Appendix A. Proof of Other Theoretical Results

A.1. Proof of Proposition 5

Proof For any $\delta \in (0,1)$, we have

$$
\begin{aligned}
&\mathbb{P}\left(\exists i \in [K], \exists s \in [t], |\hat{g}_{s,i} - g_i| > \alpha(T_i(s), \delta)\right) \\
&\quad \le \mathbb{P}\left(\exists i \in [K], \exists s \in [t], \|\hat{\boldsymbol{\mu}}_{s,i} - \boldsymbol{\mu}_i\|_\infty > \alpha(T_i(s), \delta)\right) \qquad \text{(Proposition 4)} \\
&\quad = \mathbb{P}\left(\exists i \in [K], \exists s \in [t], \exists m \in [M], \left|\hat{\mu}_{s,i}^{(m)} - \mu_i^{(m)}\right| > \alpha(T_i(s), \delta)\right) \\
&\quad \le \sum_{i \in [K]} \sum_{s \in [t]} \sum_{m \in [M]} \mathbb{P}\left(\left|\hat{\mu}_{s,i}^{(m)} - \mu_i^{(m)}\right| > \alpha(T_i(s), \delta)\right) \qquad \text{(Union Bound)} \\
&\quad \le 2KM \sum_{s \in [t]} \exp\left(-\frac{T_i(s)\alpha(T_i(s), \delta)^2}{2\sigma^2}\right) \qquad \text{(Proposition 2)} \\
&\quad \le 2KM \sum_{\tau=1}^{\infty} \exp\left(-\frac{\tau\alpha(\tau, \delta)^2}{2\sigma^2}\right) \\
&\quad = \frac{6\delta}{\pi^2} \sum_{\tau=1}^{\infty} \frac{1}{\tau^2} \qquad \text{(By our choice of } \alpha\text{)} \\
&\quad \le \delta.
\end{aligned}
$$

■

A.2. Proof of Proposition 4

Proof Fix $i \in [K]$ and $\boldsymbol{x} \in \Delta_K$ arbitrarily, and let

$$
\begin{aligned}
A_{M-1} &= \max\left\{\xi_1 - \mu_i^{(1)}, \ldots, \xi_{M-1} - \mu_i^{(M-1)}\right\} \\
\hat{A}_{M-1} &= \max\left\{\xi_1 - \hat{\mu}_i^{(1)}, \ldots, \xi_{M-1} - \hat{\mu}_i^{(M-1)}\right\}.
\end{aligned}
$$

Then, by equation 1 it suffices to show that

$$
\left|A_M - \hat{A}_M\right| \le \max\left\{\left|\mu_i^{(1)} - \hat{\mu}_i^{(1)}\right|, \ldots, \left|\mu_i^{(M)} - \hat{\mu}_i^{(M)}\right|\right\}. \tag{3}
$$

We prove this by induction on M, (Basis) For $M = 1$:

$$
\begin{aligned}
&|(\nabla_{\boldsymbol{x}} L(\boldsymbol{x}))_i - (\nabla_{\boldsymbol{x}} L(\hat{\boldsymbol{\mu}}_1, \ldots, \hat{\boldsymbol{\mu}}_K, \boldsymbol{x}))_i| \\
&\quad = \left|\max\left\{\xi_1 - \mu_i^{(1)}\right\} - \max\left\{\xi_1 - \hat{\mu}_i^{(1)}\right\}\right| \\
&\quad = \left|\hat{\mu}_i^{(1)} - \mu_i^{(1)}\right|.
\end{aligned}
$$

(Induction Step) We assume equation (3) holds for $M - 1$,

$$
\left|A_M - \hat{A}_M\right|
$$

$$
\begin{aligned}
&= \left|\max\{A_{M-1}, \xi_M - \mu_i^{(M)}\} - \max\{\hat{A}_{M-1}, \xi_M - \hat{\mu}_i^{(M)}\}\right| \\
&= \left|\frac{1}{2}[A_{M-1} + \xi_M - \mu_i^{(M)} + |A_{M-1} - (\xi_M - \mu_i^{(M)})|] \right.\\
&\quad \left. - \frac{1}{2}[\hat{A}_{M-1} + \xi_M - \hat{\mu}_i^{(M)} + |\hat{A}_{M-1} - (\xi_M - \hat{\mu}_i^{(M)})|]\right| \\
&\leq \left|\frac{1}{2}\left[A_{M-1} - \hat{A}_{M-1} + \hat{\mu}_i^{(M)} - \mu_i^{(M)}\right] + \frac{1}{2}\left|A_{M-1} - \hat{A}_{M-1} + \mu_i^{(M)} - \hat{\mu}_i^{(M)}\right|\right| \\
&= \left|\max\{A_{M-1} - \hat{A}_{M-1}, \hat{\mu}_i^{(M)} - \mu_i^{(M)}\}\right| \\
&\leq \max\left\{\left|A_{M-1} - \hat{A}_{M-1}\right|, \left|\hat{\mu}_i^{(M)} - \mu_i^{(M)}\right|\right\} \\
&\leq \max\left\{|\mu_i^{(1)} - \hat{\mu}_i^{(1)}|, \dots, |\mu_i^{(M)} - \hat{\mu}_i^{(M)}|\right\},
\end{aligned}
$$

and for any $a, b \in \mathbb{R}$,

$$
\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} \tag{4}
$$

$$
|a| - |b| \leq |a - b|. \tag{5}
$$

The second and third equations hold for (4) while the first inequality stands over (5), and others are standard algebra. ■

A.3. Proof of Theorem 6

In the following, we abuse the notations and denote $\hat{\mu}_{n,i}^{(m)} = \sum_{s \in \mathcal{T}_i(t)} z_{s,i}^{(m)}/n$, where $t = \min\{s \mid T_i(s) = n\}$ and $\mathcal{T}_i(t) = \{s \in [t] \mid i_s = i\}$. Similarly, we write $\hat{g}_{n,i} = \max\{\xi_1 - \hat{\mu}_{n,i}^{(1)}, \dots, \xi_M - \hat{\mu}_{n,i}^{(M)}\}$, $\overline{g}_{n,i} = \hat{g}_{n,i} + \alpha(n, \delta)$, and $\underline{g}_{n,i} = \hat{g}_{n,i} - \alpha(n, \delta)$, respectively. Then, we give Proposition 13 and Lemma 14 for preparation.

Proposition 13 *For any $\epsilon \geq 0$, arm $i \in [K]$ and n,*

$$
\mathbb{P}\left(|g_i - \hat{g}_{n,i}| \geq \epsilon\right) \leq 2M \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right).
$$

Proof

$$
\begin{aligned}
\mathbb{P}\left(|g_i - \hat{g}_{t,i}| \geq \epsilon\right) &\leq \mathbb{P}\left(\|\hat{\boldsymbol{\mu}}_{t,i} - \boldsymbol{\mu}_i\|_\infty \geq \epsilon\right) \quad \text{(Proposition 4)} \\
&= \mathbb{P}\left(\exists m \in [M], \left|\hat{\mu}_{t,i}^{(m)} - \mu_i^{(m)}\right| \geq \epsilon\right) \\
&\leq \sum_{m \in [M]} \mathbb{P}\left(\left|\hat{\mu}_{t,i}^{(m)} - \mu_i^{(m)}\right| \geq \epsilon\right) \quad \text{(Union Bound)} \\
&\leq 2M \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right) \quad \text{(Proposition 2)}.
\end{aligned}
$$

■

Lemma 14 *For Algorithm 2, we have*

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{\underline{g}_{n,i} > 0\}\right) \leq \frac{\delta}{K}, \quad \textit{for any good arm } i, \textit{ and}$$
$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{\overline{g}_{n,i} \leq \epsilon\}\right) \leq \frac{\delta}{K}, \quad \textit{for any non-}\epsilon\textit{-good arm } i.$$

Proof As for any non-ϵ-good arm $i \in [K]$,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{\overline{g}_{n,i} \leq \epsilon\}\right) &\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\overline{g}_{n,i} \leq \epsilon\right) \quad \text{(Union bound)} \\
&\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\overline{g}_{n,i} \leq g_i\right) \quad (g_i > \epsilon \text{ for any non-}\epsilon\text{-good arm}) \\
&\leq \sum_{n\in\mathbb{N}} 2Me^{-\frac{n\left(\sqrt{\frac{2\sigma^2 \ln(\pi^2 KMn^2/3\delta)}{n}}\right)^2}{2\sigma^2}} \quad \text{(Proposition 13)} \\
&= \sum_{n\in\mathbb{N}} \frac{6M\delta}{\pi^2 KMn^2} \\
&\leq \frac{\delta}{K} \quad \left(\text{By } \sum_{n\in\mathbb{N}} \frac{1}{n^2} \leq \frac{\pi^2}{6}\right).
\end{aligned} \tag{6}$$

Next, consider any good arm $i \in [K]$,

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{\underline{g}_{n,i} > 0\}\right) &\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\underline{g}_{n,i} > 0\right) \quad \text{(Union bound)} \\
&\leq \sum_{n\in\mathbb{N}} \mathbb{P}\left(\underline{g}_{n,i} > g_{n,i}\right) \quad (g_i \leq 0 \text{ for any good arm}) \\
&\leq \frac{\delta}{K} \quad \text{(Same arguments as (6))}.
\end{aligned}$$

■

Proof [Theorem 6] First, if no ϵ-good arm exists, the failure probability is at most

$$\mathbb{P}\left(\exists i \in [K], \bigcup_{n\in\mathbb{N}}\{\overline{g}_{n,i} \leq \epsilon\}\right) \leq \delta \tag{7}$$

by using the union bound and Lemma 14.

Similarly, if there exists a non-empty good arm set

$$[K]_{\text{good}} = \left\{i_1^{\text{good}}, \ldots, i_{|[K]_{\text{good}}|}^{\text{good}}\right\},$$

then the failure probability is given as

$$\begin{aligned}&\mathbb{P}\left(\hat{a}=\perp\cup g_{\hat{a}}>\epsilon\right)\\&\quad\leq\mathbb{P}\left(\hat{a}=\perp\right)+\mathbb{P}\left(g_{\hat{a}}>\epsilon\right)\quad\text{(Union bound)}.\end{aligned}\tag{8}$$

We give an upper bound for each term in (8). First,

$$\begin{aligned}&\mathbb{P}\left(\hat{a}=\perp\right)\\&\quad\leq\mathbb{P}\left(\forall i\in[K]_{\text{good}},\bigcup_{n\in\mathbb{N}}\{\underline{g}_{n,i}>\epsilon\}\right)\\&\quad\leq\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{\underline{g}_{n,i}>\epsilon\}\text{ for a particular good arm }i\right)\\&\quad\leq\frac{\delta}{K}\quad\text{(Lemma 14)}.\end{aligned}\tag{9}$$

Then, for the second part,

$$\begin{aligned}&\mathbb{P}\left(g_{\hat{a}}>\epsilon\right)\\&\quad\leq\mathbb{P}\left(\exists\text{non-}\epsilon\text{-good arm }i\text{ s.t. }\bigcup_{n\in\mathbb{N}}\overline{g}_i>\epsilon\right)\\&\quad\leq\frac{(K-1)\delta}{K},\end{aligned}\tag{10}$$

where the last inequality is obtained by using Lemma 14 and the fact that there are at most $K-1$ non-ϵ-good arms since a good arm exists. Combining (8), (9) and (10) leads to the result.

■

A.4. Proof of Theorem 8

Lemma 15 *The following statements hold.*

1. *For any good arm i with ϵ_0 be any number such that $0<\epsilon_0<\epsilon-g_i$ and any $n>t_i(\epsilon_0)$,*

$$\mathbb{P}\left(\overline{g}_{n,i}>\epsilon\right)\leq Me^{-\frac{n{\epsilon_0}^2}{2\sigma^2}}.\tag{11}$$

2. *For any non-ϵ-good arm i with ϵ_0 be any number such that $0<\epsilon_0<g_i-\epsilon$ and any $n>t_i(\epsilon_0)$,*

$$\mathbb{P}\left(\underline{g}_{n,i}\leq\epsilon\right)\leq Me^{-\frac{n{\epsilon_0}^2}{2\sigma^2}}.\tag{12}$$

Proof Here, we only focus on (11), the proof for the other half is similar. By assumption,

$$n>t_i(\epsilon_0)=\max\left\{\frac{4\sigma^2}{(\epsilon-g_i-\epsilon_0)^2}\ln\left(\frac{8\sqrt{3}\sigma^2\pi KM/\delta}{3(\epsilon-g_i-\epsilon_0)^2}\ln\frac{4\sqrt{3}\pi\sigma^2}{3(\epsilon-g_i-\epsilon_0)^2}\right),0\right\}.$$

First, we show

$$\sqrt{\frac{2\sigma^2 \ln \frac{\pi^2 KMn^2}{3\delta}}{n}} < \epsilon - g_i - \epsilon_0. \tag{13}$$

Let $a = (\epsilon - g_i - \epsilon_0)^2$ for simplicity. Since the left hand side of (13) is monotone decreasing w.r.t. n, it is suffices to show that (13) is satisfied for $n = t_i(\epsilon_0) = \frac{4\sigma^2}{a} \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}$ with

$$b = \max\left\{\frac{8\sqrt{3}\sigma^2}{\pi} \ln \frac{4\sqrt{3}\pi\sigma^2\sqrt{KM/\delta}}{3a}, \frac{3a}{\pi^2\sqrt{KM/\delta}}\right\}. \tag{14}$$

For the case $b = \frac{3a}{\pi^2\sqrt{KM/\delta}}$, the inequality (14) holds trivially. Then, we consider the other part and let $A = \frac{\pi^2\sqrt{KM/\delta}}{3a}$, and $B = \pi/(4\sqrt{3}\sigma^2)$. Then we have $\ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a} = \ln Ab$ and $b = \frac{2}{B}\ln\frac{A}{B}$,

$$\begin{aligned}
\ln Ab &= \ln \frac{2A}{B} \ln \frac{A}{B} \\
&= \ln \frac{A}{B} + \ln 2 \ln \frac{A}{B} \\
&\leq 2 \ln \frac{A}{B} = Bb \quad (\ln 2x < x),
\end{aligned}$$

which indicates $\ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a} < \pi b/4\sqrt{3}\sigma^2$.

$$\begin{aligned}
&\ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a} < \pi b/4\sqrt{3}\sigma^2 \\
&\Leftrightarrow \left(4\sigma^2 \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}\right)^2 < \frac{\pi^2 b^2}{3} \\
&\Leftrightarrow \frac{KM\left(4\sigma^2 \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}\right)^2}{3\delta a^2} < \frac{\pi^2 b^2 KM}{9\delta a^2} \\
&\Leftrightarrow \ln \frac{\pi^2 KM\left(\frac{4\sigma^2}{a} \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}\right)^2}{3\delta} < 2\ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}.
\end{aligned} \tag{15}$$

Then we have

$$\begin{aligned}
\sqrt{\frac{2\sigma^2 \ln \frac{\pi^2 KMn^2}{3\delta}}{n}} &= \sqrt{\frac{2\sigma^2 \ln \frac{\pi^2 KM\left(\frac{4\sigma^2}{a} \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}\right)^2}{3\delta}}{\frac{4\sigma^2}{a} \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}}} \\
&\leq \sqrt{\frac{4\sigma^2 \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}}{\frac{4\sigma^2 \ln \frac{\pi^2 b\sqrt{KM/\delta}}{3a}}{a}}} \quad \text{(Inequality (15))}
\end{aligned}$$

$$\leq \sqrt{a} = \epsilon - g_i - \epsilon_0,$$

which shows (13) holds. Next,

$$\begin{aligned}
&\overline{g}_{n,i} > \epsilon \\
&\Leftrightarrow \hat{g}_{n,i} > \epsilon - \sqrt{\frac{2\sigma^2 \ln \frac{\pi^2 K M n^2}{3\delta}}{n}} \\
&\Rightarrow \hat{g}_{n,i} \geq g_i + \epsilon_0 \quad \text{(Inequality (13))} \\
&\Rightarrow \exists j \in [M], \xi_j - \hat{\mu}^{(j)}_{n,i} \geq \xi_j - \mu^{(j)}_{n,i} + \epsilon_0.
\end{aligned}$$

Take the probability on both sides,

$$\begin{aligned}
&\mathbb{P}\left(\overline{g}_{n,i} > \epsilon\right) \\
&\leq \sum_{j \in [M]} \mathbb{P}\left(\left\{\hat{\mu}^{(j)}_{n,i} \leq \mu^{(j)}_{n,i} - \epsilon_0\right\}\right) \\
&\leq M e^{-\frac{n\epsilon_0^2}{2\sigma^2}} \quad \text{(Union bound and proposition 2).}
\end{aligned}$$

■

Lemma 16 *We have*

1. *For any good arm i with ϵ_0 be any number such that $0 < \epsilon_0 < \epsilon - g_i$,*
$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\left[\overline{g}_{n,i} > \epsilon\right]\right] \leq t_i(\epsilon_0) + \frac{2M\sigma^2}{\epsilon_0^2}. \tag{16}$$

2. *For any non-ϵ-good arm i with ϵ_0 be any number such that $0 < \epsilon_0 < g_i - \epsilon$,*
$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\left[\underline{g}_{n,i} \leq \epsilon\right]\right] \leq t_i(\epsilon_0) + \frac{2M\sigma^2}{\epsilon_0^2}. \tag{17}$$

Proof Here, we give a proof for (16).

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\left[\overline{g}_{n,i} > \epsilon\right]\right] &\leq t_i(\epsilon_0) + \sum_{n=t_i(\epsilon_0)+1}^{\infty} \mathbb{P}\left(\overline{g}_{n,i} > \epsilon\right) \\
&\leq t_i(\epsilon_0) + \sum_{n=t_i(\epsilon_0)+1}^{\infty} M e^{-\frac{n\epsilon_0^2}{2\sigma^2}} \quad (\textit{Lemma } 15) \\
&\leq t_i(\epsilon_0) + \sum_{n=1}^{\infty} M e^{-\frac{n\epsilon_0^2}{2\sigma^2}} \\
&= t_i(\epsilon_0) + \frac{M}{e^{\frac{\epsilon_0^2}{2\sigma^2}} - 1} \quad \left(\sum_{t=1}^{\infty} e^{-ta} = \frac{1}{e^a - 1} \text{ for } a > 0\right)
\end{aligned}$$

$$\leq t_i(\epsilon_0) + \frac{2M\sigma^2}{\epsilon_0^2} \quad (e^a - 1 \geq a \text{ for } a > 0)\,.$$

The proof of (17) is similar to the proof of (16).

■

Let ϵ_0 be such that $0 < \epsilon_0 < \epsilon - g_i$ for any good arm i, and

$$T_0 = K \max_{i\in[K]} \lfloor t_i(\epsilon_0) \rfloor.$$

Lemma 17 $\mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{1}\left[i_t = i^*\right]\right] \leq t_{i^*}(\epsilon_0) + \frac{2M\sigma^2}{\epsilon_0^2}$.

Proof We have

$$\begin{aligned}
\sum_{t=1}^{\infty} \mathbb{1}\left[i_t = i^*\right] &= \sum_{t=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{1}[i_t = i^*, T_{i^*}(t) = n] && (18)\\
&= \sum_{n=1}^{\infty} \mathbb{1}\left[\bigcup_{t=1}^{\infty} \{i_t = i^*, T_{i^*}(t) = n\}\right] && (19)\\
&\leq 1 + \sum_{n=2}^{\infty} \mathbb{1}\left[\overline{g}_{n-1,i^*} > \epsilon\right]\\
&= 1 + \sum_{n=1}^{\infty} \mathbb{1}\left[\overline{g}_{n,i^*,} > \epsilon\right],
\end{aligned}$$

where step (19) holds since if $i_t = i$ (arm i is pulled at round t), then i has not been considered as a good arm and $\forall s \leq t-1, \overline{g}_{s,i} > \epsilon$. By taking expectations on both sides, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{1}\left[i_t = i^*\right]\right] &\leq \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\left[\overline{g}_{n,i^*} > \epsilon\right]\right]\\
&\leq t_{i^*}(\epsilon_0) + \frac{2M\sigma^2}{\epsilon_0^2} \quad (\text{Lemma 16}).
\end{aligned}$$

■

Lemma 18

$$\mathbb{E}\left[\sum_{t=1}^{T_0} \mathbb{1}\left[i_t \neq i^*, \underline{g}_{t,i_t} \leq g_{i^*} + \epsilon_0\right]\right] \leq \sum_{i\neq i^*} \frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2} + \frac{2(K-1)M\sigma^2}{{\epsilon_0}^2}.$$

Proof

$$\sum_{t=1}^{T_0} \mathbb{1}\left[i_t \neq i^*, \underline{g}_{t,i_t} \leq g_{i^*} + \epsilon_0\right]$$

$$
\begin{aligned}
&= \sum_{i \neq i^*} \sum_{t=1}^{T_0} \sum_{n=1}^{T_0} \mathbb{1}\left[i_t = i, \tilde{g}_{t,i_t} \leq g_{i^*} + \epsilon_0, T_i(t) = n\right] \\
&= \sum_{i \neq i^*} \sum_{n=1}^{T_0} \mathbb{1}\left[\bigcup_{t=1}^{T_0} \{i_t = i, \tilde{g}_{t,i_t} \leq g_{i^*} + \epsilon_0, T_i(t) = n\}\right] \\
&\leq \sum_{i \neq i^*} \sum_{n=1}^{T_0} \mathbb{1}\left[\hat{g}_{n,i} - \sqrt{\frac{4\sigma^2 \ln(KMn)}{n}} \leq g_{i^*} + \epsilon_0\right] \\
&= \sum_{i \neq i^*} \sum_{n=1}^{T_0} \mathbb{1}\left[\hat{g}_{n,i} - \sqrt{\frac{4\sigma^2 \ln(KMn)}{n}} \leq g_i + (g_{i^*} - g_i) + \epsilon_0\right] \\
&\leq \sum_{i \neq i^*} \sum_{n=1}^{T_0} \mathbb{1}\left[\hat{g}_{n,i} - \sqrt{\frac{4\sigma^2 \ln(KMT_0)}{n}} \leq g_i + (g_{i^*} - g_i) + \epsilon_0\right] \\
&\leq \sum_{i \neq i^*} \left[\sum_{n=1}^{\frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2}} 1 + \sum_{n=\frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2}}^{T_0} \mathbb{1}\left[\hat{g}_{n,i} \leq g_i - \epsilon_0\right]\right]
\end{aligned}
$$

By taking expectations,

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T_0} \mathbb{1}\left[i_t \neq i^*, \tilde{g}_t^* \leq g_{i^*} - 2\epsilon_0\right]\right] \\
&\quad \leq \sum_{i \neq i^*} \frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2} + \sum_{i \neq i^*} \sum_{n=1}^{\infty} \mathbb{P}\left(\hat{g}_{n,i} \leq g_i - \epsilon_0\right) \\
&\quad \leq \sum_{i \neq i^*} \frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2} + \frac{(K-1)M}{e^{\frac{\epsilon_0^2}{2\sigma^2}} - 1} \qquad \text{(Proposition 13)} \\
&\quad \leq \sum_{i \neq i^*} \frac{4\sigma^2 \ln(KMT_0)}{(g_i - g_{i^*} - 2\epsilon_0)^2} + \frac{2(K-1)M\sigma^2}{\epsilon_0^2}.
\end{aligned}
$$

■

Lemma 19

$$
\mathbb{E}\left[\sum_{T_0+1}^{\infty} \mathbb{1}[t \leq T_{stop}]\right] \leq \frac{K^2 M}{2\epsilon_0^2} e^{4\epsilon_0^2}.
$$

Proof In this case, some arms are pulled at least $\lceil (t-1)/K \rceil$ times until round t.

$$
\mathbb{E}\left[\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[t \leq T_{\text{stop}}\right]\right]
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\sum_{i=1}^{K}\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[T_i(t) \geq \lceil (t-1)/K \rceil, t \leq T_{\text{stop}}\right]\right] \\
&\leq \mathbb{E}\left[K\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[T_i(t) \geq \lceil (t-1)/K \rceil, t \leq T_{\text{stop}}\right]\right] \\
&\leq \mathbb{E}\left[K\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[\overline{g}_{i,\lceil (t-1)/K \rceil} > \epsilon\right]\right].
\end{aligned}
$$

Since $T_0 \geq t_i(\epsilon_0)$ for all $i \in [K]$, then

$$
\begin{aligned}
&\mathbb{E}\left[K\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[\overline{g}_{i,\lceil (t-1)/K \rceil} > \epsilon\right]\right] \\
&\quad \leq KM \sum_{t=T_0+1}^{\infty} e^{-2{\epsilon_0}^2(\lceil (t-1)/K \rceil - 1)} \qquad \text{(Lemma 15)} \\
&\quad \leq KM \sum_{t=T_0+1}^{\infty} e^{-2{\epsilon_0}^2((t-1)/K - 1)} \\
&\quad \leq KM \int_{T_0}^{\infty} e^{-2{\epsilon_0}^2((t-1)/K - 1)} dt \\
&\quad = KMe^{4{\epsilon_0}^2}\left[-\frac{K}{2{\epsilon_0}^2} e^{-2{\epsilon_0}^2 t/K}\right]_{T_0}^{\infty} \\
&\quad = \frac{K^2M}{2{\epsilon_0}^2} e^{4{\epsilon_0}^2} e^{-2{\epsilon_0}^2 T_0/K} \\
&\quad \leq \frac{K^2M}{2{\epsilon_0}^2} e^{4{\epsilon_0}^2} e^{-2{\epsilon_0}^2 \max_{i\in[K]} \lfloor t_i(\epsilon_0) \rfloor} \\
&\quad \leq \frac{K^2M}{2{\epsilon_0}^2} e^{4{\epsilon_0}^2}.
\end{aligned}
$$

■

Lemma 20

$$
\mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{1}[i_t \neq i^*, t \leq T_{stop}, \tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0]\right] \leq \frac{2\sigma^2 M}{{\epsilon_0}^2} + \frac{(K-1)K^2M}{2{\epsilon_0}^2} e^{4{\epsilon_0}^2}.
$$

Proof

$$
\begin{aligned}
&\sum_{t=1}^{\infty} \mathbb{1}\left[i_t \neq i^*, t \leq T_{\text{stop}}, \tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0\right] \\
&\quad \leq \sum_{t=T_0+1}^{\infty} \mathbb{1}\left[i_t \neq i^*, t \leq T_{\text{stop}}\right] + \sum_{t=1}^{T_0} \mathbb{1}\left[\tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0\right] \qquad (20)
\end{aligned}
$$

Take expectation over the first part of (20),

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=T_0+1}^{\infty} \mathbb{1}\left[i_t \neq i^*, t \leq T_{\text{stop}}\right]\right] \\
&\quad\leq \sum_{t=T_0+1}^{\infty} \sum_{i \neq i^*} \mathbb{E}\left[\mathbb{1}\left[t \leq T_{\text{stop}}\right]\right] \qquad \text{(Union bound)} \\
&\quad\leq \sum_{t=T_0+1}^{\infty} (K-1)\mathbb{E}\left[\mathbb{1}\left[t \leq T_{\text{stop}}\right]\right] \\
&\quad\leq \frac{(K-1)K^2M}{2{\epsilon_0}^2} e^{4{\epsilon_0}^2} \qquad \text{(Lemma 19)}.
\end{aligned} \tag{21}$$

As for the second part of (20),

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}^{T_0} \mathbb{1}\left[\tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0\right]\right] \\
&= \sum_{t=1}^{T_0} \mathbb{E}\left[\mathbb{1}\left[\hat{g}_{t,i_t} - \sqrt{\frac{2\sigma^2 \ln\left(KMT_i(t)\right)}{T_i(t)}} > g_{i^*} + \epsilon_0\right]\right] \\
&\leq \sum_{t=1}^{T_0} \mathbb{E}\left[\mathbb{1}\left[\hat{g}_{t,i_t} > g_{i^*} + \epsilon_0\right]\right] \\
&\leq \sum_{t=1}^{T_0} Me^{-\frac{t{\epsilon_0}^2}{2\sigma^2}} \qquad \text{(Proposition 4)} \\
&\leq \frac{2\sigma^2 M}{{\epsilon_0}^2}.
\end{aligned} \tag{22}$$

Combining (21) and (22) leads to the result.

■

Proof [Theorem 8]

$$\begin{aligned}
\mathbb{E}[T_{\text{stop}}] &= \mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{1}[i_t = i^*, t \leq T_{\text{stop}}] + \sum_{t=1}^{\infty} \mathbb{1}[i_t \neq i^*, t \leq T_{\text{stop}}]\right] \\
&\leq \mathbb{E}\Bigg[\sum_{t=1}^{\infty} \mathbb{1}[i_t = i^*] \\
&\qquad + \sum_{t=1}^{\infty} \mathbb{1}[i_t \neq i^*, t \leq T_{\text{stop}}, \tilde{g}_{t,i_t} \leq g_{i^*} + \epsilon_0] \\
&\qquad + \sum_{t=1}^{\infty} \mathbb{1}[i_t \neq i^*, t \leq T_{\text{stop}}, \tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0]\Bigg] \\
&\leq \mathbb{E}\Bigg[\sum_{t=1}^{\infty} \mathbb{1}[i_t = i^*]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{T_0} \mathbb{1}[i_t \neq i^*, \tilde{g}_{t,i_t} \leq g_{i^*} + \epsilon_0] \\
&+\sum_{T_0+1}^{\infty} \mathbb{1}[t \leq T_{\mathrm{stop}}]] \\
&+\sum_{t=1}^{\infty} \mathbb{1}[i_t \neq i^*, t \leq T_{\mathrm{stop}}, \tilde{g}_{t,i_t} > g_{i^*} + \epsilon_0]\Bigg].
\end{aligned}
$$

Then, the final result is obtained by combining lemma 17 to lemma 20. ■

A.5. Proof of Theorem 9

Proof Here, we let $\mathcal{A} \subseteq [K]$ be the active set of arms that have not been deleted.

$$
\begin{aligned}
T_{\mathrm{stop}} &= \sum_{t=1}^{\infty} \mathbb{1}\left\{\text{Algorithm doesn't stop at trial } t\right\} \\
&= \sum_{t=1}^{\infty} \mathbb{1}\left[\text{Neither conditions 1 nor 2 are satisfied at trial } t\right] \\
&\leq \sum_{t=1}^{\infty} \mathbb{1}\left[\text{Condition 2 is not satisfied at trial } t\right] \\
&= \sum_{t=1}^{\infty} \mathbb{1}\left[\text{Condition 2 is not satisfied at trial } t\right] \\
&= \sum_{t=1}^{\infty} \mathbb{1}\left[\mathcal{A} \neq \emptyset \text{ at the end of trial } t\right] \\
&\leq \sum_{i=1}^{K}\sum_{t=1}^{\infty} \mathbb{1}\left[i_t = i \text{ and } i \text{ is not deleted at trial } t\right] \\
&= \sum_{i=1}^{K}\sum_{t=1}^{\infty} \mathbb{1}\left[i_t = i\right] \mathbb{1}\left[\hat{g}_{t,i} \leq \alpha(T_i(t), \delta)\right] \\
&= \sum_{i=1}^{K}\sum_{t=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{1}\left[i_t = i\right] \mathbb{1}\left[\hat{g}_{t,i} \leq \alpha(T_i(t), \delta)\right] \\
&= \sum_{i=1}^{K}\sum_{n=1}^{\infty} \mathbb{1}\left[\underline{g}_{n,i} \leq 0\right]
\end{aligned}
$$

Taking the expectation,

$$
\mathbb{E}[T_{\mathrm{stop}}] \leq \sum_{i=1}^{K}\sum_{n=1}^{\infty} \mathbb{E}\left[\mathbb{1}\left[\underline{g}_{n,i} \leq 0\right]\right]
$$

$$\leq \sum_{i\in[K]} t_i + \frac{KM\sigma^2}{\epsilon_0^2} \quad \text{(Lemma 16)}.$$

■

A.6. Proof of Theorem 12

Definition 21 (Binary entropy function) *$h:(0,1)\mapsto\mathbb{R}$ and $p\in(0,1)$,*

$$h(p) = -p\ln p - (1-p)\ln(1-p), \tag{23}$$

and $h(p)\leq \ln 2$.

Proposition 22 *Let ν and $\overline{\nu}$ be two bandits models with K arms and M objectives such that for all $i\in[K]$ and $m\in[M]$, the distributions $\nu_i^{(m)}$ and $\overline{\nu}_i^{(m)}$ are mutually absolutely continuous. For any almost-surely finite stopping time T and event $\mathcal{E}$,*

$$\sum_{i=1}^{K}\mathbb{E}[T_i(T)]\sum_{m=1}^{M} KL(\nu_i^{(m)},\overline{\nu}_i^{(m)}) \geq d(\mathbb{P}_\nu[\mathcal{E}],\mathbb{P}_{\nu'}[\mathcal{E}]),$$

where $KL(\nu_i,\nu_j)$ is the Kullback-Leibler divergence between distributions ν_i and ν_j.

Proof Denote the history until round t as $\mathcal{F}_t=\{i_1,\ldots,i_t,\boldsymbol{z}_1,\ldots,\boldsymbol{z}_t\}$ and $\{Y_{i,s}^{(m)}\}_{s=1,\ldots,T_i(t)}\sim\nu_i^{(m)}$ as the i.i.d. samples observed from arm i. Let $f_i^{(m)}$ be the density function for distribution $\nu_i^{(m)}$ with mean $\mu_i^{(m)}$ and define $\overline{f}_i^{(m)}$ similarly.

$$\begin{aligned}
&\mathbb{E}\left[\ln\frac{\mathbb{P}_\nu(\mathcal{F}_t)}{\mathbb{P}_{\overline{\nu}}(\mathcal{F}_t)}\right]\\
&\quad=\mathbb{E}\left[\sum_{i=1}^{K}\sum_{s=1}^{t}\mathbb{1}(i_s=i)\sum_{m=1}^{M}\ln\frac{f_i^{(m)}(\boldsymbol{z}_s^{(m)})}{\overline{f}_i^{(m)}(\boldsymbol{z}_s^{(m)})}\right]\\
&\quad=\mathbb{E}\left[\sum_{i=1}^{K}\sum_{s=1}^{T_i(t)}\sum_{m=1}^{M}\ln\frac{f_i^{(m)}(Y_{i,s}^{(m)})}{\overline{f}_i^{(m)}(Y_{i,s}^{(m)})}\right].
\end{aligned}\tag{24}$$

By definition,

$$\mathbb{E}_{\nu_i^{(m)}}\left[\ln\frac{f_i^{(m)}(Y_{i,s}^{(m)})}{\overline{f}_i^{(m)}(Y_{i,s}^{(m)})}\right]=KL(\nu_i^{(m)},\overline{\nu}_i^{(m)}).$$

Thus, with Wald's Lemma Wald (2004) we have

$$(24)=\sum_{i=1}^{K}\mathbb{E}\left[T_i(t)\right]\sum_{m=1}^{M}KL(\nu_i^{(m)},\overline{\nu}_i^{(m)}).$$

Then follows the analysis of Kaufmann et al. (2016) gives the conclusion. ■

Proof [Theorem 12] Here we only give details on the case when a good arm exists, and the other case can be deduced in a similar way. For any $\epsilon_1 > 0$ and fix a good arm i^*, consider a sequence of Bernoulli distributions associated with each arm $i \in [K]$ as $\{\overline{\nu}_i^{(1)}, \dots, \overline{\nu}_i^{(M)}\}$ with expectations $\{\overline{\mu}_i^{(1)}, \dots, \overline{\mu}_i^{(M)}\}$ defined as

$$\overline{\mu}_i^{(m)} = \begin{cases} \xi_m - \epsilon_1, & \text{if } i = i^*, m = m_i^* \\ \mu_i^{(m)}, & \text{otherwise,} \end{cases}$$

where $m \in [M]$ and $m_i^* = \min_{m \in [M]} d\left(\mu_i^{(m)}, \xi_m\right)$. Hence, i^* is a good arm under $\boldsymbol{\mu}$, and we denote the distributions with means $\{\mu_i^{(1)}, \dots, \mu_i^{(M)}\}$ as $v_i := \{\nu_i^{(1)}, \dots, \nu_i^{(M)}\}$ for convenience. Let $\mathcal{E}_{i^*} = \{\hat{a} = i^*\}$ and $p_{i^*} = \mathbb{P}(\hat{a} = i^*)$ under distribution ν_i with expectation $\boldsymbol{\mu}_i$. From proposition 22, we obtain that

$$\mathbb{E}[T_{i^*}(T)] KL\left(\nu_{i^*}^{(m_{i^*}^*)}, \overline{\nu}_{i^*}^{(m_{i^*}^*)}\right) \geq d(p_{i^*}, \min\{\delta, p_{i^*}\}). \tag{25}$$

The inequality (25) holds since $\mathcal{E}_{i^*}$ happens with probability $q_{i^*} \leq \delta$ under $\overline{\nu}_i$ and there is a case study as follows,

Case 1: For $\delta \leq p_{i^*}$, $d(p_{i^*}, q) \geq d(p_{i^*}, \delta)$.

Case 2: For $\delta > p_{i^*}$, $d(p_{i^*}, q) \geq d(p_{i^*}, p_{i^*}) = 0$.

Then,

$$\begin{aligned} d(p_{i^*}, \min\{\delta, p_{i^*}\}) &= \max\left\{p_{i^*} \ln \frac{1}{\min\{\delta, p_{i^*}\}} - h(p)\right. \\ &\quad \left. + (1 - p_{i^*}) \ln \frac{1}{1 - \min\{\delta, p_{i^*}\}}, 0\right\} \quad \text{(Definition 11)} \\ &\geq \max\left\{p_{i^*} \ln \frac{1}{\min\{\delta, p_{i^*}\}} - \ln 2, 0\right\} \quad (h(p) \leq \ln 2) \\ &\geq \max\left\{p_{i^*} \ln \frac{1}{\delta} - \ln 2, 0\right\}. \end{aligned}$$

Since i^* is not a good arm under means $\left\{\overline{\mu}_{i^*}^{(1)}, \dots, \overline{\mu}_{i^*}^{(M)}\right\}$, for any good arm i, by definition of $(\delta, 0)$-success

$$\begin{aligned} p_i &= \mathbb{E}_\nu[|[K]_{\text{good}} \cap \{\hat{a}\}|] \\ &\geq \mathbb{P}_\nu[\{\hat{a}\} \subseteq [K]_{\text{good}}] \\ &\geq 1 - \delta, \end{aligned}$$

where $[K]_{\text{good}} \subseteq [K]$ is the set of good arms. Thus we obtain

$$\sum_{i \in [K]} \mathbb{E}[T_i(T)] \geq \max_{i \in [K]_{\text{good}}} \mathbb{E}[T_i(T)] \geq P^*$$

and solve the lower bound as a optimization problem P^*

$$\text{minimize} \quad \frac{1}{\max\limits_{i \in [K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*} - \epsilon_1\right)} \max\left\{p_i \ln \frac{1}{\delta} - \ln 2, 0\right\},$$

$$
\begin{aligned}
\text{subject to}\quad & p_i \geq 1-\delta,\\
& 0 \leq p_i \leq 1,
\end{aligned}
$$

and reorganize the problem into

$$
\begin{aligned}
\text{minimize}\quad & \frac{q}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)},\\
\text{subject to}\quad & p_i \geq 1-\delta,\\
& q \geq p_i \ln\frac{1}{\delta} - \ln 2,\\
& q \geq 0,\\
& 0 \leq p_i \leq 1.
\end{aligned}
$$

Then, consider the dual problem

$$
\begin{aligned}
\text{maximize}\quad & (1-\delta)A - (\ln 2)B - C\\
\text{subject to}\quad & B \leq \frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)},\\
& A - B\ln\frac{1}{\delta} - C \leq 0,\\
& A, B, C \geq 0.
\end{aligned}
$$

Herein, we have the feasible solution given by

$$
\begin{aligned}
A &= \frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)} \ln\frac{1}{\delta}\\
B &= \frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)}\\
C &= 0.
\end{aligned}
$$

Then put the results back into the objective function

$$
\begin{aligned}
&(1-\delta)A - (\ln 2)B - C\\
&\quad = (1-\delta)\frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)} \ln\frac{1}{\delta} - (\ln 2)\frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)}\\
&\quad = \frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)} \ln\frac{1}{2\delta} - \frac{\delta}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)}
\end{aligned}
$$

Since the existence of the duality gap, the result of the transformed problem is smaller than the original one,

$$
\mathbb{E}[T_{\text{stop}}] \geq \frac{1}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)} \ln\frac{1}{2\delta} - \frac{\delta}{\max\limits_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m_i^*)}, \xi_{m_i^*}-\epsilon_1\right)}
$$

$$= \frac{1}{\max_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m)}, \xi_m - \epsilon_1\right)} \ln\frac{1}{2\delta} - \frac{\delta}{\max_{i\in[K]_{\text{good}}} d\left(\mu_i^{(m)}, \xi_m - \epsilon_1\right)}.$$

The final result is obtained by letting $\epsilon_1 \to 0$. ■

Appendix B. Supplementary of Experiments

B.1. Additional Results

Here, we provide other results not included in the main content. The error rate is the ratio of repetitions where the algorithm fails to output a good arm, and the symbol "–" indicates that the algorithm's error rate is higher than 50%. The exact results on stopping times are presented in Table 4, 5, 6 and 7. The error rates for medical data presented in Section 5 are shown in Table 6 and 7, and the standard deviations are in 10 and 11.

Table 4: Stopping Time w.r.t. δ with Synthetic Data

δ	MultiAPT	MultiHDoC	MultiLUCB	**MultiTUCB**
0.005	67558.19	44837.80	62850.43	**37889.20**
0.010	65445.56	43764.69	58366.11	**37184.51**
0.015	64890.75	43106.50	56017.03	**36712.40**
0.020	64149.97	42792.54	53833.43	**36359.47**
0.025	63211.24	42123.42	52583.60	**36104.91**
0.030	62911.17	42188.60	50985.18	**35887.75**
0.035	62387.45	41883.71	50186.66	**35701.17**
0.040	61927.60	41853.96	49157.64	**35557.45**
0.045	61919.50	41515.10	48519.28	**35399.62**
0.050	61754.00	41357.94	47518.73	**35263.71**

Table 5: Stopping Time w.r.t. ϵ with Synthetic Data

ϵ	MultiAPT	MultiHDoC	MultiLUCB	**MultiTUCB**
0.002	72142.25	9351.76	68861.26	**42323.90**
0.004	68899.35	46246.50	64704.05	**39260.29**
0.006	66254.14	43463.51	60994.38	**36619.35**
0.008	63817.75	41095.90	57423.84	**34091.37**
0.010	61281.93	38898.29	54010.93	**31832.58**
0.012	59506.56	37015.99	51001.18	**29879.56**
0.014	57565.94	35212.57	48344.01	**28139.68**
0.016	55942.74	33555.54	45735.95	**26371.34**
0.018	54355.53	32076.11	43349.82	**24910.16**
0.020	53031.22	30663.39	41194.27	**23509.48**

Table 6: Stopping Time w.r.t. δ with Medical Data

δ	MultiAPT	MultiHDoC	MultiLUCB	**MultiTUCB**
0.005	159612.14	1930.59	2950.43	**1512.23**
0.010	151168.96	1881.55	2664.67	**1472.03**
0.015	145659.37	1852.99	2516.41	**1450.23**
0.020	142957.35	1834.56	2406.29	**1433.91**
0.025	140266.20	1821.28	2316.79	**1422.12**
0.030	138247.26	1808.38	2221.46	**1410.64**
0.035	136898.94	1799.66	2154.06	**1399.08**
0.040	135402.13	1792.78	2079.05	**1391.86**
0.045	133913.38	1785.77	2039.52	**1385.56**
0.050	132941.90	1777.64	2006.03	**1379.53**

Table 7: Stopping Time w.r.t. ϵ with Medical Data

ϵ	MultiAPT	MultiHDoC	MultiLUCB	**MultiTUCB**
0.002	–	2041.31	3004.62	**1625.77**
0.004	173761.65	2025.90	2960.26	**1594.22**
0.006	137467.38	1986.86	2919.35	**1545.89**
0.008	108815.09	1933.00	2876.65	**1505.27**
0.010	90650.03	1884.11	2836.59	**1462.63**
0.012	76350.56	1828.88	2796.23	**1426.91**
0.014	65814.72	1778.66	2755.69	**1392.24**
0.016	57433.42	1729.29	2716.41	**1354.44**
0.018	50484.03	1683.60	2675.94	**1327.58**
0.020	45030.93	1639.93	2636.90	**1300.06**

Table 8: Error Rate (%) w.r.t. δ with Medical Data

δ	MultiAPT	MultiHDoC	MultiLUCB	MultiTUCB
0.005	10.40	0.00	0.00	0.00
0.010	8.50	0.00	0.00	0.00
0.015	7.15	0.00	0.00	0.00
0.020	6.60	0.00	0.00	0.00
0.025	5.90	0.00	0.00	0.00
0.030	5.50	0.00	0.00	0.00
0.035	5.35	0.00	0.00	0.00
0.040	5.10	0.00	0.00	0.00
0.045	4.75	0.00	0.00	0.00
0.050	4.60	0.00	0.00	0.00

Table 9: Error Rate (%) w.r.t. ϵ with Medical Data

ϵ	MultiAPT	MultiHDoC	MultiLUCB	MultiTUCB
0.002	54.35	0.00	0.00	0.00
0.004	22.45	0.00	0.00	0.00
0.006	4.55	0.00	0.00	0.00
0.008	0.75	0.00	0.00	0.00
0.010	0.00	0.00	0.00	0.00
0.012	0.00	0.00	0.00	0.00
0.014	0.00	0.00	0.00	0.00
0.016	0.00	0.00	0.00	0.00
0.018	0.00	0.00	0.00	0.00
0.020	0.00	0.00	0.00	0.00

Table 10: Standard Deviation w.r.t. δ with Medical Data

δ	MultiAPT	MultiHDoC	MultiLUCB	MultiTUCB
0.005	40630.09	5825.35	814.65	4378.73
0.010	39685.57	5680.28	776.37	4288.27
0.015	39149.34	5662.82	753.44	4277.31
0.020	38927.07	5606.34	736.62	4263.95
0.025	38662.88	5595.31	719.47	4260.32
0.030	38529.29	5590.93	712.07	4254.68
0.035	38365.54	5585.32	713.06	4252.60
0.040	38071.74	5583.01	691.61	4242.46
0.045	38105.41	5560.48	683.35	4241.99
0.050	38080.04	5546.13	675.84	4240.76

Table 11: Standard Deviation w.r.t. ϵ with Medical Data

ϵ	MultiAPT	MultiHDoC	MultiLUCB	MultiTUCB
0.002	–	4201.73	831.45	1522.96
0.004	58647.06	24170.82	815.45	1453.56
0.006	36608.01	5176.59	807.75	4141.31
0.008	31141.33	4798.52	800.43	3673.10
0.010	26136.43	4356.11	788.47	3080.87
0.012	21819.32	3823.17	777.36	2591.56
0.014	18650.05	3452.81	763.97	2289.38
0.016	16375.48	3044.30	751.69	1925.62
0.018	14093.24	2760.02	739.96	1736.17
0.020	12723.03	2581.48	730.53	1631.91