SUPPLEMENTARY MATERIAL: PROPORTIONAL RESOURCE ALLOCATION

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In the main text of the paper, we restrict our attention to exactly one individual at each time step. Now we relax this restriction by considering policies as follows:

$$\text{proportional max-U: } a_i(t) = \frac{e^{U_i(t)}}{\sum_{j=1}^N e^{U_j(t)}},$$
$$\text{proportional min-U: } a_i(t) = \frac{e^{-U_i(t)}}{\sum_{j=1}^N e^{-U_j(t)}}.$$

Theorem 1. *Under regularity (Assumption 3) and modeling conditions (Assumption 2), and assume that $f_i(x) \equiv f(x)$, $g_i(x) \equiv g(x)$, the proportional min-U policy leads to the following closed form solution of the individual rates of growth:*

$$R_i = \begin{cases} f^+,\ i = J, \\ -g^+,\ i \neq J, \end{cases} \quad a.s.$$

where J is a random variable with values in $[N]$ whose exact value depends on $U(0)$, $f(\cdot)$, and $g(\cdot)$.

Proof of Theorem 2. For $\forall i, j \in [N]$ s.t. $U_i(t) \geq U_j(t)$, we have $a_i(t) \geq a_j(t)$, then under the assumption that $f_i(x) \equiv f(x), g_i(x) \equiv g(x)$, we further obtain

$$\begin{aligned} \mathbb{E}[Z_i(t+1)] &= a_i(t) \cdot f(U_i(t)) - (1 - a_i(t)) \cdot g(U_i(t)) \\ &\geq a_j(t) \cdot f(U_i(t)) - (1 - a_j(t)) \cdot g(U_i(t)) \\ &\geq a_j(t) \cdot f(U_j(t)) - (1 - a_j(t)) \cdot g(U_j(t)) = \mathbb{E}[Z_j(t+1)]. \end{aligned}$$

where the last inequality holds because of modeling conditions (Assumption 2.(a), (b)). Consider $i \in \mathcal{M}_t$ where $\mathcal{M}_t = \arg\max_j \{U_j(t)\}$ and $i \in \mathcal{M}_t, j \in [N]$ such that $U_i(t) - U_j(t) \geq 1$, we have

$$\begin{aligned} &\mathbb{E}[U_i(t+1) - U_j(t+1) \mid \mathcal{F}_t] - (U_i(t) - U_j(t)) \\ &= \mathbb{E}[Z_i(t+1) - Z_j(t+1) \mid \mathcal{F}_t] \\ &= a_i(t) \cdot f(U_i(t)) - (1 - a_i(t)) \cdot g(U_i(t)) - (a_j(t) \cdot f(U_j(t)) - (1 - a_j(t)) \cdot g(U_j(t))) \\ &\geq a_i(t) \cdot f(U_i(t)) - (1 - a_i(t)) \cdot g(U_j(t)) - (a_j(t) \cdot f(U_i(t)) - (1 - a_j(t)) \cdot g(U_j(t))) \\ &= (a_i(t) - a_j(t)) \cdot f(U_i(t)) + (a_i(t) - a_j(t)) \cdot g(U_j(t)) \\ &\geq \frac{e^{M(t)} - e^{M(t)-1}}{\sum_{j \in [N]} e^{U_j(t)}} \cdot f^- + \frac{e^{M(t)} - e^{M(t)-1}}{\sum_{j \in [N]} e^{U_j(t)}} \cdot g^- \\ &\geq \frac{1 - e^{-1}}{N}(f^- + g^+) > 0\,. \end{aligned}$$

Now treat $U_i(t) - U_j(t)$ as the welfare process and apply Lundberg inequality for welfare process (Lemma 3), we claim that with positive probability that $U_i(t) - U_j(t) \geq 1$ for $\forall t \geq 0$ when $U_i(0) - U_j(0) \geq 1$ where $i \in \mathcal{M}_0$. Then combine with the regularity condition (Assumption 3.(c)), we have that with positive probability (lowerbounded by a constant) that $U_i(t) - U_j(t) \geq 1$ for $\forall t > 0$ where $i \in \mathcal{M}_0$. Then we apply the same reasoning for $j \in [N] \backslash i$ and conclude that with probability 1, the proportional max-U policy will fixate on one single individual asymptotically. □

Theorem 2. *Under regularity (Assumption 3) and modeling conditions (Assumption 2.(a),(b)), the survival condition (Assumption 1), the proportional max-U policy leads to the following closed form solution of the individual rates of growth:*

$$R_i = \bar{\zeta}((f_1^+, \ldots, f_N^+), (g_1^-, \ldots, g_N^-)), \quad i = 1, \ldots, N, \quad a.s.$$

Proof of Theorem 1. The result can be proved by induction, and the proof of long-term behavoir of min-U policy (Theorem 3) applies here with minor modifications. We assume for $N-1$ individuals the conclusion holds, and consider $\mathcal{M} := \arg\max_j \{U_j(0)\}$ and $\mathcal{M}^c := [N] \backslash \mathcal{M}$. For $\forall l \in \mathcal{M}$,

$$a_l(t) \leq \frac{e^{-D(t)}}{1+(N-1)e^{-D(t)}} \Rightarrow \sum_{i \in \mathcal{M}^c} a_i(t) \geq \frac{1}{1+(N-1)e^{-D(t)}},$$

where $D(t) = \max_{j \in [N]} U_j(t) - \min_{i \in [N]} U_i(t)$. Hence there exists constant C such that when $D(t) \geq C$, the survival condition for $\mathcal{M}^c$

$$\begin{aligned}
\bar{U}_{\mathcal{M}^c}(t+1) - \bar{U}_{\mathcal{M}^c}(t) &= \sum_{i \in \mathcal{M}^c} w_i^{\mathcal{M}^c} a_i(t) \cdot f_i(U_i(t)) - (1-a_i(t)) \cdot g_i(U_i(t)) \\
&\geq \sum_{i \in \mathcal{M}^c} w_i^{\mathcal{M}^c} a_i(t) \cdot f_i^- - (1-a_i(t)) \cdot g_i^+ \\
&= \left(\sum_{i \in \mathcal{M}^c} a_i(t) - \sum_{j \in \mathcal{M}^c} \frac{g_j^+}{f_j^- + g_j^+} \right) \cdot \left(\sum_{k \in \mathcal{M}^c} \frac{1}{f_k^- + g_k^+} \right)^{-1} \\
&\geq \left(\frac{1}{1+(N-1)e^{-C}} - \sum_{j \in \mathcal{M}^c} \frac{g_j^+}{f_j^- + g_j^+} \right) \cdot \left(\sum_{k \in \mathcal{M}^c} \frac{1}{f_k^- + g_k^+} \right)^{-1} > 0
\end{aligned}$$

where $\bar{U}_{\mathcal{M}^c}(t)$, $w_i^{\mathcal{M}^c}$ are defined as in equation (5) for set $\mathcal{M}^c$. Hence we apply the conclusion for $\mathcal{M}^c$ and claim that there exists constant $T_{\mathcal{M}^c}$ such that when $\sum_{i \in \mathcal{M}} a_i(t) \leq \frac{1}{1+(N-1)e^{-C}}$ for $\forall t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\left[\min_{j \in \mathcal{M}^c} U_j(t) \right] &\geq \min_{j \in \mathcal{M}^c} +1, \quad \forall t \geq T_{\mathcal{M}^c}, \\
\mathbb{E}\left[\max_{j \in \mathcal{M}^c} U_j(t) \right] &\leq \max_{j \in \mathcal{M}^c} -1, \quad \forall t \geq T_{\mathcal{M}^c}.
\end{aligned}$$

As for $i \in \mathcal{M}$,

$$\begin{aligned}
\mathbb{E}[Z_i(t+1) \mid a_i(t), \mathcal{F}_t] &\leq a_i(t) f_i^+ - (1-a_i(t)) g_i^- \\
&\leq \frac{1}{N-1+e^{-D(t)}} f_i^+ - \left(\frac{N-2+e^{-D(t)}}{N-1+e^{-D(t)}} \right) g_i^-,
\end{aligned}$$

and when $D(t) \geq C'$ for constant $C' > 0$, we have

$$\frac{1}{N-1+e^{-D(t)}} f_i^+ - \left(\frac{N-2+e^{-D(t)}}{N-1+e^{-D(t)}} \right) g_i^- < -\frac{1}{2} \min_{i \in [N]} g_i^- . \tag{1}$$

Hence for the whole population $[N]$, if $\sum_{i \in \mathcal{M}_t} a_i(t) \leq \min \left\{ \frac{1}{1+(N-1)e^{-C}}, \frac{1}{2} \min_{i \in [N]} \frac{g_i^-}{f_i^+ + g_i^-} \right\}$, there exists constant T such that

$$\begin{aligned}
\mathbb{E}\left[\min_{j \in \mathcal{M}^c} U_j(t) \right] &\geq \min_{j \in \mathcal{M}^c} U_j(0) + 1, \quad \forall t \geq T, \\
\mathbb{E}\left[\max_{j \in \mathcal{M}} U_j(t) \right] &\leq \max_{j \in \mathcal{M}} U_j(0) - 1, \quad \forall t \geq T.
\end{aligned}$$

The rest of the proof goes through with minor modifications given the above facts and omitted here. $\square$