A Control Theoretic Multi-Marginal Schrödinger Bridge Formulation

In this section, we examine the structure of the solution and the reduction of the Schrödinger Bridge problem in the multi-marginal setting. Since the case of two marginals has been thoroughly studied in prior works [12, 26], we aim to directly extend these results to the multi-marginal scenario.

In standard SBP, the optimality condition can be characterized in terms of Doob's h-transform [12, 21], with two potential functions $(\overrightarrow{\Phi}, \overleftarrow{\Phi})$ that satisfy the forward and backward time harmonic equations.

Theorem A.1 (Dynamics SB optimality [45]). *Let $(\overrightarrow{\Phi}, \overleftarrow{\Phi})$ be the solutions to the following PDEs:*

$$\partial_t \overrightarrow{\Psi} + \nabla \overrightarrow{\Phi}^\top f + \tfrac{1}{2}\sigma^2 \Delta \overrightarrow{\Psi} = 0, \quad \partial_t \overleftarrow{\Phi} + \nabla \cdot (\overleftarrow{\Phi} f) - \tfrac{1}{2}\sigma^2 \Delta \overleftarrow{\Phi} = 0, \tag{A.1}$$

$$\rho_0(\mathbf{x}) = \overrightarrow{\Phi}(0, \mathbf{x}) \overleftarrow{\Phi}(0, \mathbf{x}), \rho_T(\mathbf{x}) = \overrightarrow{\Psi}(T, \mathbf{x}) \overleftarrow{\Phi}(T, \mathbf{x}). \tag{A.2}$$

Then, the solution $\mathbb{P}^{SBP}$ of SBP is induced by following forward-backward SDEs:

$$d\overrightarrow{\mathbf{X}}_t = \left[f_t + \sigma^2 \nabla_{\mathbf{x}} \overrightarrow{\Phi}(t, \overrightarrow{\mathbf{X}}_t) \right] dt + \sigma d\mathbf{W}_t, \quad \overrightarrow{\mathbf{X}}_0 \sim \rho_0, \tag{A.3a}$$

$$d\overleftarrow{\mathbf{X}}_t = \left[-f_{T-t} + \sigma^2 \nabla_{\mathbf{x}} \overleftarrow{\Phi}(T - t, \overleftarrow{\mathbf{X}}_t) \right] dt + \sigma d\mathbf{W}_t, \quad \overleftarrow{\mathbf{X}}_0 \sim \rho_T. \tag{A.3b}$$

Note that the forward-backward stochastic process in (A.3a) and (A.3b) satisfying $\rho_t^{(A.3a)} = \rho_t^{(A.3a)} = \rho_t$ for all $t \in [0, T]$ and ρ_t obeys a factorization $\rho_t(\mathbf{x}) = \overrightarrow{\Phi}(t, \mathbf{x}) \overleftarrow{\Phi}(t, \mathbf{x})$. To solve the SB, one needs to solve the associated PDEs to estimate $(\overrightarrow{\Phi}, \overleftarrow{\Phi})$. However, due to the high-dimensional nature of the problem, directly solving the PDEs is challenging [19]. Previous works [5, 10, 30] has addressed this issue by formulating an IPF type algorithm [25], where half-bridge optimization is iteratively repeated for the two boundary conditions:

$$\mathbb{P}^{(n+1)} := \underset{\mathbb{P} \in \mathcal{P}_{[0,T]}, \mathbb{P}_T = \rho_T}{\arg\min} \mathrm{D_{KL}}(\mathbb{P} | \mathbb{P}^{(n)}), \ \mathbb{P}^{(n+2)} := \underset{\mathbb{P} \in \mathcal{P}_{[0,T]}, \mathbb{P}_0 = \rho_0}{\arg\min} \mathrm{D_{KL}}(\mathbb{P} | \mathbb{P}^{(n+1)}). \tag{A.4}$$

with initialization $\mathbb{P}^{(0)} := \mathbb{Q}$. Please refer to [5] for more details.

Now, we extend the SB optimality in A.1 to multi-marginal settings by defining appropriate potentials. Let $\mathbb{Q} \in \mathcal{P}_{[0,T]}$ be a path measure induced by following Itô SDEs $d\mathbf{X}_t = \sigma d\mathbf{W}_t$. We assume that a collection of $k+1$ marginal distributions $\rho_{\mathcal{T}} := (\rho_0, \rho_{t_1}, \cdots, \rho_{t_k})$ is specified at timestamps $\mathcal{T} = \{t_0, t_1, \cdots t_k\}$ where $= 0 = t_0 < t_1 \cdots < t_k = T$. We want to explore the most likely evolution between multiple marginals $\rho_{\mathcal{T}}$ which is the solution of multi-marginal SBP:

$$\min_{\mathbb{P} \in \mathcal{P}(\Omega)} \mathrm{D_{KL}}(\mathbb{P} | \mathbb{Q}), \quad \text{subject to} \quad \mathbb{P}_t \sim \rho_t, \quad \forall t \in \mathcal{T}. \tag{mSBP}$$

We first consider the `static` problem of mSBP. Consider the conditioning of the process on $\mathbf{X}_0 = \mathbf{x}_0, \mathbf{X}_{t_1} = \mathbf{x}_{t_1}, \cdots, \mathbf{X}_T = \mathbf{x}_{t_k}$ such that

$$\mathbb{P}_{|\mathcal{T}} = \mathbb{P}\left[\cdot | \mathbf{X}_0 = \mathbf{x}_0, \mathbf{X}_{t_1} = \mathbf{x}_{t_1}, \cdots, \mathbf{X}_T = \mathbf{x}_T \right] \tag{A.5}$$

$$\mathbb{Q}_{|\mathcal{T}} = \mathbb{W}\left[\cdot | \mathbf{X}_0 = \mathbf{x}_0, \mathbf{X}_{t_1} = \mathbf{x}_{t_1}, \cdots, \mathbf{X}_T = \mathbf{x}_T \right]. \tag{A.6}$$

These conditioned laws can be interpreted as the disintegration of path measure $\mathbb{P}, \mathbb{Q} \in \mathcal{P}_{[0,T]}$ with respect to multiple time points. Given the multi-marginal joint distributions $\mathcal{T}$, $\mathbb{P}_{\mathcal{T}}, \mathbb{Q}_{\mathcal{T}} \in \mathbb{P}_{\mathcal{T}}$ associated with $\mathbb{P}$ and $\mathbb{Q}$ at timestamps $\mathcal{T}$, respectively, the relative entropy admits the following decomposition:

$$\mathrm{D_{KL}}(\mathbb{P} | \mathbb{Q}) = \underbrace{\mathrm{D_{KL}}(\mathbb{P}_{\mathcal{T}} | \mathbb{Q}_{\mathcal{T}})}_{\texttt{static}} + \underbrace{\int_{\mathbb{R}^{d \times |\mathcal{T}|}} \mathrm{D_{KL}}(\mathbb{P}_{|\mathcal{T}} | \mathbb{Q}_{|\mathcal{T}}) d\mathbb{P}_{\mathcal{T}}}_{\texttt{dynamic}}. \tag{A.7}$$

Since $\mathbb{P}_{\mathcal{T}}$ and $\mathbb{P}_{|\mathcal{T}}$ can be chosen arbitrarily, we may set $\mathbb{P}_{|\mathcal{T}} = \mathbb{Q}_{|\mathcal{T}}$ so that the `dynamic` component of the decomposition in (A.7) vanishes. Under this choice, mSBP reduces to the `static` problem:

$$\min_{\mathbb{P}_{\mathcal{T}} \in \Pi_{\mathcal{T}}} \mathrm{D_{KL}}(\mathbb{P}_{\mathcal{T}} | \mathbb{Q}_{\mathcal{T}}), \tag{smSBP}$$

$$\text{where} \quad \Pi_{\mathcal{T}} = \{ \mathbb{P}_{\mathcal{T}} \in \mathbb{R}^{d \times |\mathcal{T}|} \Big| \int_{d \times (|\mathcal{T}| - 1)} \mathbb{P}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}^{(-i)}}) d\mathbf{x}_{\mathcal{T}^{(-i)}} = \rho_{t_i}, \forall i \in [0:k] \}, \tag{A.8}$$

where $\mathcal{T}^{(-i)} = \{0, \cdots, t_{i-1}, t_{i+1}, \cdots, T\}$ denotes the set of indices excluding t_i. In other words, this formulation seeks a multi-marginal coupling $\mathbb{P}_{\mathcal{T}}$ that matches the prescribed marginal $\rho_{\mathcal{T}}$, while minimizing its relative entropy with respect to the reference joint law $\mathbb{Q}_{\mathcal{T}}$. Therefore, once the optimal coupling $\Pi^\star_{\mathcal{T}} = \mathbb{P}^\star_{\mathcal{T}}$ solving smSBP is obtained, it directly induces the solution to mSBP via

$$\mathbb{P}^\star(\cdot) = \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \mathbb{Q}_{|\mathcal{T}}(\cdot) d\mathbb{P}^\star_{\mathcal{T}}, \tag{A.9}$$

meaning that $\mathbb{P}^\star$ constructed in this way satisfies the original mSBP.

Now, we want to derive the multi-marginal Schrödinger potential for mSBP. To do so, let us consider the Lagrangian formulation $\mathcal{L}$ of mSBP which is given by:

$$\mathcal{L}(\mathbb{P}_{\mathcal{T}}, \lambda_{\mathcal{T}}) = \mathrm{D}_{\mathrm{KL}}(\mathbb{P}_{\mathcal{T}}|\mathbb{Q}_{\mathcal{T}}) + \sum_{i=0}^{|\mathcal{T}|} \int_{\mathbb{R}^d} \lambda_{t_i}(\mathbf{x}_{t_i}) \left[\int_{\mathbb{R}^{d\times(\mathcal{T}-1)}} \mathbb{P}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) d\mathbf{x}_{\mathcal{T}^{(-i)}} - \rho_{t_i}(\mathbf{x}_{t_i}) \right] d\mathbf{x}_{t_i}, \tag{A.10}$$

where $\lambda_{\mathcal{T}} = (\lambda_0, \lambda_{t_1}, \cdots, \lambda_T)$ is the Lagrange multipliers where each λ_{t_i} enforcing the marginal constraints at $t_i \in \mathcal{T}$. By setting the first variation of $\mathcal{L}(\mathbb{P}_{\mathcal{T}}, \lambda_{\mathcal{T}})$ to zero, we obtain the optimality condition for the minimizer $\mathbb{P}^\star_{\mathcal{T}}$. Let $\delta\mathbb{P}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})$ be a perturbation around $\mathbb{P}_{\mathcal{T}}$. Consider the measure

$$\mathbb{P}^\epsilon_{\mathcal{T}} = \mathbb{P}_{\mathcal{T}} + \epsilon\delta\mathbb{P}_{\mathcal{T}}, \tag{A.11}$$

and by plugging $\mathbb{P}^\epsilon_{\mathcal{T}}$ in (A.11) into Lagrangian (A.10) and compute the first variation of each term. Then, for the first term of RHS in (A.10) we get:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{D}_{\mathrm{KL}}(\mathbb{P}^\epsilon_{\mathcal{T}}|\mathbb{Q}_{\mathcal{T}}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int \log \frac{d\left(\mathbb{P}_{\mathcal{T}} + \epsilon\delta\mathbb{P}_{\mathcal{T}}\right)}{d\mathbb{Q}_{\mathcal{T}}} d\left(\mathbb{P}_{\mathcal{T}} + \epsilon\delta\mathbb{P}_{\mathcal{T}}\right) \tag{A.12}$$

$$= \int d\delta\mathbb{P}_{\mathcal{T}} \left(\log \frac{d\mathbb{P}_{\mathcal{T}}}{d\mathbb{Q}_{\mathcal{T}}} + 1 \right). \tag{A.13}$$

Moreover, for the second term of RHS in (A.10) we get:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \sum_{i=0}^{|\mathcal{T}|} \int_{\mathbb{R}^d} \lambda_{t_i}(\mathbf{x}_{t_i}) \left[\int_{\mathbb{R}^{d\times(\mathcal{T}-1)}} \left(\mathbb{P}_{\mathcal{T}} + \epsilon\delta\mathbb{P}_{\mathcal{T}}\right)(\mathbf{x}_{\mathcal{T}}) d\mathbf{x}_{\mathcal{T}^{(-i)}} - \rho_{t_i}(\mathbf{x}_{t_i}) \right] d\mathbf{x}_{t_i} \tag{A.14}$$

$$= \sum_{i=0}^{|\mathcal{T}|} \int_{\mathbb{R}^d} \lambda_{t_i}(\mathbf{x}_{t_i}) \left[\int_{\mathbb{R}^{d\times(\mathcal{T}-1)}} \delta\mathbb{P}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) d\mathbf{x}_{\mathcal{T}^{(-i)}} \right] d\mathbf{x}_{t_i} \tag{A.15}$$

$$= \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \left[\sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i}) \right] d\delta\mathbb{P}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}). \tag{A.16}$$

Hence, we get the total first variation of $\mathcal{L}(\mathbb{P}^\epsilon_{\mathcal{T}}, \lambda_{\mathcal{T}})$ as:

$$\delta\mathcal{L} = \int d\delta\mathbb{P}_{\mathcal{T}} \left(\log \frac{d\mathbb{P}_{\mathcal{T}}}{d\mathbb{Q}_{\mathcal{T}}} + 1 + \left[\sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i}) \right] \right). \tag{A.17}$$

For the optimality, $\delta\mathcal{L}$ to be vanished for all admissible $\delta\mathbb{P}_{\mathcal{T}}$, hence we get:

$$\log \frac{d\mathbb{P}_{\mathcal{T}}}{d\mathbb{Q}_{\mathcal{T}}} + 1 + \left[\sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i}) \right] = 0. \tag{A.18}$$

Therefore, we obtain the optimality condition:

$$\mathbb{P}^\star_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) = \mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) \exp\left(-1 - \left[\sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i}) \right] \right) \tag{A.19}$$

$$\stackrel{(i)}{=} \prod_{i=1}^{|\mathcal{T}|} \mathbb{Q}_{|t_{i-1},t_i}(\mathbf{x}_{t_{i-1}}, \mathbf{x}_{t_i}) \exp\left(\log\rho_0(\mathbf{x}_0) - 1 - \left[\sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i}) \right] \right) \tag{A.20}$$

$$\stackrel{(ii)}{=} \Psi_0(\mathbf{x}_0) \prod_{i=1}^{|\mathcal{T}|} \left[\mathbb{Q}_{|t_{i-1},t_i}(\mathbf{x}_{t_{i-1}}, \mathbf{x}_{t_i}) \Psi_{t_i}(\mathbf{x}_{t_i}) \right] \tag{A.21}$$

where (i) follows from $\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) = \rho_0(\mathbf{x}_0)\prod_{i=1}^{k}\mathbb{Q}_{|t_{i-1},t_i}(\mathbf{x}_{t_{i-1}},\mathbf{x}_{t_i})$ and (ii) follows from defining

$$\exp\left(\log\rho_0(\mathbf{x}_0) - 1 - \left[\sum_{i=0}^{|\mathcal{T}|}\lambda_{t_i}(\mathbf{x}_{t_i})\right]\right) = \prod_{i=0}^{|\mathcal{T}|}\Psi_{t_i}(\mathbf{x}_{t_i}), \tag{A.22a}$$

$$\text{where } \Psi_0 = \exp\left(\log\rho_0(\mathbf{x}_0) - 1 - \lambda_0\right), \text{ and } \Psi_{t_i} = \exp(-\lambda_{t_i}),\ \forall i > 0. \tag{A.22b}$$

Observe that since $\mathbb{P}^\star_{\mathcal{T}} = \Pi^\star_{\mathcal{T}}$ is the optimal coupling, it follows from equation (A.9) that the path measure factorizes as $d\mathbb{P}^\star = d\mathbb{P}^\star_{\mathcal{T}}\cdot\mathbb{Q}^\star_{|\mathcal{T}}$. Combining this with the result from equation (A.19), we obtain the Radon–Nikodym derivative:

$$\frac{d\mathbb{P}^\star}{d\mathbb{Q}} = \frac{d\mathbb{P}^\star_{\mathcal{T}}}{d\mathbb{Q}_{\mathcal{T}}}\cdot\frac{d\mathbb{P}^\star_{|\mathcal{T}}}{d\mathbb{Q}_{|\mathcal{T}}} = \frac{d\mathbb{P}^\star_{\mathcal{T}}}{d\mathbb{Q}_{\mathcal{T}}} = \prod_{i=0}^{|\mathcal{T}|}\Psi_{t_i}(\mathbf{x}_{t_i}). \tag{A.23}$$

Moreover, due to the structure of the construction and by applying results such as those in [2, Theorem 2.10], the resulting measure $\mathbb{P}^\star$ is a Markov process.

Hence, from (A.21), we deduce that the optimal coupling $\mathbb{P}^\star_{\mathcal{T}}$ factorized into a functions of $\mathbf{x}_{t_i}$. Moreover, since the conditional measure $\mathbb{Q}_{|t_{i-1},t_i}$ is reciprocal process, it satisfies time symmetry [34] *i.e.*, $\overrightarrow{\mathbb{Q}}_{|t_{i-1},t_i} = \overleftarrow{\mathbb{Q}}_{|t_i,t_{i-1}}$. Consequently, for any increasing time sequence $\overrightarrow{\mathcal{T}} = \{t_0,\cdots,t_k\}$ and its reversed counterpart $\overleftarrow{\mathcal{T}} = \{t_k,\cdots,t_0\}$, for a given $\mathbf{x}_{\mathcal{T}}\in\mathbb{R}^{d\times|\mathcal{T}|}$, it holds

$$\overrightarrow{\mathbb{Q}}_{|\overrightarrow{\mathcal{T}}} = \prod_{i=1}^{k}\overrightarrow{\mathbb{Q}}_{|t_{i-1},t_i} = \prod_{i=k}^{1}\overleftarrow{\mathbb{Q}}_{|t_i,t_{i-1}} = \overleftarrow{\mathbb{Q}}_{|\overleftarrow{\mathcal{T}}}. \tag{A.24}$$

We now define the multi-marginal Schrödinger potentials in both the forward and backward directions.

Theorem A.2 (Multi-Marginal Schrödinger Potentials)**.** *For a finite set of timestamps $\mathcal{T}$ and with corresponding potentials $\{\Psi_i\}_{i=0}^{|\mathcal{T}|}$ defined in (A.22), let us define a pair of potentials $(\overrightarrow{\Psi},\overleftarrow{\Psi})$:*

$$\overrightarrow{\Psi}(t,\mathbf{x}) = \mathbb{E}_{\mathbb{Q}}\left[\prod_{j\geq\tau(t)}^{|\mathcal{T}|}\Psi_j(\mathbf{X}_{t_j})|\mathbf{X}_t=\mathbf{x}\right],\quad \overleftarrow{\Psi}(t,\mathbf{x}) = \mathbb{E}_{\mathbb{Q}}\left[\prod_{j<\tau(t)}^{0}\Psi_j(\mathbf{X}_{t_j})|\mathbf{X}_t=\mathbf{x}\right], \tag{A.25}$$

where $\tau(t) = \min_u\{u\geq t|t\in\mathcal{T}\}$. Then, for any $t\in[t_{j-1},t_j]$,$(\overrightarrow{\Psi},\overleftarrow{\Psi})$ satisfy following PDEs:

$$\partial_t\overrightarrow{\Psi} + \tfrac{1}{2}\sigma^2\Delta\overrightarrow{\Psi} = 0,\quad \partial_t\overleftarrow{\Psi} - \tfrac{1}{2}\sigma^2\Delta\overleftarrow{\Psi} = 0, \tag{A.26}$$

$$\rho_{t_j}(\mathbf{x}) = \overrightarrow{\Psi}(t_j,\mathbf{x})\overleftarrow{\Psi}(t_j,\mathbf{x}) = \lim_{t\uparrow t_j}\overrightarrow{\Psi}(t,\mathbf{x})\overleftarrow{\Psi}(t,\mathbf{x}), \tag{A.27}$$

$$\rho_{t_{j-1}}(\mathbf{x}) = \overrightarrow{\Psi}(t_{j-1},\mathbf{x})\overleftarrow{\Psi}(t_{j-1},\mathbf{x}) = \lim_{t\downarrow t_{j-1}}\overrightarrow{\Psi}(t,\mathbf{x})\overleftarrow{\Psi}(t,\mathbf{x}). \tag{A.28}$$

Moreover, the solution $\mathbb{P}^{mSBP}$ of mSBP is induced by following forward-backward SDEs:

$$d\overrightarrow{\mathbf{X}}_t = \sigma^2\nabla_{\mathbf{x}}\overrightarrow{\Psi}(t,\overrightarrow{\mathbf{X}}_t)dt + \sigma d\mathbf{W}_t,\quad \overrightarrow{\mathbf{X}}_0\sim\rho_0, \tag{A.29a}$$

$$d\overleftarrow{\mathbf{X}}_t = \sigma^2\nabla_{\mathbf{x}}\overleftarrow{\Psi}(t,\overleftarrow{\mathbf{X}}_t)dt + \sigma d\mathbf{W}_t,\quad \overleftarrow{\mathbf{X}}_0\sim\rho_T. \tag{A.29b}$$

Proof. We begin by proving the boundary conditions in (A.27–A.28). Without loss of generality, we consider the case where $t\in[t_{j-1},t_j)$. By invoking the factorization in (A.23), we obtain:

$$\rho(t,\mathbf{x}) = \mathbb{E}_{\mathbb{P}^\star}\left[\delta(\mathbf{X}_t=\mathbf{x})\right] = \int_\Omega\delta(\mathbf{X}_t=\mathbf{x})d\mathbb{P}^\star \tag{A.30}$$

$$= \int_\Omega\delta(\mathbf{X}_t=\mathbf{x})e^{-1-\sum_{i=0}^{|\mathcal{T}|}\lambda_{t_i}(\mathbf{x}_{t_i})}d\mathbb{Q} \tag{A.31}$$

$$= \int_{\mathbb{R}^{d\times|\mathcal{T}|}}e^{-1-\sum_{i=0}^{|\mathcal{T}|}\lambda_{t_i}(\mathbf{x}_{t_i})}d\mathbb{Q}_{t,\mathcal{T}} \tag{A.32}$$

To evaluate the marginal at time t, we consider an integral over the path space Ω, slicing the trajectory into segments for t. Due to the Markovian $\mathbb{Q}$, as consequence of [34, Proposition 1.4] and time symmetry of bridge kernel (A.23), we can express the joint distribution $\mathbb{Q}_{t,\mathcal{T}}(\mathbf{x}_t, \mathbf{x}_{\mathcal{T}})$:

$$\mathbb{Q}_{t,\mathcal{T}} = \rho_0(\mathbf{x}_0) \prod_{i=1}^{j-1} \overrightarrow{\mathbb{Q}}_{|t_{i-1},t_i} \prod_{i=j}^{|\mathcal{T}|} \overleftarrow{\mathbb{Q}}_{|t_{i-1},t_i} \left[\overrightarrow{\mathbb{Q}}_{|t_{j-1},t} \overleftarrow{\mathbb{Q}}_{|t,t_j} \right]. \tag{A.33}$$

Hence, substituting $\mathbb{Q}_{t,\mathcal{T}}$ in (A.33) into (A.32), we get[2]:

$$\rho(t,\mathbf{x}) = \int_{\mathbb{R}^{d\times|\mathcal{T}|}} e^{\log \rho_0(\mathbf{x}_0) - 1 - \sum_{i=0}^{|\mathcal{T}|} \lambda_{t_i}(\mathbf{x}_{t_i})} \prod_{i=1}^{j-1} \mathbb{Q}_{|t_{i-1},t_i} \prod_{i=j}^{|\mathcal{T}|} \mathbb{Q}_{|t_{i-1},t_i} \left[\mathbb{Q}_{|t_{j-1},t} \mathbb{Q}_{|t,t_j} \right] d\mathbf{x}_{\mathcal{T}} \tag{A.34}$$

$$= \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \prod_{i=0}^{|\mathcal{T}|} \Psi_{t_i}(\mathbf{x}_{t_i}) \prod_{i=1}^{j-1} \mathbb{Q}_{|t_{i-1},t_i} \prod_{i=j}^{|\mathcal{T}|} \mathbb{Q}_{|t_{i-1},t_i} \left[\mathbb{Q}_{|t_{j-1},t} \mathbb{Q}_{|t,t_j} \right] d\mathbf{x}_{\mathcal{T}} \tag{A.35}$$

$$= \overrightarrow{\Psi}(t,\mathbf{x}) \overleftarrow{\Psi}(t,\mathbf{x}). \tag{A.36}$$

Moreover, limiting procedures in (A.27-A.28) follows directly from the definition of (A.25): as time approaches the boundary t_i, the potential component Ψ_i, previously contained within the conditional expectation $\overrightarrow{\Psi}$ (or $\overleftarrow{\Psi}$), transitions smoothly into the complementary potential $\overleftarrow{\Psi}$ (or $\overrightarrow{\Psi}$) associated with the opposite time direction. This ensures the resulting density $\rho_t(\cdot) \in C^{1,2}([0,T], \mathbb{R}^d)$.

Now, as a consequence of [38, Theorem 4.19] for a unique minimizer of `mSBP` (if it exists), we get

$$\mathrm{D}_{\mathrm{KL}}(\mathbb{P}^\star|\mathbb{Q}) = \frac{1}{2}\mathbb{E}_{\mathbb{P}^\star}\left[\int_0^T \left\| \sigma \nabla \log \overrightarrow{\Psi}(t, \overrightarrow{\mathbf{X}}_t) \right\|^2 dt \right]. \tag{A.37}$$

In other words, by applying the Girsanov theorem [43, Theorem 8.6.5], it implies that the forward SDEs in (A.29a) induce the solution $\mathbb{P}^\star$ to `mSBP`. Consequently, the associated marginal density ρ satisfies the Fokker-Planck equation:

$$\partial_t \rho - \nabla \cdot \left(\rho(\sigma^2 \nabla \overrightarrow{\Psi}) \right) - \frac{1}{2}\sigma^2 \Delta \rho = 0, \quad \rho_t(\mathbf{x}) = \overrightarrow{\Psi}_t(\mathbf{x}) \overleftarrow{\Psi}_t(\mathbf{x}), \quad \forall t \in \mathcal{T}. \tag{A.38}$$

Furthermore, by applying the Feynman-Kac formula [1, Theorem 10.5], we obtain the PDE satisfied by $\overrightarrow{\Psi}$ as characterized in [44, Appendix B.4]:

$$\partial_t \overrightarrow{\Psi} + \tfrac{1}{2}\sigma^2 \Delta \overrightarrow{\Psi} = 0. \tag{A.39}$$

Combining equations (A.38–A.39) and using the factorization $\rho_t(\mathbf{x}) = \overrightarrow{\Psi}_t(\mathbf{x}) \overleftarrow{\Psi}_t(\mathbf{x})$, we deduce, following the derivation [30, Appendix A.4.1], that $\overleftarrow{\Psi}$ satisfies the backward PDE:

$$\partial_t \overleftarrow{\Psi} - \tfrac{1}{2}\sigma^2 \Delta \overleftarrow{\Psi} = 0. \tag{A.40}$$

Invoking Nelson's relation [42], we recover the backward dynamics described in (A.29b), which likewise induce the optimal path measure $\mathbb{P}^\star$. This concludes the proof. □

Remark A.3. *The IPF-type iteration* (A.4) *is expected to remain a valuable tool for approximating the multi-marginal potentials presented in Theorem* A.2*. Nevertheless, the challenge of enforcing boundary conditions for intermediate states, as specified in equations* (A.27) *and* (A.28)*, might necessitate more advanced optimization strategies comparable to those developed in MSBM.*

Finally, combining (A.37-A.40), we obtain the stochastic optimal control formulation of `mSBP`:

$$\min_{\alpha \in \mathbb{A}} \mathcal{J}(\alpha) = \mathbb{E}_{\mathbb{Q}^\alpha}\left[\int_0^T \frac{1}{2} \|\alpha_t\|^2 dt \right], \tag{A.41}$$

$$\text{subject to } d\mathbf{X}_t^\alpha = \sigma \alpha_t dt + \sigma d\mathbf{W}_t, \quad \mathbf{X}_t^\alpha \sim \rho_t, \forall t \in \mathcal{T}, \tag{A.42}$$

[2] We will omit the arrow above the character, as it can be inferred from the subscript.

where $\mathbb{A}$ denotes the family of finite-energy controls adapted to the filtration generated by $\mathbf{W}_t$, satisfying $\mathbb{E}_{\mathbb{Q}^\alpha}\left[\int_0^T ||\alpha_t||^2, dt\right] < \infty$. Here, $\mathbb{Q}^\alpha$ is the path measure induced associated with (A.42).

Since the cost functional (A.41) can also be derived using Girsanov's theorem with reference measure $\mathbb{Q}$ associated to the uncontrolled SDE $d\mathbf{X}_t = \sigma d\mathbf{W}_t$, we conclude that the expression in (A.37) provides the optimal cost, *i.e.*, $\min \mathcal{J} = \frac{1}{2}\mathbb{E}_{\mathbb{P}^\star}\left[\int_0^T \left\|\sigma\nabla \log \overrightarrow{\Psi}(t, \overrightarrow{\mathbf{X}}_t)\right\|^2 dt\right]$. Consequently, we identify the optimal control for the problem (A.41) as the Markovian feedback control $\alpha^\star := \sigma\nabla \log \overrightarrow{\Psi}$. Furthermore, this optimal control ensures that the marginal constraints in (A.42) are satisfied, as implied by (A.38).

B Proofs and Derivations

B.1 Proof of Proposition 1

Proposition 1 (Reciprocal Property)**.** *For any* $\mathbf{x}_\mathcal{T} := (\mathbf{x}_0, \mathbf{x}_{t_1}, \cdots, \mathbf{x}_T) \in \mathbb{R}^{d\times(k+1)}$ *and* $t \in [t_{i-1}, t_i)$*, the marginal distribution of* $\mathbb{Q}_{|\mathcal{T}}(\cdot|\mathbf{x}_\mathcal{T})$ *at* t *satisfies:*

$$\mathbb{Q}_{|\mathcal{T}}(\mathbf{x}_t|\mathbf{x}_\mathcal{T}) = \mathbb{Q}_{|t_{i-1},t_i}(\mathbf{x}_t|\mathbf{x}_{t_i}, \mathbf{x}_{t_{i-1}}). \tag{8}$$

Therefore, for any $\mathbb{P} \in \mathcal{P}_{[0,T]}$ *the reciprocal projection* $\mathcal{R}^{mm}(\mathbb{P}, \mathcal{T})$ *admits the following factorization:*

$$\mathcal{R}^{mm}(\mathbb{P}, \mathcal{T}) = \mathbb{P}_\mathcal{T}\mathbb{Q}_{|\mathcal{T}} = \mathbb{P}_{t_0,\cdots,t_k}\mathbb{Q}_{|t_0,\cdots,t_k} = \mathbb{P}_{t_0,\cdots,t_k}\prod_{i=1}^k \mathbb{Q}_{|t_{i-1},t_i}, \quad \mathbb{P}\text{-a.e.} \tag{9}$$

Proof. Let us consider a Markov measure $\mathbb{Q}$. Then following factorzation holds for $\mathbb{Q}$ [2, Definition 2.2] for any events $A_i \in \sigma(\mathbf{X}_{[t_{i-1},t_i]})$ for all $i \in [1:k]$:

$$\mathbb{Q}\left(\cap_{i=1}^k A_i \mid \mathbf{X}_\mathcal{T}\right) = \mathbb{Q}_{|0,t_1}(A_1 \mid \mathbf{X}_0, \mathbf{X}_{t_1})\mathbb{Q}_{|t_1,t_2}(A_2 \mid \mathbf{X}_{t_1}, \mathbf{X}_{t_2})\cdots\mathbb{Q}_{|t_{k-1},T}(A_k \mid \mathbf{X}_{t_{k-1}}, \mathbf{X}_T). \tag{B.1}$$

Without loss of generality, consider $t \in [t_{i-1}, t_i)$. To isolate the conditional distribution of $\mathbf{X}_t$ given endpoints $\mathbf{X}_{t_{i-1}}$ and $\mathbf{X}_{t_i}$, choose events as follows:

$$A_j = \begin{cases} \Omega, & j \neq i, \\ \{\mathbf{X}_t \in B\}, \quad B \in \sigma(\mathbf{X}_t), & j = i. \end{cases} \tag{B.2}$$

Then, substituting (B.2) into the factorization in (B.1), we obtain:

$$\mathbb{Q}_{|\mathcal{T}}(\mathbf{X}_t \in B \mid \mathbf{X}_\mathcal{T}) = \mathbb{Q}_{|t_{i-1},t_i}(\mathbf{X}_t \in B \mid \mathbf{X}_{t_{i-1}}, \mathbf{X}_{t_i}). \tag{B.3}$$

Since $B \in \sigma(\mathbf{X}_t)$ was chosen arbitrarily, this implies that:

$$\mathbb{Q}_{|\mathcal{T}}(\mathbf{X}_t \mid \mathbf{X}_\mathcal{T}) = \mathbb{Q}_{|t_{i-1},t_i}(\mathbf{X}_t \mid \mathbf{X}_{t_{i-1}}, \mathbf{X}_{t_i}), \quad t \in [t_{i-1}, t_i). \tag{B.4}$$

Now, by disintegration of the path measure, we have

$$\mathbb{Q}(\cdot) = \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \mathbb{Q}_{|\mathcal{T}}(\cdot|\mathbf{X}_\mathcal{T})d\mathbb{Q}_\mathcal{T}(\mathbf{X}_\mathcal{T}) = \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \prod_{i=1}^k \mathbb{Q}_{|t_{i-1},t_i}(\cdot|\mathbf{X}_{t_{i-1},t_i})d\mathbb{Q}_\mathcal{T}(\mathbf{X}_\mathcal{T}), \mathbb{Q}\text{-a.e.} \tag{B.5}$$

It implies that for any $\mathbb{P}_\mathcal{T} \in \mathcal{P}_\mathcal{T}$, we get $\mathbb{P}_\mathcal{T}\mathbb{Q}_{|\mathcal{T}} = \mathbb{P}_\mathcal{T}\prod_{i=1}^k \mathbb{Q}_{|t_{i-1},t_i}$.

□

B.2 Proof of Proposition 2

Proposition 2 (Multi-Marginal Markovian Projection)**.** *Let* $\Pi \in \mathcal{P}_{[0,T]}$ *admit factorzation in* (9)*. The multi-marginal Markov projection of* Π*,* $\mathbb{P}^\star := \mathcal{M}^{mm}(\Pi, \mathcal{T}) \in \mathcal{P}_{[0,T]}$*, is associated with the SDE:*

$$d\mathbf{X}_t^\star = [f_t(\mathbf{X}_t^\star) + \sigma v^\star(t, \mathbf{X}_t^\star)]\, dt + \sigma d\mathbf{W}_t, \quad \mathbf{X}_0^\star \sim \Pi_0, \tag{10}$$

$$\text{where } v^\star(t, \mathbf{x}) = \sum_{i=1}^k \mathbf{1}_{[t_{i-1},t_i)}\mathbb{E}_{\Pi_{t_i|t}}\left[\nabla \log \mathbb{Q}_{t_i|t}(\mathbf{X}_{t_i}|\mathbf{X}_t)|\mathbf{X}_t = \mathbf{x}\right]. \tag{11}$$

Moreover, $v^\star$ *satisfies the Fokker-Planck equation (FPE) [50]:*

$$\partial_t \rho_t = -\nabla\cdot(v_t^\star(\mathbf{x})\rho_t(\mathbf{x})) + \frac{\sigma^2}{2}\Delta\rho_t(\mathbf{x}) = 0, \quad \rho_t = \Pi_t, \quad \forall t \in \mathcal{T}, \tag{12}$$

where p_t *is marginal density of* Π_t*. In other words,* $\mathbb{P}_t^\star = \Pi_t$ *for all* $t \in [0, T]$*. d*

Proof. Let $\mathcal{T}_i = \{t_{i-1}, t_i\}$ denote the set of two consecutive boundary time points, $\mathcal{T}_{<i} = \{0, \dots, t_{i-2}\}$ and $\mathcal{T}_{>i} = \{t_{i+1}, \dots, T\}$ represent the set of all time points preceding and following the interval $\mathcal{T}_i$, respectively. Then, for $t \in [t_{i-1}, t_i)$, we have:

$$
\begin{aligned}
\partial_t \rho_t &= \partial_t \int \mathbb{Q}_{|\mathcal{T}}(\mathbf{x}_t) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.6)}\\
&= \partial_t \int \frac{\mathbb{Q}_{t,\mathcal{T}}(\mathbf{x}_t, \mathbf{x}_{\mathcal{T}})}{\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})} d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.7)}\\
&= \partial_t \int \frac{\mathbb{Q}_{t,\mathcal{T}_{<i},\mathcal{T}_{>i}|\mathcal{T}_i}(\mathbf{x}_t, \mathbf{x}_{\mathcal{T}_{<i}}, \mathbf{x}_{\mathcal{T}_{>i}}|\mathbf{x}_{\mathcal{T}_i}) \mathbb{Q}_{\mathcal{T}_i}(\mathbf{x}_{\mathcal{T}_i})}{\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})} d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.8)}\\
&\overset{(i)}{=} \partial_t \int \frac{\mathbb{Q}_{\mathcal{T}_{<i},\mathcal{T}_{>i}|\mathcal{T}_i}(\mathbf{x}_{\mathcal{T}_{<i}}, \mathbf{x}_{\mathcal{T}_{>i}}|\mathbf{x}_{\mathcal{T}_i}) \mathbb{Q}_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \mathbb{Q}_{\mathcal{T}_i}(\mathbf{x}_{\mathcal{T}_i})}{\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})} d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.9)}\\
&= \partial_t \int \frac{\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) \mathbb{Q}_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i})}{\mathbb{Q}_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})} d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.10)}\\
&= \partial_t \int \mathbb{Q}_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.11)}\\
&= \int \partial_t \mathbb{Q}_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.12)}\\
&= \int \left[-\nabla \cdot (v_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \rho_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i})) + \tfrac{1}{2}\sigma^2 \Delta \rho_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \right] d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.13)}\\
&= -\nabla \cdot \int v_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \rho_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) + \tfrac{1}{2}\sigma^2 \Delta \int \rho_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}}) && \text{(B.14)}\\
&\overset{(ii)}{=} -\nabla \cdot (v^i_t(\mathbf{x}_t) \rho_t(\mathbf{x}_t)) + \tfrac{1}{2}\sigma^2 \Delta p_t(\mathbf{x}_t), && \text{(B.15)}
\end{aligned}
$$

where (i) follows from the piece-wise reciprocal property of Π for each interval $[t_{i-1}, t_i)$ in (B.1), and (ii) follows by defining:

$$
\begin{aligned}
v^i_t(\mathbf{x}_t) &= \frac{\int v_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \rho_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{\mathcal{T}}(\mathbf{x}_{\mathcal{T}})}{\rho_t(\mathbf{x}_t)} && \text{(B.16)}\\
&= \int v_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) \frac{\rho_{t|\mathcal{T}_i}(\mathbf{x}_t)}{\rho_t(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i})} d\Pi_{\mathcal{T}_i}(\mathbf{x}_{\mathcal{T}_i}) && \text{(B.17)}\\
&= \int v_{t|\mathcal{T}_i}(\mathbf{x}_t|\mathbf{x}_{\mathcal{T}_i}) d\Pi_{t_i|t}(\mathbf{x}_{t_i}|\mathbf{x}_t) && \text{(B.18)}\\
&= \mathbb{E}_{\Pi_{t_i|t}} \left[\nabla_{\mathbf{x}_t} \log \mathbb{Q}_{t_i|t}(\mathbf{X}_{t_i}|\mathbf{X}_t) | \mathbf{X}_t = \mathbf{x}_t \right]. && \text{(B.19)}
\end{aligned}
$$

Hence, for arbitrary $t \in [0, T] - \mathcal{T}$, we get the desired expression:

$$
\begin{aligned}
v^\star(t, \mathbf{x}) &= \textstyle\sum_{i=1}^{|\mathcal{T}|} \mathbb{E}_{\Pi_{t_i|t}} v^i(t, \mathbf{x}) \mathbf{1}_{[t_{i-1}, t_i)}(t) && \text{(B.20)}\\
&= \textstyle\sum_{i=1}^{|\mathcal{T}|} \mathbb{E}_{\Pi_{t_i|t}} \left[\nabla_{\mathbf{x}_t} \log \mathbb{Q}_{t_i|t}(\mathbf{X}_{t_i}|\mathbf{X}_t) | \mathbf{X}_t = \mathbf{x}_t \right] \mathbf{1}_{[t_{i-1}, t_i)}(t) && \text{(B.21)}
\end{aligned}
$$

Moreover, since the measures induced by the SDEs with drifts $\{v^i\}_{i \in [1:k]}$ for each local interval $\{[t_{i-1}, t_i)\}$ share their marginal distributions at each boundary time points $t \in \mathcal{T}$, the SDEs in (10) form a Markov process. Consequently, we obtain the associated FPE with marginal constraints specified by the prescribed distributions $\{\rho_t\}_{t \in \mathcal{T}}$, which can be constructed using $v^\star$:

$$
\partial_t \rho_t = -\nabla \cdot (v^\star_t(\mathbf{x}_t) \rho_t(\mathbf{x}_t)) + \tfrac{1}{2} \Delta \rho_t(\mathbf{x}_t) = 0, \quad \rho_t = \rho_t, \quad \forall t \in \mathcal{T}. \qquad \text{(B.22)}
$$

This completes the proof. □

B.3 Proof of Proposition 3

Our results are based on [2, 38], in a manner analogous to the SBM proof for SBP in [55, Appendix C.3], which was based on [26], and serve as a natural extension to the multi-marginal setting.

Proposition 3 (Uniqueness)**.** *Let $\mathbb{P}^\star$ be a Markov measure which is reciprocal class of $\mathbb{Q}$ satisfying $\mathbb{P}^\star_t = \rho_t$ for all $t \in \mathcal{T}$. Then, $\mathbb{P}^\star$ is unique solution $\mathbb{P}^{\text{mSBP}}$ of the mSBP.*

Proof. Below, we proof that; **(A)**: If some measure $\mathbb{P}^\star \in \mathcal{P}_{[0,T]}$ satisfying the Radon-Nikodym derivative in (A.23), then it is a Markov process and reciprocal class of $\mathbb{Q}$ satisfying marginal constraints $\{\rho_t\}_\mathcal{T}$; **(B)**: If the unique solution $\mathbb{P}^{\texttt{mSBP}}$ of mSBP exists, then it will has Radon-Nikodym derivative in (A.23); **(C)**: If some measure $\mathbb{P}^\star \in \mathcal{P}_{[0,T]}$ satisfying the Radon-Nikodym derivative in (A.23), then it is unique solution $\mathbb{P}^{\texttt{mSBP}}$ of mSBP.

(A) Previously, we established that a solution $\mathbb{P}^\star$ of mSBP possesses a Radon-Nikodym derivative with respect to the reference measure $\mathbb{Q}$ of the product form $\frac{d\mathbb{P}^\star}{d\mathbb{Q}} = \prod_{i=0}^{|\mathcal{T}|} \Psi_{t_i}(\mathbf{x}_{t_i})$ as in (A.23). By combining the results in [2, Theorem 2.10], since $\prod_{i=0}^{|\mathcal{T}|} \Psi_{t_i}(\mathbf{X}_{t_i})$ is $\sigma(\mathbf{X}_\mathcal{T})$-measurable, it is a Markov process. Moreover, it satisfies $\mathbb{P}^\star = \int \mathbb{Q}_{|\mathcal{T}} d\mathbb{P}^\star_\mathcal{T}$ in (A.9), it concluded that $\mathbb{P}^\star$ reciprocal class of $\mathbb{Q}$ *i.e.*, $\mathbb{P}^\star = \mathcal{R}^{\texttt{mm}}(\mathbb{P}^\star, \mathcal{T})$.

(B) Our goal is to verify that $\mathbb{P}^\star$ is indeed the *unique* solution the the mSBP. By combining the results in [2, Theorem 4.5], (if it exists) the unique solution of $\mathbb{P}^{\texttt{mSBP}}$ is a Markov process and admit following Radon-Nikodym derivative with respect to $\mathbb{Q}$:

$$\frac{d\mathbb{P}^{\texttt{mSBP}}}{d\mathbb{Q}} = \exp\left(\mathbb{A}[0,1]\right), \tag{B.23}$$

where $\mathbb{A}[0,1]$ is $\sigma(\mathbf{X}_\mathcal{T})$-measurable function. Again, since the product form $\prod_{i=0}^{|\mathcal{T}|} \Psi_{t_i}(\mathbf{x}_{t_i})$ is $\sigma(\mathbf{X}_\mathcal{T})$-measurable, it states that the solution we found is indeed a unique solution $\mathbb{P}^{\texttt{mSBP}}$.

(C) Let us consider the set of measures satisfying the multi-marginal constraints:

$$\mathcal{C}_\mathcal{T} = \{\mathbb{P} \in \mathcal{P}_{[0,T]} : (\mathbf{X}_t)_\# \mathbb{P} = \rho_t, \ \forall\, t \in \mathcal{T}\}. \tag{B.24}$$

By leveraging results in [38, Theorem 2.6], under mild condition, the unique minimizer $\mathbb{P}^\star \in \mathcal{C}_\mathcal{T}$ of mSBP has a Radon-Nikodyn derivative that can be written as $\frac{d\mathbb{P}^\star}{d\mathbb{Q}} = \prod_{i=0}^{|\mathcal{T}|} \Psi_{t_i}(\mathbf{x}_{t_i})$ as in (A.23).

Combining the arguments from parts **(A-C)**, we establish the following: the unique minimizer $\mathbb{P}^\star \in \mathcal{C}_\mathcal{T}$ for mSBP is characterized if and only if it is a Markov process that belongs to the reciprocal class of $\mathbb{Q}$. It concludes the proof. □

B.4 Proof of Proposition 4

Our proof builds upon the work of [47, 55]. Standard SBM convergence relies on the Pythagorean property of (reverse) KL-divergence and compactness of the set $\{\mathbb{P} \in \mathcal{P}_{[0,T]} : \mathrm{D}_{\mathrm{KL}}(\mathbb{P}|\mathbb{P}^{\texttt{mSBP}}) \leq \mathrm{D}_{\mathrm{KL}}(\mathbb{P}^{(0)}|\mathbb{P}^{\texttt{mSBP}})\}$. Therefore, if our proposed multi-marginal projection operators, $\mathcal{R}^{\texttt{mm}}$ and $\mathcal{M}^{\texttt{mm}}$, also satisfy a Pythagorean law analogous to those in [47, 55], then their convergence analysis can be directly applied to our multi-marginal scenario.

Proposition 4 (Convergence)**.** $\mathbb{P}^{(n)} = \mathbb{P}^{\texttt{mSBP}}$ *of mSBP as* $n \uparrow \infty$ *with iterative procedure in* (13).

Proof. Let $\Pi := \mathcal{R}^{\texttt{mm}}(\mathbb{P}, \mathcal{T})$ denote the multi-marginal reciprocal projection of a path measure $\mathbb{P}$, and let $\mathbb{M} := \mathcal{M}^{\texttt{mm}}(\Pi, \mathcal{T})$ be the subsequent multi-marginal Markovian projection of Π. As established in Proposition 2, the marginal distributions match at each time point, i.e., $\Pi_t = \mathbb{M}_t$ for all $t \in [0,T]$, and specifically at the initial time, $\Pi_0 = \mathbb{M}_0 = \rho_0$. Following the principles outlined by [47, pp. 37-38], we can establish a Pythagorean law for the KL-divergences [15] for these multi-marginal projections $\mathcal{R}^{\texttt{mm}}$ and $\mathcal{M}^{\texttt{mm}}$. For any path measure $\mathbb{P} \in \mathcal{P}_{[0,T]}$:

$$D_{\mathrm{KL}}(\Pi|\mathbb{P}) = D_{\mathrm{KL}}(\Pi|\mathbb{M}) + D_{\mathrm{KL}}(\mathbb{M}|\mathbb{P}). \tag{B.25}$$

If we choose $\mathbb{P} = \mathbb{P}^{\texttt{mSBP}}$, the unique solution to the mSBP, the Pythagorean law implies the inequality:

$$D_{\mathrm{KL}}(\Pi|\mathbb{P}^{\texttt{mSBP}}) \geq D_{\mathrm{KL}}(\mathbb{M}|\mathbb{P}^{\texttt{mSBP}}), \tag{B.26}$$

where equality holds if and only if $\Pi = \mathbb{M}$. Furthermore, as proven in Proposition 3, $\mathbb{P}^{\texttt{mSBP}}$ is a Markov process within the set of measures $\mathcal{C}_\mathcal{T}$ (satisfying the marginal constraints at times $\mathcal{T}$) and belongs to the reciprocal class of the reference measure $\mathbb{Q}$. Consequently, the condition $\Pi = \mathbb{M}$ is met if and only if both Π and $\mathbb{M}$ are equal to the unique solution $\mathbb{P}^{\texttt{mSBP}}$.

Now, consider an iterative process for $n \geq 0$. Let $\mathbb{P}^{(n-1)}$ be a Markovian. Define the reciprocal projection $\Pi^{(n)} = \mathcal{R}^{\text{mm}}(\mathbb{P}^{(n-1)}, \mathcal{T})$. Through the disintegration of the path measure, we have that:

$$\mathrm{D}_{\mathrm{KL}}(\Pi^{(n)}|\mathbb{P}^{\text{mSBP}}) = \mathrm{D}_{\mathrm{KL}}(\Pi^{(n)}_{\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{\mathcal{T}}) + \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \mathrm{D}_{\mathrm{KL}}(\Pi^{(n)}_{|\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{|\mathcal{T}}) d\Pi^{(n)}_{\mathcal{T}} \tag{B.27}$$

$$\stackrel{(i)}{=} \mathrm{D}_{\mathrm{KL}}(\Pi^{(n)}_{\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{\mathcal{T}}), \tag{B.28}$$

where (i) follows because the reciprocal projection $\mathcal{R}^{\text{mm}}$ ensures that the resulting conditional path measure $\Pi^{(n)}_{|\mathcal{T}}$ is a mixture of bridges between adjacent marginals, identical to that of $\mathbb{P}^{\text{mSBP}}_{|\mathcal{T}}$ (as described in relation to (A.9)), i.e., $\Pi^{(n)}_{|\mathcal{T}} = \mathbb{P}^{\text{mSBP}}_{|\mathcal{T}} = \mathbb{Q}_{|\mathcal{T}}$. Next, let $\mathbb{M}^{(n)} = \mathcal{M}^{\text{mm}}(\Pi^{(n)}, \mathcal{T})$ be the Markovian projection of $\Pi^{(n)}$. The KL divergences between $\mathbb{M}^{(n)}$ and $\mathbb{P}^{\text{mSBP}}$ becomes:

$$\mathrm{D}_{\mathrm{KL}}(\mathbb{M}^{(n)}|\mathbb{P}^{\text{mSBP}}) = \mathrm{D}_{\mathrm{KL}}(\mathbb{M}^{(n)}_{\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{\mathcal{T}}) + \int_{\mathbb{R}^{d\times|\mathcal{T}|}} \mathrm{D}_{\mathrm{KL}}(\mathbb{M}^{(n)}_{|\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{|\mathcal{T}}) d\mathbb{M}^{(n)}_{\mathcal{T}} \tag{B.29}$$

$$\stackrel{(ii)}{\geq} \mathrm{D}_{\mathrm{KL}}(\Pi^{(n+1)}_{\mathcal{T}}|\mathbb{P}^{\text{mSBP}}_{\mathcal{T}}), \tag{B.30}$$

where $\Pi^{(n+1)} = \mathcal{R}^{\text{mm}}(\mathbb{M}^{(n)}, \mathcal{T})$, and (ii) is stated to follow from the IMF iteration as per (A.4). This implies the desired intermediate result for the convergence argument:

$$\mathrm{D}_{\mathrm{KL}}(\mathbb{M}^{(n)}|\mathbb{P}^{\text{mSBP}}) \geq \mathrm{D}_{\mathrm{KL}}(\Pi^{(n+1)}|\mathbb{P}^{\text{mSBP}}). \tag{B.31}$$

Consequently, under the assumption that the relevant KL divergences (such as $D_{\mathrm{KL}}(\Pi|\mathbb{M})$) are finite, the convergence proof presented by [55, Appendix C.6] can be directly adapted to our multi-marginal case, given our construction of $\mathcal{M}^{\text{mm}}$ and $\mathcal{R}^{\text{mm}}$. This concludes the proof. □

B.5 Proof of Corollary 5

Corollary 5 (Multi-Marginal Schrödinger Bridge)**.** *Assume a sequence of controls $\{v^i, u^i\}_{i\in[1:k]}$, where each v^i, u^i induced local SBs $\mathbb{P}^i$ of SBP over local interval $[t_{i-1}, t_i]$ with distributions $(\rho_{t_{i-1}}, \rho_{t_i})$ in a forward and backward direction, respectively. If $\lim_{t\uparrow t_i} v^i(t, \mathbf{x}) = v^{i+1}(t, \mathbf{x})$ and $\lim_{t\downarrow t_{i-1}} u^i(t, \mathbf{x}) = u^{i-1}(t, \mathbf{x})$ for all $i \in [1:k]$, then $\mathbb{P}^{\text{mSBP}}$ of mSBP induced by following SDEs:*

$$d\mathbf{X}^\star_t = [f_t(\mathbf{X}^\star_t) + \sigma v^\star(t, \mathbf{X}^\star_t)]\, dt + \sigma d\mathbf{W}_t, \quad \mathbf{X}^\star_0 \sim \rho_0. \tag{18a}$$

$$d\mathbf{Y}^\star_t = [-f_{T-t}(\mathbf{Y}^\star_t) + \sigma u^\star(t, \mathbf{Y}^\star_t)]\, dt + \sigma d\mathbf{W}_t, \quad \mathbf{Y}^\star_0 \sim \rho_T, \tag{18b}$$

$$\textit{where} \quad v^\star(t, \mathbf{x}) = \textstyle\sum_{i=1}^k \mathbf{1}_{[t_{i-1}, t_i)}(t) v^i(t, \mathbf{x}), \quad u^\star(t, \mathbf{x}) = \textstyle\sum_{i=1}^k \mathbf{1}_{(t_{i-1}, t_i]}(t) u^i(t, \mathbf{x}). \tag{18c}$$

Proof. Consider local forward SBs $\overrightarrow{\mathbb{P}}^i$ (governed by control v^i) and local backward SBs $\overleftarrow{\mathbb{P}}^i$ (governed by u^i) on intervals $[t_{i-1}, t_i]$. Global forward path measure $\overrightarrow{\mathbb{P}} = \rho_0 \prod_{i=1}^k \overrightarrow{\mathbb{P}}^i_{|t_{i-1}}$ and global backward path measure $\overleftarrow{\mathbb{P}} = \rho_T \prod_{i=1}^k \overleftarrow{\mathbb{P}}^i_{|t_i}$ are constructed by sequentially composing these local SBs, starting from the initial distribution ρ_0 and terminal distribution ρ_T, respectively. By this construction, $\overrightarrow{\mathbb{P}}$ and $\overleftarrow{\mathbb{P}}$ inherently satisfy all specified marginal constraints $\{\rho_t\}_{t\in\mathcal{T}}$ and belong to the reciprocal class of the reference measure $\mathbb{Q}$. The Markov property and absolute continuity constraint $\mathrm{D}_{\mathrm{KL}}(\overrightarrow{\mathbb{P}}|\mathbb{Q}) < \infty$ or $\mathrm{D}_{\mathrm{KL}}(\overleftarrow{\mathbb{P}}|\mathbb{Q}) < \infty$ for these global path measures $\overrightarrow{\mathbb{P}}$ and $\overleftarrow{\mathbb{P}}$ hinges on the continuity of their sample paths $\mathbf{X}^\star_t$. This path continuity is achieved if the composite global controls $v^\star$ in (18a) and $u^\star$ in (18b) are continuous across the entire time horizon $[0, T]$, ensuring seamless transitions at each intermediate time t_i *i.e.*, $\lim_{t\uparrow t_i} v^i(t, \mathbf{x}) = v^{i+1}(t, \mathbf{x})$ and $\lim_{t\downarrow t_{i-1}} u^i(t, \mathbf{x}) = u^{i-1}(t, \mathbf{x})$ for all $i \in [1:k]$. With continuous sample paths, $\overrightarrow{\mathbb{P}}$ and $\overleftarrow{\mathbb{P}}$ are indeed Markov processes satisfying the absolute continuity condition with respect to $\mathbb{Q}$. Given Proposition 3, which states that a Markov process satisfying all marginal constraints and belonging to the reciprocal class of $\mathbb{Q}$ is the unique solution to mSBP. It concludes the proof. □

C Experimental Details

Our evaluation of MSBM involved several datasets and followed established experimental protocols from baseline methods to ensure fair comparisons. For the petal dataset and the 100-dimensional

Table C.1: Training Hyper-parameters

Dataset	Learning Rate	Iteration (N)	Training step (S)	# of Discretization	Batch Size	T
Petal	1×10^{-3}	20	1000	30	256	4
hESC	1×10^{-3}	100	1000	30	256	5
EB(100-dim)	2×10^{-4}	10	1000	100	256	4
EB(5-dim)	2×10^{-4}	3	50000	100	256	4

EB [39], we adopted the experimental setup of DMSB [9][3]. The processed data for these experiments were inherited from TrajectoryNet [58][4]. To maintain a fair comparison with DMSB, we parameterized both the forward (v) and backward (u) controls using the residual-based networks described in [9], ensuring a similar total model parameter count of approximately 1.28M for these two datasets. For the 100-dimensional EB dataset, we further split the entire dataset into training and testing subsets with an 85%/15% ratio, respectively.

Regarding the hESC [14], our experiments mirrored the setup of SBIRR [54][5] and inherited the processed data therein. We based the parameterization of our model on the network architecture used for the petal and 100-dim EB datasets. For a consistent comparison on the hESC dataset, we maintained a model size of approximately 24k total parameters.

For the 5-dimensional EB dataset, we followed the experimental protocol of NLSB [24][6]. We inherited processed data from TrajectoryNet [58]. In this case, the forward (v) and backward (u) controls were also parameterized with the residual-based networks described in [9].

Across all experiments, models were trained using the Adam optimizer [22] with Exponential Moving Average (EMA) applied at a decay rate of 0.999. The proposed MSBM training procedure (detailed in Algorithm 1) involved N outer iterations, with each outer iteration containing S inner training steps. Cached marginal distributions were updated in each outer iteration. Impressively, the complete MSBM training for all datasets was accomplished in less than one hour using a single NVIDIA A6000 GPU. The remaining training hyper-parameters are detailed in Table C.1.

Additionally, the tables below present the complete results from experiments conducted three times, each using a different random seed.

[3] `https://github.com/TianrongChen/DMSB`, under MIT license

[4] `https://github.com/KrishnaswamyLab/TrajectoryNet`, Non-Commercial License Yale Copyright

[5] `https://github.com/YunyiShen/SB-Iterative-Reference-Refinement`

[6] `https://github.com/take-koshizuka/NLSB`, under MIT license

Table C.2: Full results over 3 different seeds. Performance on the 100-dim PCA of EB dataset. MMD and SWD are computed between test $\rho_{t_i}^{\texttt{te}}$ and generated $\hat{\rho}_{t_i}$ by simulating the dynamics from test $\rho_{t_0}^{\texttt{te}}$.

	MMD ↓				SWD ↓			
Methods	`Full`	t_1	t_2	t_3	`Full`	t_1	t_2	t_3
NLSB† [24]	0.66	0.38	0.37	0.37	0.54	0.55	0.54	0.55
MIOFlow† [20]	0.23	0.23	0.90	0.23	0.35	0.49	0.72	0.50
DMSB† [9]	0.03	**0.04**	**0.04**	**0.04**	0.16	0.20	0.19	**0.18**
MSBM	**0.018**± 4e-4	**0.049**± 3e-2	**0.038**± 5e-4	0.05± 9e-4	**0.129**± 3e-3	**0.1895**± 1e-2	**0.1772**± 2e-3	0.1997± 4e-3

† result from [9].

Table C.3: Full results over 3 different seeds. Performance on the 5-dim PCA of hESC dataset. $\mathcal{W}_2$ is compute between test ρ_{t_i} and generated $\hat{\rho}_{t_i}$ by simulating the dynamics from test ρ_{t_0}.

	$\mathcal{W}_2$ ↓		Runtime
Methods	t_1	t_3	hours
TrajectoryNet†	1.30	1.93	10.19
DMSB†	1.10	1.51	15.54
SBIRR†	**1.08**	1.33	0.36 (0.38)*
MSBM (Ours)	1.083 ± 7e-3	**1.304** ± 3e-2	**0.09** ± 1e-2

† result from [54].

Table C.4: Full results over 3 different seeds. Performance on the 5-dim PCA of EB dataset. $\mathcal{W}_1$ is computed between test $\rho_{t_i}^{\texttt{te}}$ and generated $\hat{\rho}_{t_i}$ by simulating the dynamics from previous test $\rho_{t_{i-1}}^{\texttt{te}}$.

	$\mathcal{W}_1$ ↓				
Methods	t_1	t_2	t_3	t_4	Mean
Neural SDE† [27]	0.69	0.91	0.85	0.81	0.82
TrajectoryNet† [58]	0.73	1.06	0.90	1.01	0.93
IPF (GP)† [59]	0.70	1.04	0.94	0.98	0.92
IPF (NN)† [5]	0.73	0.89	0.84	0.83	0.82
SB-FBSDE† [10]	**0.56**	0.80	1.00	1.00	0.84
NLSB† [24]	0.68	0.84	0.81	0.79	0.78
OT-CFM† [57]	0.78	0.76	0.77	0.75	0.77
WLF-SB‡ [41]	0.63	0.79	0.77	**0.75**	0.73
MSBM (Ours)	0.64± 7e-3	**0.73**± 8e-3	**0.72**± 1e-2	**0.73**± 9e-3	**0.71**± 7e-3

† result from [24], ‡ result from [41].