Proofs

proof of Theorem (2.1). By (2.4), $\mathrm{Mat}_S(X)$ is a spiked matrix with aspect ratio $N(S)^2/N_1$ and signal strength λ (as $\otimes_{j\in S}v_j$ is unit norm). There are four cases to consider:

- Let $N(S)^2/N_1 \to 0$; by the first point of (1.6), the recovery threshold lies at $(N_1/N(S)\cdot N(S))^{1/4} = N_1^{1/4}$.
- Let $N(S)^2/N_1 \to \gamma \in (0,1]$; by the first point of (1.5), partial recovery occurs for $\liminf \lambda/\sqrt{N_1/N(S)} > \gamma^{1/4}$. Equivalently, $\liminf \lambda/N_1^{1/4} > 1$. Under $\lambda = \tau(1+o(1))N_1^{1/4}$, $\lim \lambda/\sqrt{N_1/N(S)} = \tau\gamma^{1/4}$, implying $|\langle \otimes_{j\in S}v_j, w\rangle|^2 \xrightarrow{a.s.} c^2(\tau\gamma^{1/4},\gamma)$.
- Let $N(S)^2/N_1 \to \gamma \in (1,\infty]$; by the second point of (1.5), partial recovery occurs for $\liminf \lambda/\sqrt{N(S)} > \gamma^{-1/4}$. Equivalently, $\liminf \lambda/N_1^{1/4} > 1$. Under $\lambda = \tau(1+o(1))N_1^{1/4}$, $\lim \lambda/\sqrt{N(S)} = \tau\gamma^{-1/4}$, implying $|\langle \otimes_{j\in S}v_j, w\rangle|^2 \xrightarrow{a.s.} c^2(\tau\gamma^{-1/4}\cdot\gamma^{1/2},\gamma) = c^2(\tau\gamma^{1/4},\gamma)$.
- Let $N(S)^2/N_1 \to \infty$; by the second point of (1.7), the recovery threshold lies at $\sqrt{N(S)}$.

□

Proof of Theorem 3.1. We provide proof for $k=3$; the proof for higher orders is similar and omitted. For notational simplicity, we suppress the subscripts of n_1 and v_1.

Let $w_i = \mathrm{tr}(Z_i)$, $i\in[n]$, and $w=(w_1,\dots,w_n)^\top$. Expanding the partial trace matrix,

$$\begin{aligned}\mathrm{Tr}_k(X) &= \sum_{i=1}^n(\lambda v_i + w_i)X_i = \sum_{i=1}^n\big[(\lambda^2v_i^2+\lambda v_iw_i)vv^\top + (\lambda v_i+w_i)Z_i\big]\\ &= \sum_{i=1}^n\big[(\lambda^2v_i^2+\lambda v_iw_i)vv^\top + (\lambda v_i+w_i)(Z_i-\mathrm{diag}(Z_i-\widetilde{Z}_i)) + (\lambda v_i+w_i)\mathrm{diag}(Z_i-\widetilde{Z}_i)\big]\,,\end{aligned}\tag{4.4}$$

where $\widetilde{Z}$ is an independent copy of Z.

Let $M = \|\lambda v+w\|_2^{-1}\sum_{i=1}^n(\lambda v_i+w_i)(Z_i-\mathrm{diag}(Z_i-\widetilde{Z}_i))$. As w_i and $Z_i-\mathrm{diag}(Z_i-\widetilde{Z}_i)$ are independent and the Gaussian distribution is rotationally invariant, $M\stackrel{d}{=}Z_1$. Hence, we have

$$\|\lambda v+w\|_2^{-1}\mathrm{Tr}_k(X) = \alpha vv^\top + M + \|\lambda v+w\|_2^{-1}\sum_{i=1}^n(\lambda v_i+w_i)\mathrm{diag}(Z_i-\widetilde{Z}_i)\,,\tag{4.5}$$

where $\alpha = \|\lambda v+w\|_2^{-1}(\lambda^2+\lambda\langle v,w\rangle)$. The partial trace matrix is thus proportional to a perturbation of a spiked matrix with aspect ratio $\gamma=1$.

Let $\mathcal{E}$ denote the third term on the right-hand side of (4.5); we shall prove $n^{-1/2}\|\mathcal{E}\|_2\xrightarrow{a.s.}0$. Denoting $u=(\lambda v+w)/\|\lambda v+w\|_2$, the diagonal entries of $\sum_{i=1}^n u_i\mathrm{diag}(\widetilde{Z}_i)$ are i.i.d. Gaussians with variance one, implying $\|\sum_{i=1}^n u_i\mathrm{diag}(\widetilde{Z}_i)\|\lesssim\sqrt{\log n}$, almost surely. Similarly, since $w\sim\mathcal{N}(0,nI_n)$ and $\|\lambda v+w\|_2 = \Theta_{a.s.}(\lambda+n)$,

$$\lambda\|\lambda v+w\|_2^{-1}\Big\|\sum_{i=1}^n v_i\mathrm{diag}(Z_i)\Big\|_2 \lesssim \sqrt{\log n}\Big(1+\frac{\lambda}{n}\Big)\,.\tag{4.6}$$

To bound the remaining term of $\mathcal{E}$, let $\mathcal{Z}\in\mathbb{R}^{n\times n}$ denote the matrix with entries $\mathcal{Z}_{ij}=Z_{ijj}$, $i,j\in[n]$, in which case we may write

$$\Big\|\sum_{i=1}^n w_i\mathrm{diag}(Z_i)\Big\|_2 = \sup_{1\le j\le n}\Big|\sum_{i=1}^n w_iZ_{ijj}\Big| = \sup_{1\le j\le n}|e_j^\top\mathcal{Z}^\top\mathcal{Z}\mathbf{1}_n|\,,\tag{4.7}$$

where $\mathbf{1}_n$ is the length-n vector of ones. As $e_j^\top\mathcal{Z}^\top\mathcal{Z}e_j\sim\chi_n^2$ and

$$e_j^\top\mathcal{Z}^\top\mathcal{Z}(\mathbf{1}_n-e_j)\stackrel{d}{=}\sqrt{n-1}e_j^\top\mathcal{Z}^\top\widetilde{\mathcal{Z}}e_j\sim\sqrt{n-1}\cdot\mathcal{N}(0,1)\cdot\sqrt{\chi_n^2}\,,$$

standard bounds such as (2.19) in [24] yield

$$\mathbf{P}\Big(\sup_{1\le j\le n}\big|e_j^\top \mathcal{Z}^\top \mathcal{Z}\mathbf{1}_n - n\big| > c\cdot n\Big) \lesssim ne^{-Cn}\,, \tag{4.8}$$

where $c, C > 0$ are constants. As the right-hand side is summable, the Borel-Cantelli lemma implies

$$\sup_{1\le j\le n}|e_j^\top \mathcal{Z}^\top \mathcal{Z}\mathbf{1}_n| \lesssim n\,, \tag{4.9}$$

almost surely. Collecting the above bounds, we have that $n^{-1/2}\|\mathcal{E}\|_2 \xrightarrow{a.s.} 0$.

By Weyl's inequality, the limiting spectral distribution of $n^{-1/2}(M+\mathcal{E})$ equals that of $n^{-1/2}M$, the quarter circle law. The limits (1.5) from Lemma 1.1 therefore apply to $\alpha vv^\top + M + \mathcal{E}$ as well (see [7]). Basic calculations yield (1) $\liminf \alpha/\sqrt{n} > 1$ if and only if $\liminf \lambda/n^{3/4} > 1$, and (2) under $\lambda = \tau(1+o(1))n^{3/4}$, $\alpha = \tau^2(1+o_{a.s.}(1))\sqrt{n}$. Therefore, $\hat{v}$ partially recovers v if and only if $\liminf \lambda/n^{3/4} > 1$, and under $\lambda = \tau(1+o(1))n^{3/4}$,

$$|\langle v, \hat{v}\rangle|^2 \xrightarrow{a.s.} c^2(\tau^2, 1) = \tilde{c}^2(\tau)\,, \tag{4.10}$$

completing the proof. □

Proof of Theorem 4.1. By the linearity of the operators $\times_1, \ldots, \times_k$,

$$w_j = \lambda \prod_{\ell\in[k]\setminus\{j\}} \langle v_\ell, \hat{v}_\ell\rangle \cdot v_j + Z \times_1 \hat{v}_1 \cdots \times_{j-1} \hat{v}_{j-1} \times_{j+1} \hat{v}_{j+1} \cdots \times_k \hat{v}_k\,, \tag{4.11}$$

$$\langle v_j, w_j\rangle = \lambda \prod_{\ell\in[k]\setminus\{j\}} \langle v_\ell, \hat{v}_\ell\rangle + \big\langle Z, \hat{v}_1 \otimes \cdots \otimes \hat{v}_{j-1} \otimes v_j \otimes \hat{v}_{j+1} \cdots \otimes \hat{v}_k\big\rangle\,. \tag{4.12}$$

Assume that (4.1) holds for $\ell \in [j-1]$. For $\ell \in \{j+1, \ldots, k\}$, Corollary 2.1.1 implies $|\langle v_\ell, \hat{v}_\ell\rangle|$ is bounded away from zero. The first term on the right-hand side of (4.12) is therefore $\Theta_{a.s.}(\lambda)$. The second term is bounded by the spectral norm of Z: using Theorem 1 of [23],

$$\big\langle Z, \hat{v}_1 \otimes \cdots \otimes \hat{v}_{j-1} \otimes v_j \otimes \hat{v}_{j+1} \otimes \cdots \otimes \hat{v}_k\big\rangle \le \sup_{u_j \in \mathbb{S}^{j-1}, j\in[k]} \langle Z, u_1 \otimes \cdots \otimes u_k\rangle \lesssim \sqrt{n_k}\,, \tag{4.13}$$

almost surely. Thus, as $(\sqrt{n_k}/\lambda)^4 \lesssim n_k^2/N_1 \to 0$,

$$|\langle v_j, \hat{v}_j\rangle| = 1 + O_{a.s.}\Big(\frac{\sqrt{n_k}}{\lambda}\Big) \xrightarrow{a.s.} 1\,, \tag{4.14}$$

from which (4.1) follows inductively.

Equation (4.2) follows from (4.1) and (4.13), which imply $\|v_1 \otimes \cdots \otimes v_k - \hat{v}_1 \otimes \cdots \otimes \hat{v}_k\|_F \xrightarrow{a.s.} 0$ and

$$\lambda^{-1}|\hat{\lambda}| = \Bigg|\prod_{j=1}^k \langle v_j, \hat{v}_j\rangle + \lambda^{-1}\langle Z, \hat{v}_1 \otimes \cdots \hat{v}_k\rangle\Bigg| \xrightarrow{a.s.} 1\,. \tag{4.15}$$

□

Proof of Theorem 4.2. We shall prove $v_2, \ldots, v_k$ are recovered exactly; proofs for the first and last axes are similar and omitted. By the linearity of the unfolding operator,

$$\mathrm{Mat}_{j-1}(X \times_1 \hat{v}_1) = \lambda\langle v_1, \hat{v}_1\rangle(\otimes_{\ell\in[k]\setminus\{1,j\}} v_\ell)v_j^\top + \mathrm{Mat}_{j-1}(Z \times_1 \hat{v}_1)\,. \tag{4.16}$$

Observe that $Z \times_1 \hat{v}_1$ is a reshaping of the vector $\mathrm{Mat}_1(Z)\hat{v}_1$. As $\hat{v}_1$ and Z are dependent, the second term on the right-hand side is not a matrix of i.i.d. entries. Despite dependencies, we claim that appropriately scaled, $\mathrm{Mat}_1(Z)\hat{v}_1$ is Gaussian noise, in which case exact recovery thresholds are a consequence of spiked matrix model results of Section 1.3 applied to (4.16).

By definition, $\hat{v}_1$ is the first eigenvector of the symmetric matrix

$$\mathrm{Mat}_1(X)^\top \mathrm{Mat}_1(X) = \lambda^2 v_1 v_1^\top + \mathrm{Mat}_1(Z)^\top \mathrm{Mat}_1(Z) + \mathcal{E}\,, \tag{4.17}$$

where $\mathcal{E}$ is a rank-two matrix given by

$$\mathcal{E} = \lambda v_1 (\otimes_{\ell\in[k]\setminus\{1\}} v_\ell)^\top \mathrm{Mat}_1(Z) + \lambda \mathrm{Mat}_1(Z)^\top (\otimes_{\ell\in[k]\setminus\{1\}} v_\ell) v_1^\top\,.$$

Let $\tilde{v}_1$ denote the first eigenvector of $\mathrm{Mat}_1(X)^\top \mathrm{Mat}_1(X) - \mathcal{E}$; without loss of generality, we assume $\hat{v}_1^\top \tilde{v}_1 \ge 0$. Since $\mathrm{Mat}_1(Z)^\top(\otimes_{\ell\in[k]\setminus\{1\}} v_\ell) \sim \mathcal{N}(0, I_{n_1})$, we have $\|\mathcal{E}\|_2 \lesssim \lambda\sqrt{n_1}$ almost surely. By Theorem 1.1 of [13], the spectral gap of $\mathrm{Mat}_1(X)^\top \mathrm{Mat}_1(X)$ is $\Theta(\lambda^2)$. Thus, using the Davis-Kahan theorem (see Corollary 3 of [25]), we have

$$\|\hat{v}_1 - \tilde{v}_1\|_2 \lesssim \frac{\|\mathcal{E}\|_2}{\lambda^2} \xrightarrow{a.s.} 0\,. \tag{4.18}$$

Let $\mathrm{Mat}_1(Z) = U\Lambda V^\top$ be a singular value decomposition. As Z contains i.i.d. Gaussian entries, (1) U, Λ, and V are independent, (2) U and V are Haar-distributed. Moreover, as $\tilde{v}_1$ is the first eigenvector of $\lambda^2 v_1 v_1^\top + \mathrm{Mat}_1(Z)^\top \mathrm{Mat}_1(Z) = \lambda^2 v_1 v_1^\top + V\Lambda^2 V^\top$, $\tilde{v}_1$ is independent of U. Thus, $U\Lambda V^\top \tilde{v}_1 / \|\Lambda V^\top \tilde{v}_1\|_2$ is uniform on $\mathbb{S}^{N_2-1}$. Generating $\xi \sim \chi^2_{N_2}$ independent of Z, it follows that

$$\frac{\xi}{\sqrt{N_2}} \cdot \frac{\mathrm{Mat}_1(Z)\tilde{v}_1}{\|\Lambda V^\top \tilde{v}_1\|_2} \sim \mathcal{N}(0, I_{N_2})\,. \tag{4.19}$$

Defining the constant $\alpha = \xi/(\sqrt{N_2}\|\Lambda V^\top \tilde{v}_1\|_2)$, we deduce that $\alpha Z \times_1 \tilde{v}_1$ (which is a reshaping of the left-hand side of (4.19)) is distributed as a tensor with i.i.d. Gaussian entries—despite dependencies between Z and $\tilde{v}_1$. Additionally, since $\Lambda_{11}/\sqrt{N_2} \xrightarrow{a.s.} 1$ and $\Lambda_{n_1 n_1}/\sqrt{N_2} \xrightarrow{a.s.} 1$, we have $\|\Lambda V^\top \tilde{v}_1\|_2/\sqrt{N_2} \xrightarrow{a.s.} 1$ and $\alpha \xrightarrow{a.s.} 1$.

Thus, we conclude that

$$\alpha \mathrm{Mat}_{j-1}(X \times_1 \tilde{v}_1) = \lambda(1 + o_{a.s.}(1))\langle v_1, \tilde{v}_1\rangle (\otimes_{\ell\in[k]\setminus\{1,j\}} v_\ell) v_j^\top + \widetilde{Z}\,, \qquad j \in \{2, \ldots, k-1\}\,, \tag{4.20}$$

where $\widetilde{Z} \in \mathbb{R}^{n_j \times (N_2/n_j)}$ contains i.i.d. Gaussian entries, enabling us to apply spiked matrix results. Let $\tilde{v}_j$ denote the first right singular vector of $\mathrm{Mat}_{j-1}(X \times_1 \tilde{v}_1)$ (equivalently, that of $\alpha\mathrm{Mat}_{j-1}(X \times_1 \tilde{v}_1)$); without loss of generality, we assume $\hat{v}_1^\top \tilde{v}_1 \ge 0$. In particular, since $|\langle v_1, \tilde{v}_1\rangle| \asymp |\langle v_1, \hat{v}_1\rangle| \asymp 1$ by Corollary 2.1.1 and (4.18), Lemma 1.2 implies $|\langle v_j, \tilde{v}_j\rangle| \xrightarrow{a.s.} 1$ (we have $\lambda \gg (n_j \cdot N_2/n_j)^{1/4}$).

It therefore suffices to prove that $\|\hat{v}_j - \tilde{v}_j\|_2 \xrightarrow{a.s.} 0$. By Theorem 2.3 of [5] or Theorem 1.1 of [13] and Cauchy's interlacing inequality, the spectral gap of $\mathrm{Mat}_{j-1}(X \times_1 \tilde{v}_1)^\top \mathrm{Mat}_{j-1}(X \times_1 \tilde{v}_1)$ is $\Theta(\lambda^2)$. Let $\mathcal{Z}$ denote the reshaping of Z with dimensions $n_1 \times n_j \times N_2/n_j$ and slices $\mathrm{Mat}_{j-1}(Z_i)$, $i \in [n_1]$. Using the bound

$$\|\mathrm{Mat}_{j-1}(Z \times_1 (\hat{v}_1 - \tilde{v}_1))\|_2 \le \sup_{\substack{u_1\in\mathbb{S}^{n_1-1}, u_2\in\mathbb{S}^{n_j-1},\\ u_3\in\mathbb{S}^{N_2/n_j-1}}} \big(\mathcal{Z} \times_1 u_1 \times_2 u_2 \times_3 u_3\big) \cdot \|\hat{v}_1 - \tilde{v}_1\|_2$$

and Theorem 1 of [23],

$$\begin{aligned} \|\mathrm{Mat}_{j-1}(X \times_1 (\hat{v}_1 - \tilde{v}_1))\|_2 &\le \lambda\|\hat{v}_1 - \tilde{v}_1\|_2 \|\mathrm{Mat}_{j-1}(v_2 \otimes \cdots \otimes v_k)\|_2 + \|\mathrm{Mat}_{j-1}(Z \times_1 (\hat{v}_1 - \tilde{v}_1))\|_2 \\ &\lesssim \big(\lambda + (N_2/n_j)^{1/2}\big)\|\hat{v}_1 - \tilde{v}_1\|_2\,, \end{aligned} \tag{4.21}$$

almost surely. Thus, using the Davis-Kahan theorem (Theorem 4 of [25]), (4.18), and (4.21), we have

$$\|\hat{v}_j - \tilde{v}_j\|_2 \lesssim \frac{\lambda\big(\lambda + (N_2/n_j)^{1/2}\big)\|\hat{v}_1 - \tilde{v}_1\|_2}{\lambda^2} \xrightarrow{a.s.} 0\,, \tag{4.22}$$

completing the proof. □