A. Proof of GIGP Expressivity

Theorem A.1. *Let $f : F^X \to \mathbb{R}$ be some function that is invariant to a group's action, so $\forall g \in G : f(gx) = f(x)$. Here F is some set representing the possible values given to each group element. Thus, assuming:*

1. *X, F are countable.*
2. *There are a finite number of orbits.*

Given these assumptions, there exist functions ρ, ϕ, ψ such that $f(X) = \rho(\sum_{q\in Q} \phi[\sum_{x\in q} \psi(x)])$, where with some abuse of notation we denote the value given to each group element over the domain with the group element directly.

Proof: The proof goes as follows. We shall show that there exists ψ such that $\sum_{x\in q} \psi(x)$ is group invariant and constitutes an invertible function from F^X to a countable set. Afterwards we'll show that there exists a function ϕ such that $\sum_{q\in Q} \phi(h(q))$ is an invertible function from $Q^{\mathbb{N}}$ to a countable set. Finally, we'll show that we can construct a function ρ such that $f(X') = \rho(\sum_{q\in Q} \phi[\sum_{x\in q} \psi(x)])$.

Step 1: As F is a countable domain, we have that $\exists c : c(x)$ is an invertible mapping to $\mathbb{N}$. Because X is countable each orbit has countably many elements. Thus, one can define $L(x) := \ln(p_{c(x)})$, where p_i denotes the i-th prime. Note that this constitutes a one-to-one correspondence between sets $\{x\}_{x\in q}$ and logarithms of numbers as each number has a unique prime representation and $\sum_{x\in q} L(x) = \ln(\prod_{x\in q} p_{c(x)})$. Thus, we have shown that there exists a function $\psi(x) := \ln(p_{c(x)})$ that is an invertible mapping between F^X and a countable set, specifically $\ln(\mathbb{N})$. This is trivially group invariant as the action of the group simply permutes the set.

Step 2: We denote an arbitrary function from an orbit to the real numbers as $h : Q \to \mathbb{R}$. We want to show that there exists a function ϕ such that $\sum_{q\in Q} \phi(h(q))$ is invertible from $Q^{\mathbb{N}}$, all the multisets of the form $\{\{h(q)\}\}_{q\in Q}$ to $\mathbb{N}$. As there is a finite number of orbits, one can use a similar construction to the previous step - $h(q)$ can be treated as a natural number (for example, calculating $\exp(h(q))$ if following the previous construction) and define $\phi(n) := \ln(p_n)$.

Step 3. Let $g(X) := \sum_{q\in Q} \phi[\sum_{x\in q} \psi(x)]$. This is an invertible group-invariant function from F^X to $\ln(\mathbb{N})$. Thus, setting $\rho := f \circ g^{-1}$ completes the proof. □

B. Related Work

Group-Equivariant Convolution Cohen & Welling (2016a) generalized the notion of architecture equivariance to arbitrary groups. Later works extended this by using irreducible group representations to perform convolution with respect to various continuous groups and their discrete subgroups (Cohen & Welling, 2016b; Weiler & Cesa, 2019; Geiger & Smidt, 2022).

However, there is a lack of work that attempts to generate architectures equivariant to any symmetry group. Ravanbakhsh et al. (2017) achieve equivariance to any finite group by sharing weights over orbits. Finzi et al. (2020) present LieConv, which is equivariant to arbitrary Lie groups and scales well to larger datasets.

Group Invariant Pooling Laptev et al. (2016b) introduce transformation-invariant pooling for CNNs. Cohen & Welling (2016a) propose coset pooling that is equivariant to any discrete group. Sosnovik et al. (2019) propose scale-invariant pooling in CNNs. van der Ouderaa et al. (2022) obtain G-invariant global pooling for non-stationary kernels. Xu et al. (2021) propose group equivariant/invariant subsampling for discrete groups. However, there is no work that implements group invariant global pooling for arbitrary groups. We attempt to fill this gap.