Differentially Private Covariance Revisited

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Abstract

In this paper, we present two new algorithms for covariance estimation under concentrated differential privacy (zCDP). The first algorithm achieves a Frobenius error of $\tilde{O}(d^{1/4}\sqrt{\mathrm{tr}}/\sqrt{n} + \sqrt{d}/n)$, where tr is the trace of the covariance matrix. By taking $\mathrm{tr} = 1$, this also implies a worst-case error bound of $\tilde{O}(d^{1/4}/\sqrt{n})$, which improves the standard Gaussian mechanism's $\tilde{O}(d/n)$ for the regime $d > \tilde{\Omega}(n^{2/3})$. Our second algorithm offers a tail-sensitive bound that could be much better on skewed data. The corresponding algorithms are also simple and efficient. Experimental results show that they offer significant improvements over prior work.

1 Introduction

Consider a dataset represented by a matrix $\mathbf{X} \in \mathbb{R}^{d \times n}$, where each column $X_i, i = 1, \ldots, n$ corresponds to an individual's information. As standard in the literature, we assume that all the X_i's live in $\mathcal{B}_d$, the d-dimensional ℓ_2-unit ball centered at the origin. In this paper, we revisit the problem of estimating the (empirical) covariance matrix $\mathbf{\Sigma}(\mathbf{X}) := \frac{1}{n}\sum_i X_i X_i^T = \frac{1}{n}\mathbf{X}\mathbf{X}^T$ under differential privacy (DP), a fundamental problem in high-dimensional data analytics and machine learning that requires little motivation. We often write $\mathbf{\Sigma}(\mathbf{X})$ as $\mathbf{\Sigma}$ when the context is clear. As with most prior work, we use the Frobenius norm $\|\tilde{\mathbf{\Sigma}} - \mathbf{\Sigma}\|_F$ to measure the error of the estimated covariance $\tilde{\mathbf{\Sigma}}$. To better focus, in the introduction we state all results under *concentrated different privacy (zCDP)* [10]; extensions of our results to pure-DP are given in Appendix I.

1.1 A Trace-sensitive Algorithm

For any symmetric matrix $\mathbf{A}$, we use $\mathbf{P}[\mathbf{A}]$ and $\mathbf{\Lambda}[\mathbf{A}]$ to denote its matrices of eigenvectors and eigenvalues, respectively, such that $\mathbf{A} = \mathbf{P}[\mathbf{A}]\mathbf{\Lambda}[\mathbf{A}]\mathbf{P}[\mathbf{A}]^T$; we use $\lambda_i[\mathbf{A}]$ to denote its ith largest eigenvalue. When $\mathbf{A} = \mathbf{\Sigma} = \mathbf{\Sigma}(\mathbf{X})$, we simply write $\mathbf{P} = \mathbf{P}[\mathbf{\Sigma}], \mathbf{\Lambda} = \mathbf{\Lambda}[\mathbf{\Sigma}], \lambda_i = \lambda_i[\mathbf{\Sigma}]$, so that $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_d)$ and $\mathbf{\Sigma} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T$. Let $\mathbf{P} = [P_1\ P_2\ \cdots\ P_d]$, where P_i is the orthonormal basis vector corresponding to λ_i. Rudimentary linear algebra yields $\lambda_k = \frac{1}{n}\sum_i (P_k^T X_i)^2$ for $1 \leqslant k \leqslant d$ and $\|X_i\|_2^2 = \sum_k (P_k^T X_i)^2$ for $1 \leqslant i \leqslant n$. Thus, it follows that

$$\mathrm{tr}[\mathbf{\Sigma}] = \mathrm{tr}[\mathbf{\Lambda}] = \sum_k \lambda_k = \sum_k \frac{1}{n}\sum_i (P_k^T X_i)^2 = \frac{1}{n}\sum_i\sum_k (P_k^T X_i)^2 = \frac{1}{n}\sum_i \|X_i\|_2^2.$$

That is, $0 \leqslant \mathrm{tr}[\mathbf{\Lambda}] \leqslant 1$ is the average ℓ_2 norm (squared) of the X_i's, and we simply write it as tr.

Recall that it is assumed that all the X_i's live in $\mathcal{B}_d$. In practice, this is enforced by assuming an upper bound B on the norms and scaling down all X_i by B. As one often uses a conservatively large B, typical values of tr can be much smaller than 1, so a trace-sensitive algorithm would be

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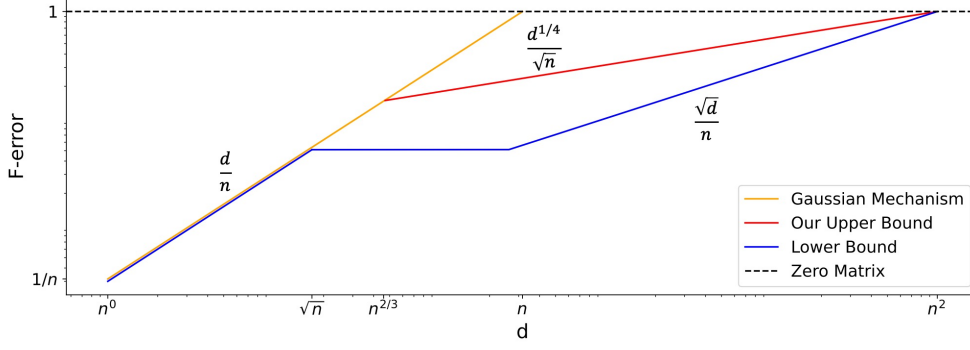
Figure 1: Currently known worst-case error bounds (both axes are in log scale).

more desirable. Indeed, Amin et al. [2] take this approach, describing an algorithm with error[1] $\tilde{O}(d^{3/4}\sqrt{\mathrm{tr}}/\sqrt{n} + \sqrt{d}/n)$ under zCDP[2]. Note that the $\sqrt{d}/n$ term inherits from mean estimation and the first term is the "extra" difficulty for covariance estimation. In this paper, we improve this term to $d^{1/4}\sqrt{\mathrm{tr}}/\sqrt{n}$ (we have a similar, albeit lesser, improvement under pure-DP; see Appendix I). Our algorithm is very simple: We first estimate $\mathbf{\Lambda}$ using the Gaussian mechanism (this is the same as in [2, 22]), then we estimate $\mathbf{P}$ by doing an eigendecomposition of $\mathbf{\Sigma}$ masked with Gaussian noise. Intuitively, we obtain a $\sqrt{d}$-factor improvement over the iterative methods of [2, 22], because we can obtain all eigenvectors from one noisy $\mathbf{\Sigma}$, while the iterative methods must allocate the privacy budget to all d eigenvectors. Our algorithm is also more efficient, performing just two eigendecompositions and one matrix multiplication, whereas the algorithm in [2, 22] needs $O(d)$ such operations.

Implication to worst-case bounds. Covariance matrix has also been studied in the traditional worst-case setting, i.e., the bound should only depend on d and n. Dwork et al. [17] show that the ℓ_2-sensitivity of $\mathbf{\Sigma}$, i.e., $\max_{\mathbf{X} \sim \mathbf{X}'} \|\mathbf{\Sigma}(\mathbf{X}) - \mathbf{\Sigma}(\mathbf{X}')\|_F$ where $\mathbf{X} \sim \mathbf{X}'$ denotes two neighboring datasets that differ by one column, is $O(1/n)$. Thus, the standard *Gaussian mechanism* achieves an error of $\tilde{O}(d/n)$ by adding an independent Gaussian noise of scale $\tilde{O}(1/n)$ to each of the d^2 entries of $\mathbf{\Sigma}$. By taking $\mathrm{tr} = 1$, our trace-sensitive bound degenerates into $\tilde{O}(d^{1/4}/\sqrt{n})$. Note that the $\sqrt{d}/n$ term is dominated by $d^{1/4}/\sqrt{n}$ for $d < \tilde{O}(n^2)$, which is the parameter regime that allows non-trivial utility (i.e., the error is less than 1).

To better understand the situation, it is instructive to compare covariance estimation with mean estimation (where data are also drawn from the ℓ_2 unit ball and the error is measured in ℓ_2 norm), as the hardness of covariance estimation lies between d-dimensional mean estimation (only estimating the diagonal entries of $\mathbf{\Sigma}$) and d^2-dimensional mean estimation (treating $\mathbf{\Sigma}$ as a d^2-dimensional vector). This observation implies a lower bound $\tilde{\Omega}(\sqrt{d}/n)$ following from the same lower bound for mean estimation [19][3], and an upper bound $\tilde{O}(d/n)$ attained by the Gaussian mechanism. For $d < O(\sqrt{n})$, Kasiviswanathan et al. [23] prove a higher lower bound[4] $\tilde{\Omega}(d/n)$, which means that the complexity of covariance estimation is same as d^2-dimensional mean estimation in the low-dimensional regime, so one cannot hope to beat the Gaussian mechanism for small d. However, in the high-dimensional regime, our result indicates that the covariance problem is strictly easier, due to the correlations of the d^2 entries of $\mathbf{\Sigma}$. Another interesting consequence is that our error bound has utility for d up to $\tilde{O}(n^2)$ (utility is lost when the error is $\tilde{O}(1)$, as returning a zero matrix can already achieve this error). This is the highest d that allows for any utility, since even mean estimation requires $d < \tilde{O}(n^2)$ to have utility under zCDP [19, 10]. We pictorially show the currently known

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[1]We use the $\tilde{O}$ notation to suppress the dependency on the privacy parameters and all polylogarithmic factors. We use e as the base of log (unless stated otherwise) and define $\log(x) = 1$ for any $x \leqslant e$.

[2]Their paper states the error bound under pure-DP and for estimating $\mathbf{X}\mathbf{X}^T$ (i.e., without normalization by $1/n$); we show how this bound is derived from their result in Appendix C.

[3]This paper proves the lower bound under the statistical setting; in Appendix D, we show how it implies the claimed lower bound under the empirical setting.

[4]Their lower bound is under approximate-DP, which also holds under zCDP.

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(worst-case) upper and lower bounds in Figure 1. It remains an interesting open problem to close the gap for $\tilde{\Omega}(\sqrt{n}) < d < \tilde{O}(n^2)$.

Through private communication with Aleksandar Nikolov, it is observed that the *projection mechanism* [31, 15] can also be shown to have error $\tilde{O}(d^{1/4}/\sqrt{n})$ when applied to the covariance problem. In Appendix E, we make this connection more explicit, while also giving an efficient implementation. However, the projection mechanism is not trace-sensitive.

1.2 A Tail-sensitive Algorithm

A trace-sensitive bound only makes use of the average ℓ_2 norm, which cannot capture the full distribution. Next, we design an algorithm with an error bound that more closely depends on the distribution of the norms. We characterize this distribution using the τ-*tail* ($\mathbb{I}(\cdot)$ is the indicator function):

$$\gamma(\mathbf{X}, \tau) = \frac{1}{n} \sum_i \|X_i\|_2^2 \cdot \mathbb{I}(\|X_i\|_2 > \tau), \tau \in [0, 1]. \tag{1}$$

Note that $\gamma(\mathbf{X}, \tau)$ decreases as τ increases. In particular, $\gamma(\mathbf{X}, 0) = \text{tr}, \gamma(\mathbf{X}, 1) = 0$.

A common technique to reduce noise, at the expense of some bias, is to clip all the X_i's so that they have norms at most τ, for some threshold τ. This yields an error of $\text{Noise}(\mathbf{X}, \tau) + \gamma(\mathbf{X}, \tau)$, where $\text{Noise}(\mathbf{X}, \tau)$ denotes the error bound of the mechanism when all the X_i's have norm bounded by τ, and $\gamma(\mathbf{X}, \tau)$ is the (additional) bias caused by clipping. Opting for the better of the Gaussian mechanism or our trace-sensitive mechanism, we have

$$\text{Noise}(\mathbf{X}, \tau) = \tilde{O}\left(\min\left(\frac{\tau^2 d}{n}, \frac{\tau d^{1/4}\sqrt{\text{tr}}}{\sqrt{n}} + \frac{\tau^2\sqrt{d}}{n}\right)\right). \tag{2}$$

The technical challenge is therefore choosing a good τ in a differentially private manner. We design a DP mechanism to choose the optimal τ up to a polylogarithmic multiplicative factor and an exponentially small additive term. It also adaptively selects the better of Gaussian mechanism or the trace-sensitive mechanism depending on the relationship between d, n, and a privatized tr. More precisely, our adaptive mechanism achieves an error of

$$\tilde{O}\left(\min_\tau\left(\text{Noise}(\mathbf{X}, \tau) + \gamma(\mathbf{X}, \tau)\right) + 2^{-dn}\right). \tag{3}$$

Note that this tail-sensitive bound is always no worse (modulo the 2^{-dn} term) than $\text{Noise}(\mathbf{X}, 1)$ (i.e., without clipping), and can be much better for certain norm distributions. In particular, the tail-sensitive bound would work very well on many real datasets with skewed distributions, e.g., most data vectors have small norms with a few having large norms. For example, suppose $d = n^{3/4}$, and a constant number of data vectors have ℓ_2 norm 1 while the others have norm $n^{-1/4}$. Then $\sqrt{\text{tr}} = \Theta(n^{-1/4})$, so $\text{Noise}(\mathbf{X}, 1)$ takes the trace-sensitive bound, which is $\tilde{O}(n^{-9/16})$. On the other hand, (3) is at most $\tilde{O}(n^{-13/16})$ by taking $\tau = n^{-1/4}$.

2 Related Work

Mean estimation and covariance estimation are perhaps the most fundamental problems in statistics and machine learning, and how to obtain the best estimates while respecting individual's privacy has attracted a lot of attention in recent years. Mean estimation under differential privacy is now relatively well understood, with the optimal worst-case error being $\tilde{\Theta}(\sqrt{d}/n)$ [19], achieved by the standard Gaussian mechanism [17]. In contrast, the covariance problem is more elusive. As indicated in Figure 1, its complexity is probably a piecewise linear (in the log-log scale) function.

When most data have norms much smaller than the upper bound given *a priori*, the worst-case bounds above are no longer optimal. In these cases, it is more desirable to have an error bound that is instance-specific. Clipping is a common technique for mean estimation [3, 18, 4, 34, 29] and it is known that running the Gaussian mechanism after clipping $\mathbf{X}$ with a certain quantile of the norms of the X_i's achieves instance-optimality in a certain sense [3, 18]. However, for covariance estimation, we show in Appendix F that no quantile can be the optimal clipping threshold achieving the bound in

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(3). Nevertheless, the bound in (3) is only achieving the optimal clipping threshold; we cannot say that is instance-optimal, since $\mathrm{Noise}(\cdot)$ is not even known to be worst-case optimal.

Closely related to covariance estimation are the PCA problem and low-rank approximation. Instead of finding all eigenvalues and eigenvectors, they only aim at finding the largest one or a few. For these problems, iterative methods [2, 22, 38, 17, 11, 36] should perform better than the Gaussian mechanism or our algorithm, both of which try to recover the full covariance matrix.

Many covariance estimation algorithms have been proposed under the statistical setting, where the X_i's are i.i.d. samples drawn from a certain distribution, e.g., a multivariate Gaussian [19, 9, 8, 1, 21, 28, 5, 25]. Instead of the Frobenius error, many of them adopt the Mahalanobis error $\|\widetilde{\Sigma} - \Sigma\|_{\Sigma} := \|\Sigma^{-1/2}\widetilde{\Sigma}\Sigma^{-1/2} - \mathbf{I}\|_F$, which can be considered as a normalized version of the former. It is known that $\lambda_d\|\mathbf{A} - \Sigma\|_{\Sigma} \leqslant \|\mathbf{A} - \Sigma\|_F \leqslant \lambda_1\|\mathbf{A} - \Sigma\|_{\Sigma}$, so when $\Sigma_{\mathbb{D}}$ is well-conditioned, i.e., $\lambda_1/\lambda_d = O(1)$, any Frobenius error directly translates to a Mahalanobis error. However, for the Mahalanobis error, the more challenging question is how to deal with an ill-conditioned Σ, for which [19, 8] have provided elegant solutions for the case where $\mathbb{D}$ is a multivariate Gaussian. It would be interesting to see if their methods can be combined with the tail-sensitive techniques in this paper to solve this problem for other distribution families, in particular, heavy-tailed distributions. For the lower bound, very recently, Kamath et al. [20] proved a similar lower bound for the low-dimensional regime as in [23] but under the statistical setting.

3 Preliminaries

3.1 Differential Privacy

We say that $\mathbf{X}, \mathbf{X}' \in \mathbb{R}^{d \times n}$ are neighbors if they differ by one column, denoted $\mathbf{X} \sim \mathbf{X}'$.

Definition 1 (Differential Privacy (DP) [16]). *For $\varepsilon > 0$ and $\delta \geqslant 0$, a randomized mechanism $\mathcal{M} : \mathbb{R}^{d \times n} \to \mathcal{Y}$ satisfies (ε, δ)-DP if for any $\mathbf{X} \sim \mathbf{X}'$ and any $\mathcal{S} \subseteq \mathcal{Y}$, $\Pr[\mathcal{M}(\mathbf{X}) \in \mathcal{S}] \leqslant e^{\varepsilon} \cdot \Pr[\mathcal{M}(\mathbf{X}') \in \mathcal{S}] + \delta$.*

In particular, we call it *pure-DP* if $\delta = 0$; otherwise *approximate-DP*.

Definition 2 (Concentrated Differential Privacy (zCDP) [10]). *For $\rho > 0$, a randomized mechanism $M : \mathbb{R}^{d \times n} \to \mathcal{Y}$ satisfies ρ-zCDP if for any $\mathbf{X} \sim \mathbf{X}'$, $D_{\alpha}(\mathcal{M}(\mathbf{X})\|\mathcal{M}(\mathbf{X}')) \leqslant \rho \cdot \alpha$ for all $\alpha > 1$, where $D_{\alpha}(\mathcal{M}(\mathbf{X})\|\mathcal{M}(\mathbf{X}'))$ is the α-Rényi divergence between $\mathcal{M}(\mathbf{X})$ and $\mathcal{M}(\mathbf{X}')$.*

The relationship between these DP definitions is as follows. Pure-DP, also written as ε-DP, implies $\frac{\varepsilon^2}{2}$-zCDP, which further implies $\left(\frac{\varepsilon^2}{2} + \varepsilon\sqrt{2\log\frac{1}{\delta}}, \delta\right)$-DP for any $\delta > 0$.

To preserve ε-DP for a query Q, a standard mechanism is to add independent Laplace noises with scale proportional to the (global) ℓ_1-sensitivity of Q to each dimension.

Lemma 1 (Laplace Mechanism [13]). *Given $Q : \mathbb{R}^{d \times n} \to \mathbb{R}^k$, let $\mathrm{GS}_Q := \max_{\mathbf{X} \sim \mathbf{X}'} \|Q(\mathbf{X}) - Q(\mathbf{X}')\|_1$. The mechanism $\mathcal{M}(\mathbf{X}) = Q(\mathbf{X}) + \frac{\mathrm{GS}_Q}{\varepsilon} \cdot \mathbf{Y}$ where $\mathbf{Y} \sim \mathrm{Lap}(1)^k$, preserves ε-DP.*

The following composition property of ε-DP allows us to design algorithms in a modular fashion.

Lemma 2 (Basic Composition). *If $\mathcal{M}$ is an adaptive composition of mechanisms $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$, where each $\mathcal{M}_i$ satisfies ε_i-DP, then $\mathcal{M}$ satisfies $(\sum_i \varepsilon_i)$-DP.*

For ρ-zCDP, the standard method is the *Gaussian mechanism*:

Lemma 3 (Gaussian Mechanism [10]). *Given $Q : \mathbb{R}^{d \times n} \to \mathbb{R}^k$, let $\mathrm{GS}_Q := \max_{\mathbf{X} \sim \mathbf{X}'} \|Q(\mathbf{X}) - Q(\mathbf{X}')\|_2$. The mechanism $\mathcal{M}(\mathbf{X}) = Q(\mathbf{X}) + \frac{\mathrm{GS}_Q}{\sqrt{2\rho}} \cdot \mathbf{Y}$ where $\mathbf{Y} \sim \mathcal{N}(0, \mathbf{I}_{k \times k})$, preserves ρ-zCDP.*

It has been shown that the covariance matrix has an ℓ_2-sensitivity of $\frac{\sqrt{2}}{n}$ [8]. Thus, the Gaussian mechanism for covariance, denoted GaussCov, simply adds an independent Gaussian noise with scale $\frac{1}{\sqrt{\rho}n}$ to each entry of Σ. Considering that Σ is symmetric, symmetric noises also suffice, which preserve the symmetry of the privatized Σ. More precisely, we draw a random noise matrix $\mathbf{W}$ where $w_{j,k} \sim \mathcal{N}(0,1)$ i.i.d. for $1 \leqslant j \leqslant k \leqslant d$ and $w_{k,j} = w_{j,k}$, denoted as $\mathbf{W} \sim \mathrm{SGW}(d)$. Then GaussCov outputs $\widetilde{\Sigma}_{\mathrm{Gau}} = \Sigma + \frac{1}{\sqrt{\rho}n} \cdot \mathbf{W}$.

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A similar composition property exists for ρ-zCDP.

Lemma 4 (Composition Theorem [10]). *If $\mathcal{M}$ is an adaptive composition of algorithms $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$, where each $\mathcal{M}_i$ satisfies ρ_i-zCDP, then $\mathcal{M}$ satisfies $(\sum_i \rho_i)$-zCDP.*

3.2 The Sparse Vector Technique

The *Sparse Vector Technique* (SVT) [14] has as input a sequence of scalar queries, $f_1(\mathbf{X}), f_2(\mathbf{X}), \ldots, f_t(\mathbf{X})$, where each has sensitivity 1, and a threshold T. It aims to find the first query (if there is) whose answer is approximately above T. See Appendix A for the detailed algorithm. The SVT has been shown to satisfy ε-DP with following utility guarantee.

Lemma 5 (Extension of Theorem 3.24 in [16]). *With probability at least $1 - \beta$, SVT returns a k such that, for any $i < k$, $f_i(\mathbf{X}) \leqslant T + \frac{6}{\varepsilon} \log(2t/\beta)$, and if $k \neq t + 1$, then $f_k(\mathbf{X}) \geqslant T - \frac{6}{\varepsilon} \log(2t/\beta)$.*

3.3 Concentration Inequalities

Lemma 6 ([26]). *Given $\mathbf{Y} \sim \mathcal{N}(0, \mathbf{I}_{d \times d})$, with probability at least $1 - \beta$,*

$$\|\mathbf{Y}\|_2 \leqslant \eta(d, \beta) := \sqrt{d + 2\sqrt{d \log(1/\beta)} + 2\log(1/\beta)}.$$

Lemma 7 ([8, 26]). *Given $\mathbf{W} \sim \mathrm{SGW}(d)$, with probability at least $1 - \beta$,*

$$\|\mathbf{W}\|_2 \leqslant \upsilon(d, \beta) := 2\sqrt{d} + 2d^{1/6}\log^{1/3} d + \frac{6(1 + (\log d/d)^{1/3})\sqrt{\log d}}{\sqrt{\log(1 + (\log d/d)^{1/3})}} + 2\sqrt{2\log(1/\beta)}.$$

Also, with probability at least $1 - \beta$,

$$\|\mathbf{W}\|_F \leqslant \omega(d, \beta) := \sqrt{d^2 + 2\sqrt{d\log(2/\beta)}(1 + \sqrt{2(d-1)}) + 6\log(2/\beta)}.$$

Ignoring polylogarithmic factors, $\eta(d, \beta)$ and $\upsilon(d, \beta)$ are both in $\tilde{O}(\sqrt{d})$, while $\omega(d, \beta)$ is in $\tilde{O}(d)$. These concentration inequalities are very useful for error analysis. For example, the bound on $\|\mathbf{W}\|_F$ immediately implies that GaussCov has error $\frac{1}{\sqrt{\rho}n} \cdot \omega(d, \beta) = \tilde{O}(d/n)$.

4 Trace-sensitive Algorithm

The state-of-the-art trace-sensitive algorithm [2] first obtains an estimate of the eigenvalues, and then iteratively finds the eigenvectors by the exponential mechanism (EM), so we denote this algorithm as EMCov. Under zCDP, it has an error of $\tilde{O}(d^{3/4}\sqrt{\mathrm{tr}}/\sqrt{n} + \sqrt{d}/n)$. Below, we present an algorithm that is simpler, faster, and more accurate, improving the trace-dependent term by a $\sqrt{d}$-factor.

The first step of our algorithm SeparateCov (shown in Algorithm 1) is basically the same as EMCov, where we obtain an estimate of the eigenvalues with half of the privacy budget. [2] uses the Laplace mechanism for pure-DP; for zCDP, we use the Gaussian mechanism, which relies on the ℓ_2-sensitivity of $\mathbf{\Lambda}$, which we provide in Lemma 10 in the Appendix B. For the eigenvectors, we use GaussCov to obtain a privatized $\widetilde{\mathbf{\Sigma}}_{\mathrm{Gau}}$ with the other half of the privacy budget, and perform an eigendecomposition. Finally, we assemble the eigenvalues of eigenvectors to obtain a privatized $\mathbf{\Sigma}$. It should be clear that, after computing $\mathbf{\Sigma}$, SeparateCov just needs two eigendecompositions and one full matrix multiplication, plus some $O(d^2)$-time operations. On the other hand, EMCov performs $O(d)$ eigendecompositions and matrix multiplications, plus a nontrivial sampling procedure for the EM.

That SeparateCov satisfies ρ-zCDP easily follows from the privacy of the Gaussian mechanism and the composition property. The utility is given by the following theorem:

Theorem 1. *Given any $\rho > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$, and any $\beta > 0$, with probability at least $1 - \beta$, SeparateCov returns a $\widetilde{\mathbf{\Sigma}}_{\mathrm{Sep}}$ such that $\|\widetilde{\mathbf{\Sigma}}_{\mathrm{Sep}} - \mathbf{\Sigma}\|_F \leqslant \frac{2^{1.25}\sqrt{\mathrm{tr}}}{\rho^{1/4}\sqrt{n}} \cdot \sqrt{\upsilon\left(d, \frac{\beta}{2}\right)} + \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \eta\left(d, \frac{\beta}{2}\right) = \tilde{O}\left(\frac{d^{1/4}\sqrt{\mathrm{tr}}}{\sqrt{n}} + \frac{\sqrt{d}}{n}\right).$*

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Algorithm 1 SeparateCov

Input: data $\mathbf{X} \in \mathcal{B}_d^n$; privacy parameter $\rho > 0$.

1: $\mathbf{\Lambda} \leftarrow$ the eigenvalues of $\mathbf{\Sigma} = \frac{1}{n}\mathbf{X}\mathbf{X}^T$

2: $\widetilde{\mathbf{\Lambda}}_{\text{Sep}} \leftarrow \mathbf{\Lambda} + \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \mathbf{Y}$, where $\mathbf{Y} \sim \mathcal{N}(0, \mathbf{I}_{d \times d})$

3: $\widetilde{\mathbf{\Sigma}}_{\text{Gau}} \leftarrow \text{GaussCov}(\mathbf{X}, \frac{\rho}{2})$

4: $\widetilde{\mathbf{P}}_{\text{Sep}} \leftarrow \mathbf{P}\left[\widetilde{\mathbf{\Sigma}}_{\text{Gau}}\right]$

5: $\widetilde{\mathbf{\Sigma}}_{\text{Sep}} \leftarrow \widetilde{\mathbf{P}}_{\text{Sep}}\widetilde{\mathbf{\Lambda}}_{\text{Sep}}\widetilde{\mathbf{P}}_{\text{Sep}}^T$

6: **return** $\widetilde{\mathbf{\Sigma}}_{\text{Sep}}$

Remark While SeparateCov strictly improves over EMCov, it does not dominate GaussCov: When $\text{tr} < \tilde{O}(d^{3/2}/n)$, SeparateCov is better; otherwise, GaussCov is better. EMCov is better than GaussCov for a smaller trace range: $\text{tr} < \tilde{O}(\sqrt{d}/n)$.

Theorem 1 implies our worst-case bound by taking $\text{tr} = 1$:

Theorem 2. *Given any $\rho > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$, and any $\beta > 0$, with probability at least $1 - \beta$,* SeparateCov *returns a* $\widetilde{\mathbf{\Sigma}}_{\text{Sep}}$ *such that* $\|\widetilde{\mathbf{\Sigma}}_{\text{Sep}} - \mathbf{\Sigma}\|_F = \tilde{O}\left(\frac{d^{1/4}}{\sqrt{n}} + \frac{\sqrt{d}}{n}\right)$.

5 Tail-sensitive Algorithm

5.1 Clipped Covariance

Clipping is a common technique to reduce the sensitivity of functions at the expense of some bias. Given $\tau \geqslant 0$ and a vector $X \in \mathbb{R}^d$, let $\text{Clip}(X, \tau) = \min\left(1, \frac{\tau}{\|X\|_2}\right) \cdot X$. Similarly, for any $\mathbf{X} \in \mathbb{R}^{d \times n}$, $\text{Clip}(\mathbf{X}, \tau)$ denotes the matrix whose columns have been clipped to have norm at most τ. Clipping can be applied to both GaussCov and SeparateCov with a given τ: just run the mechanism on $\frac{1}{\tau} \cdot \text{Clip}(\mathbf{X}, \tau)$ and scale the result back by τ^2. We denote the clipped versions of the two mechanisms as ClipGaussCov and ClipSeparateCov, respectively.

The following lemma bounds the bias caused by clipping in terms of the τ-tail as defined in (1).

Lemma 8. $\|\mathbf{\Sigma}(\mathbf{X}) - \mathbf{\Sigma}(\text{Clip}(\mathbf{X}, \tau))\|_F \leqslant \frac{1}{n}\sum_i \left(\|X_i\|_2^2 - \tau^2\right) \cdot \mathbb{I}\left(\|X_i\|_2 \geqslant \tau\right) \leqslant \gamma(\mathbf{X}, \tau)$.

Thus, running the better of ClipGaussCov and ClipSeparateCov yields a total error of $\text{Noise}(\mathbf{X}, \tau, \rho, \beta) + \gamma(\mathbf{X}, \tau)$, where

$$\text{Noise}(\mathbf{X}, \tau, \rho, \beta) = \min\left(\frac{\tau^2}{\sqrt{\rho}n} \cdot \omega(d, \beta), \frac{2^{1.25}\tau\sqrt{\text{tr}}}{\rho^{1/4}\sqrt{n}} \cdot \sqrt{\upsilon\left(d, \frac{\beta}{2}\right)} + \frac{\sqrt{2}\tau^2}{\sqrt{\rho}n} \cdot \eta\left(d, \frac{\beta}{2}\right)\right), \quad (4)$$

which is the exact version of (2). Note that the trace-sensitive term is only scaled by τ, which follows from the proof of Theorem 1 when all X_i live in $\tau \cdot \mathcal{B}_d$.

Ideally, we would like to find the optimal noise-bias trade-off, i.e., achieving an error of $\min_\tau(\text{Noise}(\mathbf{X}, \tau) + \gamma(\mathbf{X}, \tau))$. Two issues need to be addressed towards this goal: The first, minor, issue is that tr is sensitive, so we cannot use it directly to decide whether to use ClipGaussCov or ClipSeparateCov. This can be addressed by using a privatized upper bound of tr. The more challenging problem is how to find the optimal τ in a DP fashion. This problem has been well studied for the clipped mean estimator [18, 3], where it can be shown that setting τ to be the $\tilde{O}(\sqrt{d})$-th largest $\|X_i\|_2$ results in the optimal noise-bias trade-off [18]. Then the problem boils down to finding a privatized quantile, for which multiple solutions exist [18, 12, 32, 6, 37]. For the clipped mean estimator, using such a quantile of the norms results in the optimal trade-off because $\text{Noise}(\mathbf{X}, \tau)$ takes the simple form $\tilde{O}(\tau\sqrt{d}/n)$. In fact, if we only had ClipGaussCov, setting τ to be the $\tilde{O}(d)$-th largest $\|X_i\|_2$ would also yield an optimal trade-off, as ClipGaussCov is really just clipped mean in d^2 dimensions. However, due to the trace-sensitive noise term, it is no longer the case. In Appendix F, we give examples showing that no quantile, whose rank may arbitrarily depend on d, n, tr, can achieve

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an optimal trade-off even ignoring polylogarithmic factors. It thus calls for a new threshold-finding mechanism, which we describe next.

5.2 Adaptive Covariance: Finding the Optimal Clipping Threshold

Our basic idea is to try successively smaller values $\tau = 1, \frac{1}{2}, \frac{1}{4}, \ldots$. As we reduce τ, the noise decreases while the bias increases. We should stop when they are approximately balanced, which would yield a near-optimal τ.

To do so in a DP manner, we need to quantify the noise and bias. Consider the bias first. Given a τ, we divide the interval $(\tau, 1]$ into sub-intervals $(\tau, 2\tau], (2\tau, 4\tau], \ldots, (\frac{1}{2}, 1]$. For any $X \in \mathbf{X}$ such that $\|X\|_2 \in (2^s, 2^{s+1}]$, let $\check{X} = \mathrm{Clip}(X, \tau)$ and then by Lemma 8,

$$\|XX^T - \check{X}\check{X}^T\|_F \leqslant 2^{2s+2} - \tau^2. \tag{5}$$

That is, clipping X can at most lead to $\frac{1}{n} \cdot (2^{2s+2} - \tau^2)$ bias. Besides, since $\|X\|_2 \in (2^s, 2^{s+1}]$, we have

$$2^{2s+2} - \tau^2 \leqslant 2^{2s+2} \leqslant 2 \cdot \|X\|_2^2. \tag{6}$$

Then, given $\mathbf{X}$, for any $s \in \mathbb{Z}$, we define $\mathrm{Count}_s(\mathbf{X}) := \left|\left\{X_i : \|X_i\|_2 \in (2^s, 2^{s+1}]\right\}\right|$. It is easy to see for any $\mathbf{X} \sim \mathbf{X}'$, Count_s differs by at most 1, so does the sum of any subset of Count_s's. We can define an upper bound on the bias: $\widehat{\mathrm{Bias}}(\mathbf{X}, \tau) := \frac{1}{n} \cdot \sum_{s=\log_2(\tau)}^{s<0} \mathrm{Count}_s \cdot (2^{2s+2} - \tau^2)$. Let $\check{\mathbf{X}} = \mathrm{Clip}(\mathbf{X}, \tau)$. By (5) and (6), we have

$$\frac{1}{n}\|\mathbf{X}\mathbf{X}^T - \check{\mathbf{X}}\check{\mathbf{X}}^T\| \leqslant \widehat{\mathrm{Bias}}(\mathbf{X}, \tau) \leqslant 2 \cdot \gamma(\mathbf{X}, \tau). \tag{7}$$

By the property of Count_s's, given any τ, the sensitivity of $\widehat{\mathrm{Bias}}(\cdot, \tau)$ is bounded by $\frac{1}{n}$.

Now we turn to the noise. Recall that $\mathrm{Noise}(\mathbf{X}, \tau, \rho, \beta)$ is the smaller of two parts. The first part $\mathrm{GaussNoise}(\tau, \rho, \beta) := \tau^2 \cdot \frac{1}{\sqrt{\rho}n} \cdot \omega(d, \beta)$ is independent of $\mathbf{X}$, so can be used directly. The second part depends on tr, is thus sensitive. Since its sensitivity is $\frac{1}{n}$, we can easily privatize it by adding a Gaussian noise of scale $\Theta\left(\frac{1}{\sqrt{\rho}n}\right)$. For technical reasons, we need to use an upper bound, so we add $\Theta\left(\frac{\log(1/\beta)}{\sqrt{\rho}n}\right)$ to it so as to obtain a privatized $\widehat{\mathrm{tr}} \geqslant \mathrm{tr}$. Then we set

$$\mathrm{SeparateNoise}(\widehat{\mathrm{tr}}, \tau, \rho, \beta) := \tau \cdot \frac{2^{1.25}\sqrt{\widehat{\mathrm{tr}}}}{\rho^{1/4}\sqrt{n}} \cdot \sqrt{\upsilon\left(d, \frac{\beta}{2}\right)} + \tau^2 \cdot \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \eta\left(d, \frac{\beta}{2}\right),$$

and use $\widehat{\mathrm{Noise}}(\widehat{\mathrm{tr}}, \tau, \rho, \beta) := \min\left(\mathrm{GaussNoise}(\tau, \rho, \beta), \mathrm{SeparateNoise}(\widehat{\mathrm{tr}}, \tau, \rho, \beta)\right)$ as a DP upper bound of $\mathrm{Noise}(\mathbf{X}, \tau, \rho, \beta)$. Note that given $\widehat{\mathrm{tr}}$, $\widehat{\mathrm{Noise}}(\widehat{\mathrm{tr}}, \tau, \rho, \beta)$ is independent of $\mathbf{X}$.

Finally, we run SVT on the following sequence of sensitivity-1 queries with $T = 0$:

$$\mathrm{Diff}(\mathbf{X}, \widehat{\mathrm{tr}}, \tau, \rho, \beta) := n \cdot \left(\widehat{\mathrm{Bias}}(\mathbf{X}, \tau) - \widehat{\mathrm{Noise}}(\widehat{\mathrm{tr}}, \tau, \rho, \beta)\right), \tau = 1, \frac{1}{2}, \ldots, 2^{-dn}.$$

The SVT would return a τ that balances the bias and noise. After finding such a τ, we choose to run either GaussCov or SeparateCov by comparing $\mathrm{GaussNoise}(\tau, \rho, \beta)$ and $\mathrm{SeparateNoise}(\widehat{\mathrm{tr}}, \tau, \rho, \beta)$. As the sequence consists of dn queries, SVT has an error of $O(\log(dn))$, which, as we will show, affects the optimality by a logarithmic factor. Meanwhile, the smallest τ we search over will induce an additive 2^{-dn} error.

The algorithm above can almost give us the desired error bound in (3), except that one thing may go wrong: The SVT introduces an error that is a logarithmic factor larger than the optimum, but at least $\widetilde{\Omega}(1/n)$. This would be fine as long as there is one X_i with $\|X_i\|_2 \geqslant \widetilde{\Omega}(1)$, so that the optimum error is $\widetilde{\Omega}(1/n)$. However, when all the X_i's have very small norms, say $1/n^2$, the $\widetilde{\Omega}(1/n)$ error from SVT would not preserve optimality. To address this issue, we first find the radius $\mathrm{rad}(\mathbf{X}) = \max_i \|X_i\|_2$, and use it to clip $\mathbf{X}$. The following lemma shows that, under DP, it is possible to find a 2-approximation of $\mathrm{rad}(\mathbf{X})$ plus an additive b so that only $O(\log\log(1/b))$ vectors are clipped. This allows us to set $b = 2^{-dn}$ while only incurring an $O(\log dn)$ error. Nicely, they match the additive and multiplicative errors that already exist from the SVT, so there is no asymptotic degradation in the optimality.

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Algorithm 2 AdaptiveCov

Input: data $\mathbf{X} \in \mathcal{B}_d^n$; privacy parameter $\rho > 0$; high probablity parameter β.

1: $\tilde{r} \leftarrow \text{PrivRadius}(\mathbf{X}, \frac{\sqrt{\rho}}{2}, \frac{\beta}{8}, 2^{-2dn})$
2: $\widetilde{\mathbf{X}} \leftarrow \text{Clip}(\mathbf{X}, \tilde{r})$
3: $\widetilde{\text{tr}} \leftarrow \frac{1}{n} \sum_i \|\widetilde{X}_i\|_2^2$
4: $\widehat{\text{tr}} \leftarrow \min\left(\widetilde{\text{tr}} + \frac{2\tilde{r}^2}{\sqrt{\rho}n} \cdot \mathcal{N}(0,1) + \frac{2\sqrt{2}\tilde{r}^2}{\sqrt{\rho}n} \cdot \sqrt{\log(8/\beta)}, \ \tilde{r}^2\right)$
5: $\tilde{t} \leftarrow \log_2(\tilde{r}) + 1 - \text{SVT}\left(\left\{\text{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\text{tr}}, \tilde{r}, \frac{\rho}{2}, \frac{\beta}{2}\right), \text{Diff}\left((\widetilde{\mathbf{X}}, \widehat{\text{tr}}, \frac{\tilde{r}}{2}, \frac{\rho}{2}, \frac{\beta}{2}\right), \ldots, \text{Diff}\left((\widetilde{\mathbf{X}}, \widehat{\text{tr}}, 2^{-dn}, \frac{\rho}{2}, \frac{\beta}{2}\right)\right\}, 0, \frac{\sqrt{\rho}}{\sqrt{2}}\right)$
6: $\tilde{\tau} \leftarrow \min\left(2^{\tilde{t}+1}, \tilde{r}\right)$
7: **if** $\text{SeparateNoise}(\widehat{\text{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2}) \geqslant \text{GaussNoise}(\tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2})$
8: $\quad \widetilde{\mathbf{\Sigma}}_{\text{Ada}} \leftarrow \text{ClipGaussCov}(\widetilde{\mathbf{X}}, \frac{\rho}{2}, \tilde{\tau})$
9: **else**
10: $\quad \widetilde{\mathbf{\Sigma}}_{\text{Ada}} \leftarrow \text{ClipSeparateCov}(\widetilde{\mathbf{X}}, \frac{\rho}{2}, \tilde{\tau})$
11: **return** $\widetilde{\mathbf{\Sigma}}_{\text{Ada}}$

Lemma 9 ([12]). *For any $\varepsilon > 0$, $\beta > 0$ and $b > 0$, given $\mathbf{X} \in \mathcal{B}_d^n$, with probability at least $1 - \beta$, $\texttt{PrivRadius}$ returns a $\tilde{r} = \texttt{PrivRadius}(\mathbf{X}, \varepsilon, \beta, b)$ such that $\tilde{r} \leqslant 2 \cdot \text{rad}(\mathbf{X}) + b$ and $|\{\|X_i\|_2 > \tilde{r}\}| = O\left(\frac{1}{\varepsilon} \log \frac{\log(\text{rad}(\mathbf{X})/b)}{\beta}\right)$.*

The complete algorithm is given in Algorithm 2. Its privacy follows from the privacy of $\texttt{PrivRadius}$, SVT, GaussCov, SeparateCov, and the composition theorem of zCDP; its utility is analyzed in the following theorem:

Theorem 3. *Given any $\rho > 0$ and $\beta > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$, with probability at least $1 - \beta$, AdaptiveCov returns a $\widetilde{\mathbf{\Sigma}}_{\text{Ada}}$ such that*

$$\left\|\widetilde{\mathbf{\Sigma}}_{\text{Ada}} - \mathbf{\Sigma}\right\|_F = O\left(\min_\tau\left(\text{Noise}\left(\mathbf{X}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right) \cdot \frac{\log(1/\beta)^{1/4}}{\rho^{1/4}} \cdot \log(n) + \gamma(\mathbf{X}, \tau) \cdot \frac{\log(dn/\beta)}{\sqrt{\rho}}\right) + 2^{-dn}\right)$$

$$= \tilde{O}\left(\min_\tau\left(\text{Noise}\left(\mathbf{X}, \tau\right) + \gamma(\mathbf{X}, \tau)\right) + 2^{-dn}\right).$$

6 Experiments

We conducted experiments[5] to evaluate our algorithms on both synthetic and real-world datasets. We compare SeparateCov and AdaptiveCov against GaussCov [17], EMCov [2]. We implemented EMCov in Python following the pseudo-code provided in [2] and the descriptions of the sampling algorithm in [24]. We also tested CoinPress [8], but since it is designed to minimize the Mahalanobis error, it does not perform well when measured in Frobenius error. The two distance measures coincide when $\mathbf{\Sigma}$ is well-conditioned but in this case, CoinPress degenerates into GaussCov. Therefore, we omit it from the reported results. As a baseline, we include returning a zero matrix, which has error $O(\text{tr})$, hence a trivial trace-sensitive algorithm. When $\text{rad}(\mathbf{X})$ is much smaller than 1, it is unfair for GaussCov and EMCov, so we scale all datasets such that $0.5 \leqslant \text{rad}(\mathbf{X}) \leqslant 1$. As a result, we do not need the step to obtain a private radius in AdaptiveCov, either. Each experiment is repeated 50 times, and we report the average error. Furthermore, we have also conducted experiments under pure-DP; the results can be found in Appendix J.

6.1 Synthetic Datasets

We generate synthetic datasets by first following the method in [2], to obtain a matrix $\mathbf{X} = \mathbf{Z}\mathbf{U}$, where $\mathbf{U} \in \mathbb{R}^{d \times d}$ is sampled from $U(0,1)$, and $\mathbf{Z} \in \mathbb{R}^{n \times d}$ is sampled from $\mathcal{N}(0, \mathbf{I})$. Then the vectors in $\mathbf{X}$ are adjusted to be centred at 0 and their norms scaled. In [2], the vectors are scaled to have unit ℓ_2 norm; in our experiments, to better control tr and data skewness, we scale the norms so that they

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[5]The code can be found at $\texttt{https://github.com/hkustDB/PrivateCovariance}$.

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(a) $n = 1000$.   (b) $n = 4000$.   (c) $n = 16000$.
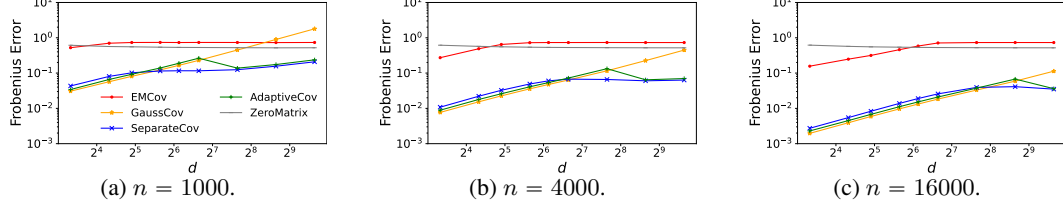
Figure 2: Results on synthetic datasets fixing $\mathrm{tr} = 1$.

(a) various tr.   (b) various d.   (c) various n.   (d) various ρ.
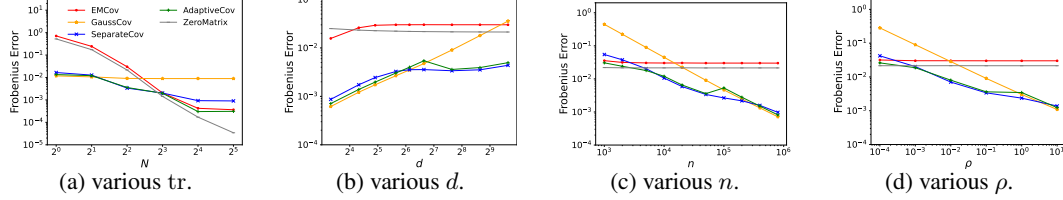
Figure 3: Results on synthetic datasets as d, n, N or ρ varies.

follow the Zipf's law. More precisely, we divide the norms into N bins. The number of vectors in the k-th bin is proportional to $1/k^s$ and their norm is 2^{k-N}. The parameter s characterizes the skewness, which we fix as $s = 3$. Note that $N = 1$ corresponds to the unit-norm case with $\mathrm{tr} = 1$.

The results on $\mathrm{tr} = 1$ case are shown in Figure 2, which correspond to the worst-case bounds. The ρ here is fixed at 0.1 and we examine the error growth w.r.t. d for $n = 1000, 4000, 16000$. The results generally agree with the theory: For low d, GaussCov is (slightly) better than SeparateCov, while the latter is much better for high d. AdaptiveCov is able to choose the better of the two adaptively, with a small cost due to allocating some privacy budget to estimate tr. Actually, if AdaptiveCov is given the precondition that all norms are 1, this extra cost can be saved.

Next, we vary one of the parameters while fixing the others at their default values $d = 200$, $n = 50000$, $N = 4$ and $\rho = 0.1$, and the results are reported in Figure 3. The most interesting case is Figure 3(a), where we increase N, hence reducing tr, which demonstrates the trace-sensitive bounds. Clearly, GaussCov is not trace-sensitive, while the other 4 methods are. In fact, returning the zero matrix is the best trace-sensitive algorithm if tr is sufficiently small. However, this may not be very meaningful in practice, as $N = 2^5$ means that most data have norm 2^{-31} but a few have norm 1. These few may be outliers and should be removed anyway. Figure 3(b)–(d) shows that higher d, smaller n, and smaller ρ all have similar effects, i.e., SeparateCov becomes better while GaussCov becomes worse, while AdaptiveCov is able to pick the better one.

(a) $n = 6000, \mathrm{tr} < 1$.   (b) $n = 20000, \mathrm{tr} < 1$.   (c) $n = 60000, \mathrm{tr} < 1$.
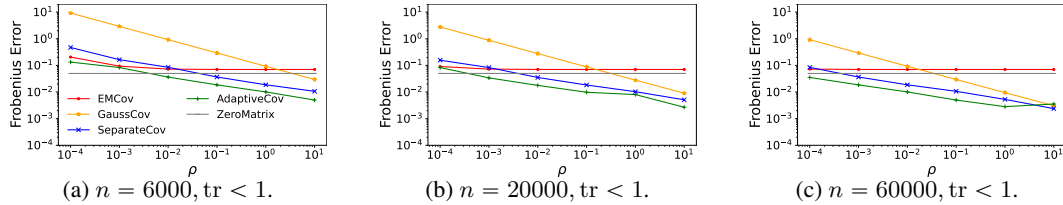
Figure 4: Results on MNIST dataset.

6.2 Real-world Datasets

We also evaluate the algorithms on two real-world datasets. The first dataset is the MNIST [27] dataset, which contains images of handwritten digits. We use its training dataset which contains $60,000$ images represented as vectors in $\mathbb{Z}_{255}^d$, where $d = 784 = 28 \times 28$. These vectors are normalized by $255\sqrt{d}$ in the experiments. We estimate $\widetilde{\Sigma}$ using samples containing all the digits, we also estimate $\widetilde{\Sigma}$ corresponding to individual digits (reported in the Appendix J). In the first case, $\widetilde{\Sigma}$ can be used for further dimensionality reduction analysis; in the second case, individual $\widetilde{\Sigma}$ can be used for modelling the distributions of individual digits, which together can be used in a collective

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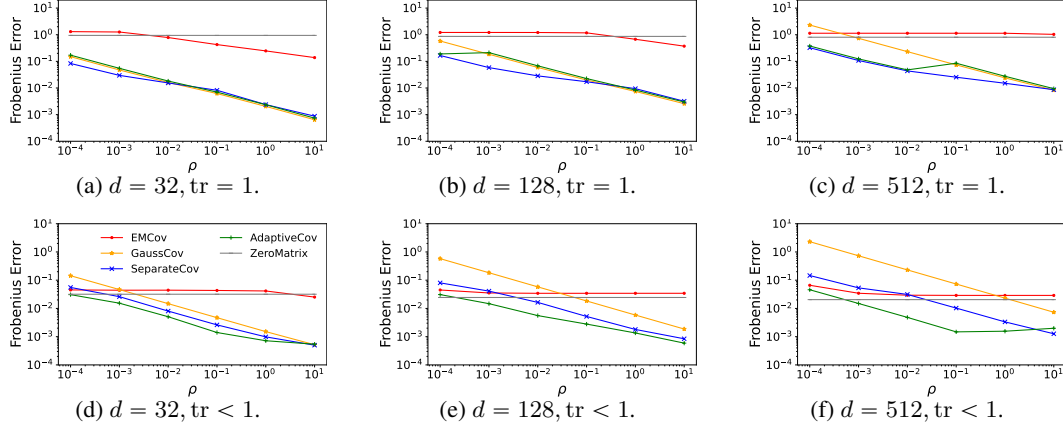
Figure 5: Results on the news dataset.

model for classification (e.g. using a mixture model). The second dataset contains news commentary data [33] consisting of approximately $15,000$ articles, each containing $500-4300$ words, which we convert into vectors of various dimensions using the hashing trick implemented in the scikit-learn package. In this case, the estimated $\widetilde{\Sigma}$ can be used to help with further feature selection for NLP models, for example. These vectors are normalized to have unit ℓ_2 norm or normalized by the max ℓ_2 norm.

The experimental results on these two real dataset are shown in Figure 4 and 5, where we vary n, d, and ρ, respectively. On these results, we see that $\mathrm{GaussCov}$ never outperforms $\mathrm{SeparateCov}$, except for a very small advantage in a few cases where we have $\mathrm{tr} = 1$, a low d, and a high ρ. Another interesting observation is that $\mathrm{AdaptiveCov}$ outperforms both $\mathrm{GaussCov}$ and $\mathrm{SeparateCov}$ in many cases, something that is not apparent on the synthetic datasets. We believe that this is because these real datasets have heavier tails than the Zipf distribution (we used $s = 3$ for Zipf), which makes the adaptive clipping threshold selection more effective. This really demonstrates the benefits of a tail-sensitive bound.

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Algorithm 3 SVT

Input: query sequence $(f_1(\mathbf{X}), \ldots, f_t(\mathbf{X}))$; threshold T; privacy parameter $\varepsilon > 0$.
Output: index k
  1: $\widetilde{T} \leftarrow T + \mathrm{Lap}(2/\varepsilon)$
  2: **for** $k \leftarrow 1, 2, \ldots, t$
  3:      **if** $f_k(\mathbf{X}) + \mathrm{Lap}(4/\varepsilon) \geqslant \widetilde{T}$
  4:          **return** k
  5: **return** $t + 1$

A   Sparse Vector Technique

The SVT is shown in Algorithm 3.

B   Technical Lemmata

Lemma 10. *For any* $\mathbf{X}, \mathbf{X}' \in \mathcal{B}_d^n$, $\mathbf{X} \sim \mathbf{X}'$,

$$\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|_F \leqslant \frac{\sqrt{2}}{n}.$$

Proof. The proof of this lemma requires the following technical result.

Fact 1. Let $a_k, b_k \geqslant c_k \geqslant 0$ for $1 \leqslant k \leqslant d$. Then

$$\sum_{k=1}^{d} (a_k - b_k)^2 \leqslant \left(\sum_{k=1}^{d} (a_k - c_k) \right)^2 + \left(\sum_{k=1}^{d} (b_k - c_k) \right)^2.$$

Let $f_k : \mathcal{B}(r)^n \to \mathbb{R}_{\geqslant 0}$ be defined as $f_k(\mathbf{X}) = \lambda_k(\frac{1}{n}\mathbf{X}\mathbf{X}^T)$. Suppose $\mathbf{X}$ and $\mathbf{X}'$ differ in the ith column. Let $\mathbf{X}_{(-i)}$ denote the matrix obtained from $\mathbf{X}$ by removing the ith column. Then $f_k(\mathbf{X}) = \lambda_k \left[\frac{1}{n}\mathbf{X}_{(-i)}\mathbf{X}_{(-i)}^T + \frac{1}{n}X_i X_i^T \right]$. Note that $X_i X_i^T$ is a rank-one matrix with eigenvalues 0 and $\|X_i\|_2^2$, so by Weyl's inequality $f_k(\mathbf{X}_{(-i)}) \leqslant f_k(\mathbf{X})$ and $f_k(\mathbf{X}_{(-i)}) \leqslant f_k(\mathbf{X}')$. We have

$$\sum_{k=1}^{d} \left(f_k(\mathbf{X}) - f_k(\mathbf{X}_{(-i)}) \right) = \sum_{k=1}^{d} f_k(\mathbf{X}) - \sum_{k=1}^{d} f_k(\mathbf{X}_{(-i)})$$

$$= \mathrm{tr} \left[\frac{1}{n}\mathbf{X}\mathbf{X}^T \right] - \mathrm{tr} \left[\frac{1}{n}\mathbf{X}_{(-i)}\mathbf{X}_{(-i)}^T \right]$$

$$= \frac{1}{n}\|X_i\|_2^2 \leqslant \frac{1}{n}.$$

Similarly, $\sum_{k=1}^{d} \left(f_k(\mathbf{X}') - f_k(\mathbf{X}_{(-i)}) \right) \leqslant \frac{1}{n}$. Now by Fact 1, with $a_k = f_k(\mathbf{X}), b_k = f_k(\mathbf{X}')$ and $c_k = f_k(\mathbf{X}_{(-i)})$, we have

$$\sum_{k=1}^{d} \left(f_k(\mathbf{X}) - f_k(\mathbf{X}') \right)^2 \leqslant \left(\sum_{k=1}^{d} \left(f_k(\mathbf{X}) - f_k(\mathbf{X}_{(-i)}) \right) \right)^2 + \left(\sum_{k=1}^{d} \left(f_k(\mathbf{X}') - f_k(\mathbf{X}_{(-i)}) \right) \right)^2$$

$$\leqslant \left(\frac{1}{n} \right)^2 + \left(\frac{1}{n} \right)^2 = \frac{2}{n^2}. \tag{8}$$

Taking the square root on both sides then gives the target inequality. $\qquad\square$

C   zCDP error bound for EMCov

The state-of-the-art trace-sensitive algorithm [2] has error $\tilde{O}\left(\sqrt{d \sum_{i=1}^{d}(\lambda_i/\varepsilon_i)} + \sqrt{d} \right)$. Under pure-DP, setting $\varepsilon_i = \varepsilon/d$ for all i turns the bound into $\tilde{O}(d\sqrt{\mathrm{tr}} + \sqrt{d})$. For zCDP, we can set

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$\varepsilon_i = \tilde{O}(\sqrt{\rho/d})$, which implies $\frac{\rho}{d}$-zCDP. Then the whole process follows ρ-zCDP by composition. Alternatively, one could adaptively set ε_i according to a privatized λ_i as mentioned in [2], but we do not consider this optimization as it does not yield a better trace-sensitive bound. Also, as observed from their experiments [2], this optimization does not have a big effect on the performance of their algorithm.

The algorithm in [2] estimates $\boldsymbol{\Sigma} = \mathbf{X}\mathbf{X}^T$. The $\frac{1}{n}$ normalization factor scales down both the error and tr by a factor of n. This yields the stated bounds in Section 1.1 and I.

D The lower bound of mean estimation from statistical setting to empirical setting

[19] considers the statistical setting, where each $X_i \sim \mathcal{P}$ and $\mathcal{P}$ is a distribution over $\{-1, 1\}^d$ and show the lower bound of (ε, δ)-DP for $\delta \leqslant \frac{\varepsilon}{64n}$ is $\Omega(\frac{d}{\varepsilon n})$ (see Lemma 6.2 in that paper). Since zCDP implies approximate-DP and each instance under statistical setting is naturally a case under empirical setting, their lower bound implies the one of zCDP under empirical setting. Besides, under their setting, data have the ℓ_2 norm equal to $\sqrt{d}$. Scaling the result by $\sqrt{d}$ and we get the lower bound under our setting.

E Applying the projection mechanism to covariance estimation

Similar as $\mathrm{SeparateCov}$, the projection mechanism first constructs $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}$ by adding Gaussian noise on $\boldsymbol{\Sigma}$. Let K be the set of covariance matrices for all possible datasets $\mathbf{X}$, i.e.,

$$K = \{\boldsymbol{\Sigma}(\mathbf{X}) : \mathbf{X} \in (\mathcal{B}_d)^n\}.$$

The projection mechanism return a $\overline{\boldsymbol{\Sigma}}$ such that,

$$\overline{\boldsymbol{\Sigma}} = \arg\min_{\overline{\boldsymbol{\Sigma}} \in K} \|\overline{\boldsymbol{\Sigma}} - \widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}\|_F. \tag{9}$$

For the privacy, projection mechanism does some post-processing on $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}$ thus naturally preserves DP. For the utility, by Lemma 1 of [31], $\|\boldsymbol{\Sigma} - \overline{\boldsymbol{\Sigma}}\|_F$ is bounded by maximum eigenvalue of noise matrix added in $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}$, which is further bounded by $\tilde{O}(\sqrt{d})$ by Lemma 7. For the efficiency, after using obtaining $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}$, which requires time $O(n + d^2)$, the projection mechanism uses Frank-Wolfe algorithm to solve the optimal projection of $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}$ on K, where there are n iterations with each one requires computing the SVD of a PSD matrix and the total running time is $O(nd^3)$. Below, we show that optimal projection can be solved by only computing one SVD thus the cost can be reduced to $O(d^3)$.

Lemma 11. *Let $\mathbf{B}$ be a symmetric $d \times d$ matrix with $\mathbf{B} = \mathbf{P}\mathbf{U}\mathbf{P}^T$, and K is defined as (9). Then there is $\mathbf{A}' \in \arg\min_{\mathbf{A} \in K} \|\mathbf{B} - \mathbf{A}\|_F$ where $\mathbf{A}' = \mathbf{P}\mathbf{U}'\mathbf{P}^T$ for some diagonal matrix $\mathbf{U}' \succeq 0$.*

Proof. It suffices to show for any $\mathbf{A} \in K$, we can find a such $\mathbf{A}'$ and $\|\mathbf{B} - \mathbf{A}'\|_F \leqslant \|\mathbf{B} - \mathbf{A}\|_F$.

Let $\mathbf{U}' = \mathrm{diag}(\mathbf{P}^T \mathbf{A}\mathbf{P})$, then, $\mathrm{tr}(\mathbf{U}') = \mathrm{tr}(\mathbf{P}^T\mathbf{A}\mathbf{P}) = \mathrm{tr}(\mathbf{A}) \leqslant 1$. Also, $U'_{i,i} = P_i^T\mathbf{A}P_i \leqslant \lambda_1(\mathbf{A}) \leqslant 1$ and $U'_{i,i} = P_i^T\mathbf{A}P_i \geqslant 0$ since $\mathbf{A}$ is PSD. Overall, $\mathbf{A}' \in K$.

Furthermore,

$$\begin{aligned}
\|\mathbf{B} - \mathbf{A}\|_F^2 &= \|\mathbf{P}\mathbf{U}\mathbf{P}^T - \mathbf{A}\|_F^2 \\
&= \|\mathbf{U} - \mathbf{P}^T\mathbf{A}\mathbf{P}\|_F^2 \\
&= 2\sum_{i<j}(\mathbf{P}^T\mathbf{A}\mathbf{P})_{i,j}^2 + \sum_i \left(U_{i,i} - (\mathbf{P}^T\mathbf{A}\mathbf{P})_{i,i}\right)^2 \\
&\geqslant \sum_i \left(U_{i,i} - (\mathbf{P}^T\mathbf{A}\mathbf{P})_{i,i}\right)^2 \\
&= \sum_i \left(U_{i,i} - U'_{i,i}\right)^2
\end{aligned}$$

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$$= \|\mathbf{B} - \mathbf{A}'\|_F^2$$

$$\square$$

Let $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}} = \widetilde{P}\widetilde{\boldsymbol{\Lambda}}\widetilde{P}^T$ and $\widetilde{\boldsymbol{\Lambda}} = \mathrm{diag}(\widetilde{\Lambda}_1, \ldots, \widetilde{\Lambda}_d)$. From Lemma 11, to find $\arg\min_{\overline{\boldsymbol{\Sigma}} \in K} \|\overline{\boldsymbol{\Sigma}} - \widetilde{\boldsymbol{\Sigma}}_{\mathrm{Gau}}\|_F$, it suffices to find a diagonal matrix $\mathbf{U}$ such that $0 \leqslant U_{i,i}$, $\sum_i U_i \leqslant 1$ and $\sum_i \left(\widetilde{\Lambda}_{i,i} - U_{i,i} \right)^2$ is minimized.

F On quantile-based truncation thresholds

Given that quantile-based thresholds have been used for mean estimation, one might ask whether such a threshold exists for covariance estimation, where τ corresponds to some $g(d, n, \mathrm{tr})$th quantile of the ℓ_2 norms of the vectors, for some function $g(\cdot)$. In this section, we provide a counterexample to show that there is no universal optimal quantile which works in all cases. Specifically, we construct two datasets with the same d, n, and tr whose optimal quantile-based thresholds are asymptotically different. Let $d = \Theta(n^{9/16})$, $\mathrm{tr} = \frac{d}{n} = \Theta(n^{-7/16})$. With such setting, SeparateCov performs better than GaussCov.

For the first dataset, we have $n^{9/16}$ number of vectors with ℓ_2 norm equal to one while all others have norm zero. In this case, we have 1 and 0 as the only candidate truncation thresholds. Using $\tau = 1$ gives an error of $\Theta(n^{-37/64})$ and while $\tau = 0$ corresponds to an error of $\Theta(n^{-7/16})$. Thus, $\tau = 1$ should be used, which corresponds a quantile between the 0th and $n^{9/16}$th.

For the second dataset, we have $O(1)$ number of vectors with ℓ_2 norm square equal to one, $n^{19/32}$ number of vectors with ℓ_2 norm square equal to $n^{-13/32}$ and all others have ℓ_2 norm square equal to $n^{-7/16}$. Here, we have three candidates: 1, $n^{-13/64}$ and $n^{-7/32}$. $\tau = 1$ corresponds to the error $\Theta(n^{-37/64})$. $\tau = n^{-13/64}$ corresponds to the error $\Theta(n^{-25/32})$. $\tau = n^{-7/32}$ corresponds to the error $\Theta(n^{-51/64})$. Thus, we should choose $\tau = n^{-7/32}$, which corresponds to a quantile between the $(\tilde{O}(1) + n^{19/32})$th and nth.

G Proofs for Trace-sensitive Algorithm

Theorem 1. *Given any $\rho > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$, and any $\beta > 0$, with probability at least $1 - \beta$,* SeparateCov *returns a $\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Sep}}$ such that* $\|\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Sep}} - \boldsymbol{\Sigma}\|_F \leqslant \frac{2^{1.25}\sqrt{\mathrm{tr}}}{\rho^{1/4}\sqrt{n}} \cdot \sqrt{\upsilon\left(d, \frac{\beta}{2}\right)} + \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \eta\left(d, \frac{\beta}{2}\right) = \tilde{O}\left(\frac{d^{1/4}\sqrt{\mathrm{tr}}}{\sqrt{n}} + \frac{\sqrt{d}}{n}\right).$

Proof. The error can be decomposed into two parts:

$$\begin{aligned}
\|\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Sep}} - \boldsymbol{\Sigma}\|_F &= \|\widetilde{\mathbf{P}}_{\mathrm{Sep}}\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Sep}}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T - \boldsymbol{\Sigma}\|_F \\
&= \|\widetilde{\mathbf{P}}_{\mathrm{Sep}}(\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Sep}} - \boldsymbol{\Lambda})\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T + \widetilde{\mathbf{P}}_{\mathrm{Sep}}\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T - \boldsymbol{\Sigma}\|_F \\
&\leqslant \|\widetilde{\mathbf{P}}_{\mathrm{Sep}}(\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Sep}} - \boldsymbol{\Lambda})\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T\|_F + \|\widetilde{\mathbf{P}}_{\mathrm{Sep}}\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T - \boldsymbol{\Sigma}\|_F.
\end{aligned}$$

For the first term, by Lemma 6, we have with probability at least $1 - \frac{\beta}{2}$,

$$\|\widetilde{\mathbf{P}}_{\mathrm{Sep}}(\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Sep}} - \boldsymbol{\Lambda})\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T\|_F = \|\widetilde{\boldsymbol{\Lambda}}_{\mathrm{Sep}} - \boldsymbol{\Lambda}\|_F = \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \|\mathbf{Y}\|_2 = \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \eta(d, \beta).$$

For the second term,

$$\begin{aligned}
\|\widetilde{\mathbf{P}}_{\mathrm{Sep}}\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T - \boldsymbol{\Sigma}\|_F^2 &= \|\widetilde{\mathbf{P}}_{\mathrm{Sep}}\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T\|_F^2 + \|\boldsymbol{\Sigma}\|_F^2 - 2 \cdot \mathrm{tr}\left(\boldsymbol{\Sigma}\widetilde{\mathbf{P}}_{\mathrm{Sep}}\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T\right) \\
&= 2\sum_j \lambda_j^2 - 2 \cdot \mathrm{tr}\left(\boldsymbol{\Lambda}\widetilde{\mathbf{P}}_{\mathrm{Sep}}^T\boldsymbol{\Sigma}\widetilde{\mathbf{P}}_{\mathrm{Sep}}\right)
\end{aligned}$$

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$$=2\sum_j \lambda_j^2 - 2\sum_j \lambda_j \left(\widetilde{P}_{\text{Sep},j}^T \boldsymbol{\Sigma} \widetilde{P}_{\text{Sep},j}\right)$$

$$=2\sum_j \lambda_j \left(\lambda_j - \widetilde{P}_{\text{Sep},j}^T \boldsymbol{\Sigma} \widetilde{P}_{\text{Sep},j}\right).$$

Now the only work is to bound the terms $\left(\lambda_j - \widetilde{P}_{\text{Sep},j}^T \boldsymbol{\Sigma} \widetilde{P}_{\text{Sep},j}\right)$ for all j. First, recall $\widetilde{\boldsymbol{\Sigma}}_{\text{Gau}} = \boldsymbol{\Sigma} + \frac{1}{\sqrt{\rho}n} \cdot \mathbf{W}$, where $\mathbf{W} \sim \text{SGW}(d)$. By Lemma 7, with probability at least $1 - \frac{\beta}{2}$, for all unit vector u,

$$\left|u^T \widetilde{\boldsymbol{\Sigma}}_{\text{Gau}} u - u^T \boldsymbol{\Sigma} u\right| = \frac{1}{\sqrt{\rho}n} \cdot \left|u^T \mathbf{W} u\right| \leqslant \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \upsilon(d, \beta/2). \tag{10}$$

For any j, let $\mathcal{S}_j = \text{span}\{P_{j+1}, \ldots, P_d\}$, and $\mathcal{S}_j' = \text{span}\{\widetilde{P}_{\text{Sep},j+1}, \ldots, \widetilde{P}_{\text{Sep},d}\}$. Let

$$\lambda_j' = \max_{u:\|u\|_2=1, u \in \mathcal{S}_j'} u^T \boldsymbol{\Sigma} u$$

and

$$P_j' = \arg\max_{u:\|u\|_2=1, u \in \mathcal{S}_j'} u^T \boldsymbol{\Sigma} u.$$

By definition,

$$\lambda_j = \max_{u:\|u\|_2=1, u \in \mathcal{S}_j} u^T \boldsymbol{\Sigma} u \leqslant \max_{u:\|u\|_2=1, u \in \mathcal{S}_j'} u^T \boldsymbol{\Sigma} u = \lambda_j'. \tag{11}$$

Finally, we have

$$\widetilde{P}_{\text{Sep},j}^T \boldsymbol{\Sigma} \widetilde{P}_{\text{Sep},j} \geqslant \widetilde{P}_{\text{Sep},j}^T \widetilde{\boldsymbol{\Sigma}}_{\text{Gau}} \widetilde{P}_{\text{Sep},j} - \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \upsilon(d, \beta)$$

$$\geqslant P_j'^T \widetilde{\boldsymbol{\Sigma}}_{\text{Gau}} P_j' - \frac{\sqrt{2}}{\sqrt{\rho}n} \cdot \upsilon(d, \beta)$$

$$\geqslant P_j'^T \boldsymbol{\Sigma} P_j' - \frac{2\sqrt{2}}{\sqrt{\rho}n} \cdot \upsilon(d, \beta)$$

$$\geqslant \lambda_j - \frac{2\sqrt{2}}{\sqrt{\rho}n} \cdot \upsilon(d, \beta).$$

The first and third inequalities are from (10). The second inequality is by the definition of $\widetilde{P}_{\text{Sep},j}$. The last inequality is by (11). $\qquad\square$

H Proofs for Tail-sensitive Algorithm

Lemma 8. $\|\boldsymbol{\Sigma}(\mathbf{X}) - \boldsymbol{\Sigma}(\text{Clip}(\mathbf{X}, \tau))\|_F \leqslant \frac{1}{n} \sum_i \left(\|X_i\|_2^2 - \tau^2\right) \cdot \mathbb{I}\left(\|X_i\|_2 \geqslant \tau\right) \leqslant \gamma(\mathbf{X}, \tau).$

Proof. The proof of this lemma can be derived by the following technical result.

Claim 1. For any $X \in \mathbf{R}^d$, and $\tau \leqslant \|X\|_2$, let $\check{X} = \text{Clip}(X, \tau)$, then,

$$\left\|XX^T - \check{X}\check{X}^T\right\|_F = \|X\|_2^2 - \tau^2.$$

By definition, $\check{X} = \frac{\tau}{\|X\|_2} X$. Then

$$\left\|XX^T - \check{X}\check{X}^T\right\|_F = \left\|XX^T - \left(\frac{\tau}{\|X\|_2}\right)^2 XX^T\right\|_F$$

$$= \frac{\|X\|_2^2 - \tau^2}{\|X\|_2^2} \|XX^T\|_F$$

$$= \|X\|_2^2 - \tau^2.$$

$\qquad\square$

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Theorem 3. *Given any $\rho > 0$ and $\beta > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$, with probability at least $1 - \beta$,* AdaptiveCov *returns a $\widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}}$ such that*

$$
\begin{aligned}
\left\| \widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}} - \mathbf{\Sigma} \right\|_F &= O\left(\min_\tau \left(\mathrm{Noise}\left(\mathbf{X}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right) \cdot \frac{\log(1/\beta)^{1/4}}{\rho^{1/4}} \cdot \log(n) + \gamma(\mathbf{X}, \tau) \cdot \frac{\log(dn/\beta)}{\sqrt{\rho}} \right) + 2^{-dn} \right) \\
&= \tilde{O}\left(\min_\tau \left(\mathrm{Noise}\left(\mathbf{X}, \tau\right) + \gamma(\mathbf{X}, \tau) \right) + 2^{-dn} \right).
\end{aligned}
$$

Proof. We start by analyzing the error from using the private estimate $\tilde{r}$ of $\mathrm{rad}(\mathbf{X})$. By Lemma 9, we have, with probability at least $1 - \frac{\beta}{8}$,

$$
\tilde{r} \leqslant 2 \cdot \mathrm{rad}(\mathbf{X}) + 2^{-2dn}, \tag{12}
$$

and

$$
|\{\|X_i\|_2 > \tilde{r}\}| = O\left(\frac{1}{\sqrt{\rho}} \log \frac{dn}{\beta} \right).
$$

This gives

$$
\left\| \mathbf{\Sigma} - \mathbf{\Sigma}[\widetilde{\mathbf{X}}] \right\|_F = O\left(\frac{\mathrm{rad}(\mathbf{X})^2}{\sqrt{\rho}n} \log \frac{dn}{\beta} \right). \tag{13}
$$

Now, we analyze the error $\left\| \widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}} - \mathbf{\Sigma}[\widetilde{\mathbf{X}}] \right\|_F$. First, by tail bound of Gaussian, with probability at least $1 - \frac{\beta}{8}$,

$$
\tilde{\mathrm{tr}} \leqslant \widehat{\mathrm{tr}} \leqslant \tilde{\mathrm{tr}} + \frac{4\sqrt{2} \cdot \tilde{r}^2}{\sqrt{\rho}n} \cdot \sqrt{\log(8/\beta)}. \tag{14}
$$

Second, as the discussion in the definitions of $\mathrm{GaussNoise}(\cdot)$, $\mathrm{SeparateNoise}(\cdot)$ and $\widehat{\mathrm{Noise}}(\cdot)$, with probability at least $1 - \frac{\beta}{2}$, we have

$$
\left\| \widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}} - \mathbf{\Sigma}[\check{\mathbf{X}}] \right\|_F = O\left(\widehat{\mathrm{Noise}}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right) \right). \tag{15}
$$

By (7),

$$
\left\| \mathbf{\Sigma}[\widetilde{\mathbf{X}}] - \mathbf{\Sigma}[\check{\mathbf{X}}] \right\|_F \leqslant \widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tilde{\tau} \right). \tag{16}
$$

Combining (15) and (16), we have

$$
\left\| \widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}} - \mathbf{\Sigma}[\widetilde{\mathbf{X}}] \right\|_F = O\left(\widehat{\mathrm{Bias}}(\mathbf{X}, \tilde{\tau}) + \widehat{\mathrm{Noise}}\left(\mathbf{X}, \widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right) \right). \tag{17}
$$

Then, let's analyze the error from SVT. There are two sub-cases: $\tilde{t} = -dn - 1$ and $\tilde{t} \geqslant -dn$. For the first case, $\tilde{\tau} = 2^{-dn}$ and no query is higher than the threshold. By Lemma 5, with probability at least $1 - \frac{\beta}{4}$, for any $\tau = \tilde{r}, \frac{\tilde{r}}{2}, \ldots, 2^{-dn}$,

$$
\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2} \right) = O\left(\frac{1}{\sqrt{\rho}} \log(dn/\beta) \right),
$$

Recall that $\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2} \right)$ increases with the decrease of τ, we have, for all $\tau = \tilde{r}, \frac{\tilde{r}}{2}, \ldots, 2^{-dn}$,

$$
\left| \mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right) \right| - \left| \mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2} \right) \right| = O\left(\frac{1}{\sqrt{\rho}} \log(dn/\beta) \right). \tag{18}
$$

Therefore, for all $\tau = \tilde{r}, \frac{\tilde{r}}{2}, \ldots, 2^{-dn}$,

$$
\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tilde{\tau} \right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right)
$$

$$
= 2 \cdot \min\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tilde{\tau} \right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right) \right) + \frac{\tilde{r}^2}{n} \cdot \left| \mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2} \right) \right|
$$

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$$\leqslant 2 \cdot \max\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right) + \frac{\tilde{r}^2}{n} \cdot \left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tilde{\tau}, \frac{\rho}{2}, \frac{\beta}{2}\right)\right|$$

$$= 2 \cdot \max\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right) + \frac{\tilde{r}^2}{n} \cdot \left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right| + O\left(\frac{\tilde{r}^2}{\sqrt{\rho}n} \log(dn/\beta)\right)$$

$$= O\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right) + \frac{\tilde{r}^2}{\sqrt{\rho}n} \log(dn/\beta)\right),$$

where the fourth line is by (18). Together with (17), finally, we have

$$\|\widetilde{\boldsymbol{\Sigma}}_{\mathrm{Ada}} - \boldsymbol{\Sigma}[\widetilde{\mathbf{X}}]\|_F$$

$$= O\left(\min_{2^{-dn} \leqslant \tau \leqslant \tilde{r}}\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right) + \frac{\tilde{r}^2}{\sqrt{\rho}n} \log(dn/\beta)\right) \qquad (19)$$

$$= O\left(\min_{2^{-dn} \leqslant \tau \leqslant 1}\left(\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right) + \frac{\tilde{r}^2}{\sqrt{\rho}n} \log(dn/\beta)\right) \qquad (20)$$

For the second case, where $\tilde{t} \geqslant -dn$, by Lemma 5, with probability at least $1 - \frac{\beta}{4}$, we have,

$$\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, 2^{\tilde{t}}, \frac{\rho}{2}, \frac{\beta}{2}\right) \geqslant -\frac{6\sqrt{2}}{\sqrt{\rho}} \cdot \tilde{r}^2 \cdot \log(8(dn + 1)/\beta),$$

and

$$\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, 2^{\tilde{t}+1}, \frac{\rho}{2}, \frac{\beta}{2}\right) \leqslant \frac{6\sqrt{2}}{\sqrt{\rho}} \cdot \tilde{r}^2 \cdot \log(8(dn + 1)/\beta).$$

Recall $\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)$ increases with the decrease of τ, thus we have, for all $\tau = \tilde{r}, \frac{\tilde{r}}{2}, \ldots, 2^{-dn}$,

$$\left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, 2^{\tilde{t}+1}, \frac{\rho}{2}, \frac{\beta}{2}\right)\right| - \left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right| = O\left(\frac{1}{\sqrt{\rho}} \log(dn/\beta)\right),$$

or

$$\left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, 2^{\tilde{t}}, \frac{\rho}{2}, \frac{\beta}{2}\right)\right| - \left|\mathrm{Diff}\left(\widetilde{\mathbf{X}}, \widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)\right| = O\left(\frac{1}{\sqrt{\rho}} \log(dn/\beta)\right).$$

Further using

$$\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, 2^{\tilde{t}+1}\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, 2^{\tilde{t}+1}, \frac{\rho}{2}, \frac{\beta}{2}\right) \leqslant \widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, 2^{\tilde{t}}\right) + \widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, 2^{\tilde{t}}, \frac{\rho}{2}, \frac{\beta}{2}\right),$$

and following the same procedure as before, we can also get (20).

Now, we are ready to show how the target error bound can be derived based on (20). Given a τ, recall in (7) we have

$$\widehat{\mathrm{Bias}}\left(\widetilde{\mathbf{X}}, \tau\right) \leqslant 2 \cdot \gamma(\widetilde{\mathbf{X}}, \tau). \qquad (21)$$

Also, recall the definitions of $\mathrm{GaussNoise}(\cdot)$, $\mathrm{SeparateNoise}(\cdot)$ and $\mathrm{Noise}(\cdot)$; combining them with (14) and the fact $\widetilde{\mathrm{tr}} \leqslant \mathrm{tr}$, we have

$$\widehat{\mathrm{Noise}}\left(\widehat{\mathrm{tr}}, \tau, \frac{\rho}{2}, \frac{\beta}{2}\right)$$

$$= O\left(\min\left(\frac{\tau^2}{\sqrt{\rho}n} \cdot \omega\left(d, \frac{\beta}{2}\right), \frac{\tau}{\rho^{1/4}\sqrt{n}} \cdot \left(\sqrt{\mathrm{tr}} + \frac{\tilde{r} \cdot \log(1/\beta)^{1/4}}{\rho^{1/4}\sqrt{n}}\right) \cdot \sqrt{\upsilon\left(d, \frac{\beta}{4}\right)} + \frac{\tau^2}{\rho^{1/2}n} \cdot \eta\left(d, \frac{\beta}{4}\right)\right)\right)$$

$$= O\left(\min\left(\frac{\tau^2}{\sqrt{\rho}n} \cdot \omega\left(d, \frac{\beta}{2}\right), \frac{\tau}{\rho^{1/4}\sqrt{n}} \cdot \left(\sqrt{\mathrm{tr}} + \frac{\mathrm{rad}(\mathbf{X}) \cdot \log(1/\beta)^{1/4}}{\rho^{1/4}\sqrt{n}}\right) \cdot \sqrt{\upsilon\left(d, \frac{\beta}{4}\right)} + \frac{\tau^2}{\rho^{1/2}n} \cdot \eta\left(d, \frac{\beta}{4}\right)\right)\right.$$

$$\left. + \frac{2^{-2dn}}{\sqrt{\rho}n} \cdot \sqrt{\upsilon\left(d, \frac{\beta}{4}\right)} \cdot \log(dn/\beta)\right)$$

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$$=O\left(\min\left(\frac{\tau^2}{\sqrt{\rho}n}\cdot\omega\left(d,\frac{\beta}{2}\right),\frac{\tau}{\rho^{1/4}\sqrt{n}}\cdot\left(\sqrt{\mathrm{tr}}+\frac{\sqrt{\mathrm{tr}}\cdot\log(1/\beta)^{1/4}}{\rho^{1/4}}\right)\cdot\sqrt{\upsilon\left(d,\frac{\beta}{4}\right)}+\frac{\tau^2}{\rho^{1/2}n}\cdot\eta\left(d,\frac{\beta}{4}\right)\right)\right.$$

$$\left.+\frac{2^{-2dn}}{\sqrt{\rho}n}\cdot\sqrt{\upsilon\left(d,\frac{\beta}{4}\right)}\cdot\log\left(dn/\beta\right)\right)$$

$$=O\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}+2^{-dn}\right). \tag{22}$$

The second equality is from (12). The third equality is by the fact $\mathrm{tr}\geqslant\frac{\mathrm{rad}^2}{n}$, and the last equality is by (4).

Finally, combining (13), (20), (21) and (22), we have

$$\|\widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}}-\mathbf{\Sigma}\|_F=O\left(\min_{2^{-dn}\leqslant\tau\leqslant\tilde{r}}\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}+\gamma(\widetilde{\mathbf{X}},\tau)\right)+2^{-dn}+\frac{\tilde{r}^2}{\sqrt{\rho}n}\log(dn/\beta)\right)$$

$$=O\left(\min_{2^{-dn}\leqslant\tau}\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}+\gamma(\widetilde{\mathbf{X}},\tau)\right)+2^{-dn}+\frac{\tilde{r}^2}{\sqrt{\rho}n}\log(dn/\beta)\right)$$

$$=O\left(\min_{2^{-dn}\leqslant\tau}\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}+\gamma(\widetilde{\mathbf{X}},\tau)\right)+2^{-dn}+\frac{\mathrm{rad}(\mathbf{X})^2}{\sqrt{\rho}n}\log(dn/\beta)\right)$$

$$=O\left(\min_{2^{-dn}\leqslant\tau}\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}\cdot\log(n)+\gamma(\mathbf{X},\tau)\cdot\frac{\log(dn/\beta)}{\sqrt{\rho}}\right)+2^{-dn}\right)$$

$$=O\left(\min_{\tau}\left(\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}\cdot\log(n)+\gamma(\mathbf{X},\tau)\cdot\frac{\log(dn/\beta)}{\sqrt{\rho}}\right)+2^{-dn}\right).$$

The second line follows from the observation that increasing τ beyond $\tilde{r}$ will increase $\mathrm{Noise}\left(\mathbf{X},\tau,\frac{\rho}{2},\frac{\beta}{2}\right)$ but will not reduce $\gamma(\widetilde{\mathbf{X}},\tau)$. The third line is by (12). The fourth line is due to $\gamma(\mathbf{X},\tau)\geqslant\gamma(\widetilde{\mathbf{X}},\tau)$ and the following fact: if $\tau\geqslant\mathrm{rad}(\mathbf{X})$, then $\frac{\tau^2}{n}\geqslant\frac{\mathrm{rad}(\mathbf{X})^2}{n}$ and $\frac{\tau\sqrt{\mathrm{tr}}}{\sqrt{n}}\geqslant\frac{\mathrm{rad}(\mathbf{X})^2}{n}$; otherwise, $\gamma(\mathbf{X},\tau)\geqslant\frac{\mathrm{rad}(\mathbf{X})^2}{n}$. The last line is because

$$2^{-dn}=\Omega\left(\mathrm{Noise}\left(\mathbf{X},2^{-dn},\frac{\rho}{2},\frac{\beta}{2}\right)\cdot\frac{\log(1/\beta)^{1/4}}{\rho^{1/4}}\cdot\log(n)\right),$$

and for any $\tau<2^{-nd}$

$$\gamma(\mathbf{X},\tau)\geqslant\gamma(\mathbf{X},2^{-dn}).$$

$\square$

I Extension to pure-DP

Our proposed algorithms can be extended to the ε-DP setting, by simply replacing Gaussian noise with Laplace noise. However, the error is enlarged since the added noise is proportional to ℓ_1 sensitivities, which can be much larger than ℓ_2 sensitivities. Using the fact that $\|X\|_2\leqslant 1$ implies $\|X\|_1\leqslant\sqrt{d}$, we immediately obtain an upperbound for the ℓ_1 sensitivity of computing $\mathbf{\Sigma}$ based on the ℓ_2 sensitivity:

Lemma 12. *For any* $\mathbf{X},\mathbf{X}'\in\mathcal{B}_d^n$, $\mathbf{X}\sim\mathbf{X}'$,

$$\|\mathbf{\Sigma}(\mathbf{X})-\mathbf{\Sigma}(\mathbf{X}')\|_1\leqslant\frac{\sqrt{2}d}{n}.$$

The ℓ_1 sensitivity for computing $\mathbf{\Lambda}$ was shown by [2] to be:

Lemma 13 ([2]). *For any* $\mathbf{X},\mathbf{X}'\in\mathcal{B}_d^n$, $\mathbf{X}\sim\mathbf{X}'$,

$$\|\mathbf{\Lambda}-\mathbf{\Lambda}'\|_1\leqslant\frac{2}{n}.$$

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Under ε-DP, our base mechanism is the Laplace mechanism. In the same spirit as the Gaussian mechanism, we draw a symmetric $d \times d$ noise matrix $\mathbf{W}$ where the entries on and above the diagonal are i.i.d. $\mathrm{Lap}(1)$. For convenience, we refer to $\mathbf{W}$ as a Symmetric Laplace Wigner matrix and write $\mathbf{W} \sim \mathrm{SLW}(d)$. Then, we scale the noise matrix by the factor $\frac{\sqrt{2}d}{\varepsilon n}$. We refer to the modified algorithm as LapCov. For SeparateCov, we use LapCov to replace GaussCov for privatizing eigenvectors and add Laplace noise in place of Gaussian noise to privatize the eigenvalues.

I.1 Concentration inequalities

Before we proceed to anaylze the error of our modified algorithms, some concentration bounds for the Laplace noise vectors and matrices are in order. Our derivations are based on two lemmata.

Lemma 14 ([35]). *Let $Y_1, ..., Y_m$ be independent sub-exponential random variables with $\mathbf{E}[Y_k] = 0$ for $1 \leqslant k \leqslant m$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex and 1-Lipschitz. Then for $Y = (Y_1, ..., Y_m)$, and some constant $c > 0$,*

$$\Pr[|f(Y) - \mathbf{E}[f(Y)]| > t] \leqslant 2 \exp\left(-c \cdot \frac{t}{\log(m)}\right).$$

Lemma 15 ([7]). *For any $d > 0$, let $\mathbf{W}$ be a $d \times d$ Symmetric Laplace Wigner matrix . Then,*

$$\mathbf{E}[\|\mathbf{W}\|_2] \leqslant 3\sqrt{d} + \frac{9\log(d)}{\sqrt{\log(1.5)}}.$$

Based on these, we have

Lemma 16. *For any $d > 0$, $\beta > 0$, let $Y = (Y_1, ..., Y_d)$ where $Y_k \sim \mathrm{Lap}(1)$, for $1 \leqslant k \leqslant d$. Then for some constant $c > 0$, with probability at least $1 - \beta$,*

$$\|Y\|_2 \leqslant \frac{3}{2}\sqrt{d} + O\left(\log(1/\beta) \cdot \log(d)\right).$$

Proof. First, note that $\mathbf{E}\left[\|Y\|_2\right] \leqslant \frac{3}{2}\sqrt{d}$. Indeed,

$$\frac{\mathbf{E}\left[\|Y\|_2\right]}{\sqrt{d}} = \mathbf{E}\left[\frac{\|Y\|_2}{\sqrt{d}}\right] \leqslant \mathbf{E}\left[\frac{1}{2}\left(1 + \frac{\|Y\|_2^2}{d}\right)\right] = \frac{1}{2}\left(1 + \frac{\sum_k \mathbf{E}\left[Y_k^2\right]}{d}\right) = \frac{1}{2}\left(1 + \frac{\sum_k 2}{d}\right) = \frac{3}{2},$$

where the inequality is due to $\sqrt{z} \leqslant \frac{1+z}{2}$ for $z \geqslant 0$ and the second last equality is due to $Y_k \sim \mathrm{Lap}(1)$ with mean 0 and variance 2. Applying Lemma 14 with $m = d$ then gives the stated inequality. $\square$

Lemma 17. *For any $d > 0$, $\beta > 0$, given $\mathbf{W} \sim \mathrm{SLW}(d)$, then, with probability at least $1 - \beta$,*

$$\|\mathbf{W}\|_2 \leqslant 3\sqrt{d} + O\left(\log(1/\beta) \cdot \log(d)\right).$$

Proof. Note that the function of computing $\|\cdot\|_2$ is convex and 1-Lipschitz for matrices [39]. Note that $\mathbf{W}$ has $m = d(d+1)/2$ independent entries. We apply Lemma 14 to get the stated inequality. $\square$

Lemma 18. *For any $d > 0$, $\beta > 0$, given $\mathbf{W} \sim \mathrm{SLW}(d)$, then, with probability at least $1 - \beta$,*

$$\|\mathbf{W}\|_F \leqslant \frac{3}{2}d + O\left(\log(1/\beta) \cdot \log(d)\right).$$

Proof. Similar to the proof of Lemma 16, we compute $\mathbf{E}\left[\|\mathbf{W}\|_F\right]$ and apply Lemma 14 with $m = d(d+1)/2$.

$$\begin{aligned}
\frac{\mathbf{E}\left[\|\mathbf{W}\|_F\right]}{d} &\leqslant \frac{1}{2} \cdot \mathbf{E}\left[1 + \frac{\|\mathbf{W}\|_F^2}{d^2}\right] \\
&= \frac{1}{2}\left(1 + \frac{\sum_k \mathbf{E}\left[W_{k,k}^2\right] + 2\sum_{j<k} \mathbf{E}\left[W_{j,k}^2\right]}{d^2}\right) \\
&= \frac{1}{2}\left(1 + \frac{2d + 2d(d-1)}{d^2}\right) = \frac{3}{2}.
\end{aligned}$$

$\square$

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I.2 Error bounds

Based on Lemma 18, LapCov achieves error $O\left(\frac{d^2}{\varepsilon n} + \frac{d}{\varepsilon n} \cdot \log(1/\beta) \cdot \log(d)\right) = \tilde{O}\left(\frac{d^2}{n}\right)$ and by Lemma 16 and 17, we have

Lemma 19. *Given any $\varepsilon > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$,* SeparateCov *preserves ε-DP and for any $\beta > 0$, with probability at least $1 - \beta$,* SeparateCov *returns a $\widetilde{\mathbf{\Sigma}}_{\mathrm{Lap}}$ such that*

$$
\begin{aligned}
\|\widetilde{\mathbf{\Sigma}}_{\mathrm{Sep}} - \mathbf{\Sigma}\|_F = & O\left(\frac{\sqrt{d \cdot \mathrm{tr}}}{\sqrt{\varepsilon n}} \cdot \sqrt{\sqrt{d} + \log(1/\beta) \cdot \log(d)} + \frac{1}{\varepsilon n}\left(\sqrt{d} + \log(1/\beta) \cdot \log(d)\right)\right) \\
= & \tilde{O}\left(\frac{d^{3/4}\sqrt{\mathrm{tr}}}{\sqrt{n}} + \frac{\sqrt{d}}{n}\right).
\end{aligned}
$$

Under ε-DP, EMCov achieves the error $\tilde{O}\left(\frac{d\sqrt{\mathrm{tr}}}{\sqrt{n}} + \frac{\sqrt{d}}{n}\right)$, so SeparateCov improves the d dependency in the first term from d to $d^{3/4}$. Note that our trace-sensitive bound also implies a worst-case bound of $\tilde{O}(\frac{d^{3/4}}{\sqrt{n}})$. Concurrently with our work, Nikolov [30] shows how to achieve a better worst-case bound of $\tilde{O}(\frac{\sqrt{d}}{\sqrt{n}})$, as an application of a new private query release mechanism based on the Johnson-Lindenstrauss transform, although his approach does not seem to yield a trace-sensitive bound.

Finally, we modify the AdaptiveCov algorithm to support ε-DP. To this end, we first replace GaussCov with LapCov and SeparateCov with its ε-DP version. Then, since PrivRadius and SVT already satisfy pure DP, the remaining modification is to use a Laplace tail bound in place of the Gaussian tail bound when constructing the DP upper bound of tr. Based on these, we have,

Theorem 4. *Given any $\varepsilon > 0$, for any $\mathbf{X} \in \mathcal{B}_d^n$,* AdaptiveCov *preserves ε-DP and for any $\beta > 0$, with probability at least $1 - \beta$,* AdaptiveCov *returns a $\widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}}$ such that*

$$
\left\|\widetilde{\mathbf{\Sigma}}_{\mathrm{Ada}} - \mathbf{\Sigma}\right\|_F = \tilde{O}\left(\min_\tau\left(\min\left(\frac{\tau^2 d^2}{n}, \frac{\tau d^{3/4}\sqrt{\mathrm{tr}}}{\sqrt{n}} + \frac{\tau^2\sqrt{d}}{n}\right) + \gamma(\mathbf{X}, \tau)\right) + 2^{-dn}\right).
$$

J   Additional Experiments

J.1 ρ-zCDP experiments on MNIST digits
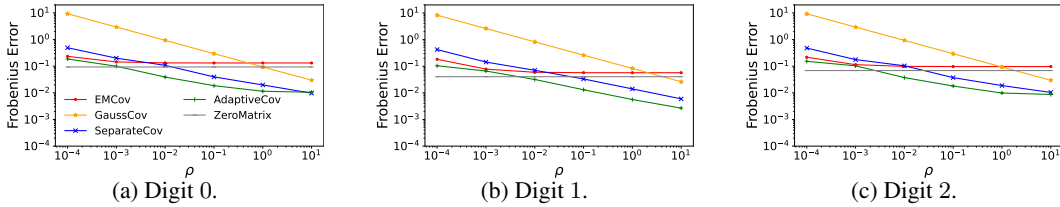
Figure 6: Results on MNIST digits.

J.2   Pure DP experiments

We also conducted experiments using the pure DP algorithms on the same datasets as in Section 6. The parameter settings are also mostly the same, except we replaced ε with ρ. For synthetic datasets, we also present an extra set of experiments with a larger sample size n. Since the dependency on d degrades for all the algorithms, a larger sample size is needed in order for them to have an advantage over the zero matrix. As mentioned in Appendix C, [2] propose two ways to allocate privacy budget for eigenvector estimation under pure-DP; one of them being uniform allocation and the other is to allocate more budget to larger (privatized) eigenvalues. We tested both and their error appeared similar, with the latter being slightly better in some cases (as consistent with their report). Here, we only report the results for the latter budget allocation scheme.

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The experimental results agree with our theoretical analysis and are similar as those in ρ-zCDP setting: SepCov always outperforms EMCov and will perform better than LapCov with higher d, smaller n, and smaller ε. Also, as expected, to outperform the zero matrix, we require larger n, smaller d, and larger ε compared with the ρ-zCDP setting. For real-world datasets, since they are highly dimensional with limited samples, none of the algorithms have better utility than the zero matrix (except when d is small and ε is large); as such we do not report the results for real-world datasets here.
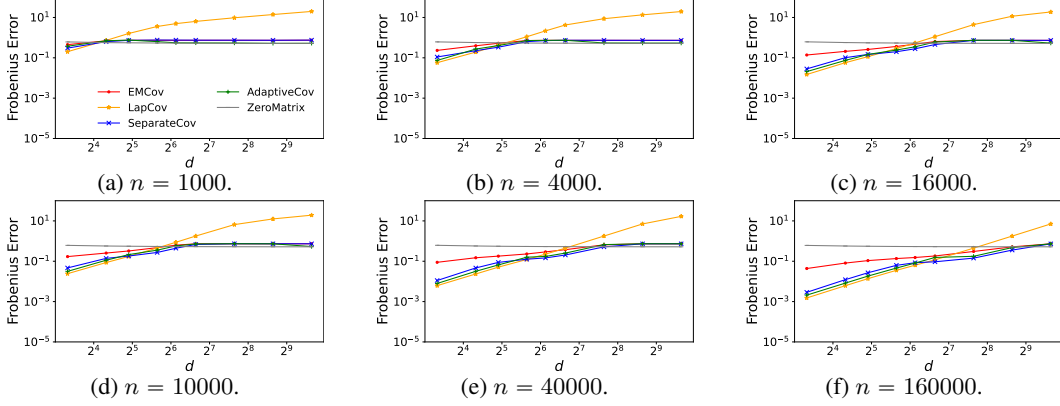
Figure 7: Results on synthetic datasets fixing $\mathrm{tr} = 1$.
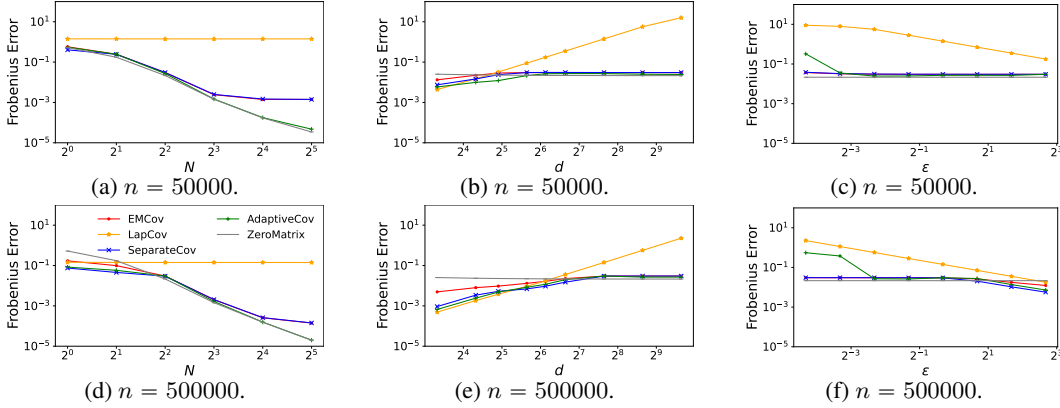
Figure 8: Results on synthetic datasets as d, N or ε varies.

Checklist

1. For all authors...
    (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
    (b) Did you describe the limitations of your work? [Yes] See Section 2.
    (c) Did you discuss any potential negative societal impacts of your work? [N/A]
    (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
    (a) Did you state the full set of assumptions of all theoretical results? [Yes]
    (b) Did you include complete proofs of all theoretical results? [Yes]

3. If you ran experiments...

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(a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See URL in Section 6.

(b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes]

(c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No]

(d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See URL in Section 6.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...

(a) If your work uses existing assets, did you cite the creators? [Yes] URLs are also provided where data were retrieved.

(b) Did you mention the license of the assets? [N/A]

(c) Did you include any new assets either in the supplemental material or as a URL? [N/A]

(d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]

(e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...

(a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]

(b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]

(c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

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