Abstract

In this supplement, we present the detailed proofs for the main theorems in the paper "Optimal testing using combined test statistics across independent studies".

A Appendix

The proofs of the main theorems (Theorem 1, 2 and 3) are divided over the subsections as follows. In Section A.1, the lower bounds of Theorem 1 and 3 are proven. Auxiliary lemmas for the proof of the lower bounds are proven in A.2. The attainability of the lower bound rates are given in Lemmas 6, 7 and 8 in Section A.4. In Section A.5, Lemmas 1 and 2 are proven.

A.1 Proof of the lower bounds (Theorems 1 and 3)

The proof is based around the following idea. If C_m satisfies the continuity condition of Assumption 2, it implies $C_m(S^{(1)}, \dots, S^{(m)})$ should not change to much if the statistics $S^{(1)}, \dots, S^{(m)}$ are replaced by finite bit approximations. If b is the number of bits used for the approximation of $S^{(j)}$, we should be able to get an approximation with accuracy of the order 2^{-b} through e.g. binary expansion. Since C_m and consequently the test based on C_m do not change (much) from passing to a finite bit approximation, tools and results from testing under bit-constrained communication apply, which finally yield the theorems.

Proof. We prove the statement for any $\alpha \in (0, 1/10]$. Since $\alpha \mapsto \kappa_\alpha$ is strictly decreasing, $\kappa_{1/8} < \kappa_{1/10} \leq \kappa_\alpha$ holds for any $\alpha \in (0, 1/10]$. Take $0 < \epsilon < \frac{1}{2}(\kappa_{1/10} - \kappa_{1/8})$. Then $|x - \kappa_\alpha| \leq \epsilon$ implies $x \geq \kappa_{1/8}$, which by the definition of the quantile function provides

$$\mathbb{P}_0 \left(|C_m(S) - \kappa_\alpha| \leq 2\epsilon \right) \leq 1/8. \tag{S.1}$$

By Lemma 3, there exist $B^{(j)}$-bit binary approximations $\tilde{S}^{(j)}$ such that

$$|S^{(j)} - \tilde{S}^{(j)}| \leq \left(\frac{\epsilon^{1/q}}{L^{1/q} m} \right)^{1/p} \tag{S.2}$$

and

$$\mathbb{E}_0 B^{(j)} \leq \mathbb{E}_0 \log_2(|S^{(j)}|) \vee 0 - \frac{1}{p} \log \left(\frac{\epsilon^{1/q}}{L^{1/q} m} \right) + 3. \tag{S.3}$$

Write $\tilde{S} = (\tilde{S}^{(1)}, \dots, \tilde{S}^{(m)})$. By combining Assumption 2 with (S.2),

$$|C_m(S) - C_m(\tilde{S})| \leq \epsilon.$$

Consequently,

$$\begin{aligned} \mathcal{R}(T_\alpha, H_\rho) &\geq \mathbb{P}_0 \left(C_m(\tilde{S}) - |C_m(S) - C_m(\tilde{S})| \geq \kappa_\alpha \right) \\ &\quad + \sup_{f \in H_\rho} \mathbb{P}_f \left(C_m(\tilde{S}) \leq \kappa_\alpha - |C_m(S) - C_m(\tilde{S})| \right) \\ &\geq \mathbb{P}_0 \left(C_m(\tilde{S}) \geq \kappa_\alpha + \epsilon \right) + \sup_{f \in H_\rho} \mathbb{P}_f \left(C_m(\tilde{S}) \leq \kappa_\alpha - \epsilon \right). \end{aligned}$$

Define the test

$$T'_\alpha := \mathbb{1} \left\{ C_m(\tilde{S}) > \kappa_\alpha - \epsilon \right\}.$$

Since

$$\mathbb{P}_0 \left(C_m(\tilde{S}) \geq \kappa_\alpha + \epsilon \right) = \mathbb{P}_0 \left(C_m(\tilde{S}) > \kappa_\alpha - \epsilon \right) - \mathbb{P}_0 \left(-\epsilon \leq C_m(\tilde{S}) - \kappa_\alpha \leq \epsilon \right),$$

the second last display can now be written as

$$\mathcal{R}(T', H_\rho) - \mathbb{P}_0 \left(|C_m(\tilde{S}) - \kappa_\alpha| \leq \epsilon \right).$$

Applying (3) again, using the reverse triangle inequality and (S.1), we obtain

$$\mathbb{P}_0\left(|C_m(\tilde{S}) - \kappa_\alpha| \leq \epsilon\right) \leq \mathbb{P}_0\left(|C_m(S) - \kappa_\alpha| \leq 2\epsilon\right) \leq 1/8.$$

It suffices to show that for ρ satisfying (5) in the case of Theorem 1 or ρ satisfying (11) in case of Theorem 3 for a small enough $c > 0$, we have

$$\mathcal{R}(T', H_\rho) \geq 7/8. \tag{S.4}$$

This follows from Lemma 4, where it is left to verify that

$$\sum_{j=1}^{m} d \wedge \mathbb{E}_0 B^{(j)} \lesssim m(d \wedge (1 \vee \log m)) \tag{S.5}$$

for a constant independent of d, n, m and $c > 0$. By (S.3) and $\mathbb{E}_0|S^{(j)}| \leq M$ for some constant $M > 0$ for $j = 1, \ldots, m$ (following from Assumption 1 or 4), we obtain that $\sum_{j=1}^{m} d \wedge \mathbb{E}_0 B^{(j)}$ is bounded by

$$m\left(d \bigwedge \left(\log_2(1 + M) + 3 - \frac{1}{p}\log\left(\frac{\epsilon^{1/q}}{L^{1/q}m}\right)\right)\right),$$

from which (S.5) follows. Putting things together, we now have that for $c > 0$ small enough we obtain (S.4), from which we conclude that (S.4) holds and the proof of the theorems is concluded. □

A.2 Auxiliary lemmas to the lower bound theorems

As a first tool, we introduce finite bit approximations of real numbers through their binary expansion. Consider the binary expansion of $x \in \mathbb{R}$; i.e. there exist digits $a_k(x), \ldots, a_1(x), a_0(x) \in \{0, 1\}$ for a $k_x \equiv k \in \mathbb{N} \cup \{0\}$ and $(b_i(x))_{i \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ such that

$$x = \text{sign}(x)\left(\sum_{i=0}^{k} 2^i a_i(x) + \sum_{i=1}^{\infty} 2^{-i} b_i(x)\right) \tag{S.6}$$

with k the largest element in $\mathbb{N} \cup \{0\}$ such that $2^k - 1 \leq |x|$. We now define $\tilde{x}_B$ to be the B-bit binary expansion giving the smallest approximation error in absolute value, where the first bit encodes $\text{sign}(x)$. That is, for $B \geq k + 2$, we have

$$|x - \tilde{x}_B| \leq \sum_{i=B-k-1}^{\infty} 2^{-i} b_i(x). \tag{S.7}$$

The following is well known, we exhibit its proof for completeness.

Lemma 3. *Let V be a random variable with a first moment. Given $1 > \epsilon > 0$, let $B_\epsilon \equiv B$ denote the number of bits required such that*

$$|V - \tilde{V}_{B_\epsilon}| \leq \epsilon. \tag{S.8}$$

It holds that

$$\mathbb{E}B \leq \mathbb{E}\log_2(|V|) \vee 0 + 1 + \log_2(1/\epsilon) + 2.$$

Proof. If $|V| < 1$, we have that

$$|V - \tilde{V}_B| \leq \sum_{i=B-1}^{\infty} 2^{-i} b_i(V).$$

So in the case that $|V| \leq 1$, since $b_i(V) \in \{0, 1\}$, for (S.8) to hold it suffices that $B \geq \log_2(1/\epsilon) + 2$. Let B' denote the amount of bits required to obtain $|V - \tilde{V}_{B'}| \leq 1$. When $2^k \leq |V| < 2^{k+1}$, it holds that $B' \leq k + 1$. Using Markov's inequality,

$$\begin{aligned}
\mathbb{E}B' &= \mathbb{E}B' \sum_{k=0}^{\infty} \mathbb{1}\left\{2^k \leq |V| < 2^{k+1}\right\} \\
&\leq \mathbb{E}\sum_{k=0}^{\infty} (k+1)\mathbb{1}\left\{k \leq \log_2(|V|) < k+1\right\} \leq \mathbb{E}\log_2(|V|) \vee 0 + 1.
\end{aligned}$$

In conclusion, $\mathbb{E}B \leq \mathbb{E}\log_2(|V|) \vee 0 + 1 + \log_2(1/\epsilon) + 2$. □

For the lemmas below, we introduce the following notation. Let π be a probability distribution on $\mathbb{R}^d$. Write $\mathbb{P}_\pi := \int \mathbb{P}_f d\pi(f)$ for the mixture distribution, where $\mathbb{P}_f$ denotes the joint distribution on X, U and S. Let F denote the draw from π. Let $\mathbb{P}_f^{\tilde{S}}$ denote the forward measure induced on the random variable $\tilde{S}$ and let $L_\pi^{\tilde{S}}$ denote the likelihood ratio of the mixture distribution and $\mathbb{P}_0$, ie

$$L_\pi^{\tilde{S}} = \int \frac{d\mathbb{P}_f^{\tilde{S}}}{d\mathbb{P}_0^{\tilde{S}}} d\pi(f). \tag{S.9}$$

Because of the Markov chain structure of $F \to (X, U) \to S$ and the independence between U and X, the joint distribution of (X, U, S) under the mixture disintegrates as

$$d\mathbb{P}_\pi^{X,U,S}(x, u, s) = \int d\mathbb{P}^{S|(X,U)}(s) d\mathbb{P}_f^X(x) d\mathbb{P}^U(u) d\pi(f) \tag{S.10}$$

where $\mathbb{P}^U$ is the marginal distribution of U. For the likelihood ratio conditionally on $U = u$, we shall write

$$L_\pi^{\tilde{S}|U=u} = \int \frac{d\mathbb{P}_f^{\tilde{S}|U=u}}{d\mathbb{P}_0^{\tilde{S}|U=u}} d\pi(f). \tag{S.11}$$

Furthermore, by the independence of the statistics given U,

$$d\mathbb{P}^{S|(X,U)} = \bigotimes_{j=1}^{m} d\mathbb{P}^{S^{(j)}|(X^{(j)},U)}. \tag{S.12}$$

Let $\tilde{S}^{(j)}$ denote the $B^{(j)}$-bit binary approximations to $S^{(j)}$ such that (S.2) holds. Note that the above displays are true for the random variable $\tilde{S} = (\tilde{S}^{(1)}, \ldots, \tilde{S}^{(m)})$ in place of S since $F \to (X, U) \to S \to \tilde{S}$ forms a Markov chain as well. The following lemma allows us to bound the chi-square divergence between the forward measure for $\tilde{S}$, which we will denote by $\mathbb{P}_\pi^{\tilde{S}}$ and $\mathbb{P}_0^{\tilde{S}}$.

The following lemma lower bounds the worst-case risk for any test T' depending only on $\tilde{S}$, the binary approximation of S as in (S.2).

Lemma 4. *Let T' be a test depending only on $\tilde{S}$ taking values in $\mathbb{R}^m$, satisfying (S.10) and where $\tilde{S}^{(j)}$ allows for an exact $B^{(j)}$-bit binary expansion as in (S.6), with $\mathbb{E}_0 B^{(j)} < \infty$ for $j = 1, \ldots, m$. There exists $c > 0$ independent of n, m and d such that*

$$\mathcal{R}(T', H_\rho) \geq 7/8$$

for all $n, m, d \in \mathbb{N}$ whenever

$$\sum_{j=1}^{m} d \wedge \mathbb{E}_0 B^{(j)} \lesssim m(d \wedge \log m) \tag{S.13}$$

in addition to

$$\rho^2 \leq c \frac{(\sqrt{m} \wedge \frac{d}{\log(m)})\sqrt{d}}{mn}, \tag{S.14}$$

if $\tilde{S}$ is generated using public randomness, or

$$\rho^2 \leq c \frac{(\sqrt{m} \wedge \sqrt{\frac{d}{\log(m)}})\sqrt{d}}{mn}, \tag{S.15}$$

in case $\tilde{S}$ is generated using only local randomness.

Proof. Consider a probability distribution π on $\mathbb{R}^d$ and $L_\pi^{\tilde{S}}$ as in (S.9). Consider the set

$$D := \left\{ u : \sum_{j=1}^{m} d \wedge \mathbb{E}_0[B^{(j)}|U = u] \leq 64 \sum_{j=1}^{m} d \wedge \mathbb{E}_0 B^{(j)} \right\},$$

whose complement, D^c, has $\mathbb{P}^U$-mass less than or equal to $1/64$ by Markov's inequality and $\mathbb{E}^U(d \wedge \mathbb{E}_0[B^{(j)}|U]) \leq d \wedge \mathbb{E}_0 B^{(j)}$. By conditioning on U (writing $\mathbb{P}_0^{|U=u} := \mathbb{P}_0(\cdot|U=u)$),

$$\begin{aligned}\mathcal{R}(T', H_\rho) &\geq \mathbb{P}_0 T' + \mathbb{P}_\pi(1-T') - \pi(f \notin H_\rho) \\ &\geq \int \left(\mathbb{P}_0^{|U=u}(T') + \mathbb{P}_\pi^{|U=u}(1-T')\right) \mathbb{1}_D(u) d\mathbb{P}^U(u) - \pi(f \notin H_\rho).\end{aligned}$$

Since $0 \leq T' \leq 1$ and $L_\pi^{\tilde{S}} \geq 0$, for all $0 < \gamma < 1$,

$$\begin{aligned}\mathbb{P}_0^{|U=u}(T') + \mathbb{P}_\pi^{|U=u}(1-T') &\geq \mathbb{P}_0^{|U=u}\left(\gamma T' + L_\pi^{\tilde{S}|U=u}(1-T')\mathbb{1}\left\{L_\pi^{\tilde{S}|U=u} > \gamma\right\}\right) \\ &\geq \gamma \mathbb{P}_0^{|U=u}\left(L_\pi^{\tilde{S}|U=u} > \gamma\right) \\ &\geq \gamma\big(1 - \mathbb{P}_0^{|U=u}(|L_\pi^{\tilde{S}|U=u} - 1| \geq 1-\gamma)\big).\end{aligned}$$

The probability on the right hand side of the above display can be bounded by applying Chebyshev's inequality and bounding the resulting chi-square divergence using the tools of [40], in particular using Lemma 10.1 from the aforementioned paper. This lemma applies if $\tilde{S}$ takes values in a space of finite, fixed cardinality.

Define $B^* = \sum_{j=1}^m 64\mathbb{E}_0|B^{(j)}|$ and the event

$$A := \left\{\sum_{j=1}^m B^{(j)} \leq B^*\right\},$$

so that A^c by Markov's inequality occurs with $\mathbb{P}_0$-probability less than $1/64$.

Let $\breve{S}^{(j)}$ be the $\breve{B}^{(j)} := B^{(j)} \wedge B^*$ binary approximation of $\tilde{S}^{(j)}$ and note that on the event A, $\breve{S}^{(j)} = \tilde{S}^{(j)}$. We have

$$\begin{gathered}\int \mathbb{P}_0^{|U=u}\left(|L_\pi^{\tilde{S}|U=u} - 1| \geq 1-\gamma\right) \mathbb{1}_D(u) d\mathbb{P}^U(u) \leq \\ \int \mathbb{P}_0^{|U=u}\left(\left\{|L_\pi^{\tilde{S}|U=u} - 1| \geq 1-\gamma\right\} \cap A\right) \mathbb{1}_D(u) d\mathbb{P}^U(u) + \mathbb{P}_0(A^c) \leq \\ \int \mathbb{P}_0^{|U=u}\left(|L_\pi^{\breve{S}|U=u} - 1| \geq 1-\gamma\right) \mathbb{1}_D(u) d\mathbb{P}^U(u) + 1/64,\end{gathered}$$

where $\breve{S} = (\breve{S}^{(1)}, \ldots, \breve{S}^{(m)})$. Using (S.10) and Chebyshev's inequality, it suffices to show that on the event D, $\mathbb{E}_0^{|U=u}|L_\pi^{\breve{S}|U=u} - 1|^2$ is smaller than $\frac{1}{32}(1-\gamma)^2$ for c small enough when ρ satisfies (S.14) or (S.15), some $\gamma \geq 5/6$ for a specific choice of π. By Lemma 5, such a distribution π exists, satisfying $\pi(f \notin H_\rho) \leq 1/32$, as long as $\mathrm{Tr}(\Xi_u)$ can be sufficiently bounded, which can be done in terms of (S.3), as we will show next.

Let $\mathcal{S}^{(j)}(b,u)$ be the space in which $\breve{S}^{(j)}|[B^{(j)} = b, U = u]$ takes values. Write

$$V_{s,u} = \mathbb{E}_0\left[X^{(j)}\middle|\breve{S}^{(j)} = s, U = u\right].$$

We have

$$\begin{aligned}\Xi_u^j &= \sum_s V_{s,u}V_{s,u}^\top \mathbb{P}_0(\breve{S}^{(j)} = s|U=u) \\ &= \sum_{b\in\mathbb{N}} \mathbb{P}_0(\breve{B}^{(j)} = b|U=u) \sum_{s=\mathcal{S}^j(b,u)} \mathbb{P}_0(\breve{S}^{(j)} = s|\breve{B}^{(j)} = b, U=u)V_{s,u}V_{s,u}^\top.\end{aligned}$$

By Lemma A.3 in [40], the trace of the matrix

$$\sum_{s\in\mathcal{S}^j(b,u)} \mathbb{P}_0\left(\breve{S}^{(j)} = s|\breve{B}^{(j)} = b, U=u\right) V_{s,u}V_{s,u}^\top$$

is bounded by $\left(2\log(2)\frac{b}{d}\bigwedge 1\right)\frac{d}{n}$. By linearity of the trace operation,

$$\begin{aligned}\text{Trace}(\Xi_u^j) &= \sum_{b\in\mathbb{N}}\mathbb{P}_0\left(\breve{B}^{(j)}=b|U=u\right)\left(2\log(2)\frac{b}{d}\bigwedge 1\right)\frac{d}{n}\\ &\leq 2\log(2)\frac{d\wedge\mathbb{E}_0[\breve{B}^{(j)}|U=u]}{n}\end{aligned}$$

and consequently, since $\breve{B}^{(j)}\leq B^{(j)}$ and $u\in D$,

$$\begin{aligned}\text{Trace}(\sum_{j=1}^m \Xi_u^j) &\leq 2\log(2)n^{-1}\sum_{j=1}^m d\wedge\mathbb{E}_0\left[\breve{B}^{(j)}|U=u\right]\\ &\leq 128\log(2)n^{-1}\sum_{j=1}^m d\wedge\mathbb{E}_0\left[B^{(j)}\right].\end{aligned}$$

The result follows after using that ρ^2 satisfies (S.15) and (S.14) in the case of local or shared randomness protocols, respectively. ☐

Lemma 5. *Let $L_\pi^{\breve{S}}$ be as defined through (S.9), with $\breve{S}=(\breve{S}^{(1)},\dots,\breve{S}^{(m)})$ taking values in a space of finite cardinality. Let $\Xi_u=\sum_{j=1}^m\Xi_u^j$ with*

$$\Xi_u^j := \mathbb{E}_0^{|U=u}\mathbb{E}_0\left[X^{(j)}\middle|\breve{S}^{(j)},U=u\right]\mathbb{E}_0\left[X^{(j)}\middle|\breve{S}^{(j)},U=u\right]^\top. \tag{S.16}$$

Let ρ^2 satisfy (S.14) or (S.15). For $c>0$ small enough (in (S.14) or (S.15)) there exists a probability distribution π on $\mathbb{R}^d$ such that

$$\pi(f\notin H_\rho)\leq 1/32 \tag{S.17}$$

and

$$\mathbb{E}_0^{|U=u}|L_\pi^{\breve{S}|U=u}-1|^2\leq\exp\left(C(\frac{mn^2\rho^4}{cd}+\frac{mn^3\rho^4}{d^2c}Tr(\Xi_u)\right)-1, \tag{S.18}$$

for a constant $C>0$ that does not depend on d,n,m or c. Furthermore, in case of private coin randomness (U is degenerate), there exists a probability distribution π on $\mathbb{R}^d$ such that (S.17) is satisfied and (the sharper bound)

$$\mathbb{E}_0|L_\pi^{\breve{S}}-1|^2\leq\exp\left(C(\frac{mn^2\rho^4}{cd}+\frac{n^4\rho^4}{d^3c}Tr(\Xi_u)^2\right)-1 \tag{S.19}$$

holds for $c>0$ small enough.

Proof. This follows from the proof of Theorem 3.1 in [40] (where it is important to note that in the notation of [40], "n" equals "nm" in this article). For completeness, we highlight the main steps here. We start by noting that

$$\mathbb{E}_0^{|U=u}|L_\pi^{\breve{S}|U=u}-1|^2=D_{\chi^2}(\mathbb{P}_0^{\breve{S}|U=u};\mathbb{P}_\pi^{\breve{S}|U=u}).$$

Let π be a $N(0,\Gamma)$-distribution with $\Gamma\in\mathbb{R}^{d\times d}$. In view of the Markov chain structure (ie (S.10) and (S.12)), the Gaussianity of π and the fact that $\breve{B}^{(j)}$ is bounded for $j=1,\dots,m$, we obtain through following the steps corresponding to displays (34) up until (42) in Section 9 of [40] that the above display is bounded by

$$\prod_{j=1}^m\mathbb{E}_0^{X^j|U=u}\left[\mathscr{L}_\pi\left(X^j\right)^2\right]\cdot\int e^{\frac{n^2}{m^2}f^\top\sum_{j=1}^m\Xi_u^j g}d(\pi\times\pi)(f,g)-1, \tag{S.20}$$

where $\mathscr{L}_\pi\left(X^j\right)=\int\frac{d\mathbb{P}_f^{X^{(j)}}}{d\mathbb{P}^{X_0^{(j)}}}d\pi(f)$ and we note that Lemma 10.1 applies by the boundedness of $\breve{B}^{(j)}$ and Gaussianity of π. Taking $\Gamma\in\mathbb{R}^{d\times d}$ equal to

$$\frac{\rho}{c^{1/4}\sqrt{d}}\bar{\Gamma}$$

for a symmetric idempotent $d \times d$ matrix $\bar{\Gamma}$ with rank (proportional to) d, we obtain (S.17) for $c > 0$ small enough (Lemma A.13 in [40]). Following the second step of the proof of Section 9 in [40], in particular the steps corresponding to displays (43) and (44), we obtain that (S.20) is bounded by

$$\exp\left(C(\frac{mn^2\rho^4}{cd} + \frac{n^4\rho^4}{d^2c}\mathrm{Tr}\left((\sqrt{\bar{\Gamma}}\Xi_u\sqrt{\bar{\Gamma}})^2 \right) \right) - 1$$

for some fixed constant $C > 0$ independent of d, n, m and ρ. The shared randomness bound of (S.18) now follows by choosing of $\bar{\Gamma} = I_d$ and using that $\mathrm{Tr}(A^\top A) \leq \|A\|\mathrm{Tr}(A)$ where $\|A\|$ is the operator norm of A, as well as by the fact that $\Xi_u \leq \frac{m}{n} I_d$ (see Lemma A.2 in [40]). In case of private randomness, we can assume that U is degenerate, so $\Xi_u = \Xi$ for $\mathbb{P}^U$-almost every u. The matrix Ξ is positive definite and symmetric, therefore it possesses a spectral decomposition $V^\top \mathrm{Diag}(\xi_1, \ldots, \xi_d)V$. Assuming that $\xi_1 \geq \xi_2 \geq \ldots \geq \xi_d$ with corresponding eigenvectors $V = (v_1 \quad \ldots \quad v_d)$, let $\check{V}$ denote the $d \times \lceil d/2 \rceil$ matrix $\left(v_{\lfloor d/2 \rfloor + 1} \quad \ldots \quad v_d\right)$. The bound of (S.19) now follows by setting $\bar{\Gamma} = \check{V}\check{V}^\top$, for a detailed computation, see page 23 of [40]. ☐

A.3 Theorem concerning necessity of signs

The theorem below tells us that in order to attain the rate of $\frac{d}{nm}$, the statistics $S^{(j)}$ need to contain at least *some* information on the signs of $X^{(j)}$, in the sense that $\sqrt{d}/(\sqrt{m}n)$ is the rate that can be attained at best when $S^{(j)}$ is measurable with respect to the absolute values of the coordinates of $X^{(j)}$. This is in particular the case for statistics based on e.g. the norm $\|X^{(j)}\|_2$ or rotation invariant statistics such as the worst-case growth rate optimal e-values (see e.g. [22]), which consequently attain the rate $\frac{\sqrt{d}}{\sqrt{m}n}$ at best and are thus suboptimal when d is small compared to m.

Theorem 4. *Suppose that $S^{(j)} = f_j(X^{(j)}, U)$ is such that $S^{(j)}$ is measurable with respect to $\sigma(U, (|X_1^{(j)}|, \ldots, |X_d^{(j)}|))$ for $j = 1, \ldots, m$. Then, for any $\alpha \in (0, 0.1]$ there exists $c > 0$ such that*

$$\sup_{f \in H_\rho} \mathbb{P}_f \left(T_\alpha = 0\right) \geq 3/4. \tag{S.21}$$

whenever

$$\rho^2 \leq c\frac{\sqrt{d}}{\sqrt{m}n}. \tag{S.22}$$

Proof. In view of Lemma 4 and the proof of the main theorems in A.1, it suffices to bound the trace of Ξ_u in (S.19) and (S.18) in Lemma 5 (the first term in the exponent is controlled by (S.22)). By assumption on $S^{(j)}$, we have

$$\sigma(S^{(j)}, U, (|X_1^{(j)}|, \ldots, |X_d^{(j)}|) = \sigma(U, (|X_1^{(j)}|, \ldots, |X_d^{(j)}|)), \tag{S.23}$$

which implies that the $\mathrm{sign}(X_i^{(j)})$ is independent of $\sigma(S^{(j)}, U, (|X_1^{(j)}|, \ldots, |X_d^{(j)}|)$. Writing $X_i^{(j)} = \mathrm{sign}(X_i^{(j)})|(X_i^{(j)})|$, we obtain that

$$\begin{aligned}\mathbb{E}_0\left[X^{(j)}\middle|S^{(j)}, U = u\right] &= \left(\mathbb{E}_0\left[\mathrm{sign}(X_i^{(j)})|(X_i^{(j)})|\middle|S^{(j)}, U = u\right]\right)_{1\leq i\leq d} \\ &= \left(\mathbb{E}_0\mathrm{sign}(X_i^{(j)})\mathbb{E}_0\left[|(X_i^{(j)})|\middle|S^{(j)}, U = u\right]\right)_{1\leq i\leq d} = 0,\end{aligned}$$

where the second last inequality follows from the fact that $\mathrm{sign}(X_i^{(j)})$ is independent of the sigma algebra in (S.23) and the final equality by the symmetry of the Gaussian distribution around the mean. Following the proof of Theorem 1 with $\Xi_u = 0$, we obtain that the testing risk is bounded from below whenever $\rho^2 \lesssim \frac{\sqrt{d}}{\sqrt{m}n}$. ☐

A.4 Lemmas related to rate attainability

Lemma 6. *Let T_α correspond to a test of level α based on Edginton's method based for p-values $p^{(j)} = \chi_d^2(\|\sqrt{n}X^{(j)}\|_2^2)$ or simply the sum of $\|\sqrt{n}X^{(j)}\|_2^2$. For all $\alpha, \beta \in (0, 1)$ if*

$$\rho^2 \geq C_{\alpha,\beta}\frac{\sqrt{d}}{\sqrt{m}n} \tag{S.24}$$

we have

$$\sup_{f \in H_\rho} \mathbb{P}_f \left(T_\alpha = 0 \right) \leq \beta$$

for $d \geq C_{\alpha,\beta} m$ *a large enough constant* $C_{\alpha,\beta}$ *depending only on* $\alpha, \beta \in (0,1)$. *The above result holds for Fisher's method also, under the additional assumption that* $\log(m) \lesssim \sqrt{d}$.

Proof. The test in (7) has level α under the null hypothesis. Under the alternative hypothesis,

$$\|\sqrt{n} X^{(j)}\|_2^2 \overset{d}{=} n\|f\|_2^2 + 2\sqrt{n}(Z^{(j)})^\top f + \|Z^{(j)}\|_2^2,$$

where $Z^{(j)} \sim N(0, I_d)$. Rearranging, the test T_α of (7) can be seen to equal

$$\mathbb{1}\left\{ 2\frac{\sqrt{n}}{\sqrt{d}} \left(m^{-1/2} \sum_{j=1}^m Z^{(j)} \right)^\top f + \frac{1}{\sqrt{md}} \sum_{j=1}^m \left(\|Z^{(j)}\|_2^2 - d \right) \geq \eta_{d,m} - \frac{\sqrt{mn}}{\sqrt{d}} \|f\|_2^2 \right\} \tag{S.25}$$

in distribution under $\mathbb{P}_f$, with

$$\eta_{d,m} := \frac{1}{\sqrt{dm}} \left(F^{-1}_{\chi^2_{dm}} (1-\alpha) - md \right).$$

By Lemma 9, $\eta_{d,m} \to \Phi^{-1}(1-\alpha)$ as both or either $d, m \to \infty$, so $\eta_{d,m}$ is bounded in d and m. Consequently, $\mathbb{P}_f(1 - T_\alpha)$ equals

$$\text{Pr}\left((1 + \frac{\sqrt{n}}{\sqrt{d}} \|f\|_2) O_P(1) \leq \eta_{d,m} - \frac{\sqrt{mn}}{\sqrt{d}} \|f\|_2^2 \right)$$

as the left hand side of the test in (S.25) is mean 0 and has constant variance. Since $\|f\|_2^2 \geq C_{\alpha,\beta} \frac{\sqrt{d}}{\sqrt{mn}}$, the latter display can be bounded from above by

$$\text{Pr}\left((1 + \frac{\sqrt{n}}{\sqrt{d}} \|f\|_2) O_P(1) \leq -\frac{\sqrt{mn}}{2\sqrt{d}} \|f\|_2^2 \right)$$

for a large enough $C_{\alpha,\beta}$. The latter display is smaller than β for $C_{\alpha,\beta} > 0$ large enough depending only on α and β.

For Edgington's method, one can take $p^{(j)} = 1 - F_{\chi^2_d}(\|\sqrt{n}X^{(j)}\|_2^2)$ and compute the test

$$T_\alpha := \mathbb{1}\left\{ m^{-1/2} \zeta_{\alpha,m} \sum_{j=1}^m (p^{(j)} - \frac{1}{2}) \geq 12^{-1/2} \Phi^{-1}(1-\alpha) \right\}, \tag{S.26}$$

where $\zeta_{\alpha,m} \to 1$ in m is such that $\mathbb{P}_0 T_\alpha = \alpha$, by Lemma 9.

Under the alternative, $\mathbb{E}_f p^{(j)} = \text{Pr}(\|\sqrt{n}f + Z^{(j)}\|_2^2 \leq \chi^2_d)$. Therefore, by Lemma 4 in [39],

$$\mathbb{E}_f p^{(j)} \geq \frac{1}{2} + \frac{1}{40} \left(d^{-1/2} n \|f\|_2^2 \bigwedge \frac{1}{2} \right),$$

where we note that we can take d larger than an arbitrary constant as the rate $\sqrt{d}/(\sqrt{mn})$ being optimal ($\sqrt{d}/(\sqrt{mn}) \lesssim d/(mn)$) implies $d \gtrsim m$ and for constant order m there is nothing to prove. We obtain that

$$\begin{aligned}
\mathbb{P}_f(1 - T_\alpha) &= \mathbb{P}_f \left(\frac{\zeta_{m,\alpha}}{\sqrt{m}} \sum_{j=1}^m (p^{(j)} - 1/2) \leq 12^{-1/2} \Phi^{-1}(1-\alpha) \right) \\
&= \mathbb{P}_f \left(\frac{\zeta_{m,\alpha}}{\sqrt{m}} \sum_{j=1}^m [(p^{(j)} - \mathbb{E}_f p^{(j)}) + \mathbb{E}_f p^{(j)} - \frac{1}{2}] \leq 12^{-1/2} \Phi^{-1}(1-\alpha) \right) \\
&\leq \text{Pr}\left(O_P(1) + \frac{\zeta_{m,\alpha}\sqrt{m}}{40} \left(d^{-1/2} n \|f\|_2^2 \bigwedge \frac{1}{2} \right) \leq 12^{-1/2} \Phi^{-1}(1-\alpha) \right),
\end{aligned}$$

where the $O_P(1)$ term in last equality follows from the fact that $\zeta_{m,\alpha}$ is bounded and the central limit theorem (the $p^{(j)}$'s are bounded and independent still under $\mathbb{P}_f$). If the minimum is taken in $1/2$, the result follows for large enough m. If the minimum is taken in the first argument,

$$\frac{\zeta_{m,\alpha}\sqrt{m}}{40}\left(d^{-1/2}n\|f\|_2^2 \bigwedge \frac{1}{2}\right) \geq \frac{C_{\alpha,\beta}\zeta_{m,\alpha}}{40}$$

so for large enough $C_{\alpha,\beta}$, we obtain that $\mathbb{P}_f(1-T_\alpha) \leq \beta$.

For Fisher's method, the test of level α is given by

$$T_\alpha := \mathbb{1}\left\{\sum_{j=1}^{m} -2\log p^{(j)} \geq F^{-1}_{\chi^2_{2m}}(1-\alpha)\right\}, \tag{S.27}$$

for the p-value $p^{(j)} := 1 - F_{\chi^2_d}(\|\sqrt{n}X^{(j)}\|_2^2)$ (or equivalently Pearson's method for the p-value $F_{\chi^2_d}(\|\sqrt{n}X^{(j)}\|_2^2)$).

For the Type II error bound, assume first that $n\|f\|_2^2 \geq 20\sqrt{d}$. We have that $\|Z^{(j)}\|_2^2 \geq d - 5\sqrt{d}$ on an event of probability at least $1 - e^{-5}$, via e.g. Theorem 3.1.1 in [46]. By using a union and a standard Gaussian concentration inequality, the event

$$\max_{1\leq j\leq m}\left|2\frac{\sqrt{n}}{\sqrt{d}}f^\top Z^{(j)}\right| \leq \frac{n}{2\sqrt{d}}\|f\|_2^2, \tag{S.28}$$

has mass at least $1 - me^{-n\|f\|_2^2/32} \geq 1 - me^{-\sqrt{d}/2}$. On the intersection of these two events,

$$\begin{aligned}
F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2) &= \Pr\left(\frac{\chi^2_d - \|Z^{(j)}\|_2^2}{\sqrt{d}} \leq 2\frac{\sqrt{n}}{\sqrt{d}}f^\top Z^{(j)} + \frac{n}{\sqrt{d}}\|f\|_2^2\right)\\
&\geq \Pr\left(\frac{\chi^2_d - d}{\sqrt{d}} \leq \frac{n}{2\sqrt{d}}\|f\|_2^2 - 5\right)\\
&\geq \Pr\left(\frac{\chi^2_d - d}{\sqrt{d}} \leq 5\right),
\end{aligned}$$

where the right-hand side tends to $\Phi(5)$ in d by the central limit theorem. As $\Phi(5) > e^{-2}$, we obtain $-\log p^{(j)} \geq 2$. Since $Z^{(1)}, \ldots, Z^{(m)}$ are independent, by binomial concentration, there are at least $(3/4)m$ indexes $j = 1, \ldots, m$ such that $\|Z^{(j)}\|_2^2 \geq d - 5\sqrt{d}$ whilst also satisfying (S.28) with probability $1 - e^{-\tau m} - me^{-\sqrt{d}/2}$ for some constant $\tau > 0$. Using that we can without loss of generality take $m \geq M_{\alpha,\beta}$ for a constant $M_{\alpha,\beta} > 0$ (otherwise the separation rate is effectively the same the one for $m = 1$) and since we consider $d \gtrsim m$, we obtain that the event the joint event occurs has mass less than $1 - \beta$. Furthermore, on this event, we have $1 - T_\alpha = 0$ for $M_{\alpha,\beta} > 0$ large enough, since

$$\sum_{j=1}^{m} -2\log p^{(j)} \geq 4m \cdot (3/4)$$

and by the fact that the chi-square quantile tends to $2m + C_\alpha\sqrt{2m}$ for some constant only depending on α, which is less than $4m \cdot (3/4) = 3m$ for $m \geq M_{\alpha,\beta}$.

Assume now that $n\|f\|_2^2 \leq 20\sqrt{d}$. Consider the following claim: for d large enough it holds that

$$-2\mathbb{E}_f \log\left(1 - F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2)\right) \geq 2 - e^{-\sqrt{d}/4} + \frac{n\|f\|_2^2}{C\sqrt{d}} \tag{S.29}$$

for a fixed constant $C > 0$. If the claim holds,

$$\mathbb{P}_f(1-T_\alpha) \leq \mathbb{P}_f\left(\frac{\sqrt{m}n\|f\|_2^2}{C\sqrt{d}} - \sqrt{m}e^{-c\sqrt{d}} + \frac{1}{\sqrt{m}}\sum_{j=1}^{m} -2(\log p^{(j)} - \mathbb{E}_f \log p^{(j)}) \leq \eta_{m,\alpha}\right)$$

with $\eta_{m,\alpha} := \frac{1}{\sqrt{2m}}(F^{-1}_{\chi^2_{2m}}(1-\alpha) - 2m)$. Since the method is rate optimal when $m \lesssim d$, the second term of the LHS in the above display may be assumed to be small. For the third term, note that

$$\begin{aligned}\mathbb{E}_f(-2\log p^{(j)})^2 &= 4\mathbb{E}_f \log(1 - F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2)) \\ &\leq 4\log(1 - F_{\chi^2_d}(n\|f\|_2^2 + d)),\end{aligned}$$

where the last inequality follows from the log-concavity of $x \mapsto 1 - F_{\chi^2_d}(x)$ (see e.g. Theorem 3.4 in [17]). For $n\|f\|_2^2 \leq 20\sqrt{d}$, the latter quanity is uniformly bounded in n, m and d. Since the second moment bounds the variance, this implies that

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m} -2(\log p^{(j)} - \mathbb{E}_f \log p^{(j)}) = O_P(1)$$

by the independence of $p^{(j)}$ and $p^{(k)}$ for $k \neq j$. Consequently, for some constant $\tau > 0$,

$$\mathbb{P}_f(1 - T_\alpha) \leq \Pr\left(\frac{\sqrt{m}n\|f\|_2^2}{C\sqrt{d}} - \sqrt{m}e^{-\tau\sqrt{d}} + O_P(1) \leq \eta_{m,\alpha}\right).$$

Since $\eta_{m,\alpha} \to \Phi^{-1}(1-\alpha)$ by Lemma 9, the fact that

$$\frac{\sqrt{m}n\|f\|_2^2}{C\sqrt{d}} \geq C_{\alpha,\beta}/C$$

for large enough $C_{\alpha,\beta} > 0$ depending only on α and β and the fact that $m \lesssim d$, we have that $\mathbb{P}_f(1 - T_\alpha) \leq \beta$.

It remains to prove the claim of (S.29). We start by writing $-2\mathbb{E}_f \log\left(p^{(j)}\right)$ as

$$-2\mathbb{E}_f \log(1 - F_{\chi^2_d}(\|Z^{(j)}\|_2^2) - 2\mathbb{E}_f \log\left(\frac{1 - F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2)}{1 - F_{\chi^2_d}(\|Z^{(j)}\|_2^2)}\right).$$

The first term equals 2. Using $\log(x) \leq |x-1|$, the second term is bounded from below by

$$2\mathbb{E}_f\left|\frac{F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2) - F_{\chi^2_d}(\|Z^{(j)}\|_2^2)}{1 - F_{\chi^2_d}(\|Z^{(j)}\|_2^2)}\right| \geq 2\mathbb{E}_f\left|F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2) - F_{\chi^2_d}(\|Z^{(j)}\|_2^2)\right|.$$

By the same argument as used for (S.28),

$$\mathbb{E}_f \mathbb{1}_{\{f^\top Z^{(j)} < 0\}} F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2) \geq \mathbb{E}_f F_{\chi^2_d}(\frac{1}{2}\|\sqrt{n}f\|_2^2 + \|Z^{(j)}\|_2^2) - e^{-\sqrt{d}/4},$$

which is larger than $\mathbb{E}_f F_{\chi^2_d}(\|Z^{(j)}\|_2^2)$ for all large enough d. Additionally, on the event that $f^\top Z^{(j)} \geq 0$, it holds that

$$F_{\chi^2_d}(\|\sqrt{n}f + Z^{(j)}\|_2^2) \geq \mathbb{E}_f F_{\chi^2_d}(\frac{1}{2}\|\sqrt{n}f\|_2^2 + \|Z^{(j)}\|_2^2) \geq \mathbb{E}_f F_{\chi^2_d}(\|Z^{(j)}\|_2^2).$$

Furthermore, we have

$$\mathbb{E}_f F_{\chi^2_d}(\frac{1}{2}\|\sqrt{n}f\|_2^2 + \|Z^{(j)}\|_2^2) - \mathbb{E}_f F_{\chi^2_d}(\|Z^{(j)}\|_2^2) = \Pr\left(0 \leq \frac{\chi^2_d - \tilde{\chi}^2_d}{\sqrt{d}} \leq \frac{n}{2\sqrt{d}}\|f\|_2^2\right),$$

where $\chi^2_d, \tilde{\chi}^2_d$ are independent chi square random variables with d degrees of freedom, which tends in d to

$$\Phi\left(\frac{n}{2\sqrt{d}}\|f\|_2^2\right) - \Phi(0) \geq \frac{n}{C\sqrt{d}}\|f\|_2^2,$$

where the inequality holds under the assumption $n\|f\|_2^2 \leq 20\sqrt{d}$ for a large enough constant $C > 0$. Putting the above lower bounds together, we obtain (S.29). □

Lemma 7. *Let T_α correspond to a test of level α considered in (8) or (9). For all $\alpha, \beta \in (0,1)$ if*

$$\rho^2 \geq C_{\alpha,\beta}\frac{d^{3/2}}{mn} \tag{S.30}$$

we have

$$\sup_{f \in H_\rho} \mathbb{P}_f\left(T_\alpha = 0\right) \leq \beta$$

for a large enough constant $C_{\alpha,\beta}$ depending only on $\alpha, \beta \in (0,1)$.

Proof. The proof follows a similar line of reasoning as e.g. the proof of Lemma A.8 in [40]. Starting with (8), note that

$$\mathbb{P}_f(1-T_\alpha) = \text{Pr}\Big(\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\Big((d^{-1/2}\sqrt{mn}f_i + Z_i)\Big)^2 \leq d^{-1/2}F^{-1}_{\chi^2_d}(1-\alpha)\Big)$$

for independent $Z_1,\dots,Z_d \sim N(0,1)$. The latter display equals

$$\begin{aligned}\text{Pr}\Big(\frac{nm}{d\sqrt{d}}\|f\|_2^2 + 2\frac{\sqrt{mn}}{d}\sum_{i=1}^{d}f_iZ_i + \frac{1}{\sqrt{d}}\sum_{i=1}^{d}\left(Z_i^2-1\right) \leq d^{-1/2}(F^{-1}_{\chi^2_d}(1-\alpha)-d)\Big) = \\ \text{Pr}\Big(\frac{nm}{d\sqrt{d}}\|f\|_2^2 + (1+\sqrt{\frac{nm}{d^2}}\|f\|_2)O_P(1) \leq d^{-1/2}(F^{-1}_{\chi^2_d}(1-\alpha)-d)\Big) \leq \\ \text{Pr}\Big((1+\sqrt{\frac{nm}{d^2}\|f\|_2^2})O_P(1) \leq -\frac{1}{2}\frac{nm}{d\sqrt{d}}\|f\|_2^2\Big),\end{aligned}$$

where the last inequality holds for large enough $C_{\alpha,\beta}$ since $\frac{nm}{d\sqrt{d}}\|f\|_2^2 \geq C_{\alpha,\beta}$ and $d^{-1/2}(F^{-1}_{\chi^2_d}(1-\alpha)-d)$ is bounded in d by Lemma 9. The resulting probability can be made arbitrarily small by taking large enough $C_{\alpha,\beta}$.

For a variation to Edgington's method, ie (9), similar reasoning applies. Under the null hypothesis, $\mathbb{E}_0\Phi(\sqrt{n}X^{(j)}) = 1/2$, so a conservative test (i.e. $\mathbb{P}_0T_\alpha \leq \alpha$) based on Edgington's method is given by

$$T_\alpha = \mathbb{1}\left\{\left|\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\left[\frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(p^{(j)}-\frac{1}{2})\Big)^2 - \text{Var}_0(p^{(j)})\right]\right| \geq c\alpha^{-1/2}\right\}$$

for a constant $c > 0$ by e.g. Chebyshev's inequality. Under the alternative hypothesis, we have $p^{(j)} = \Phi(\sqrt{n}X_i^{(j)}) = \Phi(\sqrt{n}f_i + Z_i^{(j)})$ whenever $j \in \mathcal{J}_i$. The Type II error $\mathbb{P}_f(1-T_\alpha)$ equals

$$\begin{aligned}\mathbb{P}_f\left(\left|\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\left[\frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(p^{(j)}-\Phi(Z_i^{(j)})+\Phi(Z_i^{(j)})-\frac{1}{2})\Big)^2 - \text{Var}_0(p^{(j)})\right]\right| \leq c\alpha^{-1/2}\right) = \\ \mathbb{P}_f\left(\left|\zeta+\xi+\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(\Phi(\sqrt{n}f_i+Z_i^{(j)})-\Phi(Z_i^{(j)}))\Big)^2\right| \leq c\alpha^{-1/2}\right) \quad \text{(S.31)}\end{aligned}$$

where

$$\zeta = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}\left[\frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(\Phi(Z_i^{(j)})-\frac{1}{2})\Big)^2 - \text{Var}_0(p^{(j)})\right]$$

and

$$\xi = \frac{2\sqrt{d}}{m}\sum_{i=1}^{d}\Big(\sum_{j\in\mathcal{J}_i}(\Phi(\sqrt{n}f_i+Z_i^{(j)})-\Phi(Z_i^{(j)}))\Big)\Big(\sum_{j\in\mathcal{J}_i}(\Phi(Z_i^{(j)})-\frac{1}{2})\Big).$$

By independence between $Z_i^{(j)}$ and $Z_i^{(k)}$ when $j \neq k$, the random variable ζ is mean 0 under $\mathbb{E}_f$ with constant variance (i.e. not depending on d,m,n) and is thus $O_P(1)$. Similarly, ξ has constant order variance and expectation. By Jensen's inequality

$$\mathbb{E}_f(\Phi(\sqrt{n}f_i+Z_i^{(j)})-\Phi(Z_i^{(j)}))^2 \geq (\Phi(2^{-1/2}\sqrt{n}f_i)-\Phi(0))^2$$

where it is used that

$$\mathbb{E}_f\Phi(\sqrt{n}f_i+Z_i^{(j)}) = \text{Pr}\left(\sqrt{n}f_i+Z \geq Z'\right) = \Phi(2^{-1/2}\sqrt{n}f_i).$$

By Lemma A.11 in [40], the RHS of the second last display is lower bounded by $\frac{1}{12}\min\{\frac{1}{2}nf_i^2, 1\}$. By the independence of $Z_i^{(j)}$ and $Z_i^{(k)}$ when $j \neq k$, it also holds that

$$\mathbb{E}_f(\Phi(\sqrt{n}f_i+Z_i^{(j)})-\Phi(Z_i^{(j)}))(\Phi(\sqrt{n}f_i+Z_i^{(k)})-\Phi(Z_i^{(k)})) = (\Phi(2^{-1/2}\sqrt{n}f_i)-\Phi(0))^2.$$

Therefore,

$$\mathbb{E}_f \frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(\Phi(\sqrt{n}f_i + Z_i^{(j)}) - \Phi(Z_i^{(j)}))\Big)^2 \geq \frac{m}{12d}\min\{\frac{1}{2}nf_i^2, 1\}.$$

Adding and subtracting the above expectation and noting that

$$\frac{1}{\sqrt{d}}\sum_{i=1}^{d}\frac{d}{m}\Big(\sum_{j\in\mathcal{J}_i}(\Phi(\sqrt{n}f_i + Z_i^{(j)}) - \Phi(Z_i^{(j)}))\Big)^2$$

has constant variance by the independence of $Z_i^{(j)}$ and $Z_i^{(k)}$ when $j \neq k$, we obtain that (S.31) is bounded above by

$$\mathbb{P}_f\left(O_P(1) + \frac{m}{12d\sqrt{d}}\sum_{i=1}^{d}\min\{\frac{1}{2}nf_i^2, 1\} \leq c\alpha^{-1/2}\right).$$

If the minimum is taken by 1 for any $i = 1, \ldots, d$, the proof is completed by noting that $m \gtrsim d^2$ by assumption whenever the rate $\frac{d\sqrt{d}}{nm}$ is the optimal rate and considering m large enough. Otherwise, the power is arbitrarily small for

$$\frac{mn}{d\sqrt{d}}\|f\|_2^2 \geq C_{\alpha,\beta}$$

and $C_{\alpha,\beta}$ large enough. □

Lemma 8. *Let T_α correspond to a test of level α considered in (13). For all $\alpha, \beta \in (0,1)$ if*

$$\rho^2 \geq C_{\alpha,\beta}\frac{d}{mn} \tag{S.32}$$

we have

$$\sup_{f\in H_\rho}\mathbb{P}_f\left(T_\alpha = 0\right) \leq \beta$$

for a large enough constant $C_{\alpha,\beta}$ depending only on $\alpha, \beta \in (0,1)$.

Proof. The proof follows a similar line of reasoning as e.g. the proof of Lemma A.7 in [40]. For any $f \in \mathbb{R}^d$ such that $\|f\|_2 \geq \rho$, we have

$$U\sqrt{n}X^{(j)} \overset{d}{=} \sqrt{n}Uf + Z^{(j)}$$

under $\mathbb{P}_f$ by rotational invariance of the normal distribution. The probability of a Type II error of the test of level α given in (13) is then equal to

$$\text{Pr}\left(\left|\sqrt{n}\sqrt{m}(Uf)_1 + Z\right| \leq \Phi^{-1}(1-\alpha/2)\right),$$

with $Z \sim N(0,1)$. The random variable $(Uf)_1$ is in distribution equal to $\|f\|_2 Z_1'/\|Z'\|_2$ for a d-dimensional standard Gaussian random vector Z'. For any $\beta \in (0,1)$, there exists $c' > 0$ such that $\|Z'\|_2 > c'\sqrt{d}$ occurs with probability $1-\beta/2$. Also, for $\frac{\sqrt{nm}\|f\|_2}{c'\sqrt{d}} \geq C_{\alpha,\beta}/c'$ large enough,

$$\text{Pr}\left(\left|\frac{\sqrt{nm}\|f\|_2}{c'\sqrt{d}} + Z\right| \leq \Phi^{-1}(1-\alpha/2)\right) \leq \beta/2.$$

This concludes the proof of the lemma. □

The following fact is well known and included for completeness. For a random variable V, let F_V denote its CDF.

Lemma 9. *Let $W_1, \ldots, W_m$ be random variables and let $V_m = \sum_{j=1}^m W_j$. Suppose that*

$$m^{-1/2}\sum_{j=1}^{m}(W_j - \mathbb{E}W_j) \rightsquigarrow N(0,\sigma^2).$$

Then, for all $\alpha \in (0,1)$,

$$(\sigma^2 m)^{-1/2}\left(F_{V_m}^{-1}(\alpha) - \sum_{j=1}^{m}\mathbb{E}W_j\right) \to \Phi^{-1}(\alpha),$$

where Φ is the standard Gaussian CDF.

Proof. The quantile function

$$F_{V_m}^{-1}(\alpha) = \inf\{x \in \mathbb{R} : \Pr(V_m \leq x) \geq \alpha\}$$

satisfies $z(F_{V_m}^{-1}(\alpha) - y) = F_{z(V_m - y)}^{-1}(\alpha)$. The result now follows by e.g. Lemma 21.2 in [44]. $\square$

A.5 Proof Lemma 1 and Lemma 2

Proof of Lemma 1. The lemma directly follows from Theorem 1 and Theorem 3 after verifying the corresponding conditions. Assumption 1 is satisfied if $p^{(j)}$ is generated using only local randomness, while in case of shared randomness, the same conclusion holds for Assumption 4. Below, we prove Assumptions 2 and 3 for the examples listed in the lemmas.

1. Fisher's method: let $S^{(j)} = -2\log p^{(j)} \sim^{H_0} \chi_2^2$ and consider the test of level α as

$$\mathbb{1}\Big\{\eta_{\alpha,m}\frac{1}{\sqrt{2m}}\sum_{j=1}^{m}(S^{(j)} - 2) \geq \Phi^{-1}(1-\alpha)\Big\}$$

with Φ^{-1} the inverse standard normal CDF and

$$\eta_{\alpha,m} := \Phi^{-1}(1-\alpha)\left(\frac{1}{\sqrt{2m}}\Big(F_{\chi_{2m}^2}^{-1}(1-\alpha) - 2m\Big)\right)^{-1}.$$

In view of the CLT, see Lemma 9, the sequence $\eta_{\alpha,m}$ converges to one, hence it is bounded. Furthermore, note that the corresponding combination function $C_m(s) := (\eta_{\alpha,m}/\sqrt{m})\sum_{j=1}^{m}(s_j - 1)$ with $s = (s_j) \in \mathbb{R}^m$ satisfies Assumption 2 (e.g. with $p = q = 1$). This in turn implies the moment condition for $S^{(j)}$, concluding the proof.

2. Mudholkar and George's method: The corresponding combination function $C_m(s) := |m^{-1/2}\sum_{j=1}^{m} s_j|$, by triangle inequality, satisfies Assumption 4. Since $S^{(j)} := -\log(p^{(j)}(1-p^{(j)}))$, the moment conditions are also satisfied.

3. Pearson's and Edgington's methods: the proofs follow the same reasoning as above with an additional application of the reverse triangle inequality in case of a two sided test.

4. Tippett's method: when small p-values are expected under the alternative hypothesis, a test of level $\alpha \in (0,1)$ is given by

$$T_\alpha = \mathbb{1}\left\{1 - \left(1 - \min\{p^{(1)}, \ldots, p^{(m)}\}\right)^m \leq \alpha\right\},$$

where $1 - (1 - \min\{p^{(1)}, \ldots, p^{(m)}\})^m$ is uniformly distributed under the null (see e.g. [42]). Observe that it is equivalent to

$$\mathbb{1}\left\{-m\min\left\{-\log(1-p^{(j)})\right\} \geq \log(1-\alpha)\right\}.$$

For $j = 1, \ldots, m$, take $S^{(j)} = -\log(1-p^{(j)}) \sim^{H_0} \mathrm{Exp}(1)$. The threshold $\alpha \mapsto \log(1-\alpha)$ is strictly decreasing and the combination function $C_m(s) = -m\min s_j$ satisfies

$$|C_m(s) - C_m(s')| \leq m\min|s_j - s_j'| \leq \sum_{j=1}^{m}|s_j - s_j'|.$$

Consequently, Assumptions 3 and 2 are satisfied.

5. Generalized averages: The case where $r = -\infty$ corresponds to Tippett's method above. Similarly, $r = \infty$ corresponds to the maximum of p-values, for which the proof follows by similar steps. For $r \in [\frac{1}{m-1}, \infty)$, $a_{r,m}$ can be chosen such that the test T_α in defined in Section 3.1 has precise level: $\mathbb{P}_0 T_\alpha = \alpha$, see Proposition 2 and 3 in [48]. For such $a_{r,m}$, the set $\{a_{r,m} : r \in [\frac{1}{m-1}, \infty)\}$ is bounded (see Table 1 in the aforementioned paper). This test can easily be seen to be of the form (3) and for the generalized average, we have $(m^{-1}\sum_{j=1}^{m}(s_j)^r)^{1/r} = \|m^{-1/r}s\|_r$, which yields

$$m^{-1/r}\big|\|s\|_r - \|s'\|_r\big| \leq m^{-1/r}\|s - s'\|_r \leq \max_j|s_j - s_j'|,$$

so Assumption 2 is satisfied since $a_{r,m}$ is bounded.

$\square$

Proof of Lemma 2. **Product of e-values:** The e-value test T_α for the combination function $(e_j) \mapsto \prod_{j=1}^{m} e_j$ can be written as

$$T_\alpha = \mathbb{1}\Big\{\sum_{j=1}^{m} \log E^{(j)} \geq \log(1/\alpha)\Big\}.$$

For $S^{(j)} := \log E^{(j)}$ and $C_m(s) = \sum_{j=1}^{m} s_j$ note that $E_0|\log E^{(j)}| < \infty$ and C_m satisfies (3). Since $\alpha \mapsto \log(1/\alpha)$ is strictly decreasing on $(0,1)$, the assumptions of Theorems 1 and 3 are met.

Average of e-values: Since $E^{(j)}$ is nonnegative, the moment condition is satisfied. The map $(e_j) \mapsto m^{-1}\sum_{j=1}^{m} e_j$ satisfies (3), while the map $\alpha \mapsto \alpha^{-1}$ is strictly decreasing and independent of m. Hence the conditions of Theorems 1 and 3 are satisfied. $\square$

A.6 Additional simulations

Figure A.6 shows the further improvement of the combined chi-square tests compared to the directional methods as d grows with respect to the number of trials, for signals that are around the detection threshold. Figure A.6 shows the further worsening of performance of the combined chi-square tests compared method as m grows with respect to the dimension, for signals that are around the detection threshold. For each of these simulations, $10,000$ repitions for every value $\alpha \in \{0.01, 0.02, \dots, 0.99\}$ of the level of the tests are considered.

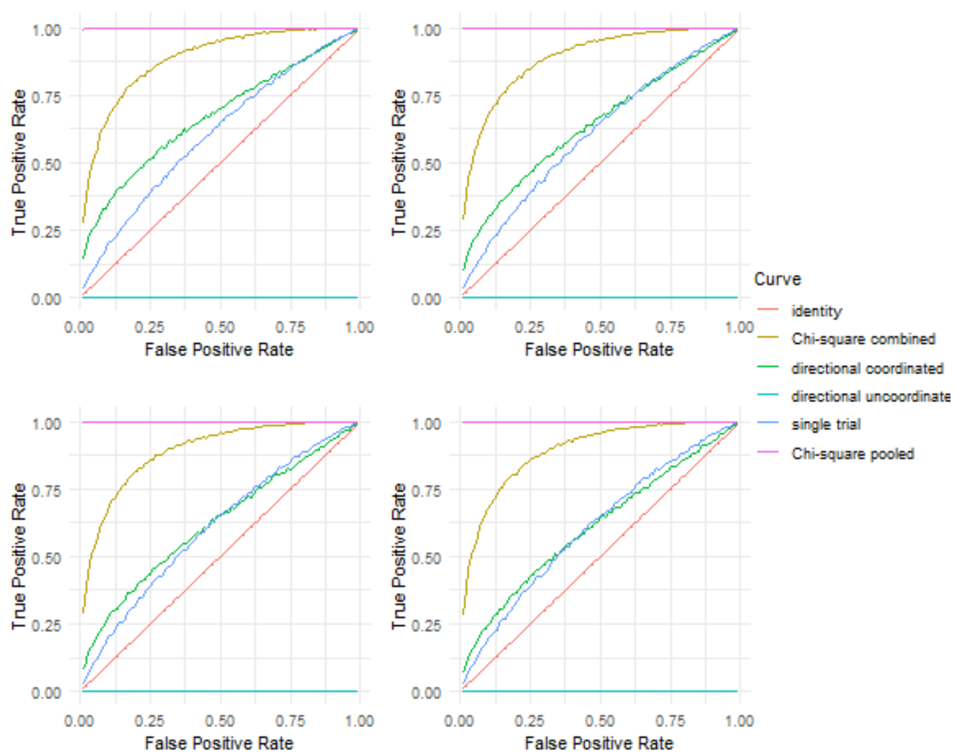

Figure 2: ROC curves for different values of d, whilst keeping $m = 20$, $n = 30$, $\rho^2 = 9\sqrt{d}/(16n)$. From left to right, top to bottom: $d = 30$, $d = 60$, $d = 90$, $d = 120$. The uncoordinated directional test requires $m \geq d$ and is therefore has TPR set to 0.

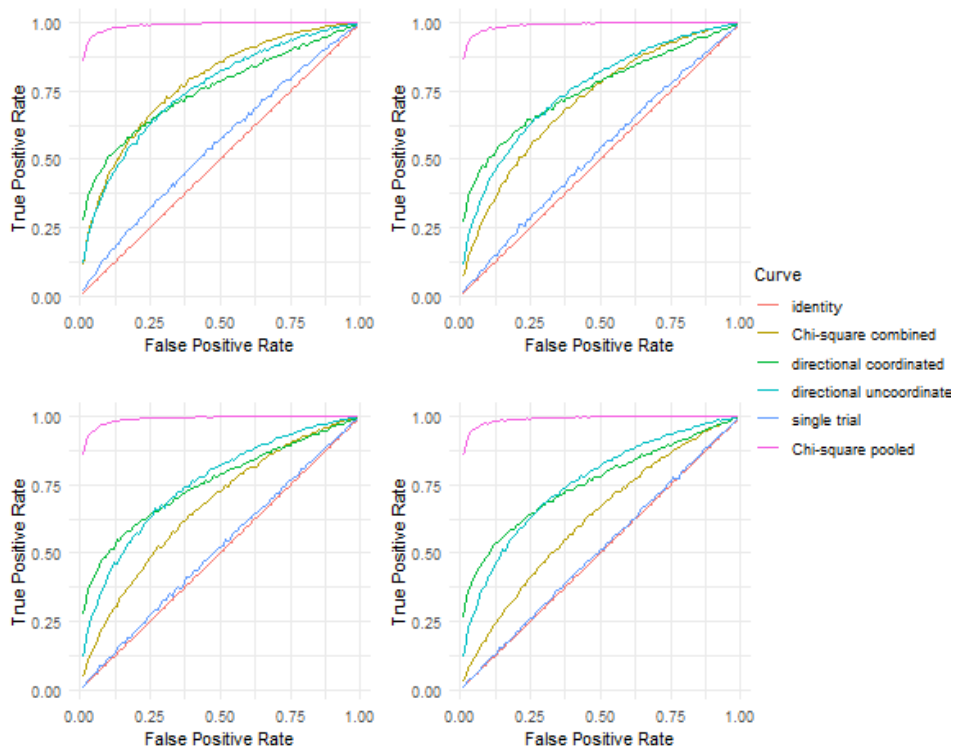

Figure 3: ROC curves for different values of m, whilst keeping $d = 5$, $n = 30$, $\rho^2 = 9d/(16nm)$. From left to right, top to bottom: $m = 30$, $m = 60$, $m = 100$, $m = 200$.