Calculus

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$$U^l_{mn}(\alpha,\beta,\gamma) = e^{im\alpha} d^l_{mn}(\beta) e^{in\gamma}$$

I require d^{mn} to **be real** and satisfy the following equation:

$$\int_0^\pi \frac{d\beta \sin(\beta)}{2} d^l_{mn}(\beta) d^{l'}_{mn}(\beta) = \frac{\delta_{ll'}}{2l+1} \tag{1}$$

From the two last equations we deduce

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^\pi \frac{d\beta \sin(\beta)}{2} \int_0^{2\pi} \frac{d\gamma}{2\pi} U^l_{mn}(\alpha,\beta,\gamma) U^{l'}_{m'n'}(\alpha,\beta,\gamma)^* \\ = \frac{\delta_{ll'}\delta_{mm'}\delta_{nn'}}{2l+1} \tag{2}$$

This equality gives the Fourier transform

$$\begin{aligned} [\mathcal{F}(f)]^l_{mn} &= \sum_{l'=0}^{\infty} \sum_{m',n'=-l'}^{l'} [\mathcal{F}(f)]^{l'}_{m'n'} \delta_{ll'}\delta_{mm'}\delta_{nn'} \\ &= \int dg \underbrace{\sum_{l'm'n'} [\mathcal{F}(f)]^{l'}_{m'n'} U^{l'}_{m'n'}(g)(2l+1)}_{f(g)} U^l_{mn}(g)^* \end{aligned} \tag{3}$$

Or with the Haar measure explicit:

$$
\begin{aligned}
[\mathcal{F}(f)]^l_{mn} &= \int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^{\pi} \frac{d\beta \sin(\beta)}{2} \int_0^{2\pi} \frac{d\gamma}{2\pi} e^{-im\alpha} d^l_{mn}(\beta) e^{-in\gamma} f(\alpha,\beta,\gamma) \\
&= \int_0^{\pi} \frac{d\beta \sin(\beta)}{2} d^l_{mn}(\beta) \left\{ \int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^{2\pi} \frac{d\gamma}{2\pi} e^{-im\alpha} e^{-in\gamma} f(\alpha,\beta,\gamma) \right\}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
f(\alpha,\beta,\gamma) &= \sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} [\mathcal{F}(f)]^l_{mn} d^l_{mn}(\beta)(2l+1) e^{im\alpha} e^{in\gamma} \\
&= \sum_{(m,n)\in\mathbb{Z}^2} \left\{ \sum_{l=\max(|m|,|n|)}^{\infty} [\mathcal{F}(f)]^l_{mn} d^l_{mn}(\beta)(2l+1) \right\} e^{im\alpha} e^{in\gamma}
\end{aligned} \tag{5}
$$

Rotation

$$L_h f(g) = f(h^{-1}g)$$

U is a unitary representation of SO(3)

$$
\begin{aligned}
U^l(gh) &= U^l(g)U^l(h) \\
U^l(g^{-1}) &= U^l(g)^\dagger
\end{aligned}
$$

$$
\begin{aligned}
[\mathcal{F}(L_h f)]^l_{mn} &= \int_g f(h^{-1}g) U^l_{mn}(g)^* = \int_g f(g) U^l_{mn}(hg)^* \\
&= \sum_i U^l_{mi}(h)^* \int_g f(g) U^l_{in}(g)^* = \sum_i U^l_{mi}(h)^* [\mathcal{F}(f)]^l_{in}
\end{aligned} \tag{6}
$$

Convolution

$$(f_1 * f_2)(g) = \int dh f_1(h) f_2^*(g^{-1}h)$$

$$
\begin{aligned}
[\mathcal{F}(f_1 * f_2)]^l &= \int dg U^l(g)^* \int dh f_1(h) f_2^*(g^{-1}h) \\
&= \int dg \int dh U^l(g^{-1})^* f_1(h) f_2^*(gh) \\
&= \int dg \int dh U^l(hg^{-1})^* f_1(h) f_2^*(g) \\
&= \int dg \int dh U^l(h)^* U^l(g^{-1})^* f_1(h) f_2^*(g) \\
&= \int dg \int dh U^l(h)^* U^l(g)^T f_1(h) f_2^*(g) \\
&= \left(\int dh U^l(h)^* f_1(h)\right) \left(\int dg U^l(g)^T f_2^*(g)\right) \\
&= [\mathcal{F} f_1]^l [\mathcal{F} f_2]^{l\dagger} \qquad (7)
\end{aligned}
$$