Appendix

A Illustrative Examples

Example 3 (Obtaining the Prolongation of $SO(2)$)**.** *We can consider If $X \times U = \mathbb{R} \times \mathbb{R}$ and the infinitesimal generator of the 2-dimensional rotation group, $SO(2)$:*

$$\begin{aligned}\mathbf{v}_{SO(2)} &= \xi(x,u)\partial_x + \phi(x,u)\partial_u \\ &= -u\partial_x + x\partial_u\end{aligned}$$

In this 2-dimensional case, the calculation of the prolonged generator is simple:

$$\begin{aligned}\phi^{(x)} &= D_x(\phi - \xi u_x) + \xi u_{xx} \\ &= D_x(x + uu_x) - uu_{xx} \\ &= (1 + u_x^2 + uu_{xx}) - uu_{xx} \\ &= 1 + u_x^2\end{aligned}$$

Therefore:

$$\mathrm{pr}^{(1)}\mathbf{v}_{SO(2)} = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x}$$

We will work through another example of obtaining the prolongation of an infinitesimal generator of the heat equation:

Example 4 (Obtaining the Prolongation of an Infinitesimal Generator)**.** *As an example, we will consider $X \times U = \mathbb{R}^2 \times \mathbb{R}$ and the following infinitesimal generator, which is a symmetry of the heat equation:*

$$\begin{aligned}\mathbf{v} &= \xi_1(x,t,u)\partial_x + \xi_2(x,t,u)\partial_t + \phi(x,t,u)\partial_u \\ &= 2\nu t\partial_x - xu\partial_u\end{aligned}$$

where x, t denote the independent variables, u is the dependent variable and ν is a positive constant. By the prolongation formula, Eq. (4), the first prolongation in t is given by:

$$\begin{aligned}\phi^t &= D_t(\phi - \xi_1 u_x - \xi_2 u_t) + \xi_1 u_{xt} + \xi_2 u_{tt} \\ &= D_t(-xu - 2\nu t u_x) + 2\nu t u_{xt} \\ &= -xu_t - 2\nu u_x\end{aligned}$$

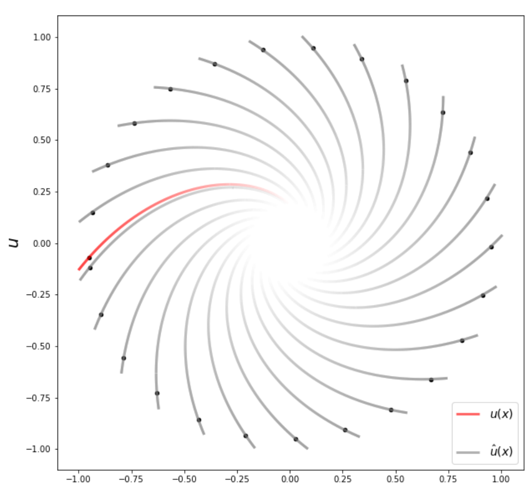

Figure 4: Various solutions of the PDE $\Delta(x, u, u_x) = (u - x)u_x + u + x = 0$ obtained via symmetry transformation (rotation) of a know solution (in red).

As a final illustrative example of the symmetry criterion, we will follow Olver's example below:

Example 5. *As an illustrative example of the infinitesimal criterion, we can consider a simple DE:*

$$\Delta(x, u, u_x) = (u - x)u_x + u + x = 0$$

We can easily see that $SO(2)$ is a symmetry group of this differential equation, using the prolongation of the generator we calculated in Example 3:

$$\begin{aligned}\mathrm{pr}^{(1)}\mathbf{v}[\Delta] &= -u\Delta_x + x\Delta_u + (1 + u_x^2)\Delta_{u_x} \\ &= -u(1 - u_x) + x(1 + u_x) + (1 + u_x^2)(u - x) \\ &= u_x\Delta\end{aligned}$$

Since $\Delta u_x = 0$ when $\Delta = 0$, we can conclude that $SO(2)$ is indeed a symmetry group of the equation. In fact, we can see that it transforms solutions of this differential equation to other solutions in Fig. 4.

B Implementation Details

We model the two networks, g_{θ_1} and e_{θ_2} in Eq. (9) with MLPs consisting of 7 hidden layers of width 100. This choice was based on the previous research using PINN and DeepONets for solving Burgers' equation [Wang et al., 2021b]. We used elu activation as differentiable activations are required for the PDE loss. The output of the embedding vectors from both networks is 100 dimensional.

For both the Heat equation and Burgers' equation experiments, we perform hyper-parameter tuning on the coefficients of the loss terms from the set $[0.1, \ldots, 1, \ldots, 10, \ldots, 100, \ldots 200]$. This is done separately for the baseline model and the model trained with symmetry loss, $\mathcal{L}_{\mathrm{sym}}$, as we varied the number of samples, N_r.

We also note that for Burgers' equation, we found that cosine similarity for $\mathcal{L}_{\mathrm{sym}}$ works better than the dot product. The results reported in Section 4 use cosine-similarity.

We will make the data and the code available on GitHub.

C Additional Results

In the figure below, we can see the behaviour of the two models, trained with and without symmetry loss for Burgers' equation, as we increase the number of training samples. It can be seen that, as expected, in the model trained with $\mathcal{L}_{\mathrm{sym}}$ performs significantly better with low samples inside the domain. The corresponding mean-squared errors are reported in Table 2.

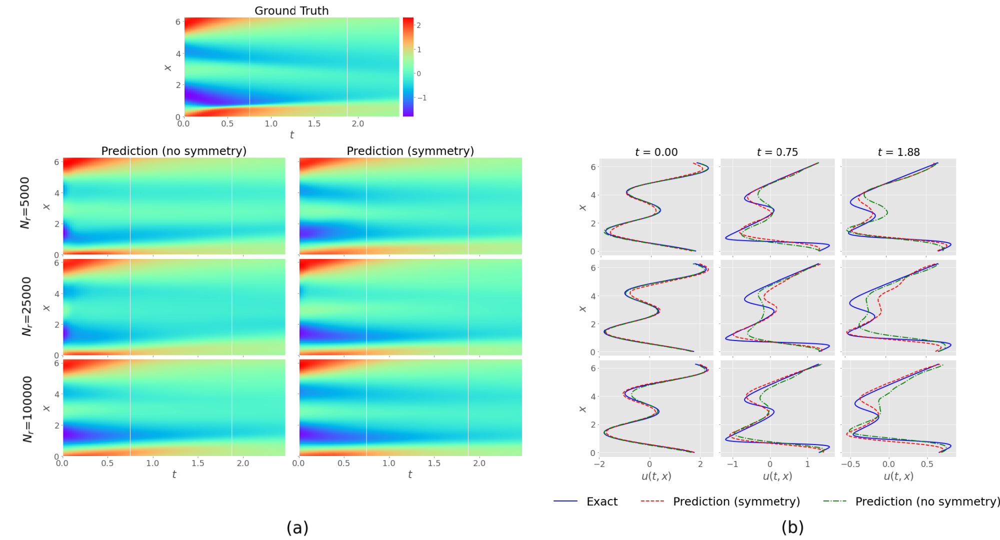

Figure 5: The effect of training the PDE solver for the Burgers' equation with and without the symmetry loss for one of the PDEs in the test dataset. (a) shows the ground truth solution and the predictions of the two models as the number of samples inside the domain increases from 5000 to 25000 and 100000.(b) shows the corresponding predictions and the ground truth solution at different time slices.