A Macaulay2 code

This appendix contains the Macaulay2 code used for computations in the paper, as well as its output. The Macaulay2 code is also separately provided as `.m2`-files, with further calculations. We change notation from λ, μ, ν to l, m, n in the computations, for compatibility of character encodings between Macaulay2 and LaTeX.

A.1 Proposition 4.5

```
i1 : --the case when c is given as a formula in terms of R: +
     clearAll
i2 : A = QQ[r_(1,1)..r_(3,3),c_3]
o2 = A
o2 : PolynomialRing
i3 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))
```

$$\text{o3} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

```
o3 : Matrix A^3 <-- A^3
i4 : c_1 = r_(3,1)*c_3/(r_(3,3)+1)
```

$$\text{o4} = \frac{r_{3,1}c_3}{r_{3,3}+1}$$

```
o4 : frac A
i5 : c_2 = r_(3,2)*c_3/(r_(3,3)+1)
```

$$\text{o5} = \frac{r_{3,2}c_3}{r_{3,3}+1}$$

```
o5 : frac A
i6 : C = matrix{{c_1},{c_2},{c_3}}
```

$$\text{o6} = \begin{pmatrix} \frac{r_{3,1}c_3}{r_{3,3}+1} \\ \frac{r_{3,2}c_3}{r_{3,3}+1} \\ c_3 \end{pmatrix}$$

```
o6 : Matrix (frac A)^3 <-- (frac A)^1
i7 : t = -R*C
```

$$\text{o7} = \begin{pmatrix} \frac{-r_{1,1}r_{3,1}c_3-r_{1,2}r_{3,2}c_3-r_{1,3}r_{3,3}c_3-r_{1,3}c_3}{r_{3,3}+1} \\ \frac{-r_{2,1}r_{3,1}c_3-r_{2,2}r_{3,2}c_3-r_{2,3}r_{3,3}c_3-r_{2,3}c_3}{r_{3,3}+1} \\ \frac{-r_{3,1}^2c_3-r_{3,2}^2c_3-r_{3,3}^2c_3-r_{3,3}c_3}{r_{3,3}+1} \end{pmatrix}$$

```
o7 : Matrix (frac A)^3 <-- (frac A)^1
i8 : T = matrix{{0,-t_(2,0),t_(1,0)},{t_(2,0),0,-t_(0,0)},{-t_(1,0),t_(0,0),0}}
```

$$\text{o8} = \begin{pmatrix} 0 & \frac{r_{3,1}^2c_3+r_{3,2}^2c_3+r_{3,3}^2c_3+r_{3,3}c_3}{r_{3,3}+1} & \frac{-r_{2,1}r_{3,1}c_3-r_{2,2}r_{3,2}c_3-r_{2,3}r_{3,3}c_3-r_{2,3}c_3}{r_{3,3}+1} \\ \frac{-r_{3,1}^2c_3-r_{3,2}^2c_3-r_{3,3}^2c_3-r_{3,3}c_3}{r_{3,3}+1} & 0 & \frac{r_{1,1}r_{3,1}c_3+r_{1,2}r_{3,2}c_3+r_{1,3}r_{3,3}c_3+r_{1,3}c_3}{r_{3,3}+1} \\ \frac{r_{2,1}r_{3,1}c_3+r_{2,2}r_{3,2}c_3+r_{2,3}r_{3,3}c_3+r_{2,3}c_3}{r_{3,3}+1} & \frac{-r_{1,1}r_{3,1}c_3-r_{1,2}r_{3,2}c_3-r_{1,3}r_{3,3}c_3-r_{1,3}c_3}{r_{3,3}+1} & 0 \end{pmatrix}$$

```
o8 : Matrix (frac A)^3 <-- (frac A)^3
i9 : F = transpose(R)*T
o9 = [very long output]
o9 : Matrix (frac A)^3 <-- (frac A)^3
i10 :
      --scale F so that its entries become polynomials
      Fscaled = sub((r_(3,3)+1)*F, A)
o10 = [very long output]
o10 : Matrix A^3 <-- A^3
i11 :
      --now use that R is a rotation matrix
      Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

$$\text{o11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
o11 : Matrix ZZ^3 <-- ZZ^3
i12 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)
```

$$\text{o12} = \text{ideal}(r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,\ -r_{1,3}r_{2,2}r_{3,1}+r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1)$$

```
o12 : Ideal of A
i13 : sub(Fscaled, A/SO3)
```

$$\text{o13} = \begin{matrix} \{-1\} \\ \{-1\} \\ \{-1\} \end{matrix} \begin{pmatrix} r_{1,2}c_3-r_{2,1}c_3 & r_{1,1}c_3+r_{2,2}c_3 & r_{3,2}c_3 \\ -r_{1,1}c_3-r_{2,2}c_3 & r_{1,2}c_3-r_{2,1}c_3 & -r_{3,1}c_3 \\ -r_{2,3}c_3 & r_{1,3}c_3 & 0 \end{pmatrix}$$

```
o13 : [very long output]
i14 :


      --the case when c is given as a formula in terms of R: -
      clearAll
i15 : A = QQ[r_(1,1)..r_(3,3),c_3]
o15 = A
o15 : PolynomialRing
i16 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))
```

$$\text{o16} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

o16 : Matrix $A^3 \longleftarrow A^3$

i17 : c_1 = r_(3,1)*c_3/(r_(3,3)-1)

o17 = $\frac{r_{3,1}c_3}{r_{3,3}-1}$

o17 : frac A

i18 : c_2 = r_(3,2)*c_3/(r_(3,3)-1)

o18 = $\frac{r_{3,2}c_3}{r_{3,3}-1}$

o18 : frac A

i19 : C = matrix{{c_1},{c_2},{c_3}}

o19 = $\begin{pmatrix} \frac{r_{3,1}c_3}{r_{3,3}-1} \\ \frac{r_{3,2}c_3}{r_{3,3}-1} \\ c_3 \end{pmatrix}$

o19 : Matrix (frac A)^3 $\longleftarrow$ (frac A)^1

i20 : t = -R*C

o20 = $\begin{pmatrix} \frac{-r_{1,1}r_{3,1}c_3-r_{1,2}r_{3,2}c_3-r_{1,3}r_{3,3}c_3+r_{1,3}c_3}{r_{3,3}-1} \\ \frac{-r_{2,1}r_{3,1}c_3-r_{2,2}r_{3,2}c_3-r_{2,3}r_{3,3}c_3+r_{2,3}c_3}{r_{3,3}-1} \\ \frac{-r_{3,1}^2c_3-r_{3,2}^2c_3-r_{3,3}^2c_3+r_{3,3}c_3}{r_{3,3}-1} \end{pmatrix}$

o20 : Matrix (frac A)^3 $\longleftarrow$ (frac A)^1

i21 : T = matrix{{0,-t_(2,0),t_(1,0)},{t_(2,0),0,-t_(0,0)},{-t_(1,0),t_(0,0),0}}

o21 = $\begin{pmatrix} 0 & \frac{r_{3,1}^2c_3+r_{3,2}^2c_3+r_{3,3}^2c_3-r_{3,3}c_3}{r_{3,3}-1} & \frac{-r_{2,1}r_{3,1}c_3-r_{2,2}r_{3,2}c_3-r_{2,3}r_{3,3}c_3+r_{2,3}c_3}{r_{3,3}-1} \\ \frac{-r_{3,1}^2c_3-r_{3,2}^2c_3-r_{3,3}^2c_3+r_{3,3}c_3}{r_{3,3}-1} & 0 & \frac{r_{1,1}r_{3,1}c_3+r_{1,2}r_{3,2}c_3+r_{1,3}r_{3,3}c_3-r_{1,3}c_3}{r_{3,3}-1} \\ \frac{r_{2,1}r_{3,1}c_3+r_{2,2}r_{3,2}c_3+r_{2,3}r_{3,3}c_3-r_{2,3}c_3}{r_{3,3}-1} & \frac{-r_{1,1}r_{3,1}c_3-r_{1,2}r_{3,2}c_3-r_{1,3}r_{3,3}c_3+r_{1,3}c_3}{r_{3,3}-1} & 0 \end{pmatrix}$

o21 : Matrix (frac A)^3 $\longleftarrow$ (frac A)^3

i22 : F = transpose(R)*T

o22 = [very long output]

o22 : Matrix (frac A)^3 $\longleftarrow$ (frac A)^3

i23 :

```
--scale F so that its entries become polynomials
Fscaled = sub((r_(3,3)-1)*F, A)
```

o23 = [very long output]

o23 : Matrix $A^3 \longleftarrow A^3$

i24 :

```
--now use that R is a rotation matrix
Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

o24 = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

o24 : Matrix $\mathbb{Z}^3 \longleftarrow \mathbb{Z}^3$

i25 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)

o25 = ideal($r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $-r_{1,3}r_{2,2}r_{3,1}+r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1$)

o25 : Ideal of A

i26 : sub(Fscaled, A/SO3)

o26 = $\begin{matrix} \{-1\} \\ \{-1\} \\ \{-1\} \end{matrix}\begin{pmatrix} -r_{1,2}c_3-r_{2,1}c_3 & r_{1,1}c_3-r_{2,2}c_3 & -r_{3,2}c_3 \\ r_{1,1}c_3-r_{2,2}c_3 & r_{1,2}c_3+r_{2,1}c_3 & r_{3,1}c_3 \\ -r_{2,3}c_3 & r_{1,3}c_3 & 0 \end{pmatrix}$

o26 : [very long output]

i27 :

```
--now we assume that the top left 2x2 block of F is a scaling of an orthogonal matrix
clearAll
```

i28 : A = QQ[r_(1,1)..r_(3,3),c_1..c_3]

o28 = A

o28 : PolynomialRing

i29 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))

o29 = $\begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$

o29 : Matrix $A^3 \longleftarrow A^3$

i30 : C = matrix{{c_1},{c_2},{c_3}}

o30 = $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

o30 : Matrix $A^3 \longleftarrow A^1$

i31 : t = -R*C

o31 = $\begin{pmatrix} -r_{1,1}c_1-r_{1,2}c_2-r_{1,3}c_3 \\ -r_{2,1}c_1-r_{2,2}c_2-r_{2,3}c_3 \\ -r_{3,1}c_1-r_{3,2}c_2-r_{3,3}c_3 \end{pmatrix}$

o31 : Matrix $A^3 \longleftarrow A^1$

i32 : T = matrix{{0,-t_(2,0),t_(1,0)},{t_(2,0),0,-t_(0,0)},{-t_(1,0),t_(0,0),0}}

o32 = $\begin{pmatrix} 0 & r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3 & -r_{2,1}c_1-r_{2,2}c_2-r_{2,3}c_3 \\ -r_{3,1}c_1-r_{3,2}c_2-r_{3,3}c_3 & 0 & r_{1,1}c_1+r_{1,2}c_2+r_{1,3}c_3 \\ r_{2,1}c_1+r_{2,2}c_2+r_{2,3}c_3 & -r_{1,1}c_1-r_{1,2}c_2-r_{1,3}c_3 & 0 \end{pmatrix}$

o32 : Matrix $A^3 \longleftarrow A^3$

```
i33 : F = transpose(R)*T
o33 = [very long output]
o33 : Matrix A^3 <-- A^3
i34 : F22 = F_{0,1}^{0,1} --the top left 2x2 block of F
```

$$\text{o34} = \begin{matrix}\{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} r_{2,2}r_{3,1}c_2 - r_{2,1}r_{3,2}c_2 + r_{2,3}r_{3,1}c_3 - r_{2,1}r_{3,3}c_3 & -r_{1,2}r_{3,1}c_2 + r_{1,1}r_{3,2}c_2 - r_{1,3}r_{3,1}c_3 + r_{1,1}r_{3,3}c_3 \\ -r_{2,2}r_{3,1}c_1 + r_{2,1}r_{3,2}c_1 + r_{2,3}r_{3,2}c_3 - r_{2,2}r_{3,3}c_3 & r_{1,2}r_{3,1}c_1 - r_{1,1}r_{3,2}c_1 - r_{1,3}r_{3,2}c_3 + r_{1,2}r_{3,3}c_3 \end{pmatrix}$$

```
o34 : Matrix A^2 <-- A^2
i35 :
      --the following conditions encode that F22 is scaling of an orthogonal matrix of determina +1 resp. -1
      O2plus = ideal(F22_(0,0)-F22_(1,1), F22_(0,1)+F22_(1,0))
```

o35 = ideal($-r_{1,2}r_{3,1}c_1+r_{1,1}r_{3,2}c_1+r_{2,2}r_{3,1}c_2-r_{2,1}r_{3,2}c_2+r_{2,3}r_{3,1}c_3+r_{1,3}r_{3,2}c_3-r_{1,2}r_{3,3}c_3-r_{2,1}r_{3,3}c_3$, $-r_{2,2}r_{3,1}c_1+r_{2,1}r_{3,2}c_1-r_{1,2}r_{3,1}c_2+r_{1,1}r_{3,2}c_2-r_{1,3}r_{3,1}c_3+r_{2,3}r_{3,2}c_3+r_{1,1}r_{3,3}c_3-r_{2,2}r_{3,3}c_3$)

```
o35 : Ideal of A
i36 : O2minus = ideal(F22_(0,0)+F22_(1,1), F22_(0,1)-F22_(1,0))
```

o36 = ideal($r_{1,2}r_{3,1}c_1-r_{1,1}r_{3,2}c_1+r_{2,2}r_{3,1}c_2-r_{2,1}r_{3,2}c_2+r_{2,3}r_{3,1}c_3-r_{1,3}r_{3,2}c_3+r_{1,2}r_{3,3}c_3-r_{2,1}r_{3,3}c_3$, $r_{2,2}r_{3,1}c_1-r_{2,1}r_{3,2}c_1-r_{1,2}r_{3,1}c_2+r_{1,1}r_{3,2}c_2-r_{1,3}r_{3,1}c_3-r_{2,3}r_{3,2}c_3+r_{1,1}r_{3,3}c_3+r_{2,2}r_{3,3}c_3$)

```
o36 : Ideal of A
i37 :
      --now we constrain R to be a rotation matrix
      Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

$$\text{o37} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
o37 : Matrix ZZ^3 <-- ZZ^3
i38 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)
```

o38 = ideal($r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $-r_{1,3}r_{2,2}r_{3,1}+r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1$)

```
o38 : Ideal of A
i39 :
      --this function eats a polynomial or a rational function and returns the numerator
      num = eq -> if (class class eq === FractionField) then numerator eq else eq
o39 = num
o39 : FunctionClosure
i40 :
      --we start by analyzing SO(2)
      decPlus = decompose(O2plus+SO3)
```

o40 = {ideal($r_{3,3}-1$, $r_{3,2}$, $r_{3,1}$, $r_{2,3}$, $r_{1,3}$, $r_{1,2}+r_{2,1}$, $r_{1,1}-r_{2,2}$, $r_{2,1}^2+r_{2,2}^2-1$), ideal($r_{3,3}-1$, $r_{2,3}c_1+r_{1,3}c_2-r_{1,2}c_3-r_{2,1}c_3$, $r_{1,3}c_1-r_{2,3}c_2-r_{1,1}c_3+r_{2,2}c_3$, $r_{2,3}r_{3,2}+r_{1,1}-r_{2,2}$, $r_{1,3}r_{3,2}-r_{1,2}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2$, $r_{2,3}r_{3,1}-r_{1,2}-r_{2,1}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}$, $r_{1,3}r_{3,1}-r_{1,1}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+1$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2$, $r_{3,1}c_1c_3+r_{3,2}c_2c_3-c_1^2-c_2^2$), ideal($r_{3,3}c_2-r_{3,2}c_3+c_2$, $r_{3,3}c_1-r_{3,1}c_3+c_1$, $r_{3,2}c_1-r_{3,1}c_2$, $r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3-c_3$, $r_{2,3}c_1+r_{1,3}c_2-r_{1,2}c_3-r_{2,1}c_3$, $r_{1,3}c_1-r_{2,3}c_2-r_{1,1}c_3+r_{2,2}c_3$, $r_{1,2}c_1-r_{2,1}c_1-r_{1,1}c_2-r_{2,2}c_2-r_{2,3}c_3$, $r_{1,1}c_1+r_{2,2}c_1+r_{1,2}c_2-r_{2,1}c_2+r_{1,3}c_3$, $r_{2,3}r_{3,2}+r_{2,2}r_{3,3}+r_{1,1}$, $r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$, $r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}$, $r_{1,3}r_{3,1}-r_{1,1}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)}

```
o40 : List
i41 : --this ideal has 3 components:
      #decPlus
```

o41 = 3

```
i42 : --the first 2 have the last entry of R equal to 1:
      decPlus#0
```

o42 = ideal($r_{3,3}-1$, $r_{3,2}$, $r_{3,1}$, $r_{2,3}$, $r_{1,3}$, $r_{1,2}+r_{2,1}$, $r_{1,1}-r_{2,2}$, $r_{2,1}^2+r_{2,2}^2-1$)

```
o42 : Ideal of A
i43 : decPlus#1
```

o43 = ideal($r_{3,3}-1$, $r_{2,3}c_1+r_{1,3}c_2-r_{1,2}c_3-r_{2,1}c_3$, $r_{1,3}c_1-r_{2,3}c_2-r_{1,1}c_3+r_{2,2}c_3$, $r_{2,3}r_{3,2}+r_{1,1}-r_{2,2}$, $r_{1,3}r_{3,2}-r_{1,2}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2$, $r_{2,3}r_{3,1}-r_{1,2}-r_{2,1}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}$, $r_{1,3}r_{3,1}-r_{1,1}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+1$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2$, $r_{3,1}c_1c_3+r_{3,2}c_2c_3-c_1^2-c_2^2$)

```
o43 : Ideal of A
i44 : --we investigate the last component:
      I = decPlus#2
```

o44 = ideal($r_{3,3}c_2-r_{3,2}c_3+c_2$, $r_{3,3}c_1-r_{3,1}c_3+c_1$, $r_{3,2}c_1-r_{3,1}c_2$, $r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3-c_3$, $r_{2,3}c_1+r_{1,3}c_2-r_{1,2}c_3-r_{2,1}c_3$, $r_{1,3}c_1-r_{2,3}c_2-r_{1,1}c_3+r_{2,2}c_3$, $r_{1,2}c_1-r_{2,1}c_1-r_{1,1}c_2-r_{2,2}c_2-r_{2,3}c_3$, $r_{1,1}c_1+r_{2,2}c_1+r_{1,2}c_2-r_{2,1}c_2+r_{1,3}c_3$, $r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$, $r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$, $r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$, $r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)

```
o44 : Ideal of A
i45 : L = flatten entries gens I
```

o45 = {$r_{3,3}c_2-r_{3,2}c_3+c_2$, $r_{3,3}c_1-r_{3,1}c_3+c_1$, $r_{3,2}c_1-r_{3,1}c_2$, $r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3-c_3$, $r_{2,3}c_1+r_{1,3}c_2-r_{1,2}c_3-r_{2,1}c_3$, $r_{1,3}c_1-r_{2,3}c_2-r_{1,1}c_3+r_{2,2}c_3$, $r_{1,2}c_1-r_{2,1}c_1-r_{1,1}c_2-r_{2,2}c_2-r_{2,3}c_3$, $r_{1,1}c_1+r_{2,2}c_1+r_{1,2}c_2-r_{2,1}c_2+r_{1,3}c_3$, $r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$, $r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$, $r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$, $r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}^2+r_{2,2}^2+$

$r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1\}$

```
o45 : List
i46 : --by investigating the first 2 equations, we see that t_1 and t_2 can be expressed in terms of R and t_3
      --(if the last entry of R is not equal to -1)
      --we substitute these expressions into all equations:
      Lsub = apply(L, eq -> sub(eq, {c_2 => (r_(3,2)*c_3)/(r_(3,3)+1), c_1 => (r_(3,1)*c_3)/(r_(3,3)+1)}))
```

o46 = $\{0,$
$0,$
$0,$
$\frac{r_{3,1}^2c_3+r_{3,2}^2c_3+r_{3,3}^2c_3-c_3}{r_{3,3}+1},$
$\frac{r_{2,3}r_{3,1}c_3+r_{1,3}r_{3,2}c_3-r_{1,2}r_{3,3}c_3-r_{2,1}r_{3,3}c_3-r_{1,2}c_3-r_{2,1}c_3}{r_{3,3}+1},$
$\frac{r_{1,3}r_{3,1}c_3-r_{2,3}r_{3,2}c_3-r_{1,1}r_{3,3}c_3+r_{2,2}r_{3,3}c_3-r_{1,1}c_3+r_{2,2}c_3}{r_{3,3}+1},$
$\frac{r_{1,2}r_{3,1}c_3-r_{2,1}r_{3,1}c_3-r_{1,1}r_{3,2}c_3-r_{2,2}r_{3,2}c_3-r_{2,3}r_{3,3}c_3-r_{2,3}c_3}{r_{3,3}+1},$
$\frac{r_{1,1}r_{3,1}c_3+r_{2,2}r_{3,1}c_3+r_{1,2}r_{3,2}c_3-r_{2,1}r_{3,2}c_3+r_{1,3}r_{3,3}c_3+r_{1,3}c_3}{r_{3,3}+1},$
$r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1},$
$r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1},$
$r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1,$
$r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2},$
$r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},$
$r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3},$
$r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2},$
$r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},$
$r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3},$
$r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},$
$r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,$
$r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},$
$r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3},$
$r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},$
$r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,$
$r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},$
$r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},$
$r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,$
$r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},$
$r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1\}$

```
o46 : List
i47 : --now we see that we obtained again the ideal of SO(3), meaning that there are no further constraints on R
      SO3 == ideal apply(Lsub, eq -> num eq)
```

o47 = true

```
i48 : --hence, for every R with last entry not equal -1, we find exactly one solution in t (up to scaling)


      --finally, we analyze the negative component of O(2)
      decMinus = decompose(O2minus+SO3)
```

o48 = {ideal($r_{3,3}+1,\ r_{3,2},\ r_{3,1},\ r_{2,3},\ r_{1,3},\ r_{1,2}-r_{2,1},\ r_{1,1}+r_{2,2},\ r_{2,1}^2+r_{2,2}^2-1$), ideal($r_{3,3}+1,\ r_{2,3}c_1-r_{1,3}c_2+r_{1,2}c_3-r_{2,1}c_3,\ r_{1,3}c_1+r_{2,3}c_2-r_{1,1}c_3-r_{2,2}c_3,\ r_{2,3}r_{3,2}+r_{1,1}+r_{2,2},\ r_{1,3}r_{3,2}+r_{1,2}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2,\ r_{2,3}r_{3,1}-r_{1,2}+r_{2,1},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}-r_{2,3},\ r_{1,3}r_{3,1}+r_{1,1}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}-r_{1,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}-1,\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}-r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}-r_{3,1},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2,\ r_{3,1}c_1c_3+r_{3,2}c_2c_3+c_1^2+c_2^2$), ideal($r_{3,3}c_2-r_{3,2}c_3-c_2,\ r_{3,3}c_1-r_{3,1}c_3-c_1,\ r_{3,2}c_1-r_{3,1}c_2,\ r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3+c_3,\ r_{2,3}c_1-r_{1,3}c_2+r_{1,2}c_3-r_{2,1}c_3,\ r_{1,3}c_1+r_{2,3}c_2-r_{1,1}c_3-r_{2,2}c_3,\ r_{1,2}c_1+r_{2,1}c_1-r_{1,1}c_2+r_{2,2}c_2+r_{2,3}c_3,\ r_{1,1}c_1-r_{2,2}c_1+r_{1,2}c_2+r_{2,1}c_2+r_{1,3}c_3,\ r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1},\ r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1,\ r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3},\ r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3},\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)}

```
o48 : List
i49 : --this ideal has 3 components:
      #decMinus
```

o49 = 3

```
i50 : --the first 2 have the last entry of R equal to -1:
      decMinus#0
```

o50 = ideal($r_{3,3}+1,\ r_{3,2},\ r_{3,1},\ r_{2,3},\ r_{1,3},\ r_{1,2}-r_{2,1},\ r_{1,1}+r_{2,2},\ r_{2,1}^2+r_{2,2}^2-1$)

```
o50 : Ideal of A
i51 : decMinus#1
```

o51 = ideal($r_{3,3}+1,\ r_{2,3}c_1-r_{1,3}c_2+r_{1,2}c_3-r_{2,1}c_3,\ r_{1,3}c_1+r_{2,3}c_2-r_{1,1}c_3-r_{2,2}c_3,\ r_{2,3}r_{3,2}+r_{1,1}+r_{2,2},\ r_{1,3}r_{3,2}+r_{1,2}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2,\ r_{2,3}r_{3,1}-r_{1,2}+r_{2,1},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}-r_{2,3},\ r_{1,3}r_{3,1}+r_{1,1}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}-r_{1,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}-1,\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}-r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}-r_{3,1},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2,\ r_{3,1}c_1c_3+r_{3,2}c_2c_3+c_1^2+c_2^2$)

```
o51 : Ideal of A
i52 : --we investigate the last component:
      I = decMinus#2
```

o52 = ideal($r_{3,3}c_2-r_{3,2}c_3-c_2,\ r_{3,3}c_1-r_{3,1}c_3-c_1,\ r_{3,2}c_1-r_{3,1}c_2,\ r_{3,1}c_1+r_{3,2}c_2+r_{3,3}c_3+c_3,\ r_{2,3}c_1-r_{1,3}c_2+r_{1,2}c_3-r_{2,1}c_3,\ r_{1,3}c_1+r_{2,3}c_2-r_{1,1}c_3-r_{2,2}c_3,\ r_{1,2}c_1+r_{2,1}c_1-r_{1,1}c_2+r_{2,2}c_2+r_{2,3}c_3,\ r_{1,1}c_1-r_{2,2}c_1+r_{1,2}c_2+r_{2,1}c_2+r_{1,3}c_3,\ r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1},\ r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1,\ r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3},\ r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3},\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)

```
o52 : Ideal of A
i53 : L = flatten entries gens I
```

$$o53 = \{r_{3,3}c_2 - r_{3,2}c_3 - c_2,\ r_{3,3}c_1 - r_{3,1}c_3 - c_1,\ r_{3,2}c_1 - r_{3,1}c_2,\ r_{3,1}c_1 + r_{3,2}c_2 + r_{3,3}c_3 + c_3,\ r_{2,3}c_1 - r_{1,3}c_2 + r_{1,2}c_3 - r_{2,1}c_3,\ r_{1,3}c_1 + r_{2,3}c_2 - r_{1,1}c_3 - r_{2,2}c_3,\ r_{1,2}c_1 + r_{2,1}c_1 - r_{1,1}c_2 + r_{2,2}c_2 + r_{2,3}c_3,\ r_{1,1}c_1 - r_{2,2}c_1 + r_{1,2}c_2 + r_{2,1}c_2 + r_{1,3}c_3,\ r_{2,3}r_{3,2} - r_{2,2}r_{3,3} + r_{1,1},\ r_{1,3}r_{3,2} - r_{1,2}r_{3,3} - r_{2,1},\ r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 - 1,\ r_{2,3}r_{3,1} - r_{2,1}r_{3,3} - r_{1,2},\ r_{2,2}r_{3,1} - r_{2,1}r_{3,2} + r_{1,3},\ r_{2,1}r_{3,1} + r_{2,2}r_{3,2} + r_{2,3}r_{3,3},\ r_{1,3}r_{3,1} - r_{1,1}r_{3,3} + r_{2,2},\ r_{1,2}r_{3,1} - r_{1,1}r_{3,2} - r_{2,3},\ r_{1,1}r_{3,1} + r_{1,2}r_{3,2} + r_{1,3}r_{3,3},\ r_{1,3}r_{2,2} - r_{1,2}r_{2,3} + r_{3,1},\ r_{2,1}^2 + r_{2,2}^2 + r_{2,3}^2 - 1,\ r_{1,3}r_{2,1} - r_{1,1}r_{2,3} - r_{3,2},\ r_{1,2}r_{2,1} - r_{1,1}r_{2,2} + r_{3,3},\ r_{1,1}r_{2,1} + r_{1,2}r_{2,2} + r_{1,3}r_{2,3},\ r_{1,3}^2 + r_{2,3}^2 + r_{3,3}^2 - 1,\ r_{1,2}r_{1,3} + r_{2,2}r_{2,3} + r_{3,2}r_{3,3},\ r_{1,1}r_{1,3} + r_{2,1}r_{2,3} + r_{3,1}r_{3,3},\ r_{1,2}^2 + r_{2,2}^2 + r_{3,2}^2 - 1,\ r_{1,1}r_{1,2} + r_{2,1}r_{2,2} + r_{3,1}r_{3,2},\ r_{1,1}^2 - r_{2,2}^2 - r_{2,3}^2 - r_{3,2}^2 - r_{3,3}^2 + 1\}$$

```
o53 : List
i54 : --by investigating the first 2 equations, we see that t_1 and t_2 can be expressed in terms of R and t_3
      --(if the last entry of R is not equal to 1)
      --we substitute these expressions into all equations:
      Lsub = apply(L, eq -> sub(eq, {c_2 => (r_(3,2)*c_3)/(r_(3,3)-1), c_1 => (r_(3,1)*c_3)/(r_(3,3)-1)}))
```

$$o54 = \{0,\ 0,\ 0,\ \frac{r_{3,1}^2c_3 + r_{3,2}^2c_3 + r_{3,3}^2c_3 - c_3}{r_{3,3}-1},\ \frac{r_{2,3}r_{3,1}c_3 - r_{1,3}r_{3,2}c_3 + r_{1,2}r_{3,3}c_3 - r_{2,1}r_{3,3}c_3 - r_{1,2}c_3 + r_{2,1}c_3}{r_{3,3}-1},\ \frac{r_{1,3}r_{3,1}c_3 + r_{2,3}r_{3,2}c_3 - r_{1,1}r_{3,3}c_3 - r_{2,2}r_{3,3}c_3 + r_{1,1}c_3 + r_{2,2}c_3}{r_{3,3}-1},\ \frac{r_{1,2}r_{3,1}c_3 + r_{2,1}r_{3,1}c_3 - r_{1,1}r_{3,2}c_3 + r_{2,2}r_{3,2}c_3 + r_{2,3}r_{3,3}c_3 - r_{2,3}c_3}{r_{3,3}-1},\ \frac{r_{1,1}r_{3,1}c_3 - r_{2,2}r_{3,1}c_3 + r_{1,2}r_{3,2}c_3 + r_{2,1}r_{3,2}c_3 + r_{1,3}r_{3,3}c_3 - r_{1,3}c_3}{r_{3,3}-1},\ r_{2,3}r_{3,2} - r_{2,2}r_{3,3} + r_{1,1},\ r_{1,3}r_{3,2} - r_{1,2}r_{3,3} - r_{2,1},\ r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 - 1,\ r_{2,3}r_{3,1} - r_{2,1}r_{3,3} - r_{1,2},\ r_{2,2}r_{3,1} - r_{2,1}r_{3,2} + r_{1,3},\ r_{2,1}r_{3,1} + r_{2,2}r_{3,2} + r_{2,3}r_{3,3},\ r_{1,3}r_{3,1} - r_{1,1}r_{3,3} + r_{2,2},\ r_{1,2}r_{3,1} - r_{1,1}r_{3,2} - r_{2,3},\ r_{1,1}r_{3,1} + r_{1,2}r_{3,2} + r_{1,3}r_{3,3},\ r_{1,3}r_{2,2} - r_{1,2}r_{2,3} + r_{3,1},\ r_{2,1}^2 + r_{2,2}^2 + r_{2,3}^2 - 1,\ r_{1,3}r_{2,1} - r_{1,1}r_{2,3} - r_{3,2},\ r_{1,2}r_{2,1} - r_{1,1}r_{2,2} + r_{3,3},\ r_{1,1}r_{2,1} + r_{1,2}r_{2,2} + r_{1,3}r_{2,3},\ r_{1,3}^2 + r_{2,3}^2 + r_{3,3}^2 - 1,\ r_{1,2}r_{1,3} + r_{2,2}r_{2,3} + r_{3,2}r_{3,3},\ r_{1,1}r_{1,3} + r_{2,1}r_{2,3} + r_{3,1}r_{3,3},\ r_{1,2}^2 + r_{2,2}^2 + r_{3,2}^2 - 1,\ r_{1,1}r_{1,2} + r_{2,1}r_{2,2} + r_{3,1}r_{3,2},\ r_{1,1}^2 - r_{2,2}^2 - r_{2,3}^2 - r_{3,2}^2 - r_{3,3}^2 + 1\}$$

```
o54 : List
i55 : --now we see that we obtained again the ideal of SO(3), meaning that there are no further constraints on R
      SO3 == ideal apply(Lsub, eq -> num eq)
o55 = true
i56 : --hence, for every R with last entry not equal 1, we find exactly one solution in t (up to scaling)

i1 : --the case when t is given as a formula in terms of R: +
     clearAll
i2 : A = QQ[r_(1,1)..r_(3,3),t_3]
o2 = A
o2 : PolynomialRing
i3 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))
```

$$o3 = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

```
o3 : Matrix A^3 <-- A^3
i4 : t_1 = r_(1,3)*t_3/(r_(3,3)+1)
```

$$o4 = \frac{r_{1,3}t_3}{r_{3,3}+1}$$

```
o4 : frac A
i5 : t_2 = r_(2,3)*t_3/(r_(3,3)+1)
```

$$o5 = \frac{r_{2,3}t_3}{r_{3,3}+1}$$

```
o5 : frac A
i6 : T = matrix{{0,-t_3,t_2},{t_3,0,-t_1},{-t_2,t_1,0}}
```

$$o6 = \begin{pmatrix} 0 & -t_3 & \frac{r_{2,3}t_3}{r_{3,3}+1} \\ t_3 & 0 & \frac{-r_{1,3}t_3}{r_{3,3}+1} \\ \frac{-r_{2,3}t_3}{r_{3,3}+1} & \frac{r_{1,3}t_3}{r_{3,3}+1} & 0 \end{pmatrix}$$

```
o6 : Matrix (frac A)^3 <-- (frac A)^3
i7 : F = transpose(R)*T
```

$$o7 = \begin{pmatrix} \{-1\} & \frac{-r_{2,3}r_{3,1}t_3 + r_{2,1}r_{3,3}t_3 + r_{2,1}t_3}{r_{3,3}+1} & \frac{r_{1,3}r_{3,1}t_3 - r_{1,1}r_{3,3}t_3 - r_{1,1}t_3}{r_{3,3}+1} & \frac{-r_{1,3}r_{2,1}t_3 + r_{1,1}r_{2,3}t_3}{r_{3,3}+1} \\ \{-1\} & \frac{-r_{2,3}r_{3,2}t_3 + r_{2,2}r_{3,3}t_3 + r_{2,2}t_3}{r_{3,3}+1} & \frac{r_{1,3}r_{3,2}t_3 - r_{1,2}r_{3,3}t_3 - r_{1,2}t_3}{r_{3,3}+1} & \frac{-r_{1,3}r_{2,2}t_3 + r_{1,2}r_{2,3}t_3}{r_{3,3}+1} \\ \{-1\} & \frac{r_{2,3}t_3}{r_{3,3}+1} & \frac{-r_{1,3}t_3}{r_{3,3}+1} & 0 \end{pmatrix}$$

```
o7 : Matrix (frac A)^3 <-- (frac A)^3
i8 :
     --scale F so that its entries become polynomials
```

```
i7 :     Fscaled = sub((r_(3,3)+1)*F, A)
```

$$\text{o8} = \begin{matrix}\{-1\}\\ \{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} -r_{2,3}r_{3,1}t_3+r_{2,1}r_{3,3}t_3+r_{2,1}t_3 & r_{1,3}r_{3,1}t_3-r_{1,1}r_{3,3}t_3-r_{1,1}t_3 & -r_{1,3}r_{2,1}t_3+r_{1,1}r_{2,3}t_3\\ -r_{2,3}r_{3,2}t_3+r_{2,2}r_{3,3}t_3+r_{2,2}t_3 & r_{1,3}r_{3,2}t_3-r_{1,2}r_{3,3}t_3-r_{1,2}t_3 & -r_{1,3}r_{2,2}t_3+r_{1,2}r_{2,3}t_3\\ r_{2,3}t_3 & -r_{1,3}t_3 & 0\end{pmatrix}$$

```
o8 : Matrix A^3 <-- A^3
i9 :
     --now use that R is a rotation matrix
     Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

$$\text{o9} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

```
o9 : Matrix ZZ^3 <-- ZZ^3
i10 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)
o10 = ideal(r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1, r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}, r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}, r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}, r_{1,2}^2+r_{2,2}^2+
r_{3,2}^2-1, r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}, r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}, r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}, r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1, -r_{1,3}r_{2,2}r_{3,1}+
r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1)
o10 : Ideal of A
i11 : sub(Fscaled, A/SO3)
```

$$\text{o11} = \begin{matrix}\{-1\}\\ \{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} -r_{1,2}t_3+r_{2,1}t_3 & -r_{1,1}t_3-r_{2,2}t_3 & -r_{3,2}t_3\\ r_{1,1}t_3+r_{2,2}t_3 & -r_{1,2}t_3+r_{2,1}t_3 & r_{3,1}t_3\\ r_{2,3}t_3 & -r_{1,3}t_3 & 0\end{pmatrix}$$

```
o11 : [very long output]

i12 : --the case when t is given as a formula in terms of R: -
      clearAll
i13 : A = QQ[r_(1,1)..r_(3,3),t_3]
o13 = A
o13 : PolynomialRing
i14 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))
```

$$\text{o14} = \begin{pmatrix}r_{1,1}&r_{1,2}&r_{1,3}\\r_{2,1}&r_{2,2}&r_{2,3}\\r_{3,1}&r_{3,2}&r_{3,3}\end{pmatrix}$$

```
o14 : Matrix A^3 <-- A^3
i15 : t_1 = r_(1,3)*t_3/(r_(3,3)-1)
```

$$\text{o15} = \frac{r_{1,3}t_3}{r_{3,3}-1}$$

```
o15 : frac A
i16 : t_2 = r_(2,3)*t_3/(r_(3,3)-1)
```

$$\text{o16} = \frac{r_{2,3}t_3}{r_{3,3}-1}$$

```
o16 : frac A
i17 : T = matrix{{0,-t_3,t_2},{t_3,0,-t_1},{-t_2,t_1,0}}
```

$$\text{o17} = \begin{pmatrix}0&-t_3&\frac{r_{2,3}t_3}{r_{3,3}-1}\\ t_3&0&\frac{-r_{1,3}t_3}{r_{3,3}-1}\\ \frac{-r_{2,3}t_3}{r_{3,3}-1}&\frac{r_{1,3}t_3}{r_{3,3}-1}&0\end{pmatrix}$$

```
o17 : Matrix (frac A)^3 <-- (frac A)^3
i18 : F = transpose(R)*T
```

$$\text{o18} = \begin{matrix}\{-1\}\\ \{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} \frac{-r_{2,3}r_{3,1}t_3+r_{2,1}r_{3,3}t_3-r_{2,1}t_3}{r_{3,3}-1} & \frac{r_{1,3}r_{3,1}t_3-r_{1,1}r_{3,3}t_3+r_{1,1}t_3}{r_{3,3}-1} & \frac{-r_{1,3}r_{2,1}t_3+r_{1,1}r_{2,3}t_3}{r_{3,3}-1}\\ \frac{-r_{2,3}r_{3,2}t_3+r_{2,2}r_{3,3}t_3-r_{2,2}t_3}{r_{3,3}-1} & \frac{r_{1,3}r_{3,2}t_3-r_{1,2}r_{3,3}t_3+r_{1,2}t_3}{r_{3,3}-1} & \frac{-r_{1,3}r_{2,2}t_3+r_{1,2}r_{2,3}t_3}{r_{3,3}-1}\\ \frac{-r_{2,3}t_3}{r_{3,3}-1} & \frac{r_{1,3}t_3}{r_{3,3}-1} & 0\end{pmatrix}$$

```
o18 : Matrix (frac A)^3 <-- (frac A)^3
i19 :
     --scale F so that its entries become polynomials
     Fscaled = sub((r_(3,3)-1)*F, A)
```

$$\text{o19} = \begin{matrix}\{-1\}\\ \{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} -r_{2,3}r_{3,1}t_3+r_{2,1}r_{3,3}t_3-r_{2,1}t_3 & r_{1,3}r_{3,1}t_3-r_{1,1}r_{3,3}t_3+r_{1,1}t_3 & -r_{1,3}r_{2,1}t_3+r_{1,1}r_{2,3}t_3\\ -r_{2,3}r_{3,2}t_3+r_{2,2}r_{3,3}t_3-r_{2,2}t_3 & r_{1,3}r_{3,2}t_3-r_{1,2}r_{3,3}t_3+r_{1,2}t_3 & -r_{1,3}r_{2,2}t_3+r_{1,2}r_{2,3}t_3\\ -r_{2,3}t_3 & r_{1,3}t_3 & 0\end{pmatrix}$$

```
o19 : Matrix A^3 <-- A^3
i20 :
     --now use that R is a rotation matrix
     Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

$$\text{o20} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

```
o20 : Matrix ZZ^3 <-- ZZ^3
i21 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)
o21 = ideal(r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1, r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}, r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}, r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}, r_{1,2}^2+r_{2,2}^2+
r_{3,2}^2-1, r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}, r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}, r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}, r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1, -r_{1,3}r_{2,2}r_{3,1}+
r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1)
o21 : Ideal of A
i22 : sub(Fscaled, A/SO3)
```

$$\text{o22} = \begin{matrix}\{-1\}\\ \{-1\}\\ \{-1\}\end{matrix}\begin{pmatrix} -r_{1,2}t_3-r_{2,1}t_3 & r_{1,1}t_3-r_{2,2}t_3 & -r_{3,2}t_3\\ r_{1,1}t_3-r_{2,2}t_3 & r_{1,2}t_3+r_{2,1}t_3 & r_{3,1}t_3\\ -r_{2,3}t_3 & r_{1,3}t_3 & 0\end{pmatrix}$$

```
o22 : [very long output]

i23 : --now we assume that the top left 2x2 block of F is a scaling of an orthogonal matrix
```

```
i24 : clearAll
i24 : A = QQ[r_(1,1)..r_(3,3),t_1..t_3]
o24 = A
o24 : PolynomialRing
i25 : R = matrix apply(3, i -> apply(3, j -> r_(i+1,j+1)))
```

$$\text{o25} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

```
o25 : Matrix A^3 <-- A^3
i26 : T = matrix{{0,-t_3,t_2},{t_3,0,-t_1},{-t_2,t_1,0}}
```

$$\text{o26} = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}$$

```
o26 : Matrix A^3 <-- A^3
i27 : F = transpose(R)*T
```

$$\text{o27} = \begin{matrix} \{-1\} \\ \{-1\} \\ \{-1\} \end{matrix} \begin{pmatrix} -r_{3,1}t_2+r_{2,1}t_3 & r_{3,1}t_1-r_{1,1}t_3 & -r_{2,1}t_1+r_{1,1}t_2 \\ -r_{3,2}t_2+r_{2,2}t_3 & r_{3,2}t_1-r_{1,2}t_3 & -r_{2,2}t_1+r_{1,2}t_2 \\ -r_{3,3}t_2+r_{2,3}t_3 & r_{3,3}t_1-r_{1,3}t_3 & -r_{2,3}t_1+r_{1,3}t_2 \end{pmatrix}$$

```
o27 : Matrix A^3 <-- A^3
i28 : F22 = F_{0,1}^{0,1} --the top left 2x2 block of F
```

$$\text{o28} = \begin{matrix} \{-1\} \\ \{-1\} \end{matrix} \begin{pmatrix} -r_{3,1}t_2+r_{2,1}t_3 & r_{3,1}t_1-r_{1,1}t_3 \\ -r_{3,2}t_2+r_{2,2}t_3 & r_{3,2}t_1-r_{1,2}t_3 \end{pmatrix}$$

```
o28 : Matrix A^2 <-- A^2
i29 :
      --the following conditions encode that F22 is scaling of an orthogonal matrix of determina +1 resp. -1
      O2plus = ideal(F22_(0,0)-F22_(1,1), F22_(0,1)+F22_(1,0))
```

$$\text{o29} = \text{ideal}(-r_{3,2}t_1-r_{3,1}t_2+r_{1,2}t_3+r_{2,1}t_3,\ r_{3,1}t_1-r_{3,2}t_2-r_{1,1}t_3+r_{2,2}t_3)$$

```
o29 : Ideal of A
i30 : O2minus = ideal(F22_(0,0)+F22_(1,1), F22_(0,1)-F22_(1,0))
```

$$\text{o30} = \text{ideal}(r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3,\ r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3)$$

```
o30 : Ideal of A
i31 :
      --now we constrain R to be a rotation matrix
      Id = matrix apply(3, i -> apply(3, j -> if i==j then 1 else 0))
```

$$\text{o31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
o31 : Matrix ZZ^3 <-- ZZ^3
i32 : SO3 = ideal flatten entries(transpose(R)*R-Id) + ideal((det R)-1)
```

$$\text{o32} = \text{ideal}(r_{1,1}^2+r_{2,1}^2+r_{3,1}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,\ -r_{1,3}r_{2,2}r_{3,1}+r_{1,2}r_{2,3}r_{3,1}+r_{1,3}r_{2,1}r_{3,2}-r_{1,1}r_{2,3}r_{3,2}-r_{1,2}r_{2,1}r_{3,3}+r_{1,1}r_{2,2}r_{3,3}-1)$$

```
o32 : Ideal of A
i33 :
      --this function eats a polynomial or a rational function and returns the numerator
      num = eq -> if (class class eq === FractionField) then numerator eq else eq
o33 = num
o33 : FunctionClosure
i34 :
      --we start by analyzing SO(2)
      decPlus = decompose(O2plus+SO3)
```

$$\text{o34} = \{\text{ideal}(r_{3,3}-1,\ r_{3,2},\ r_{3,1},\ r_{2,3},\ r_{1,3},\ r_{1,2}+r_{2,1},\ r_{1,1}-r_{2,2},\ r_{2,1}^2+r_{2,2}^2-1),$$
$$\text{ideal}(r_{3,3}-1,\ r_{3,2}t_1+r_{3,1}t_2-r_{1,2}t_3-r_{2,1}t_3,\ r_{3,1}t_1-r_{3,2}t_2-r_{1,1}t_3+r_{2,2}t_3,\ r_{2,3}r_{3,2}+r_{1,1}-r_{2,2},\ r_{1,3}r_{3,2}-r_{1,2}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2,\ r_{2,3}r_{3,1}-r_{1,2}-r_{2,1},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3},\ r_{1,3}r_{3,1}-r_{1,1}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+1,\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2,\ r_{1,3}t_1t_3+r_{2,3}t_2t_3-t_1^2-t_2^2),$$
$$\text{ideal}(r_{3,3}t_2-r_{2,3}t_3+t_2,\ r_{3,3}t_1-r_{1,3}t_3+t_1,\ r_{3,2}t_1+r_{3,1}t_2-r_{1,2}t_3-r_{2,1}t_3,\ r_{3,1}t_1-r_{3,2}t_2-r_{1,1}t_3+r_{2,2}t_3,\ r_{2,3}t_1-r_{1,3}t_2,\ r_{1,3}t_1+r_{2,3}t_2+r_{3,3}t_3-t_3,\ r_{1,2}t_1-r_{2,1}t_1+r_{1,1}t_2+r_{2,2}t_2+r_{3,2}t_3,\ r_{1,1}t_1+r_{2,2}t_1-r_{1,2}t_2+r_{2,1}t_2+r_{3,1}t_3,\ r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1},\ r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1,\ r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3},\ r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3},\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1)\}$$

```
o34 : List
i35 : --this ideal has 3 components:
      #decPlus
o35 = 3
i36 : --the first 2 have the last entry of R equal to 1:
      decPlus#0
```

$$\text{o36} = \text{ideal}(r_{3,3}-1,\ r_{3,2},\ r_{3,1},\ r_{2,3},\ r_{1,3},\ r_{1,2}+r_{2,1},\ r_{1,1}-r_{2,2},\ r_{2,1}^2+r_{2,2}^2-1)$$

```
o36 : Ideal of A
i37 : decPlus#1
```

$$\text{o37} = \text{ideal}(r_{3,3}-1,\ r_{3,2}t_1+r_{3,1}t_2-r_{1,2}t_3-r_{2,1}t_3,\ r_{3,1}t_1-r_{3,2}t_2-r_{1,1}t_3+r_{2,2}t_3,\ r_{2,3}r_{3,2}+r_{1,1}-r_{2,2},\ r_{1,3}r_{3,2}-r_{1,2}-r_{2,1},\ r_{3,1}^2+r_{3,2}^2,\ r_{2,3}r_{3,1}-r_{1,2}-r_{2,1},\ r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3},\ r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3},\ r_{1,3}r_{3,1}-r_{1,1}+r_{2,2},\ r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3},\ r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3},\ r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1},\ r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1,\ r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2},\ r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+1,\ r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3},\ r_{1,3}^2+r_{2,3}^2,\ r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2},\ r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1},\ r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1,\ r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2},\ r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2,\ r_{1,3}t_1t_3+r_{2,3}t_2t_3-t_1^2-t_2^2)$$

```
o37 : Ideal of A
i38 : --we investigate the last component:
```

```
      I = decPlus#2
o38 = ideal(r_{3,3}t_2-r_{2,3}t_3+t_2, r_{3,3}t_1-r_{1,3}t_3+t_1, r_{3,2}t_1+r_{3,1}t_2-r_{1,2}t_3-r_{2,1}t_3, r_{3,1}t_1-r_{3,2}t_2-r_{1,1}t_3+r_{2,2}t_3, r_{2,3}t_1-r_{1,3}t_2, r_{1,3}t_1+
r_{2,3}t_2+r_{3,3}t_3-t_3, r_{1,2}t_1-r_{2,1}t_1+r_{1,1}t_2+r_{2,2}t_2+r_{3,2}t_3, r_{1,1}t_1+r_{2,2}t_1-r_{1,2}t_2+r_{2,1}t_2+r_{3,1}t_3, r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}, r_{1,3}r_{3,2}-
r_{1,2}r_{3,3}-r_{2,1}, r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1, r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}, r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}, r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}, r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+
r_{2,2}, r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}, r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}, r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}, r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1, r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}, r_{1,2}r_{2,1}-
r_{1,1}r_{2,2}+r_{3,3}, r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}, r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1, r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}, r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}, r_{1,2}^2+r_{2,2}^2+
r_{3,2}^2-1, r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}, r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1)
o38 : Ideal of A
i39 : L = flatten entries gens I
o39 = {r_{3,3}t_2 - r_{2,3}t_3 + t_2,
r_{3,3}t_1 - r_{1,3}t_3 + t_1,
r_{3,2}t_1 + r_{3,1}t_2 - r_{1,2}t_3 - r_{2,1}t_3,
r_{3,1}t_1 - r_{3,2}t_2 - r_{1,1}t_3 + r_{2,2}t_3,
r_{2,3}t_1 - r_{1,3}t_2,
r_{1,3}t_1 + r_{2,3}t_2 + r_{3,3}t_3 - t_3,
r_{1,2}t_1 - r_{2,1}t_1 + r_{1,1}t_2 + r_{2,2}t_2 + r_{3,2}t_3,
r_{1,1}t_1 + r_{2,2}t_1 - r_{1,2}t_2 + r_{2,1}t_2 + r_{3,1}t_3,
r_{2,3}r_{3,2} - r_{2,2}r_{3,3} + r_{1,1},
r_{1,3}r_{3,2} - r_{1,2}r_{3,3} - r_{2,1},
r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 - 1,
r_{2,3}r_{3,1} - r_{2,1}r_{3,3} - r_{1,2},
r_{2,2}r_{3,1} - r_{2,1}r_{3,2} + r_{1,3},
r_{2,1}r_{3,1} + r_{2,2}r_{3,2} + r_{2,3}r_{3,3},
r_{1,3}r_{3,1} - r_{1,1}r_{3,3} + r_{2,2},
r_{1,2}r_{3,1} - r_{1,1}r_{3,2} - r_{2,3},
r_{1,1}r_{3,1} + r_{1,2}r_{3,2} + r_{1,3}r_{3,3},
r_{1,3}r_{2,2} - r_{1,2}r_{2,3} + r_{3,1},
r_{2,1}^2 + r_{2,2}^2 + r_{2,3}^2 - 1,
r_{1,3}r_{2,1} - r_{1,1}r_{2,3} - r_{3,2},
r_{1,2}r_{2,1} - r_{1,1}r_{2,2} + r_{3,3},
r_{1,1}r_{2,1} + r_{1,2}r_{2,2} + r_{1,3}r_{2,3},
r_{1,3}^2 + r_{2,3}^2 + r_{3,3}^2 - 1,
r_{1,2}r_{1,3} + r_{2,2}r_{2,3} + r_{3,2}r_{3,3},
r_{1,1}r_{1,3} + r_{2,1}r_{2,3} + r_{3,1}r_{3,3},
r_{1,2}^2 + r_{2,2}^2 + r_{3,2}^2 - 1,
r_{1,1}r_{1,2} + r_{2,1}r_{2,2} + r_{3,1}r_{3,2},
r_{1,1}^2 - r_{2,2}^2 - r_{2,3}^2 - r_{3,2}^2 - r_{3,3}^2 + 1}
o39 : List
i40 : --by investigating the first 2 equations, we see that t_1 and t_2 can be expressed in terms of R and t_3
      --(if the last entry of R is not equal to -1)
      --we substitute these expressions into all equations:
      Lsub = apply(L, eq -> sub(eq, {t_2 => (r_(2,3)*t_3)/(r_(3,3)+1), t_1 => (r_(1,3)*t_3)/(r_(3,3)+1)}))
o40 = {0,
0,
(r_{2,3}r_{3,1}t_3+r_{1,3}r_{3,2}t_3-r_{1,2}r_{3,3}t_3-r_{2,1}r_{3,3}t_3-r_{1,2}t_3-r_{2,1}t_3)/(r_{3,3}+1),
(r_{1,3}r_{3,1}t_3-r_{2,3}r_{3,2}t_3-r_{1,1}r_{3,3}t_3+r_{2,2}r_{3,3}t_3-r_{1,1}t_3+r_{2,2}t_3)/(r_{3,3}+1),
0,
(r_{1,3}^2t_3+r_{2,3}^2t_3+r_{3,3}^2t_3-t_3)/(r_{3,3}+1),
(r_{1,2}r_{1,3}t_3-r_{1,3}r_{2,1}t_3+r_{1,1}r_{2,3}t_3+r_{2,2}r_{2,3}t_3+r_{3,2}r_{3,3}t_3+r_{3,2}t_3)/(r_{3,3}+1),
(r_{1,1}r_{1,3}t_3+r_{1,3}r_{2,2}t_3-r_{1,2}r_{2,3}t_3+r_{2,1}r_{2,3}t_3+r_{3,1}r_{3,3}t_3+r_{3,1}t_3)/(r_{3,3}+1),
r_{2,3}r_{3,2} - r_{2,2}r_{3,3} + r_{1,1},
r_{1,3}r_{3,2} - r_{1,2}r_{3,3} - r_{2,1},
r_{3,1}^2 + r_{3,2}^2 + r_{3,3}^2 - 1,
r_{2,3}r_{3,1} - r_{2,1}r_{3,3} - r_{1,2},
r_{2,2}r_{3,1} - r_{2,1}r_{3,2} + r_{1,3},
r_{2,1}r_{3,1} + r_{2,2}r_{3,2} + r_{2,3}r_{3,3},
r_{1,3}r_{3,1} - r_{1,1}r_{3,3} + r_{2,2},
r_{1,2}r_{3,1} - r_{1,1}r_{3,2} - r_{2,3},
r_{1,1}r_{3,1} + r_{1,2}r_{3,2} + r_{1,3}r_{3,3},
r_{1,3}r_{2,2} - r_{1,2}r_{2,3} + r_{3,1},
r_{2,1}^2 + r_{2,2}^2 + r_{2,3}^2 - 1,
r_{1,3}r_{2,1} - r_{1,1}r_{2,3} - r_{3,2},
r_{1,2}r_{2,1} - r_{1,1}r_{2,2} + r_{3,3},
r_{1,1}r_{2,1} + r_{1,2}r_{2,2} + r_{1,3}r_{2,3},
r_{1,3}^2 + r_{2,3}^2 + r_{3,3}^2 - 1,
r_{1,2}r_{1,3} + r_{2,2}r_{2,3} + r_{3,2}r_{3,3},
r_{1,1}r_{1,3} + r_{2,1}r_{2,3} + r_{3,1}r_{3,3},
r_{1,2}^2 + r_{2,2}^2 + r_{3,2}^2 - 1,
r_{1,1}r_{1,2} + r_{2,1}r_{2,2} + r_{3,1}r_{3,2},
r_{1,1}^2 - r_{2,2}^2 - r_{2,3}^2 - r_{3,2}^2 - r_{3,3}^2 + 1}
o40 : List
i41 : --now we see that we obtained again the ideal of SO(3), meaning that there are no further constraints on R
      SO3 == ideal apply(Lsub, eq -> num eq)
o41 = true
i42 : --hence, for every R with last entry not equal -1, we find exactly one solution in t (up to scaling)

      --finally, we analyze the negative component of O(2)
```

```
      decMinus = decompose(O2minus+SO3)
```

o42 = {ideal($r_{3,3}+1$, $r_{3,2}$, $r_{3,1}$, $r_{2,3}$, $r_{1,3}$, $r_{1,2}-r_{2,1}$, $r_{1,1}+r_{2,2}$, $r_{2,1}^2+r_{2,2}^2-1$),
ideal($r_{3,3}+1$, $r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3$, $r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3$, $r_{2,3}r_{3,2}+r_{1,1}+r_{2,2}$, $r_{1,3}r_{3,2}+r_{1,2}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2$, $r_{2,3}r_{3,1}-r_{1,2}+r_{2,1}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}-r_{2,3}$, $r_{1,3}r_{3,1}+r_{1,1}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}-r_{1,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}-1$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}-r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}-r_{3,1}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2$, $r_{1,3}t_1t_3+r_{2,3}t_2t_3+t_1^2+t_2^2$),
ideal($r_{3,3}t_2-r_{2,3}t_3-t_2$, $r_{3,3}t_1-r_{1,3}t_3-t_1$, $r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3$, $r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3$, $r_{2,3}t_1-r_{1,3}t_2$, $r_{1,3}t_1+r_{2,3}t_2+r_{3,3}t_3+t_3$, $r_{1,2}t_1+r_{2,1}t_1-r_{1,1}t_2+r_{2,2}t_2+r_{3,2}t_3$, $r_{1,1}t_1-r_{2,2}t_1+r_{1,2}t_2+r_{2,1}t_2+r_{3,1}t_3$, $r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$, $r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$, $r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$, $r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)}

o42 : List

```
i43 : --this ideal has 3 components:
      #decMinus
```

o43 = 3

```
i44 : --the first 2 have the last entry of R equal to -1:
      decMinus#0
```

o44 = ideal($r_{3,3}+1$, $r_{3,2}$, $r_{3,1}$, $r_{2,3}$, $r_{1,3}$, $r_{1,2}-r_{2,1}$, $r_{1,1}+r_{2,2}$, $r_{2,1}^2+r_{2,2}^2-1$)

o44 : Ideal of A

```
i45 : decMinus#1
```

o45 = ideal($r_{3,3}+1$, $r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3$, $r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3$, $r_{2,3}r_{3,2}+r_{1,1}+r_{2,2}$, $r_{1,3}r_{3,2}+r_{1,2}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2$, $r_{2,3}r_{3,1}-r_{1,2}+r_{2,1}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}-r_{2,3}$, $r_{1,3}r_{3,1}+r_{1,1}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}-r_{1,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}-1$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}-r_{3,2}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}-r_{3,1}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2$, $r_{1,3}t_1t_3+r_{2,3}t_2t_3+t_1^2+t_2^2$)

o45 : Ideal of A

```
i46 : --we investigate the last component:
      I = decMinus#2
```

o46 = ideal($r_{3,3}t_2-r_{2,3}t_3-t_2$, $r_{3,3}t_1-r_{1,3}t_3-t_1$, $r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3$, $r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3$, $r_{2,3}t_1-r_{1,3}t_2$, $r_{1,3}t_1+r_{2,3}t_2+r_{3,3}t_3+t_3$, $r_{1,2}t_1+r_{2,1}t_1-r_{1,1}t_2+r_{2,2}t_2+r_{3,2}t_3$, $r_{1,1}t_1-r_{2,2}t_1+r_{1,2}t_2+r_{2,1}t_2+r_{3,1}t_3$, $r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$, $r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$, $r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$, $r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$, $r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$, $r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$, $r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$, $r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$, $r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$, $r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$, $r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$, $r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$, $r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$, $r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$, $r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$, $r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$, $r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$, $r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$, $r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$, $r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$)

o46 : Ideal of A

```
i47 : L = flatten entries gens I
```

o47 = {$r_{3,3}t_2-r_{2,3}t_3-t_2$,
$r_{3,3}t_1-r_{1,3}t_3-t_1$,
$r_{3,2}t_1-r_{3,1}t_2-r_{1,2}t_3+r_{2,1}t_3$,
$r_{3,1}t_1+r_{3,2}t_2-r_{1,1}t_3-r_{2,2}t_3$,
$r_{2,3}t_1-r_{1,3}t_2$,
$r_{1,3}t_1+r_{2,3}t_2+r_{3,3}t_3+t_3$,
$r_{1,2}t_1+r_{2,1}t_1-r_{1,1}t_2+r_{2,2}t_2+r_{3,2}t_3$,
$r_{1,1}t_1-r_{2,2}t_1+r_{1,2}t_2+r_{2,1}t_2+r_{3,1}t_3$,
$r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$,
$r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$,
$r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$,
$r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$,
$r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$,
$r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$,
$r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$,
$r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$,
$r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$,
$r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$,
$r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$,
$r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$,
$r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$,
$r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$,
$r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$,
$r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$,
$r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$,
$r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$,
$r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$,
$r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1$}

o47 : List

```
i48 : --by investigating the first 2 equations, we see that t_1 and t_2 can be expressed in terms of R and t_3
      --(if the last entry of R is not equal to 1)
      --we substitute these expressions into all equations:
      Lsub = apply(L, eq -> sub(eq, {t_2 => (r_(2,3)*t_3)/(r_(3,3)-1), t_1 => (r_(1,3)*t_3)/(r_(3,3)-1)}))
```

o48 = {0,
0,
$\frac{-r_{2,3}r_{3,1}t_3+r_{1,3}r_{3,2}t_3-r_{1,2}r_{3,3}t_3+r_{2,1}r_{3,3}t_3+r_{1,2}t_3-r_{2,1}t_3}{r_{3,3}-1}$,
$\frac{r_{1,3}r_{3,1}t_3+r_{2,3}r_{3,2}t_3-r_{1,1}r_{3,3}t_3-r_{2,2}r_{3,3}t_3+r_{1,1}t_3+r_{2,2}t_3}{r_{3,3}-1}$,
0,
$\frac{r_{1,3}^2t_3+r_{2,3}^2t_3+r_{3,3}^2t_3-t_3}{r_{3,3}-1}$,
$\frac{r_{1,2}r_{1,3}t_3+r_{1,3}r_{2,1}t_3-r_{1,1}r_{2,3}t_3+r_{2,2}r_{2,3}t_3+r_{3,2}r_{3,3}t_3-r_{3,2}t_3}{r_{3,3}-1}$,

$\frac{r_{1,1}r_{1,3}t_3-r_{1,3}r_{2,2}t_3+r_{1,2}r_{2,3}t_3+r_{2,1}r_{2,3}t_3+r_{3,1}r_{3,3}t_3-r_{3,1}t_3}{r_{3,3}-1}$,
$r_{2,3}r_{3,2}-r_{2,2}r_{3,3}+r_{1,1}$,
$r_{1,3}r_{3,2}-r_{1,2}r_{3,3}-r_{2,1}$,
$r_{3,1}^2+r_{3,2}^2+r_{3,3}^2-1$,
$r_{2,3}r_{3,1}-r_{2,1}r_{3,3}-r_{1,2}$,
$r_{2,2}r_{3,1}-r_{2,1}r_{3,2}+r_{1,3}$,
$r_{2,1}r_{3,1}+r_{2,2}r_{3,2}+r_{2,3}r_{3,3}$,
$r_{1,3}r_{3,1}-r_{1,1}r_{3,3}+r_{2,2}$,
$r_{1,2}r_{3,1}-r_{1,1}r_{3,2}-r_{2,3}$,
$r_{1,1}r_{3,1}+r_{1,2}r_{3,2}+r_{1,3}r_{3,3}$,
$r_{1,3}r_{2,2}-r_{1,2}r_{2,3}+r_{3,1}$,
$r_{2,1}^2+r_{2,2}^2+r_{2,3}^2-1$,
$r_{1,3}r_{2,1}-r_{1,1}r_{2,3}-r_{3,2}$,
$r_{1,2}r_{2,1}-r_{1,1}r_{2,2}+r_{3,3}$,
$r_{1,1}r_{2,1}+r_{1,2}r_{2,2}+r_{1,3}r_{2,3}$,
$r_{1,3}^2+r_{2,3}^2+r_{3,3}^2-1$,
$r_{1,2}r_{1,3}+r_{2,2}r_{2,3}+r_{3,2}r_{3,3}$,
$r_{1,1}r_{1,3}+r_{2,1}r_{2,3}+r_{3,1}r_{3,3}$,
$r_{1,2}^2+r_{2,2}^2+r_{3,2}^2-1$,
$r_{1,1}r_{1,2}+r_{2,1}r_{2,2}+r_{3,1}r_{3,2}$,
$r_{1,1}^2-r_{2,2}^2-r_{2,3}^2-r_{3,2}^2-r_{3,3}^2+1\}$

```
o48 : List
i49 : --now we see that we obtained again the ideal of SO(3), meaning that there are no further constraints on R
      SO3 == ideal apply(Lsub, eq -> num eq)
o49 = true
i50 : --hence, for every R with last entry not equal 1, we find exactly one solution in t (up to scaling)
```

A.2 $r_6(s)$ from Theorem 4.1

```
i1 : clearAll
i2 : R = QQ[a_1..a_2,l_1..l_4,y_1..y_4][s]
o2 = R
o2 : PolynomialRing
i3 : q_1 = a_1
o3 = a_1
o3 : QQ[a_1 .. a_2, l_1 .. l_4, y_1 .. y_4]
i4 : q_2 = -a_1
o4 = -a_1
o4 : QQ[a_1 .. a_2, l_1 .. l_4, y_1 .. y_4]
i5 : q_3 = a_2
o5 = a_2
o5 : QQ[a_1 .. a_2, l_1 .. l_4, y_1 .. y_4]
i6 : q_4 = -a_2
o6 = -a_2
o6 : QQ[a_1 .. a_2, l_1 .. l_4, y_1 .. y_4]
i7 : scan(4, i -> e_(i+1) = s*q_(i+1)*y_(i+1)/( l_(i+1)-s*q_(i+1) ))
i8 : Rfunction = sum apply(4, i -> q_(i+1)*(y_(i+1) + e_(i+1))^2)
o8 = [very long expression]
o8 : frac R
i9 : r6 = numerator Rfunction
o9 = (a_1^3a_2^4l_1^2y_1^2 - a_1^3a_2^4l_2^2y_2^2 + a_1^4a_2^3l_3^2y_3^2 - a_1^4a_2^3l_4^2y_4^2)s^6
+(2 a_1^2a_2^4l_1^2l_2y_1^2 - 2 a_1^3a_2^3l_1^2l_3y_1^2 + 2 a_1^3a_2^3l_1^2l_4y_1^2 + 2 a_1^2a_2^4l_1l_2^2y_2^2 + 2 a_1^3a_2^3l_2^2l_3y_2^2 - 2 a_1^3a_2^3l_2^2l_4y_2^2 - 2 a_1^3a_2^3l_1l_3^2y_3^2 + 2 a_1^3a_2^3l_2l_3^2y_3^2 + 2 a_1^4a_2^2l_3^2l_4y_3^2 +
2 a_1^3a_2^3l_1l_4^2y_4^2 - 2 a_1^3a_2^3l_2l_4^2y_4^2 + 2 a_1^4a_2^2l_3l_4^2y_4^2)s^5
+(a_1a_2^4l_1^2l_2^2y_1^2 - 4 a_1^2a_2^3l_1^2l_2l_3y_1^2 + a_1^3a_2^2l_1^2l_3^2y_1^2 + 4 a_1^2a_2^3l_1^2l_2l_4y_1^2 - 4 a_1^3a_2^2l_1^2l_3l_4y_1^2 + a_1^3a_2^2l_1^2l_4^2y_1^2 - a_1a_2^4l_1^2l_2^2y_2^2 - 4 a_1^2a_2^3l_1l_2^2l_3y_2^2 - a_1^3a_2^2l_2^2l_3^2y_2^2 +
4 a_1^2a_2^3l_1l_2^2l_4y_2^2 + 4 a_1^3a_2^2l_2^2l_3l_4y_2^2 - a_1^3a_2^2l_2^2l_4^2y_2^2 + a_1^2a_2^3l_1^2l_3^2y_3^2 - 4 a_1^2a_2^3l_1l_2l_3^2y_3^2 + a_1^2a_2^3l_2^2l_3^2y_3^2 - 4 a_1^3a_2^2l_1l_3^2l_4y_3^2 + 4 a_1^3a_2^2l_2l_3^2l_4y_3^2 + a_1^4a_2l_3^2l_4^2y_3^2 -
a_1^2a_2^3l_1^2l_4^2y_4^2 + 4 a_1^2a_2^3l_1l_2l_4^2y_4^2 - a_1^2a_2^3l_2^2l_4^2y_4^2 - 4 a_1^3a_2^2l_1l_3l_4^2y_4^2 + 4 a_1^3a_2^2l_2l_3l_4^2y_4^2 - a_1^4a_2l_3^2l_4^2y_4^2)s^4
+(-2 a_1a_2^3l_1^2l_2^2l_3y_1^2 + 2 a_1^2a_2^2l_1^2l_2l_3^2y_1^2 + 2 a_1a_2^3l_1^2l_2^2l_4y_1^2 - 8 a_1^2a_2^2l_1^2l_2l_3l_4y_1^2 + 2 a_1^3a_2l_1^2l_3^2l_4y_1^2 + 2 a_1^2a_2^2l_1^2l_2l_4^2y_1^2 - 2 a_1^3a_2l_1^2l_3l_4^2y_1^2 + 2 a_1a_2^3l_1^2l_2^2l_3y_2^2 +
2 a_1^2a_2^2l_1l_2^2l_3^2y_2^2 - 2 a_1a_2^3l_1^2l_2^2l_4y_2^2 - 8 a_1^2a_2^2l_1l_2^2l_3l_4y_2^2 - 2 a_1^3a_2l_2^2l_3^2l_4y_2^2 + 2 a_1^2a_2^2l_1l_2^2l_4^2y_2^2 + 2 a_1^3a_2l_2^2l_3l_4^2y_2^2 + 2 a_1a_2^3l_1^2l_2l_3^2y_3^2 - 2 a_1a_2^3l_1l_2^2l_3^2y_3^2 +
2 a_1^2a_2^2l_1^2l_3^2l_4y_3^2 - 8 a_1^2a_2^2l_1l_2l_3^2l_4y_3^2 + 2 a_1^2a_2^2l_2^2l_3^2l_4y_3^2 - 2 a_1^3a_2l_1l_3^2l_4^2y_3^2 + 2 a_1^3a_2l_2l_3^2l_4^2y_3^2 - 2 a_1a_2^3l_1^2l_2l_4^2y_4^2 + 2 a_1a_2^3l_1l_2^2l_4^2y_4^2 + 2 a_1^2a_2^2l_1^2l_3l_4^2y_4^2 -
8 a_1^2a_2^2l_1l_2l_3l_4^2y_4^2 + 2 a_1^2a_2^2l_2^2l_3l_4^2y_4^2 + 2 a_1^3a_2l_1l_3^2l_4^2y_4^2 - 2 a_1^3a_2l_2l_3^2l_4^2y_4^2)s^3
+(a_1a_2^2l_1^2l_2^2l_3^2y_1^2 - 4 a_1a_2^2l_1^2l_2^2l_3l_4y_1^2 + 4 a_1^2a_2l_1^2l_2^2l_3l_4y_1^2 + a_1a_2^2l_1^2l_2^2l_4^2y_1^2 - 4 a_1^2a_2l_1^2l_2l_3l_4^2y_1^2 + a_1^3l_1^2l_3^2l_4^2y_1^2 - a_1a_2^2l_1^2l_2^2l_3^2y_2^2 + 4 a_1a_2^2l_1^2l_2^2l_3l_4y_2^2 +
4 a_1^2a_2l_1l_2^2l_3^2l_4y_2^2 - a_1a_2^2l_1^2l_2^2l_4^2y_2^2 - 4 a_1^2a_2l_1l_2^2l_3l_4^2y_2^2 - a_1^3l_2^2l_3^2l_4^2y_2^2 + a_2^3l_1^2l_2^2l_3^2y_3^2 + 4 a_1a_2^2l_1^2l_2l_3^2l_4y_3^2 - 4 a_1a_2^2l_1l_2^2l_3^2l_4y_3^2 + a_1^2a_2l_1^2l_3^2l_4^2y_3^2 -
4 a_1^2a_2l_1l_2l_3^2l_4^2y_3^2 + a_1^2a_2l_2^2l_3^2l_4^2y_3^2 - a_2^3l_1^2l_2^2l_4^2y_4^2 + 4 a_1a_2^2l_1^2l_2l_3l_4^2y_4^2 - 4 a_1a_2^2l_1l_2^2l_3l_4^2y_4^2 - a_1^2a_2l_1^2l_3^2l_4^2y_4^2 + 4 a_1^2a_2l_1l_2l_3^2l_4^2y_4^2 - a_1^2a_2l_2^2l_3^2l_4^2y_4^2)s^2
+(2 a_1a_2l_1^2l_2^2l_3^2l_4y_1^2 - 2 a_1a_2l_1^2l_2^2l_3l_4^2y_1^2 + 2 a_1^2l_1^2l_2l_3^2l_4^2y_1^2 - 2 a_1a_2l_1^2l_2^2l_3^2l_4y_2^2 + 2 a_1a_2l_1^2l_2^2l_3l_4^2y_2^2 + 2 a_1^2l_1l_2^2l_3^2l_4^2y_2^2 + 2 a_2^2l_1^2l_2^2l_3^2l_4y_2^2 + 2 a_1a_2l_1^2l_2l_3^2l_4^2y_3^2 -
2 a_1a_2l_1l_2^2l_3^2l_4^2y_3^2 + 2 a_2^2l_1^2l_2^2l_3l_4^2y_4^2 - 2 a_1a_2l_1^2l_2l_3^2l_4^2y_4^2 + 2 a_1a_2l_1l_2^2l_3^2l_4^2y_4^2)s
+ a_1l_1^2l_2^2l_3^2l_4^2y_1^2 - a_1l_1^2l_2^2l_3^2l_4^2y_2^2 + a_2l_1^2l_2^2l_3^2l_4^2y_3^2 - a_2l_1^2l_2^2l_3^2l_4^2y_4^2
o9 : R
```

A.3 $r_4(s)$ from Theorem 4.1

```
i10 : clearAll
i11 : R = QQ[a_1..a_2,l_2,l_4,y_1..y_4,m][s]
o11 = R
o11 : PolynomialRing
i12 : q_1 = a_1
o12 = a_1
o12 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i13 : q_2 = -a_1
o13 = -a_1
o13 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i14 : q_3 = a_2
o14 = a_2
o14 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i15 : q_4 = -a_2
o15 = -a_2
o15 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i16 : l_1 = m*a_1
o16 = a_1m
o16 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i17 : l_3 = m*a_2
o17 = a_2m
o17 : QQ[a_1 .. a_2, l_2, l_4, y_1 .. y_4, m]
i18 : scan(4, i -> e_(i+1) = s*q_(i+1)*y_(i+1)/( l_(i+1)-s*q_(i+1) ))
i19 : Rfunction = sum apply(4, i -> q_(i+1)*(y_(i+1) + e_(i+1))^2)
o19 = [very long expression]
o19 : frac R
i20 : r4 = numerator Rfunction
```

$$
\begin{aligned}
\mathrm{o20} = {} & (a_1^3a_2^2y_1^2m^2 + a_1^2a_2^3y_3^2m^2 - a_1a_2^2l_2^2y_2^2 - a_1^2a_2l_4^2y_4^2)s^4 \\
& + (2\,a_1^2a_2^2l_2y_1^2m^2 + 2\,a_1^3a_2l_4y_1^2m^2 + 2\,a_1a_2^3l_2y_3^2m^2 + 2\,a_1^2a_2^2l_4y_3^2m^2 + 2\,a_1a_2^2l_2^2y_2^2m + 2\,a_1^2a_2l_4^2y_4^2m - 2\,a_1a_2l_2^2l_4y_2^2 - 2\,a_1a_2l_2l_4^2y_4^2)s^3 \\
& + (a_1a_2^2l_2^2y_1^2m^2 + 4\,a_1^2a_2l_2l_4y_1^2m^2 + a_1^3l_4^2y_1^2m^2 - a_1a_2^2l_2^2y_2^2m^2 + a_2^3l_2^2y_3^2m^2 + 4\,a_1a_2^2l_2l_4y_3^2m^2 + a_1^2a_2l_4^2y_3^2m^2 - a_1^2a_2l_4^2y_4^2m^2 + 4\,a_1a_2l_2^2l_4y_2^2m + \\
& 4\,a_1a_2l_2l_4^2y_4^2m - a_1l_2^2l_4^2y_2^2 - a_2l_2^2l_4^2y_4^2)s^2 \\
& + (2\,a_1a_2l_2^2l_4y_1^2m^2 + 2\,a_1^2l_2l_4^2y_1^2m^2 - 2\,a_1a_2l_2^2l_4y_2^2m^2 + 2\,a_2^2l_2^2l_4y_3^2m^2 + 2\,a_1a_2l_2l_4^2y_3^2m^2 - 2\,a_1a_2l_2l_4^2y_4^2m^2 + 2\,a_1l_2^2l_4^2y_2^2m + 2\,a_2l_2^2l_4^2y_4^2m)s \\
& + a_1l_2^2l_4^2y_1^2m^2 - a_1l_2^2l_4^2y_2^2m^2 + a_2l_2^2l_4^2y_3^2m^2 - a_2l_2^2l_4^2y_4^2m^2
\end{aligned}
$$

```
o20 : R
```

A.4 $r_2(s)$ from Theorem 4.1

```
i21 : clearAll
i22 : R = QQ[a_1..a_2,y_1..y_4,m,n][s]
```

$$o22 = R$$

```
o22 : PolynomialRing
i23 : q_1 = a_1
```

$$o23 = a_1$$

$$o23 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i24 : q_2 = -a_1
```

$$o24 = -a_1$$

$$o24 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i25 : q_3 = a_2
```

$$o25 = a_2$$

$$o25 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i26 : q_4 = -a_2
```

$$o26 = -a_2$$

$$o26 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i27 : l_1 = m*a_1
```

$$o27 = a_1 m$$

$$o27 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i28 : l_2 = n*a_1
```

$$o28 = a_1 n$$

$$o28 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i29 : l_3 = m*a_2
```

$$o29 = a_2 m$$

$$o29 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i30 : l_4 = n*a_2
```

$$o30 = a_2 n$$

$$o30 : \mathbb{Q}[a_1 .. a_2,\ y_1 .. y_4,\ m .. n]$$

```
i31 : scan(4, i -> e_(i+1) = s*q_(i+1)*y_(i+1)/( l_(i+1)-s*q_(i+1) ))
i32 : Rfunction = sum apply(4, i -> q_(i+1)*(y_(i+1) + e_(i+1))^2)
```

$$o32 = \text{[very long expression]}$$

```
o32 : frac R
i33 : r2 = numerator Rfunction -- coeff's A, B, C in the paper
```

$$o33 = (a_1 y_1^2 m^2 + a_2 y_3^2 m^2 - a_1 y_2^2 n^2 - a_2 y_4^2 n^2) s^2$$
$$+ (2\,a_1 y_1^2 m^2 n + 2\,a_2 y_3^2 m^2 n + 2\,a_1 y_2^2 m\,n^2 + 2\,a_2 y_4^2 m\,n^2) s$$
$$+ a_1 y_1^2 m^2 n^2 - a_1 y_2^2 m^2 n^2 + a_2 y_3^2 m^2 n^2 - a_2 y_4^2 m^2 n^2$$

```
o33 : R
```

A.5 Theorem 4.3

```
i60 : clearAll
i61 : R = QQ[a_1..a_2,y_1..y_4,n][s]
o61 = R
o61 : PolynomialRing
i62 : q_1 = a_1
o62 = a_1
o62 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i63 : q_2 = -a_1
o63 = -a_1
o63 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i64 : q_3 = a_2
o64 = a_2
o64 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i65 : q_4 = -a_2
o65 = -a_2
o65 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i66 : l_1 = a_1 -- m has been put to 1
o66 = a_1
o66 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i67 : l_2 = n*a_1
o67 = a_1n
o67 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i68 : l_3 = a_2
o68 = a_2
o68 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i69 : l_4 = n*a_2
o69 = a_2n
o69 : QQ[a_1 .. a_2, y_1 .. y_4, n]
i70 : scan(4, i -> e_(i+1) = s*q_(i+1)*y_(i+1)/( l_(i+1)-s*q_(i+1) ))
i71 : Rfunction = sum apply(4, i -> q_(i+1)*(y_(i+1) + e_(i+1))^2)
o71 = ((-a_1y_2^2n^2-a_2y_4^2n^2+a_1y_1^2+a_2y_3^2)s^2+(2a_1y_2^2n^2+2a_2y_4^2n^2+2a_1y_1^2n+2a_2y_3^2n)s+a_1y_1^2n^2-a_1y_2^2n^2+a_2y_3^2n^2-a_2y_4^2n^2)/(s^4+(2n-2)s^3+(n^2-4n+1)s^2+(-2n^2+2n)s+n^2)
o71 : frac R
i72 : r2 = numerator Rfunction
o72 = (-a_1y_2^2n^2 - a_2y_4^2n^2 + a_1y_1^2 + a_2y_3^2)s^2 + (2a_1y_2^2n^2 + 2a_2y_4^2n^2 + 2a_1y_1^2n + 2a_2y_3^2n)s + a_1y_1^2n^2 - a_1y_2^2n^2 + a_2y_3^2n^2 - a_2y_4^2n^2
o72 : R
i73 : coeffs = coefficients r2
                {2, 0}        -a_1y_2^2n^2 - a_2y_4^2n^2 + a_1y_1^2 + a_2y_3^2
o73 = ((s^2  s  1), {1, 0}  (2a_1y_2^2n^2 + 2a_2y_4^2n^2 + 2a_1y_1^2n + 2a_2y_3^2n))
                {0, 0}       a_1y_1^2n^2 - a_1y_2^2n^2 + a_2y_3^2n^2 - a_2y_4^2n^2
o73 : Sequence
i74 :
     S = QQ[a_1..a_2,y_1..y_4,n][sqdelta] -- We introduce the squareroot of delta as a variable.
o74 = S
o74 : PolynomialRing
i75 :
     coeffs = sub( coeffs#1, S )
       {2, 0}        -a_1y_2^2n^2 - a_2y_4^2n^2 + a_1y_1^2 + a_2y_3^2
o75 =  {1, 0}  (2a_1y_2^2n^2 + 2a_2y_4^2n^2 + 2a_1y_1^2n + 2a_2y_3^2n)
       {0, 0}       a_1y_1^2n^2 - a_1y_2^2n^2 + a_2y_3^2n^2 - a_2y_4^2n^2
o75 : Matrix S^3 <-- S^1
i76 : A = coeffs_(0,0)
o76 = -a_1y_2^2n^2 - a_2y_4^2n^2 + a_1y_1^2 + a_2y_3^2
o76 : S
i77 : B = coeffs_(1,0)
o77 = 2a_1y_2^2n^2 + 2a_2y_4^2n^2 + 2a_1y_1^2n + 2a_2y_3^2n
o77 : S
i78 : C = coeffs_(2,0)
o78 = a_1y_1^2n^2 - a_1y_2^2n^2 + a_2y_3^2n^2 - a_2y_4^2n^2
o78 : S
i79 : Delta = B^2 - 4*A*C
o79 = 4a_1^2y_1^2y_2^2n^4+4a_1a_2y_2^2y_3^2n^4+4a_1a_2y_1^2y_4^2n^4+4a_2^2y_3^2y_4^2n^4+8a_1^2y_1^2y_2^2n^3+8a_1a_2y_2^2y_3^2n^3+8a_1a_2y_1^2y_4^2n^3+8a_2^2y_3^2y_4^2n^3+4a_1^2y_1^2y_2^2n^2+4a_1a_2y_2^2y_3^2n^2+4a_1a_2y_1^2y_4^2n^2+4a_2^2y_3^2y_4^2n^2
o79 : S
i80 : delta = numerator ( ( Delta ) / ( 4*n^2*(n+1)^2 ) ) -- We factor out 4*n^2*(n+1)^2 from Delta to get delta.
o80 = a_1^2y_1^2y_2^2 + a_1a_2y_2^2y_3^2 + a_1a_2y_1^2y_4^2 + a_2^2y_3^2y_4^2
o80 : S
i81 :
     splus = ( - B + 2*n*( n + 1 )*sqdelta ) / ( 2*A )
o81 = ((-n^2-n)sqdelta+a_1y_2^2n^2+a_2y_4^2n^2+a_1y_1^2n+a_2y_3^2n)/(a_1y_2^2n^2+a_2y_4^2n^2-a_1y_1^2-a_2y_3^2)
o81 : frac S
i82 : sminus = ( - B - 2*n*( n + 1 )*sqdelta ) / ( 2*A )
o82 = ((n^2+n)sqdelta+a_1y_2^2n^2+a_2y_4^2n^2+a_1y_1^2n+a_2y_3^2n)/(a_1y_2^2n^2+a_2y_4^2n^2-a_1y_1^2-a_2y_3^2)
o82 : frac S
i83 : scan(4, i -> l_(i+1) = sub( l_(i+1), S ))
i84 : scan(4, i -> q_(i+1) = sub( q_(i+1), S ))
i85 : scan(4, i -> eplus_(i+1) =  splus*q_(i+1)*y_(i+1) / ( l_(i+1) - splus*q_(i+1) ) )
i86 : scan(4, i -> eminus_(i+1) =  sminus*q_(i+1)*y_(i+1) / ( l_(i+1) - sminus*q_(i+1) ) )
i87 :
     subdeltasquare = (polynom) -> ( -- This function replaces sqdelta^2 with delta
     deg = degree( sqdelta, polynom );
     coeffspoly = for i from 0 to deg list coefficient( sqdelta^i, polynom );
     subpoly = 0;

     for i from 0 to deg do (
     if mod( i, 2 ) == 0 then (
     subpoly = subpoly + coeffspoly_(i) * delta^( numerator( i / 2 ) );
     ) else (
     subpoly = subpoly + coeffspoly_(i) * delta^( numerator( ( i - 1 ) / 2 ) ) * sqdelta;
     );
     );
     subpoly
     )
o87 = subdeltasquare
```

```
o87 : FunctionClosure
i88 :
      conjugaterule = (rational) -> ( -- This function uses the conjugate rule to get rid of sqdelta in the denominator
      ratnum = numerator rational;
      ratden = denominator rational;

      ratnum = ratnum * (coefficient( sqdelta, ratden )*sqdelta - coefficient( sqdelta^0, ratden));
      ratden = ratden * (coefficient( sqdelta, ratden )*sqdelta - coefficient( sqdelta^0, ratden));
      subdeltasquare(ratnum) / subdeltasquare(ratden)
      )
o88 = conjugaterule
o88 : FunctionClosure
i89 :
      weightedsumplus = sum apply(4, i -> l_(i+1)*( eplus_(i+1) )^2 ) -- Weighted sum with the + critical point
o89 = [very long expression]
o89 : frac S
i90 : weightedsumminus = sum apply(4, i -> l_(i+1)*( eminus_(i+1) )^2 ) -- Weighted sum with the - critical point
o90 = [very long expression]
o90 : frac S
i91 : sumplus = sum apply(4, i -> ( eplus_(i+1) )^2 ) -- Non-weighted sum with the + critical point
o91 = [very long expression]
o91 : frac S
i92 : summinus = sum apply(4, i -> ( eminus_(i+1) )^2 ) -- Non-weighted sum with the - critical point
o92 = [very long expression]
o92 : frac S
i93 :
      -- We replace squares of sqdelta with delta and use the conjugate rule to remove sqdelta from the denominator:
      weightedsumplus = conjugaterule ( subdeltasquare(numerator weightedsumplus) / subdeltasquare(denominator weightedsumplus) )
```

$$\text{o93} = \frac{-2\,n\,sqdelta+a_1y_1^2n+a_1y_2^2n+a_2y_3^2n+a_2y_4^2n}{n+1}$$

```
o93 : frac S
i94 : weightedsumminus = conjugaterule ( subdeltasquare(numerator weightedsumminus) / subdeltasquare(denominator weightedsumminus) )
```

$$\text{o94} = \frac{2\,n\,sqdelta+a_1y_1^2n+a_1y_2^2n+a_2y_3^2n+a_2y_4^2n}{n+1}$$

```
o94 : frac S
i95 : sumplus = conjugaterule ( subdeltasquare(numerator sumplus) / subdeltasquare(denominator sumplus) )
o95 = [very long expression]
o95 : frac S
i96 : summinus = conjugaterule ( subdeltasquare(numerator summinus) / subdeltasquare(denominator summinus) )
o96 = [very long expression]
o96 : frac S
i97 :
      -- We get the identities from the statement by observing weighhtedsumplus and weightedsumminus directly.
      -- To examine the non-weighted sums, we first look at their denominators:
      factor denominator sumplus
```

$$\text{o97} = (n+1)^2(a_1y_2^2+a_2y_4^2)(a_1y_1^2+a_2y_3^2)$$

```
o97 : Expression of class Product
i98 : factor denominator summinus
```

$$\text{o98} = (n+1)^2(a_1y_2^2+a_2y_4^2)(a_1y_1^2+a_2y_3^2)$$

```
o98 : Expression of class Product
i99 :
      -- And next their numerators:
      numsumplus = factor numerator sumplus
```

$$\text{o99} = (a_1y_1^2y_2^2n^2+a_1y_2^2y_3^2n^2+a_2y_1^2y_4^2n^2+a_2y_3^2y_4^2n^2+a_1y_1^2y_2^2+a_2y_2^2y_3^2+a_1y_1^2y_4^2+a_2y_3^2y_4^2)$$
$$\cdot\,(2\,sqdelta-a_1y_1^2-a_1y_2^2-a_2y_3^2-a_2y_4^2)(-1)$$

```
o99 : Expression of class Product
i100 : numsumminus = factor numerator summinus
```

$$\text{o100} = (a_1y_1^2y_2^2n^2+a_1y_2^2y_3^2n^2+a_2y_1^2y_4^2n^2+a_2y_3^2y_4^2n^2+a_1y_1^2y_2^2+a_2y_2^2y_3^2+a_1y_1^2y_4^2+a_2y_3^2y_4^2)$$
$$\cdot\,(2\,sqdelta+a_1y_1^2+a_1y_2^2+a_2y_3^2+a_2y_4^2)$$

```
o100 : Expression of class Product
i101 :
       -- A factor of alpha^+, respectively alpha^-, comes out of the sum.
       -- Factoring these out, the numerators are the same: S*n^2 + T.
       -- We next factor the coefficient S and T
       T = QQ[a_1..a_2,y_1..y_4][n]
```

$$\text{o101} = T$$

```
o101 : PolynomialRing
i102 :
       numsumplus = sub(numsumplus#0#0, T)
```

$$\text{o102} = (a_1y_1^2y_2^2+a_1y_2^2y_3^2+a_2y_1^2y_4^2+a_2y_3^2y_4^2)n^2+a_1y_1^2y_2^2+a_2y_2^2y_3^2+a_1y_1^2y_4^2+a_2y_3^2y_4^2$$

```
o102 : T
i103 : S = coefficient(n^2, numsumplus)
```

$$\text{o103} = a_1y_1^2y_2^2+a_1y_2^2y_3^2+a_2y_1^2y_4^2+a_2y_3^2y_4^2$$

```
o103 : QQ[a_1 .. a_2, y_1 .. y_4]
i104 : T = coefficient(n^0, numsumplus)
```

$$\text{o104} = a_1y_1^2y_2^2+a_2y_2^2y_3^2+a_1y_1^2y_4^2+a_2y_3^2y_4^2$$

```
o104 : QQ[a_1 .. a_2, y_1 .. y_4]
i105 :
       -- Here we see that S and T are the same and in the paper:
       factor S
```

$$\text{o105} = (y_1^2+y_3^2)(a_1y_2^2+a_2y_4^2)$$

```
o105 : Expression of class Product
i106 : factor T
```

$$\text{o106} = (y_2^2+y_4^2)(a_1y_1^2+a_2y_3^2)$$

```
o106 : Expression of class Product
```