EQUAL IMPROVABILITY: A NEW FAIRNESS NOTION CONSIDERING THE LONG-TERM IMPACT

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ABSTRACT

Devising a fair classifier that does not discriminate against different groups is an important problem in machine learning. Although researchers have proposed various ways of defining group fairness, most of them only focused on the *immediate* fairness, ignoring the *long-term* impact of a fair classifier under the dynamic scenario where each individual can improve its feature over time. Such dynamic scenarios happen in real world, *e.g.*, college admission and credit loaning, where each rejected sample makes effort to change its features to get accepted afterwards. In this dynamic setting, the long-term fairness should equalize the samples' feature distribution across different groups after the rejected samples make some effort to improve. In order to promote long-term fairness, we propose a new fairness notion called *Equal Improvability* (EI), which equalizes the potential acceptance rate of the rejected samples across different groups assuming a bounded level of effort will be spent by each rejected sample. We analyze the properties of EI and its connections with existing fairness notions. To find a classifier that satisfies the EI requirement, we propose and study three different approaches that solve EI-regularized optimization problems. Through experiments on both synthetic and real datasets, we demonstrate that the proposed EI-regularized algorithms encourage us to find a fair classifier in terms of EI. Finally, we provide experimental results on dynamic scenarios which highlight the advantages of our EI metric in achieving the long-term fairness. Codes are available in anonymous GitHub repository [1].

1 INTRODUCTION

Over the past decade, machine learning has been used in a wide variety of applications. However, these machine learning approaches are observed to be unfair to individuals having different ethnicity, race, and gender. As the implicit bias in artificial intelligence tools raised concerns over potential discrimination and equity issues, various researchers suggested defining fairness notions and developing classifiers that achieve fairness. One popular fairness notion is *demographic parity* (DP), which requires the decision-making system to provide output such that the groups are equally likely to be assigned to the desired prediction classes, *e.g.*, acceptance in the admission procedure. DP and related fairness notions are largely employed to mitigate the bias in many realistic problems such as recruitment, credit lending, and university admissions (Zafar et al., 2017b; Hardt et al., 2016; Dwork et al., 2012; Zafar et al., 2017a).

However, most of the existing fairness notions are *zero-order*, *i.e.*, they only focus on immediate fairness, without taking potential follow-up inequity risk into consideration. In Fig. 1, we provide an example scenario when using a conventional zero-order fairness notion (DP in this example) has a long-term fairness issue, in a simple loan approval problem setting. Consider two groups (group 0 and group 1) with different distributions, where each individual has one label (approve the loan or not) and two features (credit score, income) that can be improved over time. Suppose each group consists of two clusters (with three samples each), and the distance between the clusters is different for two groups. Fig. 1 visualizes the distributions of two groups and the decision boundary of a classifier f which achieves DP among the groups. We observe that the rejected samples (left-hand-side of the decision boundary) in group 1 are located further away from the decision boundary than the rejected samples in group 0. As a result, the rejected applicants in group 1 need to make more effort to cross the decision boundary and get approval. This improvability gap between the two groups can make

[1] https://anonymous.4open.science/r/ei_fairness-23CA/

the rejected applicants in group 1 less motivated to improve their features, which may increase the gap between different groups in the future.

This motivated the advent of *first-order* fairness notions that consider the improvability of the rejected samples and thus defining the group fairness after such improvement is made. There are a few first-order fairness notions that have been proposed by Gupta et al. (2019); Heidari et al. (2019); Von Kügelgen et al. (2022). However, as shown in Table 1, they have some limitations *e.g.*, vulnerable to imbalanced group negative rates or outliers.

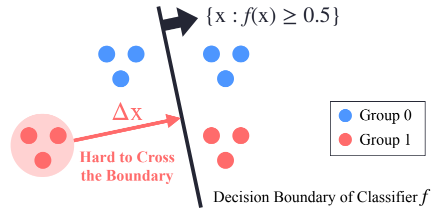


Figure 1: **Toy example showing the insufficiency of zero-order fairness notion.** We consider the binary classification (accept/reject) on 12 samples (dots), where $\mathbf{x}$ is the feature of the sample and the color of the dot represents the group. The given classifier f is fair in terms of a popular zero-order notion called demographic parity (DP), but does not have equal improvability of rejected samples ($f(\mathbf{x}) < 0.5$) in two groups; the rejected samples in group 1 needs more effort $\Delta\mathbf{x}$ to be accepted, *i.e.*, $f(\mathbf{x}+\Delta\mathbf{x}) \geq 0.5$, compared with the rejected samples in group 0.

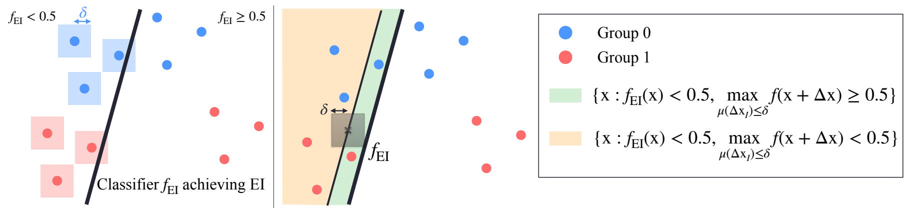
In this paper, we introduce another first-order fairness notion dubbed as *Equal Improvability* (EI), which does not suffer from these limitations. Let $\mathbf{x}$ be the feature of a sample and f be a score-based classifier, *e.g.*, predicting a sample as accepted if $f(\mathbf{x}) \geq 0.5$ holds and as rejected otherwise. We assume each rejected individual wants to get accepted in the future, thus improving its feature within a certain effort budget towards the direction that maximizes its score $f(\mathbf{x})$. Under this setting, we define EI fairness as the equity of the potential acceptance rate of the different rejected groups, once each individual makes the best effort within the predefined budget. This prevents the risk of exacerbating the gap between different groups in the long run.

Our key contributions are as follows:

- We propose a new group fairness notion called *Equal Improvability* (EI), which aims to equalize the probability of rejected samples being qualified after a certain amount of feature improvement, for different groups. EI encourages rejected individuals in different groups to have an equal amount of motivation to improve their feature to get accepted in the future. We analyze the properties of EI and the connections of EI with other existing fairness notions.
- We provide three methods to find a classifier that is fair in terms of EI, each of which uses a unique way of measuring the inequity in the improvability. Each method is solving a min-max problem where the inner maximization problem is finding the best effort to measure the EI unfairness, and the outer minimization problem is finding the classifier that has the smallest fairness-regularized loss. Experiments on synthetic/real datasets demonstrate that our algorithms find classifiers having low EI unfairness.
- We run experiments on dynamic scenarios where the data and the classifier evolve over multiple rounds, and show that training a classifier with EI constraints is beneficial for making the feature distributions of different groups identical in the long run, *i.e.*, achieving the long-term fairness.

2 Equal Improvability

Before defining our new fairness notion called *Equal Improvability* (EI), we first introduce necessary notations. For an integer n, let $[n] = \{0, \ldots, n-1\}$. We consider a binary classification setting where each data sample has an input feature vector $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^d$ and a label $\mathrm{y} \in \mathcal{Y} = \{0, 1\}$. In particular, we have a sensitive attribute $\mathrm{z} \in \mathcal{Z} = [Z]$, where Z is the number of sensitive groups. As suggested by Chen et al. (2021), we sort d features $\mathbf{x} \in \mathbb{R}^d$ into three categories: improvable features $\mathbf{x}_{\mathrm{I}} \in \mathbb{R}^{d_{\mathrm{I}}}$, mutable features $\mathbf{x}_{\mathrm{M}} \in \mathbb{R}^{d_{\mathrm{M}}}$, and immutable features $\mathbf{x}_{\mathrm{IM}} \in \mathbb{R}^{d_{\mathrm{IM}}}$, where $d_{\mathrm{I}} + d_{\mathrm{M}} + d_{\mathrm{IM}} = d$ holds. Here, improvable features $\mathbf{x}_{\mathrm{I}}$ refer to the features that can be improved and can directly affect the outcome, *e.g.*, salary in the credit lending problem, and GPA in the school's admission problem. In contrast, mutable features $\mathbf{x}_{\mathrm{M}}$ can be altered, but are not directly related to the outcome, *e.g.*, marital status in the admission problem, and communication type in the credit lending problem. Although individuals may manipulate these mutable features to get the desired outcome, we do not consider it as a way to make efforts as it does not affect the individual's true qualification status. Immutable features $\mathbf{x}_{\mathrm{IM}}$ are features that cannot be altered, such as race, age, or date of birth. Note that if sensitive attribute z is included in the feature vector, then it belongs to



(a) Original definition (b) Alternative definition

Figure 2: **Visualization of EI fairness.** For binary classification on 12 samples (dots) in two groups (red/blue), we visualize fairness notion defined in this paper: (a) shows the original definition in Def. 2.1, and (b) shows an alternative definition in Prop. 2.2. Here we assume two-dimensional features (both are improvable) and L_∞ norm for $\mu(\mathbf{x}) = \|\mathbf{x}\|_\infty$. The classifier f_{EI} achieves equal improvability (EI) since the same portion (1 out of 3) of unqualified samples in each group can be improved to qualified samples.

immutable features. For ease of notation, we write $\mathbf{x} = (\mathbf{x}_{\mathrm{I}}, \mathbf{x}_{\mathrm{M}}, \mathbf{x}_{\mathrm{IM}})$. Let $\mathcal{F} = \{f : \mathcal{X} \to [0, 1]\}$ be the set of classifiers, where each classifier is parameterized by $\boldsymbol{w}$, *i.e.*, $f = f_{\boldsymbol{w}}$. Given $f \in \mathcal{F}$, we consider the following deterministic prediction: $\hat{\mathrm{y}} \mid \mathbf{x} = \mathbf{1}\{f(\mathbf{x}) \geq 0.5\}$ where $\mathbf{1}\{A\} = 1$ if condition A holds, and $\mathbf{1}\{A\} = 0$ otherwise. We now introduce our new fairness notion.

Definition 2.1 (Equal Improvability)**.** Define a norm $\mu : \mathbb{R}^{d_{\mathrm{I}}} \to [0, \infty)$. For a given constant $\delta > 0$, a classifier f is said to achieve *equal improvability with δ-effort* if

$$\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}}) \leq \delta} f(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5 \mid f(\mathbf{x}) < 0.5, \mathrm{z} = z\right) = \mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}}) \leq \delta} f(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5 \mid f(\mathbf{x}) < 0.5\right)$$

holds for all $z \in \mathcal{Z}$, where $\Delta\mathbf{x}_{\mathrm{I}}$ is the effort for improvable features and $\Delta\mathbf{x} = (\Delta\mathbf{x}_{\mathrm{I}}, \mathbf{0}, \mathbf{0})$.

Note that the condition $f(\mathbf{x}) < 0.5$ represents that an individual is unqualified, and $f(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5$ implies that the effort $\Delta\mathbf{x}$ allows the individual to become qualified. The above definition of fairness in *equal improvability* requires that unqualified individuals from different groups $z \in [Z]$ are equally likely to become qualified if appropriate effort is made. Note that μ can be defined on a case-by-case basis. For example, we can use $\|\mathbf{x}_{\mathrm{I}}\| = \sqrt{\mathbf{x}_{\mathrm{I}}^\top \boldsymbol{C} \mathbf{x}_{\mathrm{I}}}$, where $\boldsymbol{C} \in \mathbb{R}^{d_{\mathrm{I}} \times d_{\mathrm{I}}}$ is a cost matrix that is diagonal and positive definite. Here, the diagonal terms of $\boldsymbol{C}$ characterize how difficult to improve each feature. For instance, consider the graduate school admission problem where $\mathbf{x}_{\mathrm{I}}$ contains features "number of publications" and "GPA". Since publishing more papers is harder than raising the GPA, the corresponding diagonal term for the number of publications feature in $\boldsymbol{C}$ should be greater than that for the GPA feature. The constant δ in Definition 2.1 can be selected depending on the classification task and the features. Appendix B.1 contains the interpretation of each term in Def. 2.1.

We introduce an alternative equivalent definition of EI fairness below.

Proposition 2.2. *The EI fairness notion defined in Def. 2.1 has an equivalent format: a classifier f achieves equal improvability with δ-effort if and only if*

$$\mathbb{P}(\mathbf{x} \in \mathcal{X}_-^{\mathrm{imp}} \mid \mathbf{x} \in \mathcal{X}_-, \mathrm{z} = z) = \mathbb{P}(\mathbf{x} \in \mathcal{X}_-^{\mathrm{imp}} \mid \mathbf{x} \in \mathcal{X}_-)$$

holds for all $z \in \mathcal{Z}$ where $\mathcal{X}_- = \{\mathbf{x} : f(\mathbf{x}) < 0.5\}$ is the set of features $\mathbf{x}$ for unqualified samples, and $\mathcal{X}_-^{\mathrm{imp}} = \{\mathbf{x} : f(\mathbf{x}) < 0.5, \max_{\mu(\Delta\mathbf{x}_I) \leq \delta} f(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5\}$ is the set of features $\mathbf{x}$ for unqualified samples that can be improved to qualified samples by adding $\Delta\mathbf{x}$ satisfying $\mu(\Delta\mathbf{x}_I) \leq \delta$.

The proof of this proposition is trivial from the definition of $\mathcal{X}_-$ and $\mathcal{X}_-^{\mathrm{imp}}$. Note that $\mathbb{P}(\mathbf{x} \in \mathcal{X}_-^{\mathrm{imp}} \mid \mathbf{x} \in \mathcal{X}_-)$ in the above equation indicates the probability that unqualified samples can be improved to qualified samples by changing the features within budget δ. This is how we define the "improvability" of unqualified samples, and the EI fairness notion is equalizing this improvability for all groups.

Visualization of EI. Fig. 2 shows the geometric interpretation of EI fairness notion in Def. 2.1 and Prop. 2.2, for a simple two-dimensional dataset having 12 samples in two groups $z \in \{\text{red}, \text{blue}\}$. Consider a linear classifier f_{EI} shown in the figure, where the samples at the right-hand-side of the decision boundary is classified as qualified samples ($f_{\mathrm{EI}}(\mathbf{x}) \geq 0.5$). In Fig. 2a, we have L_∞ ball at each unqualified sample, representing that these samples have a chance to improve their feature in a way that the improved feature $\mathbf{x} + \Delta\mathbf{x}$ allows the sample to be classified as qualified, *i.e.*, $f_{\mathrm{EI}}(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5$. One can confirm that $\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}}) \leq \delta} f(\mathbf{x} + \Delta\mathbf{x}) \geq 0.5 \mid f(\mathbf{x}) < 0.5, \mathrm{z} = z\right) = \frac{1}{3}$ holds for each group $z \in \{\text{red}, \text{blue}\}$, thus satisfying equal improvability according to Def. 2.1. In Fig. 2b, we check this in an alternative way by using the EI fairness definition in Prop. 2.2. Here, instead of making a set of improved features at each sample, we partition the feature domain $\mathcal{X}$ into

Table 1: Comparison of our EI fairness with existing fairness notions.

Name of fairness	Definition	First order?	Limitations
Equal Improvability (Ours)	$\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right) = \mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right)$	Yes	-
Demographic Parity	$\mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{z}=z\right) = \mathbb{P}\left(f(\mathbf{x}) \geq 0.5\right)$	No	-
Equal Opportunity (Hardt et al., 2016)	$\mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{y}=1, \mathrm{z}=z\right) = \mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{y}=1\right)$	No	-
Equalized Odds (Hardt et al., 2016)	$\mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{y}=y, \mathrm{z}=z\right) = \mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{y}=y\right)$	No	-
Bounded Effort (Heidari et al., 2019)	$\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right) = \mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right)$	Yes	Cannot handle imbalanced group negative rates
Equal Recourse (Gupta et al., 2019)	$\mathbb{E}\left[\min_{f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5} \mu(\Delta\mathbf{x}) \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right] = \mathbb{E}\left[\min_{f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5} \mu(\Delta\mathbf{x}) \mid f(\mathbf{x})<0.5\right]$	Yes	Vulnerable to outliers
Individual-level ER (Von Kügelgen et al., 2022)	$\min_{\mathbf{x}':f(\mathbf{x}')\geq 0.5} \mu_z(\mathbf{x}',\mathbf{x}) = \min_{\mathbf{x}':f(\mathbf{x}')\geq 0.5} \mu_{z'}(\mathbf{x}',\mathbf{x}),$ for all rejected individuals and $z, z' \in [Z]$	Yes	Limitations of counterfactual fairness

three parts: (i) the features for qualified samples $\mathcal{X}_+ = \{\mathbf{x} : f_\mathrm{EI}(\mathbf{x}) \geq 0.5\}$, (ii) the features for unqualified samples that can be improved $\mathcal{X}_-^\mathrm{imp} = \{\mathbf{x} : f_\mathrm{EI}(\mathbf{x}) < 0.5, \max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x}) \geq 0.5\}$ and (iii) the features for unqualified samples that cannot be improved $\mathcal{X}_-^\mathrm{imp} = \{\mathbf{x} : f_\mathrm{EI}(\mathbf{x}) < 0.5, \max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x}) < 0.5\}$. In the figure, (ii) is represented as the green region and (iii) is shown as the yellow region. From Prop. 2.2, EI fairness means that $\frac{\text{\# samples in (ii)}}{\text{\# samples in (ii) + \# samples in (iii)}}$ is identical at each group $z \in \{\text{red}, \text{blue}\}$, which is true for the example in Fig. 2b.

Comparison of EI with other fairness notions. The suggested fairness notion *equal improvability* (EI) is in stark difference with existing popular zero-order fairness notions, *e.g.*, demographic parity, which can be "myopic" and focus only on achieving classification fairness in the current status. Our notion instead, uses classification fairness as a tool to equalize the true qualification status of different groups in the long run, thereby promoting social fairness. We here also note that EI has differences with existing first-order fairness notions that capture the dynamics of samples (Heidari et al., 2019; Huang et al., 2019; Gupta et al., 2019). Table 1 compares our fairness notion with the related existing notions. In particular, Bounded Effort (BE) fairness proposed by Heidari et al. (2019) equalizes 'the available reward after each individual making a bounded effort' for different groups, which is very similar to EI when we set a proper reward function. To be more specific, the BE fairness can be represented as in the Table 1. Comparing this BE expression with EI in Definition 2.1, one can confirm the difference: the inequality $f(\mathbf{x}) < 0.5$ is located at the conditional part for EI, which is not true for BE. EI and BE are identical if the negative prediction rates are equal across the groups, but in general they are different. The condition $f(\mathbf{x}) < 0.5$ here is very important since only looking into the unqualified members makes more sense when we consider *improvability*. More importantly, the BE definition is based on reward functions and we are presenting BE in a form that is closest to our EI fairness expression. Besides, Equal Recourse (ER) fairness proposed by Gupta et al. (2019) suggests to equalize the average effort of different groups without limiting the amount of effort that each sample can make. Note that ER is vulnerable to outliers. For example, when we have an unqualified outlier sample that is located far way from the decision boundary, ER disparity will be dominated by this outlier and fail to reflect the overall unfairness. Huang et al. (2019) proposed a causal-based fairness notion to equalize the minimum level of effort such that the expected prediction score of the groups is equal to each other. Note that, their definition is specific to causal settings and it considers the sensitive groups not the rejected samples of the sensitive groups. The comparison with (Von Kügelgen et al., 2022) is given in Section B.4 of Appendix. In addition to fairness notions, we also discuss other related works such as fairness-aware algorithms in Sec. 5.

Compatibility of EI with other fairness notions. Here we prove the compatibility of three fairness notions (EI, DP, and BE), under two mild assumptions. Assumption 2.3 ensures that EI is well-defined, while Assumption 2.4 implies that the norm μ and the effort budget δ are chosen such that we have nonzero probability that unqualified individuals can become qualified after making efforts.

Assumption 2.3. For any classifier f, the probability of unqualified samples for each demographic group is not equal to 0, *i.e.*, $\mathbb{P}\left(f(\mathbf{x}) < 0.5, \mathrm{z}=z\right) \neq 0$ for all $z \in \mathcal{Z}$.

Assumption 2.4. For any classifier f, the probability of being qualified after the effort for unqualified samples is not equal to 0, *i.e.*, $\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x}) \geq 0.5, f(\mathbf{x}) < 0.5\right) \neq 0$.

Under these assumptions, the following theorem reveals the relationship between DP, EI and BE.

Theorem 2.5. *If a classifier f achieves two of the following three fairness notions, DP, EI, and BE; then it has to achieve the remaining fairness notion as well.*

The proof of the Theorem 2.5 is provided in Appendix A. This theorem immediately implies the following corollary, which provides a condition such that EI and BE conflict with each other.

Corollary 2.6. *The above theorem says that if a classifier f achieves EI and BE, it has to achieve DP. Thus, by contraposition, if f does not achieve DP, then it cannot achieve EI and BE simultaneously.*

3 ACHIEVING EQUAL IMPROVABILITY

In this section, we discuss methods for finding a classifier that achieves EI fairness. Following existing in-processing techniques (Zafar et al., 2017c; Donini et al., 2018; Zafar et al., 2017a; Cho et al., 2020), we focus on finding a fair classifier by solving a fairness-regularized optimization problem. To be specific, we first derive a differentiable penalty term U_δ that approximates the unfairness with respect to EI, and then solve a regularized empirical minimization problem having the unfairness as the regularization term. This optimization problem can be represented as

$$\min_{f\in\mathcal{F}}\left\{\frac{(1-\lambda)}{N}\sum_{i=1}^{N}\ell(\mathrm{y}_i, f(\mathbf{x}_i))+\lambda U_\delta\right\}, \tag{1}$$

where $\{(\mathbf{x}_i,\mathrm{y}_i)\}_{i=1}^N$ is the given dataset, $\ell:\{0,1\}\times[0,1]\to\mathbb{R}$ is the loss function, $\mathcal{F}$ is the set of classifiers we are searching over, and $\lambda\in[0,1)$ is a hyperparameter that balances fairness and prediction loss. Here we consider three different ways of defining the penalty term U_δ, which are (a) covariance-based, (b) kernel density estimator (KDE)-based, and (c) loss-based methods. We first introduce how we define U_δ in each method, and then discuss how we solve (1).

Covariance-based EI Penalty. Our first method is inspired by Zafar et al. (2017c), which measures the unfairness of a score-based classifier f by the covariance of the sensitive attribute z and the score $f(\mathbf{x})$, when the demographic parity (DP) fairness condition ($\mathbb{P}(f(\mathbf{x})>0.5|\mathrm{z}=z)=\mathbb{P}(f(\mathbf{x})>0.5)$ holds for all z) is considered. The intuition behind this idea of measuring the covariance is that a perfect fair DP classifier should have zero correlation between z and $f(\mathbf{x})$. By applying similar approach to our fairness notion in Def. 2.1, the EI unfairness is measured by the covariance between the sensitive attribute z and the maximally improved score of rejected samples within the effort budget. In other words, $(\mathrm{Cov}(\mathrm{z},\max_{\|\Delta\mathbf{x}_\mathrm{I}\|\le\delta}f(\mathbf{x}+\Delta\mathbf{x})\mid f(\mathbf{x})<0.5))^2$ represents the EI unfairness of a classifier f where we took the square to penalize negative correlation case as well. Let $I_-=\{i:f(\mathbf{x}_i)<0.5\}$ be the set of indices of unqualified samples, and $\bar{\mathrm{z}}=\sum_{i\in I_-}\mathrm{z}_i/|I_-|$. Then, EI unfairness can be approximated by the square of the empirical covariance, *i.e.*,

$$U_\delta\triangleq\left(\frac{1}{|I_-|}\sum_{i\in I_-}(\mathrm{z}_i-\bar{\mathrm{z}})\left(\max_{\|\Delta\mathbf{x}_{\mathrm{I}_i}\|\le\delta}f(\mathbf{x}_i+\Delta\mathbf{x}_i)-\sum_{j\in I_-}\max_{\|\Delta\mathbf{x}_{\mathrm{I}_j}\|\le\delta}f(\mathbf{x}_j+\Delta\mathbf{x}_j)/|I_-|\right)\right)^2.$$

Since $\sum_{i\in I_-}(\mathrm{z}_i-\bar{\mathrm{z}})\left(\sum_{j\in I_-}\max_{\|\Delta\mathbf{x}_{\mathrm{I}_j}\|\le\delta}f(\mathbf{x}_j+\Delta\mathbf{x}_j)/|I_-|\right)=0$ from $\sum_{i\in I_-}(\mathrm{z}_i-\bar{\mathrm{z}})=0$, we have $U_\delta=\left(\frac{1}{|I_-|}\sum_{i\in I_-}(\mathrm{z}_i-\bar{\mathrm{z}})\max_{\|\Delta\mathbf{x}_{\mathrm{I}_i}\|\le\delta}f(\mathbf{x}_i+\Delta\mathbf{x}_i)\right)^2$.

KDE-based EI Penalty. The second method is inspired by Cho et al. (2020), which suggests to first approximate the probability density function of the score $f(\boldsymbol{x})$ via kernel density estimator (KDE) and then put the estimated density formula into the probability term in the unfairness penalty. Recall that given m samples $\mathrm{y}_1,\dots,\mathrm{y}_m$, the true density g_y on y is estimated by KDE as $\hat{g}_\mathrm{y}(\hat{\mathrm{y}})\triangleq\frac{1}{mh}\sum_{i=1}^m g_k\left(\frac{\hat{\mathrm{y}}-\mathrm{y}_i}{h}\right)$, where g_k is a kernel function and h is a smoothing parameter.

Here we apply this KDE-based method for estimating the EI penalty term in Def. 2.1. Let $\mathrm{y}_i^{\max}=\max_{\|\Delta\mathbf{x}_{\mathrm{I}_i}\|\le\delta}f(\mathbf{x}_i+\Delta\mathbf{x}_i)$ be the maximum score achievable by improving feature $\mathbf{x}_i$ within budget δ, and $I_{-,z}=\{i:f(\mathbf{x}_i)<0.5,\mathrm{z}_i=z\}$ be the set of indices of unqualified samples of group z. Then, the density of $\mathrm{y}_i^{\max}$ for the unqualified samples in group z can be approximated as[2]

$$\hat{g}_{\mathrm{y}^{\max}|f(\mathbf{x})<0.5,z}(\hat{\mathrm{y}}^{\max})\triangleq\frac{1}{|I_{-,z}|h}\sum_{i\in I_{-,z}}g_k\left(\frac{\hat{\mathrm{y}}^{\max}-\mathrm{y}_i^{\max}}{h}\right).$$

Then, the estimate on the left-hand-side (LHS) probability term in Def. 2.1 is represented as $\hat{\mathbb{P}}\left(\max_{\mu(\Delta\mathbf{x}_\mathrm{I})\le\delta}f(\mathbf{x}+\Delta\mathbf{x})\ge0.5\mid f(\mathbf{x})<0.5,\mathrm{z}=z\right)=\int_{0.5}^\infty\hat{g}_{\mathrm{y}^{\max}|f(\mathbf{x})<0.5,z}(\hat{\mathrm{y}}^{\max})\mathrm{d}\hat{\mathrm{y}}^{\max}=\frac{1}{|I_{-,z}|h}\sum_{i\in I_{-,z}}G_k\left(\frac{0.5-\mathrm{y}_i^{\max}}{h}\right)$ where $G_k(\tau)\triangleq\int_\tau^\infty g_k(y)\mathrm{d}y$. Similarly, we can estimate the right-hand-side (RHS) probability term in Def. 2.1, and the EI-penalty U_δ is computed as the summation of the absolute difference of the two probability values (LHS and RHS) among all groups z.

[2]This term is differentiable with respect to model parameters, since g_k is differentiable with respect to $\mathrm{y}_i^{\max}$, and $\mathrm{y}_i^{\max}=\max_{\|\Delta\mathbf{x}_{\mathrm{I}_i}\|\le\delta}f(\mathbf{x}_i+\Delta\mathbf{x}_i)$ is differentiable w.r.t. model parameters from (Danskin, 1967).

Loss-based EI Penalty. Another common way of approximating the fairness violation as a differentiable term is to compute the absolute difference of group-specific losses (Roh et al., 2021; Shen et al., 2022). Following the spirit of EI notion in Def. 2.1, we define EI loss of group z as $\tilde{L}_z \triangleq \frac{1}{|I_{-,z}|}\sum_{i\in I_{-,z}} \ell\left(1, \max_{\|\Delta \mathbf{x}_{\mathrm{I}i}\|\le\delta} f(\mathbf{x}_i+\Delta\mathbf{x}_i)\right)$. Here, $\tilde{L}_z$ measures how far the rejected samples in group z are away from being accepted after the feature improvement within budget δ. Similarly, EI loss for all groups is written as $\tilde{L} \triangleq \sum_{z\in\mathcal{Z}} \frac{I_{-,z}}{I_-}\tilde{L}_z$. Finally, the EI penalty term is defined as $U_\delta \triangleq \sum_{z\in\mathcal{Z}} \left|\tilde{L}_z - \tilde{L}\right|$.

Solving (1). For each approach defined above (covariance-based, KDE-based and loss-based), the penalty term U_δ is defined uniquely. Note that in all cases, we need to solve a maximization problem $\max_{\|\Delta \mathbf{x}_{\mathrm{I}}\|\le\delta} f(\mathbf{x}+\Delta\mathbf{x})$ in order to get U_δ. Since (1) is a minimization problem containing U_δ in the cost function, it is essentially a minimax problem. We leverage adversarial training techniques to solve (1). The inner maximization problem is solved using one of two methods: (i) derive the closed-form solution for generalized linear models, (ii) use projected gradient descent for general settings. The details can be found in Appendix B.2.

4 Experiments

In this section, we provide empirical results on our EI fairness notion. To measure the fairness violation, we use EI disparity$=\max_{z\in[Z]} |\mathbb{P}(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})<\delta} f(\mathbf{x}+\Delta\mathbf{x}) \ge 0.5 \mid f(\mathbf{x}) < 0.5, \mathrm{z}=z) - \mathbb{P}(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})<\delta} f(\mathbf{x}+\Delta\mathbf{x}) \ge 0.5 \mid f(\mathbf{x}) < 0.5)|$. First, we show that our methods suggested in Sec. 3 achieve EI fairness in various real/synthetic datasets. Second, focusing on the dynamic scenario where each individual can make effort to improve its outcome, we demonstrate that training an EI classifier at each time step promotes achieving the long-term fairness, *i.e.*, the feature distribution of two groups become identical in the long run. Codes are available in anonymous GitHub repository [3].

4.1 Suggested methods achieve EI fairness

Recall that Sec. 3 provided three approaches for achieving EI fairness. Here we check whether such methods successfully find a classifier with small EI disparity, compared with ERM which does not have fairness constraints. Due to the space limitation, the main body contains the results for logistic regression (LR) only, and in Appendix C contains the performance for multi-layer perceptron (MLP) as well as the effect of over-parameterization.

Experiment setting. For all experiments, we use the Adam optimizer and cross entropy loss. We perform cross-validation on the training set to find the best hyperparameter. We provide statistics for five trials having different random seeds. For KDE-based approach, we use Gaussian kernel.

Datasets. We perform the experiments on one synthetic dataset, and two real datasets: German Statlog Credit (Dua & Graff, 2017), and ACSIncome-CA (Ding et al., 2021). The synthetic dataset has two non-sensitive attributes $\mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2)$, one binary sensitive attribute z, and a binary label y. Both features x_1 and x_2 are assumed to be improvable. We generate 20,000 samples where $(\mathbf{x}, \mathrm{y}, \mathrm{z})$ pair of each sample is generated independently as below. We define z and $(\mathrm{y}|\mathrm{z}=z)$ as Bernoulli random variables for all $z \in \{0,1\}$, and define $(\mathbf{x}|\mathrm{y}=y, \mathrm{z}=z)$ as multivariate Gaussian random variables for all $y, z \in \{0,1\}$. The numerical details are in Appendix C.1 The maximum effort δ for this dataset is set to 0.5. The ratio of the training versus test data is 4:1.

German Statlog Credit Data contains 1,000 samples and the ratio of the training versus test data is 4:1. The task is to predict the credit risk of an individual given its financial status. Following Jiang & Nachum (2020), we divide the samples into two groups using the age of thirty as the boundary, *i.e.*, $z = 1$ for samples with age above thirty. Four features $\mathbf{x}$ are considered as improvable: `checking account`, `saving account`, `housing` and `occupation`, all of which are ordered categorical features. For example, the `occupation` feature has four levels: (1) unemployed, (2) unskilled, (3) skilled, and (4) highly qualified. We set the maximum effort $\delta = 1$, meaning that an unskilled man (with level 2) can become a skilled man, but cannot be a highly qualified man.

ACSIncome-CA dataset consists of data for 195,665 people and is split into training/test set in the ratio of 4:1. The task is predicting whether a person's income would exceed 50K USD per year. We use sex as the sensitive attribute; we have two sensitive groups, male and female. We select education level (ordered categorical feature) as the improvable feature. We set the maximum effort $\delta = 3$.

[3] https://anonymous.4open.science/r/ei_fairness-23CA/

Table 2: **Error rate and EI disparities of ERM and three proposed EI-regularized methods on logistic regression (LR).** For each dataset, the lowest EI disparity (disp.) value is in boldface. Classifiers obtained by our three methods have much smaller EI disparity values than the ERM solution, without having much additional error.

Dataset	Metric	Methods: ERM	Covariance-based	KDE-based	Loss-based
Synthetic	Error Rate(↓)	.221 ± .001	.253 ± .003	.250 ± .001	.246 ± .001
	EI Disp.(↓)	.117 ± .007	.003 ± .001	.005 ± .003	**.002 ± .001**
German Stat.	Error Rate(↓)	.220 ± .009	.262 ± .009	.243 ± .024	.237 ± .008
	EI Disp.(↓)	.041 ± .008	.021 ± .019	.035 ± .026	**.015 ± .009**
ACSIncome-CA	Error Rate(↓)	.184 ± .000	.200 ± .000	.196 ± .000	.193 ± .000
	EI Disp.(↓)	.031 ± .001	.008 ± .001	**.005 ± .001**	.006 ± .001

Results. Table 2 shows the test error rate and test EI disparity (disp.) for ERM and our three EI-regularized methods (covariance-based, KDE-based and loss-based) suggested in Sec. 3. For all three datasets, our EI-regularized methods successfully reduce the EI disparity without increasing the error rate too much, compared with ERM. Figure 3 shows the tradeoff between error rate and EI disparity of our EI-regularized methods. We marked the dots after running each method multiple times with different penalty coefficient λ, and plotted the frontier line. For the synthetic dataset with Gaussian feature, we numerically obtained the performance of the optimal EI classifier, which is added in the yellow line at the bottom left corner of the first column plot. The details of finding the optimal EI classifier is in Appendix B.3. One can confirm that our methods regularizing EI are having similar tradeoff curves for the synthetic dataset. Especially, for synthetic dataset, the tradeoff curve of our methods nearly achieves that of the optimal EI classifier. For German and ACSIncome-CA datasets, the loss-based method is having a slightly better tradeoff curve than other methods.

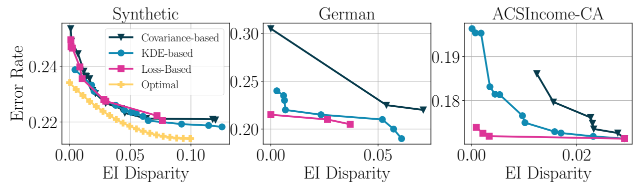


Figure 3: **Tradeoff between EI disparity and error rate.** We run three EI-regularized methods suggested in Sec. 3 for different regularizer coefficient λ and plot the frontier lines. For the synthetic dataset, the tradeoff curve for the ideal classifier is located at the bottom left corner, which is similar to the curves of proposed EI-regularized methods. This shows that our methods successfully find classifiers balancing EI disparity and error rate.

4.2 EI promotes long-term fairness in dynamic scenarios

Recall that the motivation for proposing EI is to achieve *long-term fairness*, which is equalizing the feature distribution of samples in different groups in the long run, under the dynamic scenario where each individual can improve its feature. In this section, we provide simulation results on dynamic setting, which show that training a classifier with EI constraint encourages the long-term fairness.

4.2.1 Dynamic System Description

We consider a binary classification problem under the dynamic scenario with T rounds, where the improvable feature $\mathbf{x}_t \in \mathbb{R}$ and the label $\mathbf{y}_t \in \{0,1\}$ of each sample as well as the classifier f_t evolve at each round $t \in \{0, \cdots, T-1\}$. We denote the sensitive attribute as $\mathrm{z} \in \{0,1\}$, and the estimated label as $\hat{\mathrm{y}}_t$. We assume $\mathrm{z} \sim \mathrm{Bern}(0.5)$ and $(\mathrm{x}_t \mid \mathrm{z} = z) \sim \mathcal{P}_t^{(z)} = \mathcal{N}(\mu_t^{(z)}, \{\sigma_t^{(z)}\}^2)$. To mimic the admission problem, we only accept a fraction $\alpha \in (0,1)$ of the population, *i.e.*, the true label is modeled as $\mathrm{y}_t = \mathbf{1}_{\mathrm{x}_t \geq \chi_\alpha^{(t)}}$, where $\chi_\alpha^{(t)}$ is the $(1-\alpha)$ percentile of the feature distribution at round t. We consider z-aware linear classifier outputting $\hat{\mathrm{y}}_t = \mathbf{1}_{\mathrm{x}_t \geq \tau_t^{(z)}}$, which is parameterized by the thresholds $(\tau_t^{(0)}, \tau_t^{(1)})$ for two groups. Note that this classification rule is equivalent to defining score function $f_t(x,z) = 1/(\exp(\tau_t^{(z)} - x) + 1)$ and $\hat{\mathrm{y}}_t = \mathbf{1}_{f(\mathbf{x}_t, z) \geq 0.5}$.

Updating data parameters $(\mu_t^{(z)}, \sigma_t^{(z)})$. At each round t, we allow each sample can improve its feature from x to $x + \epsilon(x)$. Here we model $\epsilon(x) = \nu(x;z) = \frac{1}{(\tau_t^{(z)} - x + \beta)^2} \mathbf{1}_{x < \tau_t^{(z)}}$ for a constant $\beta > 0$. In this model, the rejected samples with larger gap $(\tau_t^{(z)} - x)$ with the decision boundary are making less effort Δx, which is inspired by the intuition that a rejected sample is less motivated to improve its feature if it needs to take a large amount of effort to get accepted in one scoop. After such effort is made, we compute the mean and standard deviation of each group: $\mu_{t+1}^{(z)} = \int_{-\infty}^{\infty} (x + \nu(x;z)) \phi(x; \mu_t^{(z)}, \sigma_t^{(z)}) \mathrm{d}x$ and $\sigma_{t+1}^{(z)} = \sqrt{\int_{-\infty}^{\infty} (x + \nu(x;z) - \mu_{t+1}^{(z)})^2 \phi(x; \mu_t^{(z)}, \sigma_t^{(z)}) \mathrm{d}x}$,

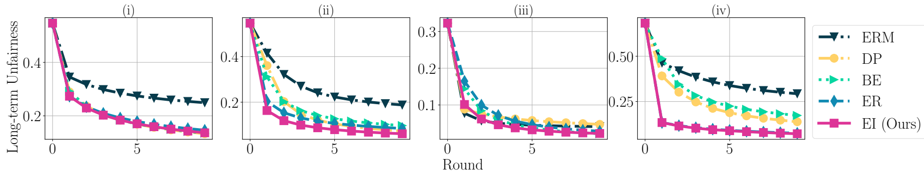


Figure 4: **Long-term unfairness $d_{\mathrm{TV}}(\mathcal{P}_t^{(0)}, \mathcal{P}_t^{(1)})$ at each round t for various algorithms.** Consider the binary classification problem over two groups, under the dynamic scenario where the data distribution and the classifier for each group evolves over multiple rounds. We plot how the long-term unfairness (measured by the total variation distance between two groups) changes as round t increases. Here, each column shows the result for different initial feature distribution, details of which are given in Sec. 4.2.2. The long-term unfairness of EI classifier reduces faster than other existing fairness notions, showing that EI proposed in this paper is helpful for achieving long-term fairness.

where $\phi(\cdot;\mu,\sigma)$ is the pdf of $\mathcal{N}(\mu,\sigma^2)$. We assume that the feature $\mathbf{x}_{t+1}$ in the next round follows a Gaussian distribution parameterized by $(\mu_{t+1}^{(z)}, \sigma_{t+1}^{(z)})$ for ease of simulation.

Updating classifier parameters $(\tau_t^{(0)}, \tau_t^{(1)})$. At each round t, we update the classifier depending on the current feature distribution $\mathbf{x}_t$. The EI classifier considered in this paper updates $(\tau_t^{(0)}, \tau_t^{(1)})$ as below. Note that the maximized score $\max_{\|\Delta \mathbf{x}_{\mathrm{I}}\| \le \delta} f(\mathbf{x}+\Delta\mathbf{x})$ in Def. 2.1 can be written as $f_t(\mathrm{x}_t+\delta_t, \mathrm{z})$, and the equation $\max_{\|\Delta \mathbf{x}_{\mathrm{I}}\| \le \delta} f(\mathbf{x}+\Delta\mathbf{x}) \ge 0.5$ is equivalent to $\mathrm{x}_t + \delta_t > \tau_t^{(\mathrm{z})}$. Consequently, EI classifier obtains $(\tau_t^{(0)}, \tau_t^{(1)})$ by solving

$$\min_{\tau_t^{(0)},\tau_t^{(1)}} \left| \mathbb{P}(\mathrm{x}_t+\delta_t>\tau_t^{(0)} \mid \mathrm{z}=0, \mathrm{x}_t<\tau_t^{(0)}) - \mathbb{P}(\mathrm{x}_t+\delta_t>\tau_t^{(1)} \mid \mathrm{z}=1, \mathrm{x}_t<\tau_t^{(1)}) \right| \text{ s.t. } \mathbb{P}(\hat{\mathrm{y}}_t \neq \mathrm{y}_t) \le c,$$

where $c \in [0,1)$ is the maximum classification error rate we allow, and δ_t is the effort level at iteration t. In our experiments, δ_t is chosen as the mean efforts the population makes, *i.e.*, $\delta_t = 0.5 \sum_{z=0}^{1} \int_{-\infty}^{\infty} \nu(x;z)\phi(x;\mu_t^{(z)},\sigma_t^{(z)})\mathrm{d}x$. We can similarly obtain the classifier for DP, BE and ER constraints, details of which are in Appendix C.4. In the experiments, we numerically obtain the solution of this optimization problem.

4.2.2 Experiments on long-term fairness

We first initialize the feature distribution in a way that both sensitive groups have either different mean (*i.e.*, $\mu_0^{(0)} \neq \mu_0^{(1)}$) or different variance (*i.e.*, $\sigma_0^{(0)} \neq \sigma_0^{(1)}$). At each round $t \in \{1, \cdots, T\}$, we update the data parameter $(\mu_t^{(z)}, \sigma_t^{(z)})$ for group $z \in \{0,1\}$ and the classifier parameter $(\tau_t^{(0)}, \tau_t^{(1)})$, following the rule described in Sec. 4.2.1. At each round $t \in \{1, \cdots, T\}$, we measure the *long-term unfairness* defined as the total variation distance between the two group distributions: $d_{\mathrm{TV}}(\mathcal{P}^{(0)}, \mathcal{P}^{(1)}) = \frac{1}{2}\int_{-\infty}^{\infty} |\phi(x;\mu_t^{(0)},\sigma_t^{(0)}) - \phi(x;\mu_t^{(1)},\sigma_t^{(1)})|\mathrm{d}x$. We run experiments on four different initial feature distributions: (i) $(\mu_0^{(0)},\sigma_0^{(0)},\mu_0^{(1)},\sigma_0^{(1)}) = (0,1,1,0.5)$, (ii) $(\mu_0^{(0)},\sigma_0^{(0)},\mu_0^{(1)},\sigma_0^{(1)}) = (0,0.5,1,1)$, (iii) $(\mu_0^{(0)},\sigma_0^{(0)},\mu_0^{(1)},\sigma_0^{(1)}) = (0,2,0,1)$, and (iv) $(\mu_0^{(0)},\sigma_0^{(0)},\mu_0^{(1)},\sigma_0^{(1)}) = (0,0.5,1,0.5)$, respectively. We set $\alpha = 0.2, c = 0.1, \beta = 0.25$.

Baselines. We compare our EI classifier with multiple baselines, including the empirical risk minimization (ERM) and algorithms with fairness constraints: demographic parity (DP), bounded effort (BE) (Heidari et al., 2019), and equal recourse (ER) (Gupta et al., 2019).

Results. Fig. 4 shows how the long-term unfairness $d_{\mathrm{TV}}(\mathcal{P}_t^{(0)}, \mathcal{P}_t^{(1)})$ changes as a function of round t, for cases (i) – (iv) having different initial feature distribution. Note that fairness-constrained algorithms (DP, BE, ER and EI) enjoys lower long-term unfairness compared with ERM, for case (i), (ii) and (iv). More importantly, EI accelerates the process of mitigating long-term unfairness, compared to other fairness notions. This observation highlights the benefit of EI in promoting true equity of groups in the long-term. Fig. 5 visualizes the initial distribution (at the leftmost column) and the evolved distribution at round $t = 3$ for multiple algorithms (at the rest of the columns). Each row represents different initial feature distribution, for cases (i) – (iv). One can confirm that EI brings the distribution of the two groups closer, compared with ERM, DP, BE, and ER.

5 Related Works

Fairness metric. Group fairness is a class of fairness notions requiring the classifier to treat different groups similarly. Most of the existing group fairness notions are myopic, measuring fairness by only comparing the positive rate of certain groups or subgroups at the current snapshot (Hardt et al., 2016; Zafar et al., 2017a). In contrast, EI suggested in this paper is more farsighted by taking improvability

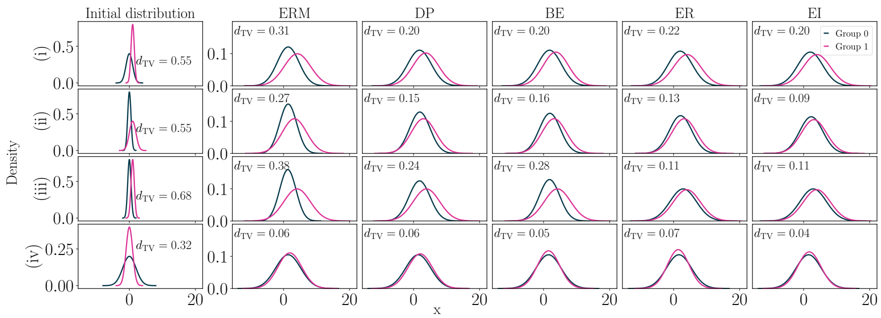


Figure 5: **Evolution of the feature distribution, when we apply each algorithm for $t = 3$ rounds.** At each row, the leftmost column shows the initial distribution and the rest of the columns show the evolved distribution for each algorithm, under the dynamic setting. Compared with existing fairness notions (DP, BE, and ER), our EI fairness achieves smaller feature distribution gap between groups.

into consideration. There exist a few works that propose effort-based fairness notions (Heidari et al., 2019; Huang et al., 2019; Gupta et al., 2019; Von Kügelgen et al., 2022) that capture the potential follow-up impact. EI is closely related to, but also clearly distinct from these effort-based fairness notions, particularly in terms of motivation. Note that Heidari et al. (2019) aims to propose fairness notions that capture the dynamics, while Gupta et al. (2019) and Huang et al. (2019) aim to propose reasonable fairness notions that take effort into consideration. Our work takes one-step further, focusing on promoting the long-term fairness. Besides, Huang et al. (2019) only consider the causal setting and are restricted to certain models. Without the restricted assumptions introduced by Huang et al. (2019), our work applies to more general settings, and provides a fairness metric that is easy to measure. We provide a more systematic comparison between EI and related work in Sec. 2.

Fairness-aware algorithms. Most of the existing fair learning techniques fall into three categories: i) pre-processing approaches (Kamiran & Calders, 2012; 2010; Gordaliza et al., 2019; Jiang & Nachum, 2020), which primarily involves massaging the dataset to remove the bias; ii) in-processing approaches (Fukuchi et al., 2013; Kamishima et al., 2012; Calders & Verwer, 2010; Zafar et al., 2017c;a; Zhang et al., 2018; Cho et al., 2020; Roh et al., 2020; 2021; Shen et al., 2022), adjusting the model training for fairness; iii) post-processing approaches (Calders & Verwer, 2010; Alghamdi et al., 2020; Wei et al., 2020; Hardt et al., 2016) which achieve fairness by modifying a given unfair classifier. Prior work (Woodworth et al., 2017) showed that the in-processing approach generally outperforms other approaches due to its flexibility. Hence, we focus on the in-processing approach and propose three methods to achieve EI. These methods achieve EI by solving fairness-regularized optimization problems. In particular, our proposed fairness regularization terms are inspired by Zafar et al. (2017c); Cho et al. (2020); Roh et al. (2021); Shen et al. (2022).

Fairness dynamics. There are also a few attempts to study the long-term impact of different fairness policies (Zhang et al., 2020; Heidari et al., 2019; Hashimoto et al., 2018). In particular, Hashimoto et al. (2018) studies how ERM amplifies the unfairness of a classifier in the long run. The key idea is that if the classifier of the previous iteration favors a certain candidate group, then the candidate groups will be more unbalanced since fewer individuals from the unfavored group will apply for this position. Thus, the negative feedback leads to a more unfair classifier. In contrast, Heidari et al. (2019) and Zhang et al. (2020) focus more on long-term impact instead of classification fairness. To be specific, Heidari et al. (2019) studies how fairness intervention affects the different groups in terms of evenness, centralization, and clustering by simulating the population's response through effort. Zhang et al. (2020) investigates how different fairness policies affect the gap between the qualification rates of different groups under a partially observed Markov decision process. Besides, there are a few works which study how individuals may take strategic actions to improve their outcomes given a classifier (Chen et al., 2021). However, Chen et al. (2021) aims to address this problem by designing an optimization problem that is robust to strategic manipulation, which is orthogonal to our focus.

6 Conclusion

In this paper, we proposed a group fairness notion called Equal Improvability (EI), which equalizes the potential acceptance of rejected samples in different groups, when appropriate effort is made by the rejected samples. We analyzed the properties of EI fairness, and provided three approaches to find a classifier that achieves EI. Experimental results showed that the proposed approaches reduce the EI disparity. Lastly, we formulated a dynamic model to showcase the benefit of EI in promoting the equity of feature distribution of different groups. Extending our work to settings with multiple sensitive attributes and high-dimensional features is remained as a future direction.

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A THEORETICAL RESULTS

A.1 CONNECTIONS BETWEEN EI, DP, AND BE

In this section, we provide the proof of Theorem 2.5 and Corollary 2.6.

Proof of Theorem 2.5. All we need to prove are three statements:

1. Prove that EI and BE imply DP
2. Prove that DP and EI imply BE
3. Prove that BE and DP imply EI

Below we prove each statement.

1. EI, BE $\Rightarrow$ DP Suppose a classifier f achieves EI and BE. Recall that a classifier achieves EI if

$$\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right) \tag{2}$$

and a classifier achieves BE if

$$\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right) \tag{3}$$

By dividing both sides of 3 by the both sides of 2, we have

$$\frac{\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)}{\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right)}=\frac{\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right)}{\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right)}$$

Then, it can be simplified as

$$\mathbb{P}\left(f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x})<0.5\right),$$

which implies that the classifier achieves demographic parity,

$$\mathbb{P}\left(f(\mathbf{x})\geq 0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x})\geq 0.5\right)$$

2. DP, EI $\Rightarrow$ BE Suppose a classifier f achieves DP and EI. Recall that a classifier achieves DP if

$$\mathbb{P}\left(f(\mathbf{x})\geq 0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x})\geq 0.5\right),$$

which implies

$$\mathbb{P}\left(f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x})<0.5\right). \tag{4}$$

Recall that a classifier achieves EI if

$$\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right) \tag{5}$$

By multiplying both sides of 4 and 5, we have

$$\begin{aligned}\mathbb{P}&\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right)\mathbb{P}\left(f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)\\ &=\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right)\mathbb{P}\left(f(\mathbf{x})<0.5\right)\end{aligned}$$

Then, it can be simplified as

$$\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta \mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right),$$

which implies that the classifier f achieves BE.

3. BE, DP ⇒ EI Suppose a classifier f achieves BE and DP. Recall that a classifier achieves DP if

$$\mathbb{P}\left(f(\mathbf{x}) \geq 0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x}) \geq 0.5\right),$$

which implies

$$\mathbb{P}\left(f(\mathbf{x}) < 0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(f(\mathbf{x}) < 0.5\right) \tag{6}$$

Recall that a classifier achieves BE if

$$\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right). \tag{7}$$

By dividing both sides of 7 by the both sides of 6, we have

$$\frac{\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5 \mid \mathrm{z}=z\right)}{\mathbb{P}\left(f(\mathbf{x}) < 0.5 \mid \mathrm{z}=z\right)}=\frac{\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5, f(\mathbf{x})<0.5\right)}{\mathbb{P}\left(f(\mathbf{x}) < 0.5\right)}$$

Then, it can be simplified as

$$\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5, \mathrm{z}=z\right)=\mathbb{P}\left(\max_{\mu(\Delta\mathbf{x}_{\mathrm{I}})\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})\geq 0.5 \mid f(\mathbf{x})<0.5\right),$$

which implies that the classifier f achieves EI. □

Proof of Corollary 2.6. The Corollary 2.6 can be proved directly from Theorem 2.5. □

A.2 Connections between EI and ER

Lemma A.1. *Consider $\mathrm{x} \mid \mathrm{z}=z \sim \mathcal{N}(\mu_z,\sigma^2)$ for $z\in\{0,1\}$, $\mu_z,\sigma\in\mathbb{R}$, and classifiers characterized by two accepting thresholds (τ_0,τ_1), where $\tau_0,\tau_1\in\mathbb{R}$. If a classifier satisfies EI, then it satisfies ER.*

Proof. Here we use $\Phi = 1-Q$ and ϕ to denote the CDF and PDF of standard Gaussian distribution, respectively. We consider the cost function $\mu=|\cdot|$.

Recall the definition of EI disparity and ER disparity

$$\begin{aligned}
\text{EI Disparity} = \Big|&\mathbb{P}\Big(\underbrace{\max_{\mu(\Delta\mathrm{x})<\delta} f(\mathrm{x}+\Delta\mathrm{x}) > 0.5}_{\mathrm{x}>\tau_0-\delta} \mid \underbrace{f(\mathrm{x}) < 0.5}_{\mathrm{x}\leq\tau_0}, \mathrm{z}=0\Big),\\
&-\mathbb{P}\left(\max_{\mu(\Delta\mathrm{x})<\delta} f(\mathrm{x}+\Delta\mathrm{x}) > 0.5 \mid f(\mathrm{x}) < 0.5, \mathrm{z}=1\right)\Big|\\
\text{ER Disparity} = \Big|&\mathbb{E}\Big[\underbrace{\min_{f(\mathrm{x}+\Delta\mathrm{x})\geq 0.5}\mu(\Delta\mathrm{x})}_{\tau_0-\mathrm{x}} \mid \underbrace{f(\mathrm{x}) < 0.5}_{\mathrm{x}\leq\tau_0}, \mathrm{z}=0\Big]\\
&-\mathbb{E}\Big[\min_{f(\mathrm{x}+\Delta\mathrm{x})\geq 0.5}\mu(\Delta\mathrm{x}) \mid f(\mathrm{x}) < 0.5, \mathrm{z}=1\Big]\Big|.
\end{aligned}$$

Consequently, the EI constraint and ER constraint can be written as

$$\text{EI Disparity }(\tau_0,\tau_1) = \left|\Phi\left(\frac{\tau_0-\delta-\mu_0}{\sigma}\right)/\Phi\left(\frac{\tau_0-\mu_0}{\sigma}\right) - \Phi\left(\frac{\tau_1-\delta-\mu_1}{\sigma}\right)/\Phi\left(\frac{\tau_1-\mu_1}{\sigma}\right)\right| = 0, \tag{8}$$

$$\text{ER Disparity }(\tau_0,\tau_1) = \left|\frac{1}{\Phi((\tau_0-\mu_0)/\sigma)}\int_{-\infty}^{\tau_0}(\tau_0-t)\phi\left(\frac{t-\mu_0}{\sigma}\right)\mathrm{d}t - \frac{1}{\Phi((\tau_1-\mu_1)/\sigma)}\int_{-\infty}^{\tau_1}(\tau_1-t)\phi\left(\frac{t-\mu_1}{\sigma}\right)\mathrm{d}t\right| = 0 \tag{9}$$

In this proof, we will first show that achieving EI is equivalent to $\tau_0 - \mu_0 = \tau_1 - \mu_1$, and then show that the classifier with $\tau_0 - \mu_0 = \tau_1 - \mu_1$ satisfies the ER constraint.

1. EI constraint.

   Let $\varphi(x) = \Phi(\frac{x-\delta}{\sigma})/\Phi(\frac{x}{\sigma})$. First, we show that φ is a strictly increasing function. Note that

   $$\varphi'(x) = \frac{1}{\sigma\Phi\left(\frac{x}{\sigma}\right)^2}\left(\phi\left(\frac{x-\delta}{\sigma}\right)\Phi\left(\frac{x}{\sigma}\right) - \Phi\left(\frac{x-\delta}{\sigma}\right)\phi\left(\frac{x}{\sigma}\right)\right).$$

   Therefore, to show that φ is strictly increasing, it is sufficient to show that

   $$\phi\left(\frac{x-\delta}{\sigma}\right)\Phi\left(\frac{x}{\sigma}\right) > \Phi\left(\frac{x-\delta}{\sigma}\right)\phi\left(\frac{x}{\sigma}\right). \tag{10}$$

   We show that (10) is equivalent as the following inequality by dividing both the left-hand side and right-hand side by $\phi(\frac{x-\delta}{\sigma})\phi(\frac{x}{\sigma})$:

   $$\Phi\left(\frac{x}{\sigma}\right)/\phi\left(\frac{x}{\sigma}\right) > \Phi\left(\frac{x-\delta}{\sigma}\right)/\phi\left(\frac{x-\delta}{\sigma}\right). \tag{11}$$

   Note that $\frac{1-\Phi(\cdot)}{\phi(\cdot)}$ is known in literatures as Mills' ratio (Mitrinovic & Vasic, 1970), which is strictly decreasing on $\mathbb{R}$. Therefore, $\frac{\Phi(\cdot)}{\phi(\cdot)}$ is strictly increasing on $\mathbb{R}$. Since $\frac{x}{\sigma} > \frac{x-\delta}{\sigma}$, (11) holds, thereby (10) holds and φ is strictly increasing.

   Given that $\varphi(x) = \Phi(\frac{x-\delta}{\sigma})/\Phi(\frac{x}{\sigma})$ is a strictly increasing function on $\mathbb{R}$,

   $$(8) = |\varphi(\tau_0 - \mu_0) - \varphi(\tau_1 - \mu_1)| = 0$$

   yields that

   $$\tau_0 - \mu_0 = \tau_1 - \mu_1.$$

2. ER constraint.

   We first note that

   $$\int_{-\infty}^{\tau_0}(\tau_0 - t)\phi\left(\frac{t-\mu_0}{\sigma}\right)\mathrm{d}t \overset{t'=t-\mu_0}{=\!=\!=} \int_{-\infty}^{\tau_0-\mu_0}(\tau_0 - \mu_0 - t')\phi\left(\frac{t'}{\sigma}\right)\mathrm{d}t'$$

   Let $\psi(x) = \frac{1}{\Phi(x/\sigma)}\int_{-\infty}^{x}(x-t)\phi(\frac{t}{\sigma})\mathrm{d}t$. It is clear that ER constraint is equivalent to

   $$(9) = |\psi(\tau_0 - \mu_0) - \psi(\tau_1 - \mu_1)|.$$

   Therefore, the classifier with $\tau_0 - \mu_0 = \tau_1 - \mu_1$ clearly satisfies the ER constraint.

Combining all the discussion above completes the proof. □

B Supplementary Materials on the EI fairness notion

Recall that in Sec. 2 and Sec. 3, this paper proposes a new fairness notion called equal improvability (EI) and finds a classifier by solving a EI-constrained optimization which is formulated as a minimax problem. In Sec. B.1, we first explain what each term in the definition of EI means. Then in Sec. B.2, we provide how we solved the inner maximization problem in the EI-constrained optimization. Finally, in Sec. B.3, we provide numerical methods for finding the optimal solution for the EI-constrained problem, under simple synthetic dataset setting.

B.1 MEANING OF EACH TERM IN THE DEFINITION OF EI

To help readers better understand our EI definition, here we explain what each term means in EI definition means. Let the data sample $(\mathbf{x}, \mathrm{y}, \mathrm{z})$ follows the distribution $\mathcal{P}_{(\mathbf{x},\mathrm{y},\mathrm{z})}$. The meaning of each term of EI defined in Def. 2.1 is detailed below.

$$\mathbb{P}_{\underbrace{\mathbf{x},\mathrm{y},\mathrm{z}\sim\mathcal{P}_{\mathbf{x},\mathrm{y},\mathrm{z}}}_{\substack{\text{Randomness is over}\\\text{the data distribution}}}}\left(\underbrace{\underbrace{\max_{\mu(\Delta\mathbf{x}_I)\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})}_{\substack{\text{Maximum score}\\\text{after improvement}}} \geq 0.5}_{\substack{\text{Event in which the sample can}\\\text{be accepted after improvement}}} \,\middle|\, \underbrace{f(\mathbf{x}) < 0.5, \mathrm{z}=z}_{\substack{\text{Event in which the sample}\\\text{comes from group } z\\\text{and gets rejected}}}\right)$$

$$= \mathbb{P}_{\underbrace{\mathbf{x},\mathrm{y},\mathrm{z}\sim\mathcal{P}_{\mathbf{x},\mathrm{y},\mathrm{z}}}_{\substack{\text{Randomness is over}\\\text{the data distribution}}}}\left(\underbrace{\underbrace{\max_{\mu(\Delta\mathbf{x}_I)\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})}_{\substack{\text{Maximum score}\\\text{after improvement}}} \geq 0.5}_{\substack{\text{Event in which the sample can}\\\text{be accepted after improvement}}} \,\middle|\, \underbrace{f(\mathbf{x}) < 0.5}_{\substack{\text{Event in which the}\\\text{sample gets rejected}}}\right),$$

B.2 SOLVING THE INNER MAXIMIZATION PROBLEM

As explained in Sec.3, finding a EI classifier can be formulated as a minimax problem (1), where solving the inner maximization problem $\max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} f(\mathbf{x}+\Delta\mathbf{x})$ is required to compute U_δ in the regularization term, and the outer problem is the regularized-loss minimization finding the optimal model parameter $\boldsymbol{w}$ for the classifier $f = f_{\boldsymbol{w}}$. In this section, we provide two ways of solving the inner maximization problem. In particular, in Sec. B.2.1 we give the explicit expression of the optimizer $\Delta\mathbf{x}_\mathrm{I}$ when generalized linear model is considered. In Sec. B.2.2, we solve the problem under a more general setting via adversarial training.

B.2.1 CLOSED-FORM SOLUTION FOR GENERALIZED LINEAR MODEL

Consider a Generalized Linear Model (GLM) written as $f(\mathbf{x}) = g^{-1}(\boldsymbol{w}^\top\mathbf{x})$, where $g : [0,1] \to \mathbb{R}$ is a strictly increasing link function, and $\boldsymbol{w}$ is the model parameter. Denote the weights corresponding to $\mathbf{x}_\mathrm{I}$ as $\boldsymbol{w}_\mathrm{I}$. Then, the inner maximization problem can be written as:

$$\begin{aligned}\max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} f(\mathbf{x}+\Delta\mathbf{x}) &= \max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} g^{-1}(\boldsymbol{w}^\top(\mathbf{x}+\Delta\mathbf{x}))\\ &= \max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} g^{-1}(\boldsymbol{w}^\top\mathbf{x}+\boldsymbol{w}_\mathrm{I}^\top\Delta\mathbf{x}_\mathrm{I}) \qquad (\because \Delta\mathbf{x} = (\Delta\mathbf{x}_I, 0, 0))\\ &= g^{-1}\left(\boldsymbol{w}^\top\mathbf{x} + \max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} \boldsymbol{w}_\mathrm{I}^\top\Delta\mathbf{x}_\mathrm{I}\right) \qquad (\because g \text{ is strictly increasing})\end{aligned}$$

When $\|\cdot\| = \|\cdot\|_\infty$, the maximum is achieved by letting $\Delta\mathbf{x}_\mathrm{I} = \delta\,\mathrm{sign}(\boldsymbol{w}_\mathrm{I}), \boldsymbol{w}^\top\Delta\mathbf{x}_\mathrm{I} = \delta\,\|\boldsymbol{w}_\mathrm{I}\|_1$. When $\|\cdot\| = \|\cdot\|_2$, the maximum can be achieved by letting $\Delta\mathbf{x}_\mathrm{I} = \delta\boldsymbol{w}_\mathrm{I}/\,\|\boldsymbol{w}_\mathrm{I}\|_2\,, \boldsymbol{w}^\top\Delta\mathbf{x}_\mathrm{I} = \delta\,\|\boldsymbol{w}_\mathrm{I}\|_2$.

B.2.2 ADVERSARIAL TRAINING BASED APPROACH FOR GENERAL SETUP

Here we discuss how we solve the inner maximization problem under a more general setting. Following popular adversarial training methods, we apply projected gradient descent (PGD) for multiple times to update $\Delta\mathbf{x}_\mathrm{I}$, *i.e.*, set

$$\Delta\mathbf{x}_\mathrm{I} = \mathcal{P}(\Delta\mathbf{x}_\mathrm{I} + \gamma\nabla_{\Delta\mathbf{x}_\mathrm{I}} f(\mathbf{x}+\Delta\mathbf{x})), \tag{12}$$

where $\gamma > 0$ is the step size, and $\mathcal{P}$ is the projection onto the constrained space $\|\Delta\mathbf{x}_\mathrm{I}\| \leq \delta$. For instance, $\mathcal{P}$ is equivalent to the clipping process when we use ℓ_∞ norm. Denote the maximizer of the inner maximization problem as $\Delta\mathbf{x}^\star = (\Delta\mathbf{x}_\mathrm{I}{}^\star, \mathbf{0}, \mathbf{0})$. Then, from Danskin's theorem Danskin (1967), we have $\nabla_{\boldsymbol{w}} \max_{\|\Delta\mathbf{x}_\mathrm{I}\|\leq\delta} f_{\boldsymbol{w}}(\mathbf{x}+\Delta\mathbf{x}) = \nabla_{\boldsymbol{w}} f_{\boldsymbol{w}}(\mathbf{x}+\Delta\mathbf{x}^\star)$. We can use this derivative to update $\boldsymbol{w}$ in the outer loss minimization problem. The pseudocode of this adversarial training based method is shown in Algorithm 1.

Algorithm 1 Pseudocode for achieving EI

Input : Dataset $\mathcal{D}$
Output : Model parameter $\boldsymbol{w}$ for the classifier f.
Initialize $\boldsymbol{w}$;
for *each iteration* **do**
  for *each* $(\mathbf{x}_i, \mathrm{y}_i) \in \mathcal{D}$ **do**
    Initialize $\Delta\mathbf{x}_i^\star$;
    for *each PGD iteration* **do**
      Update $\Delta\mathbf{x}_i^\star$ according to (12);
  Update $\boldsymbol{w}$ according to the regularized loss function defined in (1);

B.3 Derivation of Optimal EI classifier for Synthetic Dataset

This section shows how we obtain the optimal EI classifier (that minimizes the cost function in (1)) for a synthetic dataset having two features $\mathbf{x} = [\mathrm{x}_1, \mathrm{x}_2]$ sampled from a Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}_{\mathrm{z,y}}, \boldsymbol{\Sigma}_{\mathrm{z,y}})$ where the mean $\boldsymbol{\mu}_{\mathrm{z,y}}$ and the standard deviation $\boldsymbol{\Sigma}_{\mathrm{z,y}}$ depends on the label $\mathrm{y} \in \{0, 1\}$ and the group attribute $\mathrm{z} \in \{0, 1\}$. Note that the performance curve of the optimal EI classifier obtained in this section is provided in the yellow line in Fig. 3.

The optimal EI classifier is obtained in the following steps: (i) define mathematical notations used for analysis (Sec. B.3.1), (ii) compute the error probability (Sec. B.3.2), (iii) compute EI disparity (Sec. B.3.3), and (iv) solve the EI-regularized optimization problem and find the optimal EI classifier (Sec. B.3.4).

B.3.1 Notations

We consider finding a z-aware linear classifier which predicts the label y from two features $\mathrm{x}_1, \mathrm{x}_2$ and one sensitive attribute z. In other words, given $\mathbf{x}$ and z, the output of a model is represented as $f(\mathbf{x}) = w_1\mathrm{x}_1 + w_2\mathrm{x}_2 + w_3\mathrm{z} + b$ [4] where $[w_1, w_2, w_3]$ is the weight vector, b is the bias. For group $\mathrm{z} = 0$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad w_1\mathrm{x}_1 + w_2\mathrm{x}_2 > -b \\ 0 & \text{else} \end{cases}$$

For group $\mathrm{z} = 1$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad w_1\mathrm{x}_1 + w_2\mathrm{x}_2 > -w_3 - b \\ 0 & \text{else} \end{cases}$$

Without the loss of generality, let $\sqrt{w_1^2 + w_2^2} = 1$, and parameterize them as $w_1 = \sin\theta, w_2 = \cos\theta$. Then, for group $\mathrm{z} = 0$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad (\sin\theta)\mathrm{x}_1 + (\cos\theta)\mathrm{x}_2 > b_0 \\ 0 & \text{else} \end{cases}$$

and for group $\mathrm{z} = 1$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad (\sin\theta)\mathrm{x}_1 + (\cos\theta)\mathrm{x}_2 > b_1 \\ 0 & \text{else} \end{cases}$$

where $b_0 = -b$ and $b_1 = -w_3 - b$. Since the linear combination of multivariate Gaussian is a univariate Gaussian, we have

$$\boldsymbol{w}_{\boldsymbol{\theta}}^\top \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{\mathrm{z,y}}, \boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{\mathrm{z,y}} \boldsymbol{w}_\theta) \tag{13}$$

where $\boldsymbol{w}_{\boldsymbol{\theta}} = [\sin\theta, \cos\theta]$. The decision rules can be written in terms of the $\boldsymbol{w}_\theta$. For group $\mathrm{z} = 0$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad \boldsymbol{w}_{\boldsymbol{\theta}}^\top \boldsymbol{x} > b_0 \\ 0 & \text{else} \end{cases}$$

For group $\mathrm{z} = 1$,

$$\hat{\mathrm{y}} = \begin{cases} 1 & \text{if} \quad \boldsymbol{w}_{\boldsymbol{\theta}}^\top \boldsymbol{x} > b_1 \\ 0 & \text{else} \end{cases}$$

[4] In this case, the decision boundary is $\{\mathbf{x} : f(\mathbf{x}) = 0\}$ instead of $\{\mathbf{x} : f(\mathbf{x}) = 0.5\}$ used in the main paper.

Now, the question is, what is the optimal parameters θ, b_0, b_1 that solve the optimization problem in (1). In order to answer this question, we need to understand how the equal improvability condition is represented in terms of the model parameters. Suppose we use 0-1 loss function l, and use l_∞ norm $\mu(\mathbf{x}) = \|\mathbf{x}\|_\infty$. From the result in Appendix B.2, given the effort budget δ, the maximum score improvement $(\max_{\mu(\Delta\mathbf{x}_I)\leq\delta} f(\mathbf{x}+\Delta\mathbf{x}) - f(\mathbf{x})) = \delta\|\boldsymbol{w}_I\|_1 = \delta(|\sin\theta| + |\cos\theta|)$ where $\boldsymbol{w}_I$ is the weights for improvable features $\mathrm{x}_1, \mathrm{x}_2$. Thus, if we denote the $\hat{\mathrm{y}}^{\max}$ as the estimated label after the improvement, we have

$$\hat{\mathrm{y}}^{\max} = \begin{cases} 1 & \text{if} \quad (\sin\theta)\mathrm{x}_1 + (\cos\theta)\mathrm{x}_2 > b_0 - \delta(|\sin\theta| + |\cos\theta|) = b_0' \\ 0 & \text{else} \end{cases}$$

for group $\mathrm{z} = 0$ and

$$\hat{\mathrm{y}}^{\max} = \begin{cases} 1 & \text{if} \quad (\sin\theta)\mathrm{x}_1 + (\cos\theta)\mathrm{x}_2 > b_1 - \delta(|\sin\theta| + |\cos\theta|) = b_1' \\ 0 & \text{else} \end{cases}$$

for group $\mathrm{z} = 1$.

B.3.2 Compute Error Probability

The error probability can be written as,

$$\Pr(\hat{\mathrm{y}} \neq \mathrm{y}) = \sum_{i=0}^{1} \Pr(\mathrm{z} = i) \Pr(\hat{\mathrm{y}} \neq \mathrm{y} | \mathrm{z} = i)$$

We can derive the term $\Pr(\hat{\mathrm{y}} \neq \mathrm{y}|\mathrm{z} = 0)$ as below:

$$\begin{aligned} \Pr(\hat{\mathrm{y}} \neq \mathrm{y}|\mathrm{z} = 0) = & \Pr(\mathrm{y} = 0|\mathrm{z} = 0) \Pr(\hat{\mathrm{y}} = 1|\mathrm{y} = 0, \mathrm{z} = 0) \\ & + \Pr(\mathrm{y} = 1|\mathrm{z} = 0) \Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) \end{aligned}$$

We can look each term $\Pr(\hat{\mathrm{y}} = 1|\mathrm{y} = 0, \mathrm{z} = 0), \Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0)$ and write those terms in terms of Q-functions because,

$$\begin{aligned} \Pr(\hat{\mathrm{y}} = 1|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} > b_0|\mathrm{y} = 0, \mathrm{z} = 0) \\ \Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 1, \mathrm{z} = 0) \end{aligned}$$

From (13), we have

$$\Pr(\hat{\mathrm{y}} = 1|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} > b_0|\mathrm{y} = 0, \mathrm{z} = 0) = Q\left(\frac{b_0 - \boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0}}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0} \boldsymbol{w}_\theta}}\right)$$

$$\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_{\boldsymbol{\theta}}^\top \mathbf{x} < b_0|\mathrm{y} = 1, \mathrm{z} = 0) = Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0} \boldsymbol{w}_\theta}}\right)$$

One can derive the error rates for group $\mathrm{z} = 1$ similarly. So, the total error rate can be written as

$$\begin{aligned} \Pr(\hat{\mathrm{y}} \neq \mathrm{y}) = & \Pr(\mathrm{z} = 0) \left[\Pr(\mathrm{y} = 0|\mathrm{z} = 0) Q\left(\frac{b_0 - \boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0}}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0} \boldsymbol{w}_\theta}}\right) \right. \\ & \left. + \Pr(\mathrm{y} = 1|\mathrm{z} = 0) Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0} \boldsymbol{w}_\theta}}\right) \right] \\ & + \Pr(\mathrm{z} = 1) \left[\Pr(\mathrm{y} = 0|\mathrm{z} = 1) Q\left(\frac{b_1 - \boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0}}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0} \boldsymbol{w}_\theta}}\right) \right. \\ & \left. + \Pr(\mathrm{y} = 1|\mathrm{z} = 1) Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,1} - b_1}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,1} \boldsymbol{w}_\theta}}\right) \right] \end{aligned}$$

We have three parameters to optimize the error rate θ, b_0, b_1. All the other terms are known.

B.3.3 Compute EI Disparity

To compute EI disparity, we start with computing

$$\Pr(\hat{\mathrm{y}}^{\max} = 1|\hat{\mathrm{y}} = 0, \mathrm{z} = 0) = \frac{\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{z} = 0)}{\Pr(\hat{\mathrm{y}} = 0|\mathrm{z} = 0)} \tag{14}$$

The denominator of (14) can be expanded as

$$\begin{aligned}\Pr(\hat{\mathrm{y}} = 0|\mathrm{z} = 0) = &\Pr(\mathrm{y} = 0|\mathrm{z} = 0)\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0)\\ &+ \Pr(\mathrm{y} = 1|\mathrm{z} = 0)\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0).\end{aligned}$$

We can look each term $\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0), \Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0)$ and write those terms in terms of Q-function because,

$$\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 0, \mathrm{z} = 0)$$
$$\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 1, \mathrm{z} = 0)$$

From (13), we have

$$\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 0, \mathrm{z} = 0) = Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right)$$

$$\Pr(\hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(\boldsymbol{w}_\theta^\top \mathbf{x} > b_0|\mathrm{y} = 1, \mathrm{z} = 0) = Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right)$$

Then,

$$\begin{aligned}\Pr(\hat{\mathrm{y}} = 0|\mathrm{z} = 0) = &\Pr(\mathrm{y} = 0|\mathrm{z} = 0)Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right)\\ &+ \Pr(\mathrm{y} = 1|\mathrm{z} = 0)Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right)\end{aligned}$$

The numerator of (14) can be expanded as

$$\begin{aligned}\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{z} = 0) = &\Pr(\mathrm{y} = 0|\mathrm{z} = 0)\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0)\\ &+ \Pr(\mathrm{y} = 1|\mathrm{z} = 0)\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0).\end{aligned}$$

We can look each term $\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0), \Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0)$ and write those terms in terms of Q-function because,

$$\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(b_0' < \boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 0, \mathrm{z} = 0)$$
$$\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(b_0' < \boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 1, \mathrm{z} = 0)$$

From (13), we have

$$\begin{aligned}\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 0, \mathrm{z} = 0) = \Pr(b_0' < \boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 0, \mathrm{z} = 0) =&\\ Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right) - Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{0,0} - b_0'}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right)&\end{aligned}$$

$$\begin{aligned}\Pr(\hat{\mathrm{y}}^{\max} = 1, \hat{\mathrm{y}} = 0|\mathrm{y} = 1, \mathrm{z} = 0) = \Pr(b_0' < \boldsymbol{w}_\theta^\top \mathbf{x} < b_0|\mathrm{y} = 1, \mathrm{z} = 0) =&\\ Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right) - Q\left(\frac{\boldsymbol{w}_\theta^\top \boldsymbol{\mu}_{1,0} - b_0'}{\sqrt{\boldsymbol{w}_\theta^\top \boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right)&\end{aligned}$$

Then,

$$\begin{aligned}\Pr(\hat{y}^{\max}=1,\hat{y}=0|z=0)=\Pr(y=0|z=0)&\left[Q\left(\frac{\boldsymbol{w}_\theta^\top\boldsymbol{\mu}_{0,0}-b_0}{\sqrt{\boldsymbol{w}_\theta^\top\boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right)-Q\left(\frac{\boldsymbol{w}_\theta^\top\boldsymbol{\mu}_{0,0}-b_0'}{\sqrt{\boldsymbol{w}_\theta^\top\boldsymbol{\Sigma}_{0,0}\boldsymbol{w}_\theta}}\right)\right]\\+\Pr(y=1|z=0)&\left[Q\left(\frac{\boldsymbol{w}_\theta^\top\boldsymbol{\mu}_{1,0}-b_0}{\sqrt{\boldsymbol{w}_\theta^\top\boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right)-Q\left(\frac{\boldsymbol{w}_\theta^\top\boldsymbol{\mu}_{1,0}-b_0'}{\sqrt{\boldsymbol{w}_\theta^\top\boldsymbol{\Sigma}_{1,0}\boldsymbol{w}_\theta}}\right)\right]\end{aligned}$$

It can be derived for group $z=1$ similarly. So, we derived EI disparity in terms of θ, b_0, b_1.

B.3.4 Solve the Optimization Problem

In the previous two sections, we derived the error rate and EI disparity in terms of Q-functions containing parameters θ, b_0, b_1. Therefore, we can construct an EI-constrained optimization problem (which is essentially same as (1)):

$$\begin{aligned}\min_{\theta,b_0,b_1}\quad&\Pr(\hat{y}\neq y)\\ \text{s.t.}\quad&\max_{i\in\{0,1\}}|\Pr(\hat{y}^{\max}=1|\hat{y}=0,z=i)-\Pr(\hat{y}^{\max}=1|\hat{y}=0)|<c\end{aligned}$$

where c is a hyperparameter we can choose to balance error rate and EI disparity.

After writing error rate and EI disparity in terms of Q-functions (using the derivations in Sec. B.3.2 and Sec. B.3.3), we numerically solve the constrained optimization problems above with a popular python module `scipy.optimize`. To get the experimental results in Fig. 3, we numerically solved the above problem for 20 different c values, where the maximum c is picked as the EI disparity of the unconstrained optimization problem.

B.4 Comparison with Individual-Level Equal Recourse

In this part, we compare EI with individual-level equal recourse, which is suggested by Von Kügelgen et al. (2022). Individual-level equal recourse considers a more general setting that allows causal influence between the features. It aims to equalize the cost of recourse (*i.e.*, effort) required for a rejected individual to obtain an improved outcome if the individual is from a different group. Formally, it aims at finding a classifier f satisfying

$$\min_{\mathbf{x}':f(\mathbf{x}')\geq 0.5}\mu_z(\mathbf{x}',\mathbf{x})=\min_{\mathbf{x}':f(\mathbf{x}')\geq 0.5}\mu_{z'}(\mathbf{x}',\mathbf{x}),\quad\text{for all rejected individuals and } z,z'\in[Z],$$

where $\mu_z(\mathbf{x}',\mathbf{x})$ denotes the cost of improving feature from $\mathbf{x}$ to $\mathbf{x}'$ within a causal model when the individual has sensitive attribute $z\in[Z]$. This means that the minimum effort needed to improve the decision outcome is identical irrespective of the sensitive attribute, for all rejected samples.

Individual-level equal recourse shares a similar spirit with EI since both of them are taking care of equalizing the potential to improve the decision outcome for the rejected samples. However, at the same time, introducing individual-level fairness with respect to different groups inherently requires counterfactual fairness, which has its own limitation, as described in Wu et al. (2019).

C Supplementary Materials on Experiments

In this section, we provide additional experimental results and details of experimental setup.

C.1 Synthetic Dataset

We define y, z as $z\sim\text{Bern}(0.4)$, $(y\mid z=0)\sim\text{Bern}(0.3)$, and $(y\mid z=1)\sim\text{Bern}(0.5)$. The feature $\mathbf{x}$ follows the conditional distribution $(\mathbf{x}\mid y=y,z=z)\sim\mathcal{N}(\boldsymbol{\mu}_{y,z},\boldsymbol{\Sigma}_{y,z})$ where the mean of each cluster is

$$\boldsymbol{\mu}_{0,0}=[-0.1,-0.2],\boldsymbol{\mu}_{0,1}=[-0.2,-0.3],\boldsymbol{\mu}_{1,0}=[0.1,0.4],\boldsymbol{\mu}_{1,1}=[0.4,0.3]$$

and the covariance matrix of each cluster is

$$\boldsymbol{\Sigma}_{0,0}=\begin{bmatrix}0.4&0.0\\0.0&0.4\end{bmatrix},\boldsymbol{\Sigma}_{1,0}=\boldsymbol{\Sigma}_{0,1}=\begin{bmatrix}0.2&0.0\\0.0&0.2\end{bmatrix},\boldsymbol{\Sigma}_{1,1}=\begin{bmatrix}0.1&0.0\\0.0&0.1\end{bmatrix}.$$

Table 3: **Comparison of error rate and EI disparities of ERM baseline and proposed methods on the synthetic, German Statlog Credit and ACSIncome-CA datasets on Multi-Layer Perceptron (MLP).** For each dataset, we boldfaced the lowest EI disparity value. Compared with ERM, all three methods proposed in this paper enjoys much lower EI disparity without losing the accuracy much. All reported numbers are evaluated on the test set.

Dataset	Metric	Methods ERM	 Covariance-based	 KDE-based	 Loss-based
Synthetic	Error Rate(↓)	.205 ± .003	.242 ± .006	.227 ± .008	.229 ± .012
	EI Disp.(↓)	.141 ± .036	**.004 ± .002**	.011 ± .006	.018 ± .009
German Stat.	Error Rate(↓)	.221 ± .010	.299 ± .012	.232 ± .018	.238 ± .035
	EI Disp.(↓)	.059 ± .045	**.013 ± .025**	.041 ± .025	**.013 ± .019**
ACSIncome-CA	Error Rate(↓)	.181 ± .002	.202 ± .002	.182 ± .002	.185 ± .001
	EI Disp.(↓)	.042 ± .002	.010 ± .006	.010 ± .002	**.006 ± .003**

C.2 Additional Experimental Results on Algorithm Evaluation

Table 3 shows the performance of ERM baseline and our three approaches (covariance-based, KDE-based, loss-based) introduced in Sec. 3, for a multi-layer perceptron (MLP) network having one hidden layer with four neurons. Similar to the result in Table 2 for the logistic regression model, our methods can reduce the EI disparity without losing the classification accuracy (*i.e.*, increasing the error rate) much.

Meanwhile, according to a previous work (Cherepanova et al., 2021), large deep learning models are observed to overfit to fairness constraints during training and therefore produce unfair predictions on the test data. To confirm whether our method is also having such limitations, we investigate the performance of our algorithms on over-parameterized models. Specifically, we conduct experiments on a five-layer ReLU network with 200 hidden neurons per layer, which is over-parameterized for German dataset. Table 4 reports the performance of EI-constrained classifiers on such over-parameterized setting. One can confirm that our methods (covariance-based, KDE-based, loss-based) perform well in both training and test dataset, and we do not observe the overfitting problem.

Table 4: **Error rate and EI disparities for ERM baseline and proposed methods, for an over-parameterized neural network on German Statlog Credit dataset.** Performances on train/test dataset are presented. Note that we don't observe the overfitting issue in the over-parameterized setting.

Dataset	Metric	Methods ERM	 Covariance-based	 KDE-based	 Loss-based
German Stat.	Train Err.(↓)	.117 ± .004	.133 ± .003	.125 ± .008	.132 ± .011
	Test Err.(↓)	.117 ± .010	.118 ± .010	.121 ± .010	.130 ± .009
	Train EI Disp.(↓)	.022 ± .017	.018 ± .011	.018 ± .009	.015 ± .013
	Test EI Disp.(↓)	.060 ± .032	.049 ± .024	.057 ± .028	.047 ± .025

In addition, in Table 5, we include ER and BE as baselines for the synthetic dataset experiment provided in Table 2. We leverage the algorithm suggested by Gupta et al. (2019) for mitigating ER disparity. We extend the loss-based to reduce BE disparity, by redefining the BE loss of group z as

$$\tilde{L}_z^{\text{BE}} \triangleq \frac{1}{\text{number of samples in group } z} \sum_{i \in I_{-,z}} \ell(1, \max_{\|\Delta \mathbf{x}_{\mathrm{I}i}\| \leq \delta} f(\mathbf{x}_i + \Delta \mathbf{x}_i)),$$

where

$$\tilde{L}_z^{\text{EI}} \triangleq \frac{1}{\text{number of rejected samples in group } z} \sum_{i \in I_{-,z}} \ell(1, \max_{\|\Delta \mathbf{x}_{\mathrm{I}i}\| \leq \delta} f(\mathbf{x}_i + \Delta \mathbf{x}_i)),$$

and $I_{-,z}$ is the set of rejected samples in group z for $z \in [Z]$.

Table 5 shows that the minimum EI disparity is achieved by our methods. Hence, if the EI fairness needs to be achieved, then it cannot be replaced with the existing other fairness notions for some datasets.

Table 5: **Comparison of error rate and EI disparities of ERM, ER, and BE baseline and proposed methods on the synthetic dataset.** We boldfaced the lowest EI disparity value. The three EI-regularized approaches achieve the lowest EI disparity while maintaining low error rates. All reported numbers are evaluated on the test set.

Dataset	Metric	Methods					
		ERM	ER (Gupta et al. (2019))	BE (Loss-based)	Covariance-based	KDE-based	Loss-based
Synthetic	Error Rate(↓)	$.221 \pm .001$	$.235 \pm .009$	$.252 \pm .006$	$.253 \pm .003$	$.250 \pm .001$	$.246 \pm .001$
	EI Disp.(↓)	$.117 \pm .007$	$.036 \pm .018$	$.006 \pm .004$	$.003 \pm .001$	$.005 \pm .003$	$\mathbf{.002 \pm .001}$

C.3 Hyperparameter Selection

The selected hyperparameter for each experiment is provided in our anonymous Github. In all our experiments, we perform cross-validation to select the learning rate from $\{0.0001, 0.001, 0.01, 0.1\}$. In addition, for each penalty term we did two-step cross-validation to choose λ. In the first step, we used a set $\lambda \in \{0, 0.2, 0.4, 0.6, 0.8, 0.9\}$. In the second step, we generate a second set around the best $\lambda^\star$ found in the first step, *i.e.*, the second set is $\{\max\{\lambda^\star + \varepsilon, 0\} : \varepsilon \in \{-0.1, -0.05, 0, 0.05, 0.1\}\}$. For example, if $\lambda^\star = 0.4$ is the best at the first step, then at the second step we use the set $\lambda \in \{0.3, 0.35, 0.4, 0.45, 0.5\}$.

C.4 Obtaining Baseline Classifiers for Dynamic Scenarios

Continued from Sec. 4.2, this section describes how we compute the classifiers that satisfy demographic parity (DP), bounded effort (BE) (Heidari et al., 2019) and equal recourse (ER) (Gupta et al., 2019), respectively. Similar to EI classifier, we obtain the best DP classifier by considering the following constrained optimization problem:

$$\min |\mathbb{P}(\hat{\mathrm{y}}_t = 1 \mid \mathrm{z} = 0) - \mathbb{P}(\hat{\mathrm{y}}_t = 1 \mid \mathrm{z} = 1)| \quad \text{s.t. } \mathbb{P}(\hat{\mathrm{y}}_t \neq \mathrm{y}_t) \leq c.$$

The best BE classifier can be obtained by solving the following problem:

$$\min \left|\mathbb{P}(\tau_t^{(0)} - \delta_t < \mathrm{x}_t < \tau_t^{(0)} \mid \mathrm{z} = 0) - \mathbb{P}(\tau_t^{(1)} - \delta_t < \mathrm{x}_t < \tau_t^{(1)} \mid \mathrm{z} = 1)\right| \quad \text{s.t. } \mathbb{P}(\hat{\mathrm{y}}_t \neq \mathrm{y}_t) \leq c.$$

Similarly, the optimization problem for obtaining the best ER classifier is written as:

$$\min \left|\mathbb{E}\left[\tau_t^{(0)} - \mathrm{x}_t \mid \mathrm{x}_t < \tau_t^{(0)}, \mathrm{z} = 0\right] - \mathbb{E}\left[\tau_t^{(1)} - \mathrm{x}_t \mid \mathrm{x}_t < \tau_t^{(1)}, \mathrm{z} = 1\right]\right| \quad \text{s.t. } \mathbb{P}(\hat{\mathrm{y}}_t \neq \mathrm{y}_t) \leq c.$$

Given the data distribution at each round t, we numerically solve the constrained optimization problems above using a popular python module `scipy.optimize`.

C.5 Additional Experimental Results on Fairness Notions Comparison

In this section, we highlight the advantages of EI over BE and ER in terms of robustness to outliers and imbalanced negative prediction rates. In doing so, we consider certain data distributions and follow the method discussed in Appendix B.3 for solving the classifiers.

C.5.1 EI versus ER: Robustness to Outliers

As we claimed in Table 1, ER is vulnerable to outliers. In this experiment, we systematically study the robustness of EI and ER to outliers.

Data Distributions **(Clean)** Let sensitive attribute $\mathrm{z} \sim \mathrm{Bern}(0.5)$ and label $\mathrm{y} \sim \mathrm{Bern}(0.5)$ be independent of sensitive attribute z. Given the sensitive attribute z and label y, feature x follows the conditional distribution $\mathbf{x} \mid \mathrm{y} = y, \mathrm{z} = z \sim \mathcal{N}(\boldsymbol{\mu}_{y,z}, \boldsymbol{\Sigma}_{y,z})$, where the mean and covariance of the four Gaussian clusters are

$$\boldsymbol{\mu}_{0,0} = [1, -6], \boldsymbol{\mu}_{0,1} = [-1, -2], \boldsymbol{\mu}_{1,0} = [2, 1.5], \boldsymbol{\mu}_{1,1} = [1, 2.5],$$

and

$$\boldsymbol{\Sigma}_{0,0} = \boldsymbol{\Sigma}_{1,0} = \boldsymbol{\Sigma}_{0,1} = \boldsymbol{\Sigma}_{1,1} = \begin{bmatrix} 0.25 & 0.0 \\ 0.0 & 0.25 \end{bmatrix}.$$

(Contaminated) We contaminate the distribution by introducing additional 5% outliers to group $\mathrm{z} = 0$. The outliers follow Gaussian distribution with mean and covariance matrix

$$\boldsymbol{\mu}_{\text{outlier},0} = [-1, -20], \boldsymbol{\Sigma}_{\text{outlier},0} = \begin{bmatrix} 0.05 & 0.0 \\ 0.0 & 0.05 \end{bmatrix}.$$

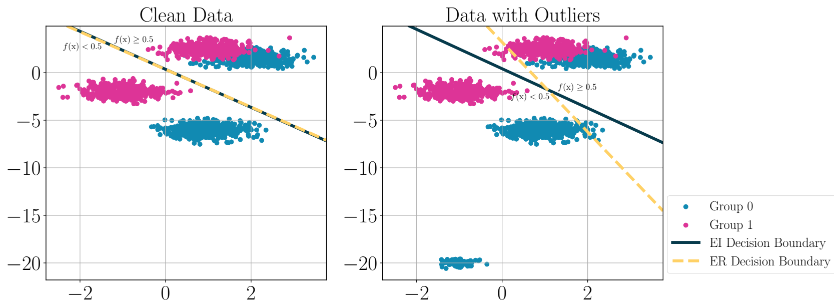

Figure 6: **Visualizations of the EI and ER decision boundaries without and with the presence of outliers.** We observe that the decision boundary of ER changes a lot in the presence of outliers, while the decision boundary of EI is not affected. This phenomenon implies the robustness of EI to outliers.

Results The decision boundaries of EI and ER for both the clean dataset and contaminated dataset are depicted in Fig. 6. These decision boundaries are the optimal linear decision boundaries based on the distributional information, we followed the same procedure as we mentioned in B.3. The δ for the EI classifier is picked as 1.5. We observe that the introduction of outliers makes the ER decision boundary rotate a lot, leading to a drop in classification accuracy and ER disparity *w.r.t.*the clean data distribution. Moreover, we note that the newly added outliers fail to destroy EI classifier, which implies the robustness of EI to outliers.

C.5.2 EI VERSUS BE: ROBUSTNESS TO IMBALANCED NEGATIVE RATE

In this experiment, we investigate the robustness of EI and BE to an imbalanced negative rate.

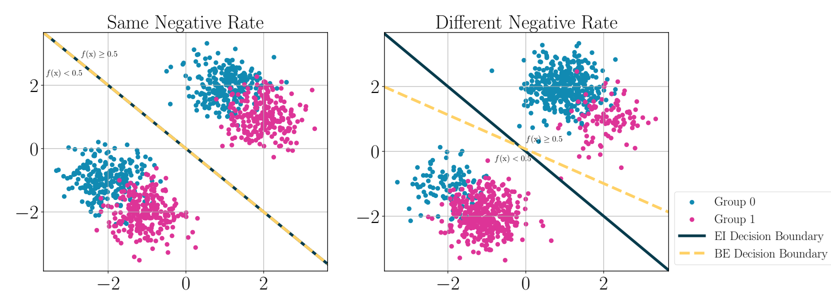

Figure 7: **Visualizations of the EI and BE decision boundaries given the data distribution with the same negative rates and different negative rates.** The decision boundary of BE rotates a lot when the negative rate of the dataset becomes different, implying the sensitivity of BE to imbalanced negative rates. In contrast, the consistency of EI decision boundaries showcases the robustness of EI *w.r.t.*imbalanced negative rates.

Data Distributions **(Same Negative Rate)** We consider the distribution with balanced subgroups. In other words, let sensitive attribute $\mathrm{z} \sim \mathrm{Bern}(0.5)$, and label $\mathrm{y} \sim \mathrm{Bern}(0.5)$ be independent of sensitive attribute z. Given the sensitive attribute z and label y, feature, feature $\mathbf{x}$ follows the Gaussian distribution $(\mathbf{x} \mid \mathrm{y} = y, \mathrm{z} = z) \sim \mathcal{N}(\boldsymbol{\mu}_{y,z}, \boldsymbol{\Sigma}_{y,z})$ where the mean of each cluster is

$$\boldsymbol{\mu}_{0,0} = [-2, -1], \boldsymbol{\mu}_{0,1} = [-1, -2], \boldsymbol{\mu}_{1,0} = [1, 2], \boldsymbol{\mu}_{1,1} = [2, 1]$$

and the covariance matrix of each cluster is

$$\mathbf{\Sigma}_{0,0} = \mathbf{\Sigma}_{1,0} = \mathbf{\Sigma}_{0,1} = \mathbf{\Sigma}_{1,1} = \begin{bmatrix} 0.25 & 0.0 \\ 0.0 & 0.25 \end{bmatrix}.$$

(Different Negative Rate) We manipulate the distribution of label y for constructing data distribution with different negative rates. To be more specific, we let $\mathrm{y} \mid \mathrm{z} = 0 \sim \mathrm{Bern}(0.7)$, and $\mathrm{y} \mid \mathrm{z} = 1 \sim \mathrm{Bern}(0.3)$.

Results Figure 7 shows the decision boundaries of EI and BE when (i) the dataset has the same negative rate, and (ii) the dataset has a different negative rate. These decision boundaries are the optimal linear decision boundaries based on the distributional information, we followed the same procedure as we mentioned in B.3. The δ for the EI and BE classifiers is picked as 1.5. The huge difference in BE decision boundaries under the two cases verifies our claim that BE cannot handle imbalanced negative prediction rates. In contrast, the decision boundaries of EI learned with two datasets with different group proportions are consistent.

D The effect of fairness notions on the long-term fairness

D.1 Basic setup and Preliminaries

D.1.1 Classifiers

We consider the $z-$aware classifier, having

$$f(x) = \begin{cases} \mathbf{1}_{x \geq \tau_0}, & z = 0, \\ \mathbf{1}_{x \geq \tau_1}, & z = 1, \end{cases}$$

which is parameterized by the threshold pair (τ_0, τ_1). We assume the effort budget is $\delta = \frac{m}{2}$ where m is defined in the dataset.

Recall the zero equal improvability (EI) disparity condition is:

$$\mathbb{P}\left(\max_{\mu(\Delta x) \leq \delta} f(x + \Delta x) \geq 0.5 \mid f(x) < 0.5, z = 0\right) = \mathbb{P}\left(\max_{\mu(\Delta x) \leq \delta} f(x + \Delta x) \geq 0.5 \mid f(x) < 0.5, z = 1\right)$$

where $\mu(\Delta x) = |\Delta x|$. This condition is equivalent to

$$\frac{\int_{\tau_0 - \delta}^{\tau_0} p_0(x) dx}{\int_{-\infty}^{\tau_0} p_0(x) dx} = \frac{\int_{\tau_1 - \delta}^{\tau_1} p_1(x) dx}{\int_{-\infty}^{\tau_1} p_1(x) dx}. \tag{15}$$

We denote the improvability ratio of each group as

$$r_0(\tau_0) = \frac{\int_{\tau_0 - \delta}^{\tau_0} p_0(x) dx}{\int_{-\infty}^{\tau_0} p_0(x) dx}, \tag{16}$$

$$r_1(\tau_1) = \frac{\int_{\tau_1 - \delta}^{\tau_1} p_1(x) dx}{\int_{-\infty}^{\tau_1} p_1(x) dx} \tag{17}$$

The classifier that satisfies this zero EI disparity condition and minimizes the error rate is denoted as the optimal EI classifier.

Recall that the bounded effort (BE) fairness constraint is:

$$\mathbb{P}\left(\max_{\mu(\Delta x) \leq \delta} f(x + \Delta x) \geq 0.5, f(x) < 0.5 \mid z = 0\right) = \mathbb{P}\left(\max_{\mu(\Delta x) \leq \delta} f(x + \Delta x) \geq 0.5, f(x) < 0.5 \mid z = 1\right) \tag{18}$$

where $\mu(\Delta x) = |\Delta x|$. Meanwhile, the Equal Recourse (ER) constraint is defined as

$$\mathbb{E}\left[\min_{f(x + \Delta x) \geq 0.5} \mu(\Delta x) \mid f(x) < 0.5, z = 0\right] = \mathbb{E}\left[\min_{f(x + \Delta x) \geq 0.5} \mu(\Delta x) \mid f(x) < 0.5, z = 1\right] \tag{19}$$

D.1.2 DYNAMIC SCENARIO

Suppose each rejected sample improves its feature by

$$\varepsilon(x) = \begin{cases} \delta \cdot \mathbf{1}_{x \in [\tau_0 - \delta, \tau_0)}, & \text{if } z = 0, \\ \delta \cdot \mathbf{1}_{x \in [\tau_1 - \delta, \tau_1)}, & \text{if } z = 1 \end{cases}$$

Note that depending on the classifier we are using, the threshold pair (τ_0, τ_1) changes, thus the formulation for $\varepsilon(x)$ also changes. Here, we use $\varepsilon^{\text{ERM}}(x)$ to denote the improvement of features given ERM classifier, and similarly define $\varepsilon^{\text{EI}}(x), \varepsilon^{\text{BE}}(x)$ and $\varepsilon^{\text{ER}}(x)$.

Let $p_z^{\text{ERM}}(x) = p_z(x + \varepsilon^{\text{ERM}}(x))$ for $z \in \{0, 1\}$, which represent the data distribution after the features are improved based on ERM classifier. Similarly, we define $p_z^{\text{EI}}(x)$, $p_z^{\text{BE}}(x)$ and $p_z^{\text{ER}}(x)$ for EI/BE/ER classifiers, respectively. In the upcoming sections, we measure the total-variance (TV) distance

$$d_{TV}(p_0, p_1) = \frac{1}{2} \int_{\mathbb{R}} |p_0(x) - p_1(x)| dx$$

between two groups after a single step of feature improvement, and provide how this measurement differs for various classifiers.

D.2 EI VERSUS BE & ERM

D.2.1 FINDING THE OPTIMAL CLASSIFIER FOR EACH FAIRNESS CRITERION

Setup Let $p_z(x)$ be the data distribution of each group $z \in \{0, 1\}$, shown in Fig. 8. We consider the case of each sample having one feature x, and the label is assigned as

$$y = \begin{cases} \mathbf{1}_{x \geq m/2}, & \text{if } z = 0 \\ \mathbf{1}_{x \geq 0}, & \text{if } z = 1 \end{cases}$$

Note that we have $\mathbb{P}(y = 0 | z = 0) = \frac{3}{4}$, $\mathbb{P}(y = 1 | z = 0) = \frac{1}{4}$, $\mathbb{P}(y = 0 | z = 1) = \frac{1}{2}$, and $\mathbb{P}(y = 1 | z = 1) = \frac{1}{2}$. We set $\mathbb{P}(z = 0) = \frac{1}{4}$ and $\mathbb{P}(z = 1) = \frac{3}{4}$.

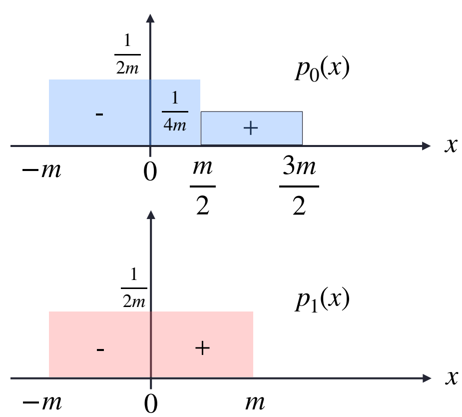

Figure 8: The data distribution $p_z(x)$ for each group $z \in \{0, 1\}$. Samples with $(+)$ sign have the true label $y = 1$, while samples with $(-)$ sign have the true label $y = 0$.

ERM classifier We have

$$(\tau_0^{\text{ERM}}, \tau_1^{\text{ERM}}) = \left(\frac{m}{2}, 0\right)$$

since this threshold pair has zero classification error.

BE classifier We have

$$(\tau_0^{\text{BE}}, \tau_1^{\text{BE}}) = \left(\frac{m}{2}, 0\right) \tag{20}$$

since this threshold pair has zero classification error and satisfy the BE condition in (18).

EI classifier The optimal EI classifier for dataset in Fig. 8 is

$$(\tau_0^{\text{EI}}, \tau_1^{\text{EI}}) = (0, 0). \tag{21}$$

Proof. Note that $(\tau_0^{\text{EI}}, \tau_1^{\text{EI}}) = (0, 0)$ has error rate $\mathbb{P}(\text{error}) = \mathbb{P}(z = 0)\mathbb{P}(\text{error}|z = 0) + \mathbb{P}(z = 1)\mathbb{P}(\text{error}|z = 1) = \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot 0 = \frac{1}{16}$. We prove that no other classifier satisfing EI condition in (15) is having error rate less than $\frac{1}{16}$. Note that when $|\tau_1| > \frac{m}{6}$, the error rate is larger than $\frac{1}{16}$. Thus it is sufficient to consider cases when $|\tau_1| \le \frac{m}{6}$. Combining this with the fact that

- $r_0(\tau_0)$ in (16) and $r_1(\tau_1)$ in (17) are monotonically decreasing,
- EI condition in (15) is satisfied when $r_0(\tau_0) = r_1(\tau_1)$ holds,
- $r_0(\tau_0) = r_1(\tau_1)$ holds for $\tau_0 = \tau_1 \le \frac{m}{2}$,

we can see that the optimal EI classifier satisfies $\tau_0 = \tau_1$ and $|\tau_1| \le \frac{m}{6}$. The error rate for these classifiers is represented as $\mathbb{P}(\text{error}) = \mathbb{P}(z = 0)\mathbb{P}(\text{error}|z = 0) + \mathbb{P}(z = 1)\mathbb{P}(\text{error}|z = 1) = \frac{1}{4} \cdot (\frac{m}{2} - \tau_0) \cdot \frac{1}{2m} + \frac{3}{4} \cdot |\tau_1| \cdot \frac{1}{2m}$. Plugging in $\tau_0 = \tau_1$ and optimizing the error probability over $|\tau_1| \le \frac{m}{6}$ completes the proof. □

D.2.2 TOTAL-VARIATION DISTANCE BETWEEN TWO GROUPS

For the dataset given in Fig. 8, the total-variation distance between two groups for each classifier is:

$$d_{TV}(p_0^{\text{ERM}}, p_1^{\text{ERM}}) = 0.5,$$
$$d_{TV}(p_0^{\text{BE}}, p_1^{\text{BE}}) = 0.5,$$
$$d_{TV}(p_0^{\text{EI}}, p_1^{\text{EI}}) = 0.125.$$

Proof. Since ERM solution is identical to BE solution, proving the above equation for BE and EI is sufficient. Recall that the expression of BE/EI classifiers are in (20) and (21). Using this expression, we can derive the distribution of each group:

$$p_0^{\text{BE}}(x) = \begin{cases} \frac{3}{4m}, & \text{if } x \in [\frac{m}{2}, m] \\ \frac{1}{2m}, & \text{if } x \in [-m, 0] \\ \frac{1}{4m}, & \text{if } x \in [m, \frac{3m}{2}] \\ 0, & \text{if } x \in [0, \frac{m}{2}] \text{ or } x \notin [-m, \frac{3m}{2}] \end{cases},$$

$$p_1^{\text{BE}}(x) = \begin{cases} \frac{1}{m}, & \text{if } x \in [0, \frac{m}{2}] \\ \frac{1}{2m}, & \text{if } x \in [-m, -\frac{m}{2}] \cup [\frac{m}{2}, m] \\ 0, & \text{if } x \in [-\frac{m}{2}, 0] \text{ or } x \notin [-m, m] \end{cases},$$

$$p_0^{\text{EI}}(x) = \begin{cases} \frac{1}{m}, & \text{if } x \in [0, \frac{m}{2}] \\ \frac{1}{2m}, & \text{if } x \in [-m, -\frac{m}{2}] \\ \frac{1}{4m}, & \text{if } x \in [\frac{m}{2}, \frac{3m}{2}] \\ 0, & \text{if } x \in [-\frac{m}{2}, 0] \text{ or } x \notin [-m, \frac{3m}{2}] \end{cases},$$

$$p_1^{\text{EI}}(x) = p_1^{\text{BE}}(x) \quad \forall x$$

From this expression, we can derive the total-variation distance, which completes the proof. □

D.3 EI VERSUS ER

D.3.1 FINDING THE OPTIMAL CLASSIFIER FOR EACH FAIRNESS CRITERION

Setup Let $p_z(x)$ be the data distribution of each group $z \in \{0, 1\}$, show in Fig. 9. We consider the case of each sample having one feature x, and the label is assigned as

$$y = \begin{cases} \mathbf{1}_{x \ge 0}, & \text{if } z = 0 \\ \mathbf{1}_{x \ge 0}, & \text{if } z = 1 \end{cases}.$$

Note that we have $\mathbb{P}(y = 0 \mid z = 0) = \mathbb{P}(y = 0 \mid z = 1) = \mathbb{P}(y = 1 \mid z = 0) = \mathbb{P}(y = 1 \mid z = 1) = \frac{1}{2}$. We set $\mathbb{P}(z = 0) = \mathbb{P}(z = 1) = \frac{1}{2}$.

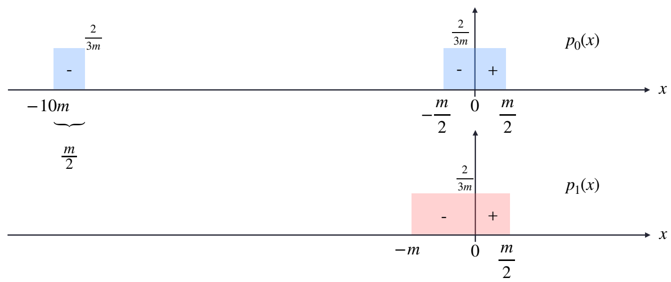

Figure 9: The data distribution $p_z(x)$ for each group $z \in \{0, 1\}$. Samples with $(+)$ sign have the true label $y = 1$, while samples with $(-)$ sign have the true label $y = 0$.

ER classifier The optimal ER classifier for the dataset in Fig. 9 is

$$(\tau_0^{\mathrm{ER}}, \tau_1^{\mathrm{ER}}) = (-9m, 0) . \tag{22}$$

Proof. Note that the accepting thresholds $(-9m, 0)$ has error rate $\mathbb{P}(\text{error}) = \mathbb{P}(\text{error} \mid z = 0)\mathbb{P}(z = 0) + \mathbb{P}(\text{error} \mid z = 1)\mathbb{P}(z = 1) = \frac{1}{3} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{6}$. We prove that no other classifier satisfying ER constraint (19) is having error rate less than $\frac{1}{6}$.

The necessary condition for the classifier to have an error rate less than $\frac{1}{6}$ is $\tau_0, \tau_1 \in (-\frac{m}{2}, \frac{m}{2})$. Note that when $\tau_0 \in (-\frac{m}{2}, \frac{m}{2})$, we have the recourse of the group 0:

$$\begin{aligned}
&\mathbb{E}\left[\min_{f(x+\Delta x)\geq 0.5} \mu(\Delta x) \mid f(x) < 0.5, z = 0\right] \\
&= \mathbb{E}\left[\tau_0 - x \mid x < \tau_0, z = 0\right] \\
&= \tau_0 - \underbrace{\mathbb{E}\left[x \mid x < \tau_0, z = 0\right]}_{\varphi(\tau_0)},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\varphi(\tau_0) &= \mathbb{E}\left[x \mid x < \tau_0, z = 0\right] \qquad (24) \\
&= \underbrace{\mathbb{P}\left(x \in \left[-10m, -\frac{19}{2}m\right] \mid x < \tau_0, z = 0\right)}_{\triangleq \lambda} \mathbb{E}\left[x \mid x \in \left[-10m, -\frac{19}{2}m\right], z = 0\right] \\
&\quad + \underbrace{\mathbb{P}\left(x \in \left[-\frac{m}{2}, \tau_0\right] \mid x < \tau_0, z = 0\right)}_{1-\lambda} \mathbb{E}\left[x \mid x \in \left[-\frac{m}{2}, \tau_0\right], z = 0\right],
\end{aligned}$$

for all $\tau_0 \in [-\frac{m}{2}, \frac{m}{2}]$, where $\lambda \in [\frac{1}{3}, 1]$. Since $\mathbb{E}\left[x \mid x \in \left[-10m, -\frac{19}{2}m\right], z = 0\right] < \mathbb{E}\left[x \mid x \in \left[-\frac{m}{2}, \tau_0\right], z = 0\right]$, (24) is a decreasing function. Consequently,

$$\begin{aligned}
&\leq \frac{1}{3}\mathbb{E}\left[x \mid x \in \left[-10m, -\frac{19}{2}m\right], z = 0\right] + \frac{2}{3}\mathbb{E}\left[x \mid x \in \left[-\frac{m}{2}, \tau_0\right], z = 0\right] \\
&= -\frac{39}{12}m + \frac{\tau_0}{3} - \frac{m}{6} = \frac{\tau_0}{3} - \frac{41}{12}m.
\end{aligned}$$

Thus, the recourse of group 0 is

$$(23) \geq \frac{2\tau_0}{3} + \frac{41}{12}m \geq \frac{37}{12}m \tag{25}$$

Meanwhile, when $\tau_1 \in (-\frac{m}{2}, \frac{m}{2})$, we have the recourse of the group 1:

$$
\begin{aligned}
&\mathbb{E}\left[\min_{f(x+\Delta x)\geq 0.5} \mu(\Delta x) \mid f(x) < 0.5, z = 1\right] \\
&= \mathbb{E}\left[\tau_1 - x \mid x < \tau_1, z = 0\right] \\
&= \tau_1 - \mathbb{E}\left[x \mid x < \tau_1, z = 0\right] \\
&= \tau_1 - \frac{\tau_1 - m}{2} \\
&= \frac{\tau_1 + m}{2} \in \left(\frac{m}{4}, \frac{3}{4}m\right),
\end{aligned} \tag{26}
$$

Therefore, combining (25) and (26) implies that when $\tau_0, \tau_1 \in (-\frac{m}{2}, \frac{m}{2})$, the ER constraint (19) cannot be satisfied. Thus, the way for achieving error rate $< \frac{1}{6}$ is $\tau_0 < -\frac{m}{2}$ and $\tau_1 = 0$. Note that the recourse of group 0 which is written in (23) is a strictly increasing function when $\tau_0 < -\frac{m}{2}$. Therefore, there exists a unique classifier that achieves both EI while maintaining error rate $< \frac{1}{6}$. One can easily verify that $(-9m, 0)$ is the optimal ER classifier. □

EI classifier We have

$$
(\tau_0^{\mathrm{EI}}, \tau_1^{\mathrm{EI}}) = (0, 0) \tag{27}
$$

since this threshold pair has zero classification error and satisfies the EI constraint (15).

D.3.2 Total-variation distance between two groups

For the dataset given in Fig. 9, the total variation distance between two groups for EI and ER classifier is

$$
\begin{aligned}
d_{TV}(p_0^{\mathrm{ER}}, p_1^{\mathrm{ER}}) &= \frac{2}{3}, \\
d_{TV}(p_0^{\mathrm{EI}}, p_1^{\mathrm{EI}}) &= \frac{1}{3}.
\end{aligned}
$$

Proof. By (27) and (22),

$$
p_0^{\mathrm{ER}}(x) = \begin{cases} \frac{2}{3m}, & \text{if } x \in [-\frac{19m}{2}, -8m] \\ \frac{2}{3m}, & \text{if } x \in [-\frac{m}{2}, \frac{m}{2}] \\ 0, & \text{o.w.} \end{cases},
$$

$$
p_1^{\mathrm{ER}}(x) = \begin{cases} \frac{2}{3m}, & \text{if } x \in [-m, -\frac{m}{2}] \\ \frac{4}{3m}, & \text{if } x \in [0, \frac{m}{2}] \\ 0, & \text{o.w.} \end{cases},
$$

$$
p_0^{\mathrm{EI}}(x) = \begin{cases} \frac{2}{3m}, & \text{if } x \in [-10m, -\frac{19m}{2}] \\ \frac{4}{3m}, & \text{if } x \in [0, \frac{m}{2}] \\ 0, & \text{o.w.} \end{cases},
$$

$$
p_1^{\mathrm{EI}}(x) = \begin{cases} \frac{2}{3m}, & \text{if } x \in [-m, -\frac{m}{2}] \\ \frac{4}{3m}, & \text{if } x \in [0, \frac{m}{2}] \\ 0, & \text{o.w.} \end{cases}.
$$

For this expression, we can derive the total-variation distance, which completes the proof. □