements of summations as follows;

$$
\begin{aligned}
&\sum_{i_1=1}^{p_1} \cdots \sum_{i_N=1}^{p_N} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{N-1}=1}^{r_{N-1}} G_1(:, i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \cdots G_N(\alpha_{N-1}, i_N, 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad g(x_{i_1}^{(1)}, \ldots, x_{i_N}^{(N)}) \\
&= \sum_{i_1=1}^{p_1} \cdots \sum_{i_N=1}^{p_N} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{N-1}=1}^{r_{N-1}} G_1(:, i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \cdots G_N(\alpha_{N-1}, i_N, 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum_{k=1}^{K} g_1^{(k)}(x_{i_1}^{(1)}) g_2^{(k)}(x_{i_2}^{(2)}) \cdots g_N^{(k)}(x_{i_N}^{(N)}) \\
&= \sum_{k=1}^{K} \sum_{i_1=1}^{p_1} \cdots \sum_{i_N=1}^{p_N} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{N-1}=1}^{r_{N-1}} G_1(:, i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \cdots G_N(\alpha_{N-1}, i_N, 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad g_1^{(k)}(x_{i_1}^{(1)}) g_2^{(k)}(x_{i_2}^{(2)}) \cdots g_N^{(k)}(x_{i_N}^{(N)}) \\
&= \sum_{k=1}^{K} \sum_{i_1=1}^{p_1} \cdots \sum_{i_N=1}^{p_N} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{N-1}=1}^{r_{N-1}} g_1^{(k)}(x_{i_1}^{(1)}) G_1(:, i_1, \alpha_1) g_2^{(k)}(x_{i_2}^{(2)}) G_2(\alpha_1, i_2, \alpha_2) \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad g_N^{(k)}(x_{i_N}^{(N)}) G_N(\alpha_{N-1}, i_N, 1) \\
&= \sum_{k=1}^{K} \sum_{i_1=1}^{p_1} \sum_{\alpha_1=1}^{r_1} g_1^{(k)}(x_{i_1}^{(1)}) G_1(:, i_1, \alpha_1) \sum_{i_2=1}^{p_2} \sum_{\alpha_2=1}^{r_2} g_2^{(k)}(x_{i_2}^{(2)}) G_2(\alpha_1, i_2, \alpha_2) \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sum_{i_N=1}^{p_N} g_N^{(k)}(x_{i_N}^{(N)}) G_N(\alpha_{N-1}, i_N, 1).
\end{aligned}
$$

□