Supplement to: Optimal community detection in dense bipartite graphs

Contents

A Proofs for lower bound

A.1 Proof of Theorem 1

In this section, we prove the lower bound on the minimax rate of separation δ^*. Our general strategy is to lower bound the minimax risk of testing with the Bayes risk defined with respect to a well-chosen prior over the parameter space. We then invoke the Neyman-Pearson lemma and carefully control the second moment of the resulting likelihood ratio statistic.

Let $\eta \in (0,1)$ be given. Recall that the minimax risk is defined as

$$\begin{aligned}\mathcal{R}^*(k_1,k_2,n_1,n_2,\delta) &= \inf_{\Delta} \mathcal{R}(\Delta,\delta)\\ &= \inf_{\Delta}\Big\{\mathbb{P}_0(\Delta=1) + \sup_{\mathbf{P}\in\Theta(k_1,k_2,n_1,n_2,\mu)} \mathbb{P}_{\mathbf{P}}(\Delta=0)\Big\}.\end{aligned}$$

Define the reduced parameter space $\overline{\Theta}(k_1, k_2, n_1, n_2, \delta)$ as

$$\overline{\Theta}(k_1, k_2, n_1, n_2, \delta) = \{\mathbf{P} \in \Theta(k_1, k_2, n_1, n_2, \delta) : P_{ij} \neq 0 \implies P_{ij} = p_0 + \delta\}.$$

We let π denote the uniform distribution over $\overline{\Theta}(k_1, k_2, n_1, n_2, \delta)$, meaning that for any $\mathbf{M} \in \overline{\Theta}(k_1, k_2, n_1, n_2, \delta)$ it holds $\mathbb{P}_{\mathbf{P}\sim\pi}(\mathbf{P} = \mathbf{M}) = \frac{1}{\binom{n_1}{k_1}\binom{n_2}{k_2}}$. Using π as our prior on $\overline{\Theta}(k_1, k_2, n_1, n_2, \delta)$, we define the mixture distribution $\mathbb{P}_\pi$ on $\mathbb{R}^{n_1\times n_2}$ as

$$\mathbb{P}_\pi(B) = \int_{\mathbf{P}\in\overline{\Theta}(k_1,k_2,n_1,n_2,\delta)} \mathbb{P}_{\mathbf{P}}(B)\pi(\mathrm{d}\mathbf{P}),$$

where B is a measurable set. With this infrastructure in hand, we can lower bound the minimax risk as follows:

$$\begin{aligned}
\mathcal{R}^*(k_1, k_2, n_1, n_2, \delta) &= \inf_\Delta \Big\{\mathbb{P}_0(\Delta = 1) + \sup_{\mathbf{P}\in\Theta(k_1,k_2,n_1,n_2,\delta)} \mathbb{P}_{\mathbf{P}}(\Delta = 0)\Big\} \\
&\geq \inf_\Delta \Big\{\mathbb{P}_0(\Delta = 1) + \sup_{\mathbf{P}\in\overline{\Theta}(k_1,k_2,n_1,n_2,\mu)} \mathbb{P}_{\mathbf{P}}(\Delta = 0)\Big\} \\
&\geq \inf_\Delta \Big\{\mathbb{P}_0(\Delta = 1) + \mathbb{P}_\pi(\Delta = 0)\Big\}.
\end{aligned}$$

The final expression above is the minimax risk of a simple versus simple hypothesis testing problem, and we can characterize it precisely using the Neyman-Pearson lemma ([3], Lemma 4.3). Combining this result with standard equivalent formulations of the total variation distance ([3], Proposition 4.4), we have

$$\begin{aligned}
\inf_\Delta \Big\{\mathbb{P}_0(\Delta = 1) + \mathbb{P}_\pi(\Delta = 0)\Big\} &= 1 - \mathbf{TV}(\mathbb{P}_0, \mathbb{P}_\pi) \\
&= 1 - \frac{1}{2}\int_{\mathbf{A}\in\mathbb{R}^{n_1\times n_2}} |\mathrm{d}\mathbb{P}_0(\mathbf{A}) - \mathrm{d}\mathbb{P}_\pi(\mathbf{A})| \\
&= 1 - \frac{1}{2}\int_{\mathbf{A}\in\mathbb{R}^{n_1\times n_2}} \Big|\frac{\mathrm{d}\mathbb{P}_\pi(\mathbf{A})}{\mathrm{d}\mathbb{P}_0(\mathbf{A})} - 1\Big|\mathrm{d}\mathbb{P}_0(\mathbf{A}),
\end{aligned}$$

where $\mathrm{d}\mathbb{P}_0$ and $\mathrm{d}\mathbb{P}_\pi$ denote the Radon-Nikodym derivatives with respect to the counting measure of $\mathbb{P}_0$ and $\mathbb{P}_\pi$ respectively. Letting $L_\pi = \frac{\mathrm{d}\mathbb{P}_\pi}{\mathrm{d}\mathbb{P}_0}$ denote the likelihood ratio, we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\int_{\mathbf{A}\in\mathbb{R}^{n_1\times n_2}} \Big|\frac{\mathrm{d}\mathbb{P}_\pi(\mathbf{A})}{\mathrm{d}\mathbb{P}_0(\mathbf{A})} - 1\Big|\mathrm{d}\mathbb{P}_0(\mathbf{A}) &= \int_{\mathbf{A}\in\mathbb{R}^{n_1\times n_2}} \Big|L_\pi(\mathbf{A}) - 1\Big|\mathrm{d}\mathbb{P}_0(\mathbf{A}) \\
&\leq \sqrt{\int_{\mathbf{A}\in\mathbb{R}^{n_1\times n_2}} \Big(L_\pi(\mathbf{A}) - 1\Big)^2 \mathrm{d}\mathbb{P}_0(\mathbf{A})} \\
&= \sqrt{\mathbb{E}_0\big[L_\pi^2(\mathbf{A})\big] - 1},
\end{aligned}$$

where we arrive at the final expression by expanding the square and using $\mathbb{E}_0[L_\pi(\mathbf{A})] = 1$. This chain of calculations reveals

$$\mathcal{R}^*(k_1, k_2, n_1, n_2) \geq 1 - \frac{1}{2}\sqrt{\mathbb{E}_0\big[L_\pi^2(\mathbf{A})\big] - 1}.$$

Therefore, to prove that $\mathcal{R}*(k_1, k_2, n_1, n_2) \geq \eta$, it suffices to show

$$\begin{aligned}
\mathbb{E}_0\big[L_\pi^2\big] &\leq 1 + 4(1-\eta)^2 \\
&= 1 + \varepsilon,
\end{aligned}$$

where we define $\varepsilon := 4(1-\eta)^2$. By direct calculation, we have

$$\begin{aligned}
\mathbb{E}_0\big[L_\pi^2\big] &= \int_{\mathbf{A}} \frac{\big(\mathrm{d}\mathbf{P}_\pi(\mathbf{A})\big)^2}{\mathrm{d}\mathbb{P}_0(\mathbf{A})} \\
&= \int_{\mathbf{A}} \frac{\int \mathrm{d}\mathbb{P}_{\mathbf{P}}(\mathbf{A})\pi(\mathrm{d}\mathbf{P}) \int \mathrm{d}\mathbb{P}_{\mathbf{P}'}(\mathbf{A})\pi(\mathrm{d}\mathbf{P}')}{\mathrm{d}\mathbb{P}_0(\mathbf{A})}
\end{aligned}$$

$$= \int_{\mathbf{P}} \int_{\mathbf{P}'} \Big[\int_{\mathbf{A}} \frac{\mathrm{d}\mathbb{P}_{\mathbf{P}}(\mathbf{A}) \mathrm{d}\mathbb{P}_{\mathbf{P}'}(\mathbf{A})}{\mathrm{d}\mathbb{P}_0(\mathbf{A})} \Big] \pi(\mathrm{d}\mathbf{P})\pi(\mathrm{d}\mathbf{P}'),$$

where $\mathbf{P}$ and $\mathbf{P}'$ are two mean matrices drawn independently from π. Recalling that the entries of $\mathbf{A}$ are Bernoulli random variables, we have

$$\mathrm{d}\mathbb{P}_{\mathbf{P}}(\mathbf{A}) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} P_{ij}^{A_{ij}} (1 - P_{ij})^{1 - A_{ij}}.$$

Using this, as well as the definition of our prior π, we can compute the above triple integral as follows. Here, we let $\mathcal{A} = \{0,1\}^{n_1 \times n_2}$ denote the set of all possible values of the adjacency matrix $\mathbf{A}$.

$$\begin{aligned}
\mathbb{E}_0\big[L_\pi^2\big] &= \int_{\mathbf{P}} \int_{\mathbf{P}'} \Big[\int_{\mathbf{A}} \frac{\mathrm{d}\mathbb{P}_{\mathbf{P}}(\mathbf{A}) \mathrm{d}\mathbb{P}_{\mathbf{P}'}(\mathbf{A})}{\mathrm{d}\mathbb{P}_0(\mathbf{A})} \Big] \pi(\mathrm{d}\mathbf{P})\pi(\mathrm{d}\mathbf{P}') \\
&= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \sum_{\mathbf{A} \in \mathcal{A}} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \frac{P_{ij}^{A_{ij}} (1 - P_{ij})^{1 - A_{ij}} {P'_{ij}}^{A_{ij}} (1 - P'_{ij})^{1 - A_{ij}}}{p_0^{A_{ij}} (1 - p_0)^{1 - A_{ij}}} \\
&= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \sum_{\mathbf{A} \in \mathcal{A} : A_{ij} = 0}^{1} \left(\frac{P_{ij}^{A_{ij}} (1 - P_{ij})^{1 - A_{ij}} {P'_{ij}}^{A_{ij}} (1 - P'_{ij})^{1 - A_{ij}}}{p_0^{A_{ij}} (1 - p_0)^{1 - A_{ij}}} \right) \\
&= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \left(\frac{P_{ij} \cdot P'_{ij}}{p_0} + \frac{(1 - P_{ij}) \cdot (1 - P'_{ij})}{1 - p_0} \right).
\end{aligned}$$

Now, fix two mean matrices $\mathbf{P}$ and $\mathbf{P}'$ with elevated entries on $K_1 \times K_2$ and $K_1' \times K_2'$ respectively. Letting $(i, j) \in [n_1] \times [n_2]$, suppose that $(i, j) \notin (K_1 \cap K_1') \times (K_2 \cap K_2')$. Without loss of generality, say $(i, j) \notin K_1 \times K_2$. Then it holds that $P_{ij} = p_0$, and we have

$$\frac{P_{ij} \cdot P'_{ij}}{p_0} + \frac{(1 - P_{ij}) \cdot (1 - P'_{ij})}{1 - p_0} = P'_{ij} + (1 - P'_{ij}) = 1.$$

On the other hand, if $(i, j) \in (K_1 \cap K_1') \times (K_2 \cap K_2')$, it holds

$$\begin{aligned}
\frac{P_{ij} \cdot P'_{ij}}{p_0} + \frac{(1 - P_{ij}) \cdot (1 - P'_{ij})}{1 - p_0} &= \frac{(p_0 + \delta)^2}{p_0} + \frac{(1 - p_0 - \delta)^2}{1 - p_0} \\
&= p_0 - 2\delta + \frac{\delta^2}{p_0} + 1 - p_0 - 2\delta + \frac{\delta^2}{1 - p_0} \\
&= 1 + \frac{\delta^2}{p_0(1 - p_0)}.
\end{aligned}$$

With these calculations in hand, we have

$$\begin{aligned}
\mathbb{E}_0\big[L_\pi^2\big] &= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \left(\frac{P_{ij} \cdot P'_{ij}}{p_0} + \frac{(1 - P_{ij}) \cdot (1 - P'_{ij})}{1 - p_0} \right) \\
&= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \left(1 + \frac{\delta^2}{p_0(1 - p_0)} \mathbf{1}\{(i, j) \in (K_1 \cap K_1') \times (K_2 \cap K_2')\} \right) \\
&\leq \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} \exp\left(\frac{\delta^2}{p_0(1 - p_0)} \mathbf{1}\{(i, j) \in (K_1 \cap K_1') \times (K_2 \cap K_2')\} \right) \\
&= \frac{1}{\binom{n_1}{k_1}^2 \binom{n_2}{k_2}^2} \sum_{\mathbf{P}} \sum_{\mathbf{P}'} \exp\left(\frac{\delta^2}{p_0(1 - p_0)} |K_1 \cap K_1'||K_2 \cap K_2'| \right) \\
&= \mathbb{E}_{\substack{U \sim \text{HypGeom}(n_1, k_1, k_1) \\ V \sim \text{HypGeom}(n_2, k_2, k_2)}} \left[\exp\left(\frac{\delta^2}{p_0(1 - p_0)} UV \right) \right] \\
&\leq \mathbb{E}_{\substack{X \sim \text{Bin}(k_1, \frac{k_1}{n_1 - k_1}) \\ Y \sim \text{Bin}(k_2, \frac{k_2}{n_2 - k_2})}} \left[\exp\left(\frac{\delta^2}{p_0(1 - p_0)} XY \right) \right],
\end{aligned}$$

where the final inequality follows from Lemma 3 of [1].Therefore, it suffices to show that

$$\mathbb{E}[\exp(\mu^2 XY)] \leq 1+\varepsilon, \tag{1}$$

where $X \sim \text{Bin}(k_1, \frac{k_1}{n_1-k_1}), Y \sim \text{Bin}(k_2, \frac{k_2}{n_2-k_2})$, and $\mu^2 = \frac{\delta^2}{p_0(1-p))}$. The remainder of the proof is structured as follows.

1. In the *general sparsity* setting, meaning that $k_1 \leq c_1 n_1$ and $k_2 \leq c_2 n_2$ for sufficiently small constants $c_1, c_2 > 0$, we show that there exists a constant $c_\mu > 0$ such that if $\mu^2 \leq c_\mu R$, then (1) holds. The proof of this claim constitutes the primary technical difficulty of the derivation of our lower bound, and relies on a precise analysis of the moment generating function in the left hand side of (1). The details are given in Section A.2, with the key lemmas collected in Section A.4.
2. Otherwise, we place ourselves in the *very dense* setting and assume without loss of generality that $k_1 \geq cn_1$ for a constant $c \in (0,1)$. In this case, the problem roughly reduces to the sparse signal detection problem in a standard Gaussian sequence model. In Section A.3, we prove that there exists a constant $c_\mu > 0$ such that if $\mu^2 \leq c_\mu \frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right)$, then (1) holds. We then show that $\frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right) \asymp R$ in this setting, which completes the proof.

A.2 Proof of lower bound in the general sparsity setting

Throughout this proof, we assume that there exist sufficiently small constants $c_1, c_2 \in (0,1)$ which depend on η such that $k_1 \leq c_1 n_1$ and $k_2 \leq c_2 n_2$. In this case, $\frac{k_1}{n_1-k_1} \asymp \frac{k_1}{n_1}$ and $\frac{k_2}{n_2-k_2} \asymp \frac{k_2}{n_2}$, and it suffices to control $\mathbb{E}[\exp(\mu^2 XY)]$ for $X \sim \text{Bin}(k_1, \frac{k_1}{n_1})$ and $Y \sim \text{Bin}(k_2, \frac{k_2}{n_2})$. We will also assume that $\frac{n_1}{k_1} \geq e\log(\frac{n_2}{k_2})$. In fact, this assumption may be made without loss of generality due to Lemma 27. Recall that we aim to show that there exists a constant $c_\mu > 0$ such that if $\mu^2 \leq c_\mu R$, then $\mathbb{E}[\exp(\mu^2 XY)] \leq 1+\varepsilon$. We divide our analysis into two cases.

<u>Case 1:</u> Suppose that $k_1^2 \geq (2e)^{-4} n_1 k_2$. Let $C_* \geq 1$ and $c_{1,\mu} > 0$ be the constants obtained from applying Lemma 8 with $\alpha = \varepsilon/2$. We can write

$$\mathbb{E}[\exp(\mu^2 XY)] = \mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \leq C_*\frac{k_1^2}{n_1})] + \mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X > C_*\frac{k_1^2}{n_1})]$$

By Lemma 8, if $\mu^2 \leq c_{1,\mu} R$, then $\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X > C_*\frac{k_1^2}{n_1})] < \varepsilon/2$. Now we turn our attention to the first term. Suppose that $C_*\frac{k_1^2}{n_1} < 1$. Then

$$\begin{aligned}\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \leq C_*\frac{k_1^2}{n_1})] &= \mathbb{P}(X=0) \\ &\leq 1,\end{aligned}$$

and it holds that $\mathbb{E}[\exp(\mu^2 XY)] \leq 1+\varepsilon/2$ and the proof is complete. Otherwise, suppose that $C_*\frac{k_1^2}{n_1} \geq 1$. Then by Lemma 1, there exists a constant $c_{2,\mu} > 0$ such that if $\mu^2 \leq c_{2,\mu} R$, it holds $\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \leq C_*\frac{k_1^2}{n_1})] < 1+\varepsilon/2$. Letting $c_\mu = \min(c_{1,\mu}, c_{2,\mu})$, it follows that if $\mu^2 \leq c_\mu R$, then

$$\mathbb{E}[\exp(\mu^2 XY)] \leq 1+\varepsilon.$$

The proof in this case is complete.

<u>Case 2:</u> Suppose that there exists a constant $\bar{c} \in (0, (2e)^{-4})$ such that $k_1^2 \leq \bar{c} n_1 k_2$. We split our analysis into two sub-cases. Assume first that $k_2 < k_1 \log\left(\frac{n_1 k_2}{k_1^2}\right)$. For a constant $C_* \geq 1$ whose value will be determined later, we form the partition

$$\mathbb{E}[\exp(\mu^2 XY)] = \mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \leq C_*\frac{k_1^2}{n_1})]$$

$$
\begin{aligned}
&+\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(C_*\frac{k_1^2}{n_1} < X < k_2^{-1}\log(\frac{n_1k_2}{k_1^2}))] \\
&+\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \geq C_*\frac{k_1^2}{n_1} \vee k_2^{-1}\log(\frac{n_1k_2}{k_1^2}))] \\
&= \mathrm{I}(C_*) + \mathrm{II}(C_*) + \mathrm{III}(C_*).
\end{aligned}
$$

By Lemma 12, there exist constants $C_{1,*} \geq 1$ and $c_{1,\mu} > 0$ such that if $\mu^2 \leq c_{1,\mu}R$, then $\mathrm{III}(C_{1,*}) < \varepsilon/3$. Suppose that $\lceil C_{1,*}\frac{k_1^2}{n_1}\rceil \geq \lfloor k_2^{-1}\log(\frac{n_1k_2}{k_1^2})\rfloor$. In this case, we take $C_* = C_{1,*}$, which gives us $\mathrm{II}(C_*) = 0$. If $C_*\frac{k_1^2}{n_1} < 1$, it immediately follows that $\mathrm{I}(C_*) \leq 1$ from an elementary calculation in the proof of Case 1, and thus $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \varepsilon/3$ which completes the proof. Otherwise, if $C_*\frac{k_1^2}{n_1} \geq 1$, then by Lemma 1 there exists a constant $c_{2,\mu} > 0$ such that if $\mu^2 \leq c_{2,\mu}R$, then $\mathrm{I}(C_*) \leq 1 + \frac{2}{3}\varepsilon$. In this case, we take $c_\mu = \min(c_{1,\mu}, c_{2,\mu})$ and for $\mu^2 \leq c_\mu R$ it holds that $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \varepsilon$. This completes the proof in the case $\lceil C_{1,*}\frac{k_1^2}{n_1}\rceil \geq \lfloor k_2^{-1}\log(\frac{n_1k_2}{k_1^2})\rfloor$.

Now suppose that $\lceil C_{1,*}\frac{k_1^2}{n_1}\rceil \leq \lfloor k_2^{-1}\log(\frac{n_1k_2}{k_1^2})\rfloor$. From here, we consider two further subcases. First, suppose that $2e\frac{k_1^2}{n_1} \geq \frac{k_2^2}{n_2}$. Then, by Lemma 9, there exist constants $C_{2,*} \geq 1$ and $c_{2,\mu}$ such that if $\mu^2 \leq c_{2,\mu}R$, then $\mathrm{II}(C_{2,*}) < \varepsilon/3$. Now we take $C_* = \max(C_{1,*}, C_{2,*})$. If $C_*\frac{k_1^2}{n_1} < 1$, it immediately follows that $\mathrm{I}(C_*) \leq 1$, and thus $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \frac{2}{3}\varepsilon$ which completes the proof. Otherwise, if $C_*\frac{k_1^2}{n_1} \geq 1$, then by Lemma 1 there exists a constant $c_{3,\mu} > 0$ such that if $\mu^2 \leq c_{3,\mu}R$, then $\mathrm{I}(C_*) \leq 1 + \varepsilon/3$. We then take $c_\mu = \min(c_{1,\mu}, c_{2,\mu}, c_{3,\mu})$ and for $\mu^2 \leq c_\mu R$ it holds that $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \varepsilon$. Next, suppose that $2e\frac{k_1^2}{n_1} \leq \frac{k_2^2}{n_2}$. In this case, we partition $\mathrm{II}(C_*)$ as follows:

$$
\begin{aligned}
\mathrm{II}(C_*) &= \mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(C_*\frac{k_1^2}{n_1} < X < k_2^{-1}\log(\frac{n_1k_2}{k_1^2}))] \\
&= \mathbb{E}\left[\exp\left(\mu^2 XY\right)\mathbf{1}\left(1 \vee C_*\frac{k_1^2}{n_1} \leq X \leq \frac{C_*k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)} \wedge k_1\right)\right] \\
&\quad+\mathbb{E}\left[\exp(\mu^2 XY)\mathbf{1}\left(1 \vee C_*\frac{k_1^2}{n_1} \vee \frac{C_*k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)} \leq X \leq \frac{k_2}{\log\left(\frac{n_1k_2}{k_1^2}\right)}\right)\right] \\
&= \mathrm{II}^{(a)}(C_*) + \mathrm{II}^{(b)}(C_*).
\end{aligned}
$$

By Lemmas 10 and 11, there exist constants $C_{2,*}, C_{3,*} \geq 1$ and $c_{2,\mu}, c_{3,\mu} > 0$ such that if $\mu^2 \leq \min(c_{2,\mu}, c_{3,\mu})R$, then $\mathrm{II}^{(a)}(C_{2,*}) + \mathrm{II}^{(b)}(C_{3,*}) < \varepsilon/3$. We take $C_* = \max(C_{1,*}, C_{2,*}, C_{3,*})$. If $C_*\frac{k_1^2}{n_1} < 1$, then $\mathrm{I}(C_*) \leq 1$ and thus $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \frac{2}{3}\varepsilon$ which completes the proof. Otherwise, if $C_*\frac{k_1^2}{n_1} \geq 1$, then by Lemma 1 there exists a constant $c_{4,\mu} > 0$ such that if $\mu^2 \leq c_{4,\mu}R$, then $\mathrm{I}(C_*) \leq 1 + \varepsilon/3$. We then take $c_\mu = \min(c_{1,\mu}, c_{2,\mu}, c_{3,\mu}, c_{4,\mu})$ and for $\mu^2 \leq c_\mu R$ it holds that $\mathbb{E}[\exp(\mu^2 XY)] \leq 1 + \varepsilon$. This completes the proof in the case $k_2 < k_1\log\left(\frac{n_1k_2}{k_1^2}\right)$.

We now turn our attention to the case $k_2 \geq k_1\log\left(\frac{n_1k_2}{k_1^2}\right)$. In this case, we perform the partition

$$
\begin{aligned}
\mathbb{E}[\exp(\mu^2 XY)] &= \mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(X \leq C_*\frac{k_1^2}{n_1})] \\
&\quad+\mathbb{E}[\exp(\mu^2 XY)\mathbf{1}(C_*\frac{k_1^2}{n_1} < X \leq k_1))] \\
&= \mathrm{I}(C_*) + \mathrm{II}(C_*)
\end{aligned}
$$

The remainder of the proof follows exactly as in the $k_2 < k_1\log\left(\frac{n_1k_2}{k_1^2}\right)$ setting. If $2e\frac{k_1^2}{n_1} \geq \frac{k_2^2}{n_2}$, we control $\mathrm{II}(C_*)$ using Lemma 9; otherwise, we use Lemmas 10 and 11. We then control $\mathrm{I}(C_*)$ using Lemma 1 if needed. We omit the details for brevity. The proof of the lower bound in the general sparsity setting is complete.

A.3 Proof of lower bound in the very dense setting

Now suppose that there exists a constant $c > 0$ such that $k_1 > cn_1$ and that $k_2 < \bar{c}n_2$ for a constant $\bar{c} < 1$. This allows us to consider $Y \sim \text{Bin}(k_2, \frac{k_2}{n_2})$ rather than $Y \sim \text{Bin}(k_2, \frac{k_2}{n_2-k_2})$. If, on the other hand, $k_2 > cn_2$, the steps of the proof follow identically by plugging in $Y \leq k_2$ and noticing $\frac{n_2}{k_2^2} \asymp \log(1+\frac{n_2}{k_2^2})$. We omit the details for brevity. Suppose that $\mu^2 \leq c\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right)$ for $c' = \log(1+\varepsilon)$. By the definition of c, it immediately holds that

$$\mu^2 \leq c\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) \leq \frac{1}{k_1}\log\left(1+c'\frac{n_2}{k_2^2}\right)$$

By direct calculation, we have

$$\begin{aligned}
\mathbb{E}[\exp(\mu^2 XY)] &\leq \mathbb{E}[\exp(\mu^2 k_1 Y)] \quad \text{(since } X \leq k_1 \text{ almost surely)}\\
&\leq \mathbb{E}\Big[\exp\Big(\log\left(1+c'\frac{n_2}{k_2^2}\right)Y\Big)\Big] \quad \text{(since } \mu^2 \leq \frac{1}{k_1}\log\left(1+c'\frac{n_2}{k_2^2}\right))\\
&= \Big(1+\frac{k_2}{n_2}\big(e^{\log\left(1+c'\frac{n_2}{k_2^2}\right)}-1\big)\Big)^{k_2}\\
&= \left(1+\frac{c'}{k_2}\right)^{k_2}\\
&\leq e^{c'}\\
&= 1+\varepsilon.
\end{aligned}$$

It just remains to show that $\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) \asymp R$. Let $C' = \frac{1}{c}$. First, note that $\frac{n_1}{k_1^2} \leq \frac{C'}{k_1} \leq C'$, so hence if R is defined with $C \geq C'$, it holds $R \leq \frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right)$ by definition. It remains to show that $\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) \lesssim R$. Since $\frac{n_1}{k_1} \leq C'$, we have

$$\begin{aligned}
\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) &\leq \frac{C'}{k_1}\log\left(1+c'\frac{n_2}{k_2^2}\right)\\
&\leq \frac{C'}{k_1}\log\left(1+c'\frac{n_2}{k_2^2}\log\left(e\binom{n_1}{k_1}\right)\right)\\
&= \psi_{12}\\
&\leq \psi_{12}+\psi_{21}.
\end{aligned}$$

Furthermore, using the inequality $\log(1+x) \leq x$ for any $x > 0$, we have

$$\begin{aligned}
\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) &\leq c'\frac{n_1 n_2}{k_1^2 k_2^2}\\
&\leq c'C'\frac{n_2}{k_1 k_2^2}\\
&\leq C''\frac{n_2}{k_2^2}\log\left(1+\frac{1}{k_1}\right) \quad \text{(since } x \lesssim \log(1+x) \text{ for } x < 1)\\
&\leq C''\frac{n_2}{k_2^2}\log\left(1+\frac{n_1}{k_1^2}\right)\\
&\leq C''\phi_{21}.
\end{aligned}$$

Clearly, $\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) \leq \phi_{12}$. Therefore we have $\frac{n_1}{k_1^2}\log\left(1+c'\frac{n_2}{k_2^2}\right) \lesssim R$, and the proof is complete.

A.4 Lemmas for proof of lower bound

Lemma 1. *Let $C > 0$ be a constant such that that $k_1^2 \geq n_1/C$. Then for any $\alpha > 0$, there exists a constant $c_\mu > 0$ such that if $\mu^2 \leq c_\mu\frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right)$, then*

$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \leq C\frac{k_1^2}{n_1}\big)\Big] < 1+\alpha.$$

Proof. By direct calculation, we have

$$\begin{aligned}\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \le C\frac{k_1^2}{n_1}\big)\Big] &\le \mathbb{E}\Big[\exp(\mu^2 C\frac{k_1^2}{n_1}Y)\Big] \\ &= \Big(1+\frac{k_2}{n_2}\big(e^{\mu^2 C(k_1^2/n_1)}-1\big)\Big)^{k_2} \\ &\le \exp\Big(\frac{k_2^2}{n_2}\big(e^{\mu^2 C(k_1^2/n_1)}-1\big)\Big).\end{aligned}$$

Now since $\mu^2 \le c_\mu \frac{n_1}{k_1^2}\log\big(1+\frac{n_2}{k_2^2}\big)$, we have

$$\begin{aligned}\exp\Big(\frac{k_2^2}{n_2}\big(e^{\mu^2 C(k_1^2/n_1)}-1\big)\Big) &\le \exp\Big(\frac{k_2^2}{n_2}\big(e^{c_\mu \frac{n_1}{k_1^2}\log\big(1+\frac{n_2}{k_2^2}\big)C(k_1^2/n_1)}-1\big)\Big) \\ &= \exp\Big(\frac{k_2^2}{n_2}\big((1+\frac{n_2}{k_2^2})^{c_\mu C}-1\big)\Big) \\ &\le \exp\Big(\frac{k_2^2}{n_2}\big(c_\mu C\frac{n_2}{k_2^2}\big)\Big) \\ &= \exp\big(c_\mu C\big)\end{aligned}$$

where the second inequality holds for $c_\mu < C^{-1}$, by the inequality $(1+x)^y \le 1+yx$ which holds for any $x \ge 0$ and $y \in [0,1]$. This final expression is at most $1+\alpha$ for c_μ taken sufficiently small. ☐

For any $C \ge 1$, we define the set of indices

$$\bar{\mathcal{A}}_C = \Big\{\Big\lceil C\frac{k_1^2}{n_1}\Big\rceil, \dots, k_1\Big\} \tag{2}$$

The results in this section will depend largely on the following lemma.

Lemma 2. *Suppose that $k_1 \le \frac{1}{2}n_1$. Then for any $\alpha > 0$, there exist constants $C_* \ge 1$ and $c_\mu > 0$ such that for any $\mathcal{A} \subset \bar{\mathcal{A}}_{C_*}$, if*

$$\mu^2 \le \min_{k\in\mathcal{A}} \frac{1}{k}\log\left(1+\frac{n_2}{k_2}\left(\exp\Big[\frac{c_\mu k}{k_2}\log\big(\frac{kn_1}{2ek_1^2}\big)\Big]-1\right)\right)$$

then

$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\Big(X \in \mathcal{A}\Big)\Big] < \alpha.$$

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\Big(X \in \mathcal{A}\Big)\Big] &= \sum_{k\in\mathcal{A}} \mathbb{E}\Big[\exp(\mu^2 kY)\Big]\Pr(X=k) \\ &= \sum_{k\in\mathcal{A}} \Big(1+\frac{k_2}{n_2}\big(e^{k\mu^2}-1\big)\Big)^{k_2}\Pr(X=k) \\ &\le \sum_{k\in\mathcal{A}} \Big(1+\frac{k_2}{n_2}\big(e^{k\mu^2}-1\big)\Big)^{k_2}\Big(\frac{k_1^2 2e}{kn_1}\Big)^k e^{-k_1^2/n_1}\end{aligned}$$

where the inequality follows from Lemma 28. Since we have assumed that

$$\mu^2 \le \min_{k\in\mathcal{A}} \frac{1}{k}\log\left(1+\frac{n_2}{k_2}\left(\exp\Big[\frac{c_\mu k}{k_2}\log\big(\frac{kn_1}{2ek_1^2}\big)\Big]-1\right)\right),$$

we can substitute this into the above calculation to obtain

$$\sum_{k\in\mathcal{A}} \Big(1+\frac{k_2}{n_2}\big(e^{k\mu^2}-1\big)\Big)^{k_2}\Big(\frac{k_1^2 2e}{kn_1}\Big)^k$$

$$\leq \sum_{k\in\mathcal{A}} \left(\left(\frac{kn_1}{k_1^2 2e}\right)^{c_\mu}\right)^k \left(\frac{k_1^2 2e}{kn_1}\right)^k$$
$$= \sum_{k\in\mathcal{A}} \left(\left(\frac{k_1^2 2e}{kn_1}\right)^{1-c_\mu}\right)^k.$$

Since $\mathcal{A} \subset \bar{A}_{C_*}$, for every $k \in \mathcal{A}$ we have

$$\frac{k_1^2}{kn_1} \leq \frac{1}{C_*}.$$

Therefore, by taking C_* sufficiently large, we can control the above sum with a geometric series

$$\sum_{k\in\mathcal{A}} \left(\left(\frac{k_1^2 2e}{kn_1}\right)^{1-c_\mu}\right)^k$$
$$\leq \sum_{k\in\mathcal{A}} \left(\left(\frac{2e}{C_*}\right)^{1-c_\mu}\right)^k$$
$$\leq \frac{\tilde{c}}{1-\tilde{c}}$$

where $\tilde{c} = (2e/C_*)^{1-c_\mu}$. This quantity can be made arbitrarily small by taking c_μ sufficiently small and C_* sufficiently large. This completes the proof. $\square$

Lemma 3. *Suppose that there exists a constant $\bar{c} \in (0, (2e)^{-4})$ such that $k_1^2 \leq \bar{c} n_1 k_2$, and $k_1 \leq \frac{1}{2} n_1$. Furthermore, suppose that $k_2 < k_1 \log\left(\frac{n_1 k_2}{k_1^2}\right)$. Then for any $\alpha > 0$, there exist constants $c_\mu > 0$ and $C_* \geq 1$ such that if*

$$\mu^2 \leq c_\mu \frac{1}{k_2} \log\left(\frac{k_2 n_1}{2e k_1^2} \log\left(\frac{n_2}{k_2}\right)\right),$$

then

$$\mathbb{E}\left[\exp(\mu^2 XY) \mathbf{1}\Big(X \geq \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \vee \frac{C_* k_1^2}{n_1}\Big)\right] < \alpha.$$

If we further assume that $\frac{n_2}{k_2} \geq 2(1-e^{-c_\mu})^{-1}$ and that $k_1 < k_2 \log(n_2/k_2)$, then if $\mu^2 \leq \frac{c_\mu}{k_2} \log\left(\frac{n_1}{2ek_1}\right) + \frac{1}{k_1} \log\left((1-e^{-c_\mu})\frac{n_2}{k_2}\right)$ it holds

$$\mathbb{E}\left[\exp(\mu^2 XY) \mathbf{1}\Big(X > \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \vee \frac{C_* k_1^2}{n_1}\Big)\right] < \alpha.$$

Proof of Lemma 3. Note that the set $\mathcal{A} = \{1 \vee \lceil \frac{k_2}{\log\left((n_1 k_2)/k_1^2\right)} \rceil \vee \lceil C_* \frac{k_1^2}{n_1} \rceil, ..., k_1\}$ is a subset of $\bar{A}_{C_*}$, and to prove the claim it suffices to show

$$\mu^2 \leq \min_{k\in\mathcal{A}} \frac{1}{k} \log\left(1 + \frac{n_2}{k_2}\left(\exp\left[\frac{c_\mu k}{k_2} \log\left(\frac{kn_1}{2ek_1^2}\right)\right] - 1\right)\right)$$

and invoke Lemma 2. To this end, for $k \in \mathcal{A}$ we define

$$g(k) = \frac{1}{k} \log\left(1 + \frac{n_2}{k_2}\left(\exp\left[\frac{c_\mu k}{k_2} \log\left(\frac{kn_1}{2ek_1^2}\right)\right] - 1\right)\right). \tag{3}$$

Notice that for $k \in \mathcal{A}$, we have

$$\frac{c_\mu k}{k_2} \log\left(\frac{kn_1}{2ek_1^2}\right) \geq \frac{c_\mu}{\log\left(\frac{k_2 n_1}{k_1^2}\right)} \log\left(\frac{k_2 n_1}{2ek_1^2 \log\left(\frac{k_2 n_1}{k_1^2}\right)}\right)$$

$$= c_\mu\Big(1 - \frac{\log\big(2e\log(\frac{n_1k_2}{k_1^2})\big)}{\log\big(\frac{n_1k_2}{k_1^2}\big)}\Big).$$

Recall that there exists a constant $\bar{c} \in (0, (2e)^{-4})$ such that $k_1^2 \leq \bar{c}n_1k_2$. In particular, this implies that $n_1k_2/k_1^2 \geq 16e^4$. Using this, as well as the bound $\log(x)/x \leq 1/2$ for $x > 1$, we have

$$\begin{aligned}\frac{\log\big(2e\log(\frac{n_1k_2}{k_1^2})\big)}{\log\big(\frac{n_1k_2}{k_1^2}\big)} &= \frac{\log(2e)}{\log\big(\frac{n_1k_2}{k_1^2}\big)} + \frac{\log\big(\log(\frac{n_1k_2}{k_1^2})\big)}{\log\big(\frac{n_1k_2}{k_1^2}\big)} \\ &\leq \frac{\log(2e)}{4\log(2e)} + \frac{1}{2} \\ &= \frac{3}{4}.\end{aligned}$$

Combining this with our calculation above, we have that, for any $k \in \mathcal{A}$,

$$\begin{aligned}\frac{c_\mu k}{k_2}\log\big(\frac{kn_1}{2ek_1^2}\big) &\geq \frac{c_\mu}{4} \\ &> 0.\end{aligned}$$

Thus, $\exp\big(\frac{c_\mu k}{k_2}\log\big(\frac{kn_1}{2ek_1^2}\big)\big)$ is bounded away from 1, and hence for $c = 1 - e^{-c_\mu}$ we have

$$f(k) =: \frac{1}{k}\log\left(c\frac{n_2}{k_2}\exp\Big[\frac{c_\mu k}{k_2}\log\big(\frac{kn_1}{2ek_1^2}\big)\Big]\right) \leq g(k)$$

for each $k \in \mathcal{A}$. So the prove the claim, it suffices to show that μ^2 is at most the minimum value of $f(k)$ over $\mathcal{A}$. We can write $f(k)$ as

$$f(k) = \frac{1}{k}\log\Big(c\frac{n_2}{k_2}\Big) + \frac{c_\mu}{k_2}\log\Big(\frac{kn_1}{2ek_1^2}\Big).$$

By direct calculation, we have

$$\frac{\mathrm{d}f}{\mathrm{d}k}(k) = -\frac{\log\big(c\frac{n_2}{k_2}\big)}{k^2} + \frac{c_\mu}{k_2k}$$

from which we deduce that f is minimized at $k^* = c_\mu^{-1}k_2\log\big(c\frac{n_2}{k_2}\big)$ and decreasing for $k < k^*$. Then if $\mu^2 < c_\mu\frac{1}{k_2}\log\big(\frac{k_2n_1\log(n_2/k_2)}{k_1^2}\big)$, it follows that

$$\begin{aligned}\mu^2 &< c_\mu\frac{1}{k_2}\log\big(\frac{k_2n_1\log(n_2/k_2)}{2ek_1^2}\big) \\ &\leq c_\mu\frac{1}{k_2}\Big(\log\big(\frac{k_2n_1\log(n_2/k_2)}{2ec_\mu k_1^2}\big) + 1\Big) \\ &= f(k^*) \\ &\leq \min_{k\in\mathcal{A}} f(k) \\ &\leq \min_{k\in\mathcal{A}} g(k)\end{aligned}$$

and we may apply Lemma 2 to complete the proof. Now if $c\frac{n_2}{k_2} \geq 2$ and $k_1 < k_2\log(n_2/k_2)$, then, for some sufficiently small c_μ, we have

$$\begin{aligned}k_1 &< k_2\log(n_2/k_2) \\ &< \frac{k_2}{c_\mu}\frac{\log(2)}{\log(2) + \log\big(\frac{2e}{c}\big)}\log\big(\frac{n_2}{k_2}\big) \\ &\leq \frac{k_2\log\big(\frac{cn_2}{k_2}\big)}{c_\mu\log\big(\frac{2en_2}{k_2}\big)}\log\big(\frac{n_2}{k_2}\big)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{k_2}{c_\mu} \log\big(c\frac{n_2}{k_2}\big) \\
&= k^*
\end{aligned}
$$

where the third inequality uses the fact that $x \mapsto \frac{x}{x+c}$ is increasing in x. Therefore since $k_1 < k^*$, the minimum value of $f(k)$ over the set $\mathcal{A}$ is attained at $k = k_1$. Thus for $\mu^2 < \frac{c_\mu}{k_2} \log\big(\frac{n_1}{2ek_1}\big) + \frac{1}{k_1} \log\big(c\frac{n_2}{k_2}\big)$, we have

$$
\begin{aligned}
\mu^2 &< \frac{c_\mu}{k_2} \log\big(\frac{n_1}{2ek_1}\big) + \frac{1}{k_1} \log\big(c\frac{n_2}{k_2}\big) \\
&= f(k_1) \\
&= \min_{k \in \mathcal{A}} f(k) \\
&\leq \min_{k \in \mathcal{A}} g(k).
\end{aligned}
$$

Applying Lemma 2 completes the proof. $\square$

Lemma 4. *Suppose that $k_1^2 \leq \bar{c} n_1 k_2$ for some $\bar{c} \in (0, (2e)^{-4})$ and $k_1 \leq \frac{1}{2} n_1$. Furthermore, suppose that $\frac{k_2^2}{n_2} \leq 2e\frac{k_1^2}{n_1}$. For any small constant $c > 0$, we define the quantity*

$$
M(c) = \begin{cases} \frac{1}{k_1} \log\Big(1 + c\frac{n_2 k_1}{k_2^2} \log(\frac{n_1}{2ek_1})\Big) & \text{if } k_1 \leq \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)} \\ \frac{1}{k_2} \log\big(\frac{n_1 k_2}{k_1^2}\big) \log\Big(1 + c\frac{n_2}{k_2 \log\big(\frac{n_1 k_2}{k_1^2}\big)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big) & \text{otherwise.} \end{cases}
$$

Then for any $\alpha > 0$, there exist constants $c_\mu > 0$ and $C_* \geq 1$ such that if $\mu \leq M(c_\mu)$ and $1 \vee C_* \frac{k_1^2}{n_1} \leq k_1 \wedge \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)}$, then

$$
\mathbb{E}\Big[\exp(\mu^2 XY) \mathbf{1}\Big(1 \vee C_* \frac{k_1^2}{n_1} \leq X \leq k_1 \wedge \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)}\Big)\Big] < \alpha.
$$

Moreover, it holds that

$$
2M(c_\mu) \geq \frac{1}{k_1} \log\Big(1 + c_\mu \frac{n_2 k_1}{k_2^2} \log(\frac{n_1}{2ek_1})\Big) + \frac{\log\big(\frac{n_1 k_2}{k_1^2}\big)}{k_2} \log\Big(1 + c_\mu \frac{n_2}{k_2 \log\big(\frac{n_1 k_2}{k_1^2}\big)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big).
$$

Proof of Lemma 4. Define $\mathcal{A} = \{1 \vee \lceil C_* \frac{k_1^2}{n_1} \rceil, ... k_1 \wedge \lfloor \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)} \rfloor\}$. Clearly $\mathcal{A} \subset \bar{\mathcal{A}}_{C_*}$, and hence it suffices to show that $\mu^2 \leq \min_{k \in \mathcal{A}} g(k)$ where $g(k)$ is defined as in (3). Using the inequality $e^x \geq 1 + x$, we have

$$
\begin{aligned}
g(k) &= \frac{1}{k} \log\Bigg(1 + \frac{n_2}{k_2}\Big(\exp\Big[\frac{c_\mu k}{k_2} \log\big(\frac{kn_1}{2ek_1^2}\big)\Big] - 1\Big)\Bigg) \\
&\geq \frac{1}{k} \log\Bigg(1 + c_\mu \frac{n_2}{k_2^2} k \log\Big(\frac{kn_1}{2ek_1^2}\Big)\Bigg) \\
&=: f^{(1)}(k).
\end{aligned}
$$

We define $\widetilde{\mathcal{A}} = \{1 \vee \lceil C_* \frac{k_1^2}{n_1} \rceil, \ldots\}$ and we will now show that $f^{(1)}(k)$ is decreasing in k over the set $\widetilde{\mathcal{A}}$. To do so, it suffices to consider the function $f(k) = \frac{1}{k} \log\Big(c_\mu \frac{n_2}{k_2^2} k \log\big(\frac{kn_1}{2ek_1^2}\big)\Big)$ and apply Lemma 29. Direct calculation reveals

$$
\frac{\mathrm{d}f}{\mathrm{d}k}(k) = -\frac{\log\big(c_\mu \frac{n_2}{k_2^2}\big)}{k^2} - \frac{\log(k)}{k^2} + \frac{1}{k^2} - \frac{\log\log\big(\frac{kn_1}{2ek_1^2}\big)}{k^2} + \frac{1}{k^2 \log\big(\frac{kn_1}{2ek_1^2}\big)}.
$$

The right-hand side is less than zero if

$$1 + \frac{1}{\log\big(\frac{kn_1}{2ek_1^2}\big)} < \log\Big(c_\mu \frac{n_2 k}{k_2^2}\log\big(\frac{kn_1}{2ek_1^2}\big)\Big).$$

For $k \in \widetilde{\mathcal{A}}$, it holds

$$\begin{aligned}\log\Big(c_\mu \frac{n_2 k}{k_2^2}\log\big(\frac{kn_1}{2ek_1^2}\big)\Big) &\geq \log\Big(c_\mu C_* \frac{n_2 k_1^2}{k_2^2 n_1}\log\big(\frac{C_*}{2e}\big)\Big)\\ &\geq \log\Big(\frac{c_\mu C_*}{2e}\log\big(\frac{C_*}{2e}\big)\Big)\\ &\geq 2\end{aligned}$$

where the second inequality uses the assumption $\frac{k_2^2}{n_2} \leq 2e\frac{k_1^2}{n_1}$ and the third holds as long as C_* is taken sufficiently large. Furthermore, for $k \in \widetilde{\mathcal{A}}$ we have

$$\begin{aligned}1 + \frac{1}{\log\big(\frac{kn_1}{2ek_1^2}\big)} &\leq 1 + \frac{1}{\log\big(\frac{C_*}{2e}\big)}\\ &< 2.\end{aligned}$$

Therefore, $f(k)$ is decreasing over $\widetilde{\mathcal{A}}$, and by Lemma 29, $f^{(1)}(k)$ is decreasing over $\widetilde{\mathcal{A}}$ as well. We also observe that

$$M = \begin{cases} f^{(1)}(k_1) & \text{if } k_1 \leq \frac{k_2}{\log(\frac{n_1 k_2}{k_1^2})} \\ f^{(1)}(\frac{k_2}{\log(\frac{n_1 k_2}{k_1^2})}) & \text{otherwise,}\end{cases}$$

that is, $M = f^{(1)}\Big(k_1 \wedge \frac{k_2}{\log(\frac{n_1 k_2}{k_1^2})}\Big)$. Moreover, since $f^{(1)}$ is decreasing over $\widetilde{\mathcal{A}}$, we also have

$$\begin{aligned}M &= \max\Big(f^{(1)}(k_1),\ f^{(1)}\Big(\frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)}\Big)\Big)\\ &\geq \frac{1}{2}\Big[f^{(1)}(k_1) + \ f^{(1)}\Big(\frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)}\Big)\Big]\\ &= \frac{1}{2}\Big[\frac{1}{k_1}\log\Big(1 + c_\mu\frac{n_2 k_1}{k_2^2}\log(\frac{n_1}{2ek_1})\Big) + \frac{\log\big(\frac{n_1 k_2}{k_1^2}\big)}{k_2}\log\Big(1 + c_\mu \frac{n_2}{k_2\log\big(\frac{n_1 k_2}{k_1^2}\big)}\log\big(\frac{n_1 k_2}{2ek_1^2\log(\frac{n_1 k_2}{k_1^2})}\big)\Big)\Big]\end{aligned}$$

as claimed. Using our assumed upper bound on μ^2, we have

$$\begin{aligned}\mu^2 \leq M &= f^{(1)}\Big(k_1 \wedge \frac{k_2}{\log(\frac{n_1 k_2}{k_1^2})}\Big)\\ &\leq \min_{k\in\mathcal{A}} f^{(1)}(k)\\ &\leq \min_{k\in\mathcal{A}} g(k).\end{aligned}$$

We conclude the proof by invoking Lemma 2. □

Lemma 5. *Suppose that $k_1^2 \leq \bar{c} n_1 k_2$ for some $\bar{c} \in (0, (2e)^{-4})$ and $k_1 < \frac{1}{2}n_1$. Furthermore, suppose that $\frac{k_2^2}{n_2} \geq 2e\frac{k_1^2}{n_1}$. Then for any $\alpha > 0$, there exist constants $c_\mu' > 0$ and $C_* > 0$ such that if $1 \vee \lceil C_* \frac{k_1^2}{n_1}\rceil \leq \lfloor \frac{C_* k_2^2/n_2}{\log\big(\frac{n_1 k_2^2}{k_1^2 n_2}\big)}\rfloor \wedge k_1$ and*

$$\mu^2 \leq c_\mu' \frac{n_2}{k_2^2}\log\big(\frac{C_*}{2e} \vee \frac{n_1}{2ek_1^2}\big),$$

then

$$\mathbb{E}\left[\exp\left(\mu^2 XY\right)\mathbf{1}\left(1\vee C_*\frac{k_1^2}{n_1}\leq X\leq \frac{C_* k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)}\wedge k_1\right)\right]<\alpha.$$

Proof of Lemma 5. Let $g(k)$ be defined as in (3). For $\mathcal{A}=\{1\vee\lceil C_*\frac{k_1^2}{n_1}\rceil,\dots,\left\lfloor\frac{k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)}\right\rfloor\wedge k_1\}$, it suffices to show that $\mu^2\leq\min_{k\in\mathcal{A}}g(k)$. Using $e^x\geq x+1$, $\forall x\in\mathbb{R}$, we have

$$\begin{aligned}
g(k)&=\frac{1}{k}\log\left(1+\frac{n_2}{k_2}\left(\exp\left[\frac{c_\mu k}{k_2}\log\left(\frac{kn_1}{2ek_1^2}\right)\right]-1\right)\right)\\
&\geq\frac{1}{k}\log\left(1+c_\mu\frac{n_2}{k_2^2}k\log\left(\frac{kn_1}{2ek_1^2}\right)\right).
\end{aligned}$$

For $k\in\mathcal{A}$, we can control the term within the logarithm as

$$\begin{aligned}
c_\mu\frac{n_2}{k_2^2}k\log\left(\frac{kn_1}{2ek_1^2}\right)&\leq c_\mu C_*\frac{\log\left(\frac{C_*n_1k_2^2}{2ek_1^2n_2\log\left(n_1k_2^2/(k_1^2n_2)\right)}\right)}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)}\\
&\leq c_\mu C_*\big(\log(C_*)+1\big)\\
&<2c_\mu C_*\log(C_*)
\end{aligned}$$

where the second inequality follows from $\log(x/\log(x))/\log(x)\leq1$ for $x\geq2e$, and the final inequality holds for $C_*>e$. Therefore, there exists a constant $c\in(0,1)$ which depends on c_μ and C_* such that

$$\begin{aligned}
g(k)&\geq\frac{1}{k}\log\left(1+c_\mu\frac{n_2}{k_2^2}k\log\left(\frac{kn_1}{2ek_1^2}\right)\right)\\
&\geq cc_\mu C_*\frac{1}{k}\frac{n_2}{k_2^2}k\log\left(\frac{kn_1}{2ek_1^2}\right)\\
&=c_\mu'\frac{n_2}{k_2^2}\log\left(\frac{kn_1}{2ek_1^2}\right)\\
&=:f(k)
\end{aligned}$$

for $c_\mu'=cc_\mu C_*$. The function $f(k)$ is clearly increasing in k, and hence is minimized at the smallest value of k in $\mathcal{A}$. Now for $\mu^2\leq c_\mu'\frac{n_2}{k_2^2}\log\left(\frac{C_*}{2e}\vee\frac{n_1}{2ek_1^2}\right)$, we have

$$\begin{aligned}
\mu^2&\leq c_\mu'\frac{n_2}{k_2^2}\log\left(\frac{C_*}{2e}\vee\frac{n_1}{2ek_1^2}\right)\\
&\leq\min_{k\in\mathcal{A}}f(k)\\
&\leq\min_{k\in\mathcal{A}}g(k).
\end{aligned}$$

We apply Lemma 2 to complete the proof. $\square$

Lemma 6. *Suppose that $k_1^2\leq\bar{c}n_1k_2$ for some $\bar{c}\in(0,(2e)^{-4})$ and $k_1\leq\frac{1}{2e}n_1$. Furthermore, suppose that $\frac{k_2^2}{n_2}\geq2e\frac{k_1^2}{n_1}$. Then for any $\alpha>0$, there exist constants $c_\mu>0$ and $C_*\geq1$ such that if $1\vee C_*\frac{k_1^2}{n_1}\vee\frac{C_*k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)}\leq k_1\wedge\frac{k_2}{\log\left(\frac{n_1k_2}{k_1^2}\right)}$ and*

$$\mu^2\leq\frac{1}{2}\left(\frac{1}{k_1}\log\left(1+c_\mu\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{2ek_1}\right)\right)+\frac{\log\left(\frac{n_1k_2}{k_1^2}\right)}{k_2}\log\left(1+c_\mu\frac{n_2}{k_2\log\left(\frac{n_1k_2}{k_1^2}\right)}\log\left(\frac{n_1k_2}{2ek_1^2\log\left(\frac{n_1k_2}{k_1^2}\right)}\right)\right)\right),$$

then

$$\mathbb{E}\left[\exp(\mu^2XY)\mathbf{1}\left(1\vee C_*\frac{k_1^2}{n_1}\vee\frac{C_*k_2^2/n_2}{\log\left(\frac{n_1k_2^2}{k_1^2n_2}\right)}\leq X\leq k_1\wedge\frac{k_2}{\log\left(\frac{n_1k_2}{k_1^2}\right)}\right)\right]<\alpha.$$

Proof of Lemma 6. The structure of this proof is similar to that of Lemma 4. Define $\mathcal{A} = \{1 \vee \lceil C_* \frac{k_1^2}{n_1} \rceil \vee \left\lceil \frac{C_* k_2^2 / n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \right\rceil, \ldots, k_1 \wedge \lfloor \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \rfloor\}$. With $g(k)$ defined as in (3), we have

$$
\begin{aligned}
g(k) &= \frac{1}{k} \log\left(1 + \frac{n_2}{k_2}\left(\exp\left[\frac{c_\mu k}{k_2} \log\left(\frac{k n_1}{2 e k_1^2}\right)\right] - 1\right)\right) \\
&\geq \frac{1}{k} \log\left(1 + c_\mu \frac{n_2}{k_2^2} k \log\left(\frac{k n_1}{2 e k_1^2}\right)\right) \\
&=: f^{(1)}(k).
\end{aligned}
$$

As in the proof of Lemma 4, will now show that $f^{(1)}(k)$ is decreasing in k over the set $\mathcal{A}$. It suffices to consider the function $f(k) =: \frac{1}{k} \log\left(c_\mu \frac{n_2}{k_2^2} k \log\left(\frac{k n_1}{2 e k_1^2}\right)\right)$ and apply Lemma 29. By direct calculation, we have

$$
\frac{\mathrm{d}f}{\mathrm{d}k}(k) = -\frac{\log\left(c_\mu \frac{n_2}{k_2^2}\right)}{k^2} - \frac{\log(k)}{k^2} + \frac{1}{k^2} - \frac{\log\log\left(\frac{k n_1}{2 e k_1^2}\right)}{k^2} + \frac{1}{k^2 \log\left(\frac{k n_1}{2 e k_1^2}\right)}.
$$

The right hand side is less than zero if

$$
1 + \frac{1}{\log\left(\frac{k n_1}{2 e k_1^2}\right)} < \log\left(c_\mu \frac{n_2 k}{k_2^2} \log\left(\frac{k n_1}{2 e k_1^2}\right)\right)
$$

For $k \in \mathcal{A}$, it holds

$$
\begin{aligned}
\log\left(c_\mu \frac{n_2 k}{k_2^2} \log\left(\frac{k n_1}{2 e k_1^2}\right)\right) &\geq \log\left(\frac{c_\mu C_*}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \log\left(\frac{C_* k_2^2 n_1}{2 e k_1^2 n_2 \log\left(\frac{k_2^2 n_1}{k_1^2 n_2}\right)}\right)\right) \\
&\geq \log\left(\frac{c_\mu C_*}{2}\right) \\
&\geq 2
\end{aligned}
$$

where the second inequality uses that $\frac{\log\log x}{\log x} < \frac{1}{2}$ for $x > 1$ to conclude

$$
\frac{\log\left(\frac{C_* k_2^2 n_1}{2 e n_2 k_1^2}\right) - \log\log\left(\frac{k_2^2 n_1}{n_2 k_1^2}\right)}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \geq \frac{1}{2},
$$

and the final inequality holds as long as C_* is taken sufficiently large such that $C_* \geq 2e^2/c_\mu$. Furthermore, for $k \in \mathcal{A}$ we have

$$
\begin{aligned}
1 + \frac{1}{\log\left(\frac{k n_1}{2 e k_1^2}\right)} &\leq 1 + \frac{1}{\log\left(\frac{C_*}{2e}\right)} \\
&< 2
\end{aligned}
$$

Therefore, $f(k)$ is decreasing over $\mathcal{A}$, and by Lemma 29, $f^{(1)}(k)$ is decreasing over $\mathcal{A}$ as well. Using our assumed upper bound on μ^2, we have

$$
\begin{aligned}
\mu^2 &\leq \frac{1}{2}\left(\frac{1}{k_1} \log\left(1 + c_\mu \frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{2 e k_1}\right)\right) + \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\left(1 + c_\mu \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\left(\frac{n_1 k_2}{2 e k_1^2 \log\left(\frac{n_1 k_2}{k_1^2}\right)}\right)\right)\right) \\
&= \frac{1}{2}\left(f^{(1)}(k_1) + f^{(1)}\left(\frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}\right)\right) \\
&\leq \max\left(f^{(1)}(k_1), f^{(1)}\left(\frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \min_{k\in\mathcal{A}} f^{(1)}(k) \\
&\le \min_{k\in\mathcal{A}} g(k).
\end{aligned}
$$

We conclude the proof by invoking Lemma 2. $\square$

Lemma 7. *Suppose that $k_1^2 \ge (2e)^{-4} n_1 k_2$ and that $k_1 < c' n_1$ for some small enough constant $c' > 0$. Then for any $\alpha > 0$, there exist constants $c_\mu > 0$ and $C_* \ge 1$ such that the following claims hold. In what follows, we define $c = 1 - \exp\Big(-c_\mu C_* (2e)^{-4} \log\Big(\frac{C_*}{2e}\Big)\Big)$.*

1. *Suppose that $c_\mu^{-1} k_2 \log\big(c\frac{n_2}{k_2}\big) \le C_* \frac{k_1^2}{n_1}$. Then if*
$$\mu^2 < C_*^{-1} \frac{n_1}{k_1^2} \log\Big(c\frac{n_2}{k_2}\Big) + \frac{c_\mu}{k_2} \log\Big(\frac{C_*}{2e}\Big),$$
it holds
$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \ge \lceil C_* \frac{k_1^2}{n_1}\rceil\big)\Big] < \alpha.$$

2. *Suppose that $C_* \frac{k_1^2}{n_1} < c_\mu^{-1} k_2 \log\big(c\frac{n_2}{k_2}\big) \le k_1$. Then if*
$$\mu^2 < \frac{c_\mu}{k_2} \log\Big(\frac{k_2 n_1 \log(c\frac{n_2}{k_2})}{c_\mu 2e k_1^2}\Big),$$
it holds
$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \ge \lceil C_* \frac{k_1^2}{n_1}\rceil\big)\Big] < \alpha.$$

3. *Suppose that $k_1 < c_\mu^{-1} k_2 \log\big(c\frac{n_2}{k_2}\big)$. Then if*
$$\mu^2 < \frac{1}{k_1} \log\big(c\frac{n_2}{k_2}\big) + \frac{c_\mu}{k_2} \log\big(\frac{n_1}{2ek_1}\big),$$
it holds
$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \ge \lceil C_* \frac{k_1^2}{n_1}\rceil\big)\Big] < \alpha.$$

Proof. Define $\mathcal{A} = \{\lceil C_* \frac{k_1^2}{n_1}\rceil, ..., k_1\}$ and g as in (3). For $k \in \mathcal{A}$, it holds

$$
\begin{aligned}
\frac{c_\mu k}{k_2} \log\Big(\frac{k n_1}{2ek_1^2}\Big) &\ge \frac{c_\mu C_* k_1^2}{k_2 n_1} \log\Big(\frac{C_*}{2e}\Big) \\
&\ge c_\mu C_* (2e)^{-4} \log\Big(\frac{C_*}{2e}\Big) \\
&> 0
\end{aligned}
$$

where the second inequality follows from the assumption $k_1^2 \ge (2e)^{-4} n_1 k_2$ and the final inequality holds for $C_* > 2e$. This implies that $\exp\Big(\frac{c_\mu k}{k_2} \log\Big(\frac{kn_1}{2ek_1^2}\Big)\Big)$ is bounded away from 1 for $k \in \mathcal{A}$. Therefore, for $c = 1 - \exp\Big(-c_\mu C_* (2e)^{-4} \log\Big(\frac{C_*}{2e}\Big)\Big)$, it holds

$$
\begin{aligned}
g(k) &= \frac{1}{k} \log\left(1 + \frac{n_2}{k_2}\left(\exp\Big[\frac{c_\mu k}{k_2} \log\big(\frac{kn_1}{2ek_1^2}\big)\Big] - 1\right)\right) \\
&\ge \frac{1}{k} \log\left(1 + c\frac{n_2}{k_2} \exp\Big[\frac{c_\mu k}{k_2} \log\big(\frac{kn_1}{2ek_1^2}\big)\Big]\right) \\
&\ge \frac{1}{k} \log\left(c\frac{n_2}{k_2} \exp\Big[\frac{c_\mu k}{k_2} \log\big(\frac{kn_1}{2ek_1^2}\big)\Big]\right)
\end{aligned}
$$

$$=: f(k).$$

Therefore, it suffices to show that μ^2 is at most the minimum of f over $\mathcal{A}$. We write $f(k)$ as

$$f(k) = \frac{1}{k}\log\left(c\frac{n_2}{k_2}\right) + \frac{c_\mu}{k_2}\log\left(\frac{kn_1}{2ek_1^2}\right).$$

By direct calculation, we have

$$\frac{\mathrm{d}f}{\mathrm{d}k}(k) = -\frac{\log\left(c\frac{n_2}{k_2}\right)}{k^2} + \frac{c_\mu}{k_2 k}$$

from which we deduce that f is minimized at $k^* = c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right)$ over $\mathbb{R}^+$, decreasing for $k < k^*$, and increasing for $k > k^*$. We now proceed by cases.

1. Suppose that $c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right) \leq C_*\frac{k_1^2}{n_1}$. Then f is increasing over $\mathcal{A}$, and by our assumption on μ^2, it holds

$$\begin{aligned}\mu^2 &< C_*^{-1}\frac{n_1}{k_1^2}\log\left(c\frac{n_2}{k_2}\right) + \frac{c_\mu}{k_2}\log\left(\frac{C_*}{2e}\right)\\ &\leq \min_{k\in\mathcal{A}} f(k)\\ &\leq \min_{k\in\mathcal{A}} g(k).\end{aligned}$$

2. Suppose that $C_*\frac{k_1^2}{n_1} < c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right) \leq k_1$. Then by our assumption on μ^2, we have

$$\begin{aligned}\mu^2 &< \frac{c_\mu}{k_2}\log\left(\frac{k_2 n_1\log(c\frac{n_2}{k_2})}{c_\mu 2ek_1^2}\right)\\ &\leq \min_{k\in\mathcal{A}} f(k)\\ &\leq \min_{k\in\mathcal{A}} g(k).\end{aligned}$$

3. Suppose that $k_1 < c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right)$. Then f is decreasing over $\mathcal{A}$ and is minimized at $k = k_1$. Therefore by our assumption on μ^2, we have

$$\begin{aligned}\mu^2 &< \frac{1}{k_1}\log\left(c\frac{n_2}{k_2}\right) + \frac{c_\mu}{k_2}\log\left(\frac{n_1}{2ek_1}\right)\\ &= \min_{k\in\mathcal{A}} f(k)\\ &\leq \min_{k\in\mathcal{A}} g(k).\end{aligned}$$

By Lemma 2, the proof is complete. $\square$

A.4.1 Simplification of the rate

For any $k_1, k_2, n_1, n_2 \in \mathbb{N}$, we define the following quantities

$$\psi(k_1,k_2,n_1,n_2) = \frac{1}{k_1}\log\left(1 + \frac{n_2 k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) \tag{4}$$

$$\phi(k_1,k_2,n_1,n_2) = \begin{cases}\frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right) & \text{if } \frac{n_1}{k_1^2}\leq 1\\ \infty & \text{otherwise.}\end{cases} \tag{5}$$

$$\nu(k_1,k_2,n_1,n_2) = \frac{1}{k_1}\log\left(\frac{n_2}{k_2}\right)\mathbf{1}_{\left\{\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)>1\right\}}. \tag{6}$$

To alleviate the notation, we will write

$$\begin{aligned}
\phi_{12} &= \phi(k_1, k_2, n_1, n_2) \\
\phi_{21} &= \phi(k_2, k_1, n_2, n_1) \\
\psi_{12} &= \psi(k_1, k_2, n_1, n_2) \\
\psi_{21} &= \psi(k_2, k_1, n_2, n_1) \\
\nu_{12} &= \nu(k_1, k_2, n_1, n_2) \\
\nu_{21} &= \nu(k_2, k_1, n_2, n_1).
\end{aligned}$$

Below, we show that the lower bound can be rewritten as

$$R := R(k_1, k_2, n_1, n_2) = \Big(\psi_{12} + \psi_{21}\Big) \wedge \phi_{12} \wedge \phi_{21}.$$

The following lemma simplifies the rate emerging from Lemma 7.

Lemma 8. *Assume that $k_1^2 \geq \bar{c} n_1 k_2$ for some constant $\bar{c} > 0$ and $k_j \leq c' n_j$, $\forall j \in \{1, 2\}$ for some sufficiently small constant $c' > 0$. Assume also that $\frac{n_1}{k_1} \geq e \log\left(\frac{n_2}{k_2}\right)$.*

1. *Then the following two properties hold*

$$\Big(\psi_{12} + \psi_{21}\Big) \wedge \phi_{12} \asymp \psi_{21} \wedge \phi_{12} \tag{7}$$

$$\psi_{21} \wedge \phi_{12} \asymp \Big(\psi_{21} + \nu_{12}\Big) \wedge \phi_{12}. \tag{8}$$

2. *It follows that, for any $\alpha > 0$, there exists a constant $c_\mu > 0$ such that, whenever $\mu^2 \leq c_\mu R$, we have*

$$\mathbb{E}\Big[\exp(\mu^2 XY)\, \mathbf{1}\Big(X \geq C_* \frac{k_1^2}{n_1}\Big)\Big] < \alpha.$$

3. *Moreover, it holds that $R \asymp \big(\psi_{21} + \nu_{12}\big) \wedge \big(\psi_{12} + \nu_{21}\big) \wedge \phi_{12} \wedge \phi_{21}$.*

Proof of Lemma 8.

1. For any k_1, k_2, n_1, n_2, we let

$$\widetilde{\phi}_{12} = \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right).$$

We start by showing that

$$\psi_{21} \wedge \phi_{12} \asymp \psi_{21} \wedge \widetilde{\phi}_{12}.$$

This is clear if $\frac{n_1}{k_1^2} \leq 1$ by definition of ϕ_{12}. Assume now that $\frac{n_1}{k_1^2} > 1$, which implies that $\phi_{12} = \infty$ and $\psi_{21} \wedge \phi_{12} = \psi_{21}$. By the assumption $k_1^2 \geq \bar{c} n_1 k_2$, we obtain $k_2 \leq \frac{k_1^2}{n_1 \bar{c}} \leq \frac{1}{\bar{c}}$. Therefore, we have

$$\begin{aligned}
\phi_{12} = \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right) &\geq \frac{n_1}{k_1^2} \log\left(1 + \bar{c}^2 n_2\right) \\
&\geq \bar{c}^2 \frac{n_1}{k_1^2} \log\left(1 + n_2\right) && \text{by Lemma 31.(i)} \\
&\geq \bar{c}^2 \log\left(1 + \frac{n_1}{k_1^2} k_2 \log\left(\frac{n_2}{k_2}\right)\right) && \text{by Lemma 31.(ii)} \\
&\geq \bar{c}^2 \frac{1}{k_2} \log\left(1 + \frac{n_1}{k_1^2} k_2 \log\left(\frac{n_2}{k_2}\right)\right) \\
&= \bar{c}^2 \psi_{21}.
\end{aligned}$$

Therefore, it follows that

$$
\begin{aligned}
\psi_{21} \wedge \phi_{12} &\asymp \psi_{21} \wedge \widetilde{\phi}_{12} \\
\Big(\psi_{12} + \psi_{21}\Big) \wedge \phi_{12} &\asymp \Big(\psi_{12} + \psi_{21}\Big) \wedge \widetilde{\phi}_{12} \\
\Big(\psi_{21} + \nu_{12}\Big) \wedge \phi_{12} &\asymp \Big(\psi_{21} + \nu_{12}\Big) \wedge \widetilde{\phi}_{12}
\end{aligned}
$$

and the equations (7) and (8) become equivalent to proving

$$
\Big(\psi_{12} + \psi_{21}\Big) \wedge \widetilde{\phi}_{12} \asymp \psi_{21} \wedge \widetilde{\phi}_{12}. \tag{9}
$$

$$
\psi_{21} \wedge \widetilde{\phi}_{12} \asymp \Big(\psi_{21} + \nu_{12}\Big) \wedge \widetilde{\phi}_{12} \tag{10}
$$

In the rest of the proof, we focus on proving (9) and (10).

Assume first that $\frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right) \leq 1$. Using the inequality $\log(1+x) \geq x\log(2)$ for any $x \leq 1$, we have

$$
\begin{aligned}
\psi_{21} = \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right) &\geq \log(2) \frac{n_1}{k_1^2} \log\left(\frac{n_2}{k_2}\right) \\
&\geq \frac{\log(2)}{2} \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right) \qquad \text{if } c' \text{ is small enough} \\
&= \frac{\log(2)}{2} \widetilde{\phi}_{12},
\end{aligned}
$$

which guarantees that both (9) and (10) hold. From now on, we will therefore assume that $\frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right) > 1$. Now, assume that $\frac{n_2}{k_2^2} \leq 1$. This implies the following inequalities

$$
\begin{aligned}
\psi_{21} = \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right) &> \frac{\log(2)}{k_2} \\
&\geq \frac{\log(2)}{k_2} \frac{\bar{c} n_1 k_2}{k_1^2} \frac{n_2}{k_2^2} \\
&= \bar{c}\log(2) \frac{n_1 n_2}{k_1^2 k_2^2} \\
&\geq \bar{c}\log(2) \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right) \\
&= \bar{c}\log(2) \widetilde{\phi}_{12},
\end{aligned}
$$

which ensures that (9) and (10) hold as well and yields the result. Thus, we will now assume that $\frac{n_2}{k_2^2} > 1$, which also implies $\frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right) > 1$.

Now, suppose for the sake of contradiction that $k_1 > \frac{n_1}{\log(n_1)}$. Note that the assumption $\frac{n_1}{k_1} \geq e \log\left(\frac{n_2}{k_2}\right)$ implies that

$$
\log(n_1) > e \log\left(\frac{n_2}{k_2}\right) > e \log(\sqrt{n_2}) = \frac{e}{2} \log(n_2),
$$

that is $n_1 > n_2^{e/2}$. Therefore,

$$
\begin{aligned}
\frac{k_2 n_1}{k_1^2} \log\left(\frac{n_2}{k_2}\right) &< \frac{k_2 n_1}{n_1^2 / \log^2(n_1)} \log\left(n_2\right) < \frac{\sqrt{n_2}}{n_1} \cdot \log(n_2) \log^2(n_1) \\
&< \frac{2}{e} \frac{\log^3(n_1)}{n_1^{1-1/e}} < 1,
\end{aligned}
$$

which contradicts the assumed inequality $\frac{k_2 n_1}{k_1^2} \log\left(\frac{n_2}{k_2}\right) > 1$.

Therefore, it holds that $k_1 \le \frac{n_1}{\log(n_1)}$. Since $\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right) > 1$, we have

$$
\begin{aligned}
\psi_{12} = \frac{1}{k_1}\log\left(1+\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) &\le \frac{1}{k_1}\log\left(2\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right)\\
&\le \frac{\log(n_2/k_2)+2\log(n_1)}{k_1} \qquad \text{provided } k_1 \le c'n_1\\
&\le \frac{1}{\bar{c}k_2}\frac{k_1}{n_1}\left[\log\left(\frac{n_2}{k_2}\right)+2\log(n_1)\right]\\
&\le \frac{1}{\bar{c}k_2}\left(1/e+2\right)\\
&\le \frac{e^{-1}+2}{\bar{c}\log(2)}\cdot\frac{1}{k_2}\log\left(1+\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right),\\
&= \frac{e^{-1}+2}{\bar{c}\log(2)}\psi_{21},
\end{aligned}
$$

which ensures that (9) holds. Moreover, we verify that equation (10) holds as well. This is clear if $\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right) \le 1$ since we have $\nu_{12} = 0$. Otherwise, we have

$$
\psi_{21} = \frac{1}{k_2}\log\left(1+\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) > \log(2)\frac{1}{k_2} \ge \frac{\log(2)\bar{c}}{k_1}\frac{n_1}{k_1} \ge \frac{e\log(2)\bar{c}}{k_1}\log\left(\frac{n_2}{k_2}\right) \ge e\bar{c}\log(2)\nu_{12},
$$

which ensures (10) holds and concludes the proof of the first claim.

2. Now, let $\alpha > 0$ and let $c_\mu, c > 0$ be the constants defined in Lemma 7.

   Assume first that $c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right) \le C_*\frac{k_1^2}{n_1}$. Recalling that $\frac{n_2}{k_2} \ge \frac{1}{c'}$ where c' can be taken arbitrarily small, we have

$$
\begin{aligned}
R \le \phi_{12} &= \frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right)\\
&\le \frac{n_1}{k_1^2}\log\left(2\frac{n_2}{k_2}\right) &&\text{provided } c' \text{ is small enough}\\
&\le 2\frac{n_1}{k_1^2}\log\left(c\frac{n_2}{k_2}\right) &&\text{provided } c' \text{ is small enough}\\
&\le C_*^{-1}\frac{n_1}{k_1^2}\log\left(c\frac{n_2}{k_2}\right)+\frac{c_\mu}{k_2}\log\left(\frac{C_*}{2e}\right),
\end{aligned}
$$

   which implies that if $\mu^2 \le R$, then we have by Lemma 7.1. that

$$
\mathbb{E}\Big[\exp(\mu^2XY)\mathbf{1}\big(X \ge C_*\frac{k_1^2}{n_1}\big)\Big] < \alpha.
$$

   Now, assume $C_*\frac{k_1^2}{n_1} < c_\mu^{-1}k_2\log\left(c\frac{n_2}{k_2}\right) \le k_1$. Then we have

$$
\begin{aligned}
R \le \psi_{21} &= \frac{1}{k_2}\log\left(1+\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right)\\
&\le \frac{1}{k_2}\log\left(2\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) &&\text{since } \frac{n_1k_2}{k_1^2} \ge 1\\
&\le \frac{1}{k_2}\log\Big(\frac{k_2n_1\log(c\frac{n_2}{k_2})}{c_\mu 2ek_1^2}\Big) &&\text{if } c' \text{ is small enough.}
\end{aligned}
$$

   Therefore, whenever $\mu \le c_\mu R$, we have by Lemma 7.2. that

$$
\mathbb{E}\Big[\exp(\mu^2XY)\mathbf{1}\big(X \ge \lceil C_*\frac{k_1^2}{n_1}\rceil\big)\Big] < \alpha.
$$

Finally, assume that $k_1 < c_\mu^{-1} k_2 \log\left(c\frac{n_2}{k_2}\right)$. Then we have

$$\frac{1}{c_\mu}\log\left(c\frac{n_2}{k_2}\right) > \frac{k_1}{k_2} \geq \frac{1}{16e^4}\frac{n_1}{k_1} \geq \frac{1}{16e^3}\log\left(\frac{n_2}{k_2}\right),$$

so that $\frac{k_2}{k_1}\log\left(\frac{n_2}{k_2}\right) \leq 16e^3$. Therefore, we have

$$\begin{aligned}
R \leq \psi_{21} &= \frac{1}{k_2}\log\left(1+\frac{n_1}{k_1^2}k_2\log\left(\frac{n_2}{k_2}\right)\right)\\
&\leq \frac{1}{k_2}\log\left(1+16e^3\frac{n_1}{k_1}\right)\\
&\leq \frac{1}{2k_2}\log\left(\frac{n_1}{2ek_1}\right) \qquad \text{if } c' \text{ is small enough}\\
&\leq \frac{1}{c_\mu}\left[\frac{1}{k_1}\log\left(c\frac{n_2}{k_2}\right)+\frac{c_\mu}{k_2}\log\left(\frac{n_1}{2ek_1}\right)\right]
\end{aligned}$$

By Lemma 7.3, if $\mu^2 \leq c_\mu R \leq \frac{1}{k_1}\log\left(c\frac{n_2}{k_2}\right)+\frac{c_\mu}{k_2}\log\left(\frac{n_1}{2ek_1}\right)$, we obtain

$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\big(X \geq \lceil C_*\frac{k_1^2}{n_1}\rceil\big)\Big] < \alpha,$$

and the proof is complete.

3. We immediately deduce from (7) that $R \gtrsim \big(\psi_{21}+\nu_{12}\big)\wedge\big(\psi_{12}+\nu_{21}\big)\wedge\phi_{12}\wedge\phi_{21}$. We now prove the converse bound. Note that if $\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right) \leq 1$, then we have $\nu_{21}=0$ and $\psi_{12}+\nu_{21} = \psi_{12} \lesssim \phi_{21}$ by Lemma 13 and the result follows. We assume that $\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right) > 1$ from now on.

   To prove that $R \asymp \big(\psi_{21}+\nu_{12}\big)\wedge\big(\psi_{12}+\nu_{21}\big)\wedge\phi_{12}\wedge\phi_{21}$, it suffices to show that $\nu_{21} \gtrsim \psi_{21}+\nu_{12}$. We have

$$\begin{aligned}
\psi_{21} &= \frac{1}{k_2}\log\left(1+\frac{n_1}{k_1}\frac{k_2}{k_1}\log\left(\frac{n_2}{k_2}\right)\right)\\
&\leq \frac{1}{k_2}\log\left(1+\frac{n_1}{k_1}\frac{k_1}{\bar{c}n_1}\frac{n_1}{ek_1}\right)\\
&\lesssim \frac{1}{k_2}\log\left(\frac{n_1}{k_1}\right)\\
&= \nu_{21}
\end{aligned}$$

   which concludes the proof.

$\square$

The Lemma below shows that R is a lower bound on the rate emerging from Lemma 4.

Lemma 9. *Suppose that $k_1^2 \leq \bar{c}n_1k_2$ for some $\bar{c} \in (0,(2e)^{-4})$ and $k_1 \leq c'n_1$, $k_2 \leq c'n_2$ for some sufficiently small $c'>0$. Furthermore, suppose that $\frac{k_2^2}{n_2} \leq 2e\frac{k_1^2}{n_1}$. For some small enough constant c_μ, we define the quantity*

$$M = \begin{cases} \frac{1}{k_1}\log\left(1+c_\mu\frac{n_2k_1}{k_2^2}\log(\frac{n_1}{2ek_1})\right) & \text{if } k_1 \leq \frac{k_2}{\log\left(\frac{n_1k_2}{k_1^2}\right)}\\ \frac{1}{k_2}\log\left(\frac{n_1k_2}{k_1^2}\right)\log\left(1+c_\mu\frac{n_2}{k_2\log\left(\frac{n_1k_2}{k_1^2}\right)}\log\left(\frac{n_1k_2}{2ek_1^2\log(\frac{n_1k_2}{k_1^2})}\right)\right) & \text{otherwise.}\end{cases}$$

1. *Then for some sufficiently small constant $c>0$ depending only on μ, it holds that $M \geq cR$. Hence, for any $\alpha>0$, there exists a constant $c'_\mu>0$ such that, whenever $\mu^2 \leq c'_\mu R$, we have*

$$\mathbb{E}\Big[\exp(\mu^2XY)\mathbf{1}\Big(1\vee C_*\frac{k_1^2}{n_1} \leq X \leq k_1 \wedge \frac{k_2}{\log\left(\frac{n_1k_2}{k_1^2}\right)}\Big)\Big] < \alpha.$$

2. *Moreover, we have*

$$R \geq \begin{cases} \frac{1}{2}(\psi_{12} + \nu_{21}) \wedge \phi_{12} \wedge \phi_{21} & \text{if } k_1 \leq \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \\ \frac{1}{2}(\psi_{21} + \nu_{12}) \wedge \phi_{12} \wedge \phi_{21} & \text{otherwise.} \end{cases} \tag{11}$$

3. *Moreover, it holds that* $R \asymp \big(\psi_{21} + \nu_{12}\big) \wedge \big(\psi_{12} + \nu_{21}\big) \wedge \phi_{12} \wedge \phi_{21}$.

Proof of Lemma 9.

1. We start by showing that $M \geq c(\psi_{21} + \psi_{12})$. Using Lemma 4 and Lemma 31.(i), we obtain

$$\begin{aligned} 2M &\geq \frac{1}{k_1} \log\Big(1 + c_\mu \frac{n_2 k_1}{k_2^2} \log(\frac{n_1}{2ek_1})\Big) + \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(1 + c_\mu \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big) \\ &\geq \frac{c_\mu}{k_1} \log\Big(1 + \frac{n_2 k_1}{k_2^2} \log(\frac{n_1}{2ek_1})\Big) + c_\mu \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(1 + \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big) \\ &\geq c_\mu \left[\frac{1}{2}\psi_{12} + \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(1 + \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big)\right]. \end{aligned}$$

To obtain $M \geq c(\psi_{12} + \psi_{21})$, it now remains to show

$$c\psi_{21} \leq \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(1 + \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big).$$

Since $\frac{n_1 k_2}{k_1^2} \geq \bar{c}^{-1} \geq 2$, it follows that $\frac{1}{2}\log(1 + \frac{n_1 k_2}{k_1^2}) \leq \log(\frac{n_1 k_2}{k_1^2})$ by the inequality $\sqrt{1+x} \leq x$ that holds true for any $x \geq 2$. Furthermore, the inequality $\frac{n_1 k_2}{k_1^2} \geq \bar{c}^{-1} \geq 16e^4$ implies that

$$\begin{aligned} \frac{\log\left(\frac{n_1 k_2}{2ek_1^2 \log\left(\frac{n_1 k_2}{k_1^2}\right)}\right)}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} &= 1 - \frac{\log\Big(2e \log\left(\frac{n_1 k_2}{k_1^2}\right)\Big)}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \\ &\geq \frac{1}{2}. \end{aligned}$$

Moreover, since $n_2/k_2 \geq 1/c' \geq 4$ provided $c' \leq 1/4$, we also have that $\log\Big(\frac{n_2}{2k_2}\Big) \geq \frac{1}{2}\log\Big(\frac{n_2}{k_2}\Big)$. Combining these observations yields

$$\begin{aligned} \frac{\log\left(\frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(1 + \frac{n_2}{k_2 \log\left(\frac{n_1 k_2}{k_1^2}\right)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big) &\geq \frac{\frac{1}{2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(\frac{1}{2}\frac{n_2}{k_2}\Big) \\ &\geq \frac{1}{4} \frac{\log\left(1 + \frac{n_1 k_2}{k_1^2}\right)}{k_2} \log\Big(\frac{n_2}{k_2}\Big) \\ &\geq \frac{1}{4}\frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\Big(\frac{n_2}{k_2}\Big)\right) \quad \text{by Lemma 31.(ii)} \\ &= \frac{1}{4}\psi_{21}. \end{aligned} \tag{12}$$

It remains to invoke Lemma 4 to conclude the proof of the first claim.

2. We now prove equation (11). Assume first that $k_1 \leq \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}$. Then we obtain

$$\frac{n_1}{k_1} \leq \frac{\frac{n_1 k_2}{k_1^2}}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}, \quad \text{hence} \quad \frac{n_1 k_2}{k_1^2} \geq \frac{n_1}{k_1} \log\left(\frac{n_1}{k_1}\right), \quad \text{i.e.} \quad \frac{1}{k_1} \geq \frac{1}{k_2} \log\left(\frac{n_1}{k_1}\right) \tag{13}$$

by Lemma 30.(ii). Note that, since $\frac{n_2 k_1}{k_2^2} \geq \frac{n_1}{k_1} \geq \frac{1}{c'}$ can be made arbitrarily large by taking c' small enough, we have

$$\nu_{21} \leq \frac{1}{k_2} \log\left(\frac{n_1}{k_1}\right) \leq \frac{1}{k_1} \leq \frac{1}{k_1} \log\left(1 + \frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right)\right) = \psi_{12}.$$

Hence,

$$\begin{aligned} R &= (\psi_{12} + \psi_{21}) \wedge \phi_{12} \wedge \phi_{21} \\ &\geq \psi_{12} \wedge \phi_{12} \wedge \phi_{21} \\ &\geq \frac{1}{2}(\psi_{12} + \nu_{21}) \wedge \phi_{12} \wedge \phi_{21} \end{aligned}$$

as claimed. Assume now that $k_1 > \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}$. By assumption, we have that $\frac{n_1 k_2}{k_1^2} \geq \bar{c}^{-1} \geq 2e$. Therefore,

$$\frac{\frac{n_1 k_2}{k_1^2}}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \leq \frac{n_1}{k_1}, \quad \text{hence} \quad \frac{n_1 k_2}{k_1^2} \leq 2\frac{n_1}{k_1} \log\left(\frac{2n_1}{k_1}\right), \quad \text{i.e.} \quad k_2 \leq 3k_1 \log\left(\frac{n_1}{k_1}\right)$$

by Lemma 30.(i) and for c' sufficiently small. This yields

$$\psi_{12} = \frac{1}{k_1} \log\left(1 + \frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right)\right) \geq \frac{1}{k_1} \log\left(1 + \frac{n_2}{k_2}\right) \geq \frac{1}{k_1} \log\left(\frac{n_2}{k_2}\right) \geq \nu_{12}.$$

Therefore, we obtain

$$\begin{aligned} R &= (\psi_{12} + \psi_{21}) \wedge \phi_{12} \wedge \phi_{21} \\ &\geq \frac{1}{2}(\psi_{21} + \nu_{12}) \wedge \phi_{12} \wedge \phi_{21}. \end{aligned}$$

This concludes the proof of equation (11).

3. Assume that $k_1 \leq \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)}$, which implies $k_2 \geq k_1 \log\left(\frac{n_1}{k_1}\right)$ by (13). We aim to prove that $\psi_{21} + \nu_{12} \gtrsim \psi_{12} + \nu_{21}$. We have

$$\begin{aligned} \psi_{12} &= \frac{1}{k_1} \log\left(1 + \frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right)\right) \\ &\leq \frac{1}{k_1} \log\left(1 + \frac{n_2}{k_2}\right) \\ &\lesssim \nu_{12}. \end{aligned}$$

Moreover,

$$\psi_{21} = \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right)$$

□

The lemma below shows that R is a lower bound on the rate emerging from Lemma 5.

Lemma 10. *Suppose that $k_1^2 \leq \bar{c} n_1 k_2$ for some $\bar{c} \in (0, (2e)^{-4})$ and $k_1 < \frac{1}{2} n_1$. Furthermore, suppose that $\frac{k_2^2}{n_2} \geq 2e \frac{k_1^2}{n_1}$ and that $\lceil C_* \frac{k_1^2}{n_1} \rceil \leq \left\lfloor \frac{C_* k_2^2 / n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \right\rfloor \wedge k_1$. Then for any $\alpha > 0$, there exists a constant $c > 0$ such that if $\mu^2 \leq cR$,*

$$\mathbb{E}\left[\exp\left(\mu^2 XY\right) \mathbf{1}\left(C_* \frac{k_1^2}{n_1} \leq X \leq \frac{C_* k_2^2 / n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \wedge k_1\right)\right] < \alpha.$$

Proof of Lemma 10. First, observe that the assumption $\lceil C_* \frac{k_1^2}{n_1} \rceil \leq \lfloor \frac{C_* k_2^2/n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \rfloor \wedge k_1$ implies $1 \leq \frac{C_* k_2^2/n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)}$. Rearranging terms, this implies

$$\frac{n_2}{k_2^2} \leq C_* \log^{-1}\big(\frac{n_1 k_2^2}{n_2 k_1^2}\big) \leq \frac{C_*}{\log(2e)},$$

and therefore $\phi_{21} = \frac{n_2}{k_2^2} \log\left(1 + \frac{n_1}{k_1^2}\right)$ as long as we take the C in the definition of R to be at least $\frac{C_*}{\log(2e)}$. Now, assume that $\mu^2 \leq \widetilde{c}_\mu R$ for some $\widetilde{c}_\mu > 0$. Then, we have

$$\begin{aligned}
\frac{1}{\widetilde{c}_\mu}\mu^2 &\leq R \\
&\leq \phi_{21} \\
&= \frac{n_2}{k_2^2} \log\big(1 + \frac{n_1}{k_1^2}\big) \\
&\leq \frac{n_2}{k_2^2} \log\big(4e\big) \quad \text{(this follows from } \frac{n_1}{k_1^2} \leq 2e\frac{n_2}{k_2^2}\text{)} \\
&\leq \frac{n_2}{k_2^2} \log\big(4e\big) \log\big(\frac{C_*}{2e} \vee \frac{n_1}{2ek_1^2}\big).
\end{aligned}$$

Letting c'_μ denote the constant from Lemma 5, we can now choose $\widetilde{c}_\mu = \frac{c'_\mu}{\log(4e)}$ sufficiently small so that, whenever $\mu^2 \leq \widetilde{c}_\mu R$, we have $\mu^2 \leq c'_\mu \frac{n_2}{k_2^2} \log\big(\frac{C_*}{2e} \vee \frac{n_1}{2ek_1^2}\big)$. The conclusion follows by Lemma 5, and the proof is complete. ☐

The lemma below ensure that R is a lower bound on the rate obtained in Lemma 6.

Lemma 11. *Suppose that $k_1^2 \leq \bar{c} n_1 k_2$ for some $\bar{c} \in (0, (2e)^{-4})$ and $k_1 < c' n_1$, $k_2 \leq c' n_2$ for some sufficiently small constant c'. Furthermore, suppose that $\frac{k_2^2}{n_2} \geq 2e\frac{k_1^2}{n_1}$.*

1. *Then for any $\alpha > 0$, there exist constants $c_\mu > 0$ and $C_* \geq 1$ such that if $1 \vee \lceil C_* \frac{k_1^2}{n_1} \rceil \vee \left\lceil \frac{C_* k_2^2/n_2}{\log\left(\frac{n_1 k_2^2}{k_1^2 n_2}\right)} \right\rceil \leq k_1 \wedge \lfloor \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \rfloor$ and $\mu^2 \leq c_\mu R$,*

$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\Big(C_* \frac{k_1^2}{n_1} \vee \frac{C_* k_2^2/n_2}{\log\big(\frac{n_1 k_2^2}{k_1^2 n_2}\big)} \leq X \leq k_1 \wedge \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)}\Big)\Big] < \alpha.$$

2. *Moreover, we have*

$$R \geq \begin{cases} \big(\psi_{21} + \nu_{12}\big) \wedge \phi_{12} \wedge \phi_{21} & \text{if } k_2 \leq k_1 \log\left(\frac{n_1}{k_1}\right) \\ \big(\psi_{12} + \nu_{21}\big) \wedge \phi_{12} \wedge \phi_{21} & \text{otherwise.} \end{cases}$$

   Consequently, we have $R \geq \big(\psi_{21} + \nu_{12}\big) \wedge \big(\psi_{12} + \nu_{21}\big) \wedge \phi_{12} \wedge \phi_{21}$.

Proof of Lemma 11.

1. By the assumption $\frac{n_1 k_2}{k_1^2} \geq \bar{c}^{-1} \geq 16e^4$, we can repeat the steps leading to equation (12) to obtain

$$\begin{aligned}
\frac{\log\big(\frac{n_1 k_2}{k_1^2}\big)}{k_2} \log\Big(1 + c_\mu \frac{n_2}{k_2 \log\big(\frac{n_1 k_2}{k_1^2}\big)} \log\big(\frac{n_1 k_2}{2ek_1^2 \log(\frac{n_1 k_2}{k_1^2})}\big)\Big) &\geq \frac{c_\mu}{4} \frac{\log\Big(1 + \frac{n_1 k_2}{k_1^2} \log\big(\frac{n_2}{k_2}\big)\Big)}{k_2} \\
&= \frac{c_\mu}{4}\psi_{21}.
\end{aligned}$$

It immediately follows that, for some small enough constant c, we have

$$\begin{aligned}
\mu^2 &\leq cR \\
&\leq c(\psi_{12} + \psi_{21}) \\
&\leq \frac{1}{2}\frac{1}{k_1}\log\Big(1 + c_\mu \frac{n_2 k_1}{k_2^2}\log(\frac{n_1}{2ek_1})\Big) + \frac{1}{2}\frac{\log\big(\frac{n_1 k_2}{k_1^2}\big)}{k_2}\log\Big(1 + c_\mu \frac{n_2}{k_2 \log\big(\frac{n_1 k_2}{k_1^2}\big)}\log\big(\frac{n_1 k_2}{2ek_1^2\log(\frac{n_1 k_2}{k_1^2})}\big)\Big).
\end{aligned}$$

The conclusion follows from Lemma 6.

2. Assume now that $k_2 \leq k_1 \log\left(\frac{n_1}{k_1}\right)$. Then we have

$$\begin{aligned}
\psi_{12} &= \frac{1}{k_1}\log\left(1 + \frac{n_2 k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) \\
&\geq \frac{1}{k_1}\log\left(1 + \frac{n_2}{k_2}\right) \\
&\geq \nu_{12},
\end{aligned}$$

which yields the desired result. Assume now that $k_2 > k_1 \log\left(\frac{n_1}{k_1}\right)$. Then

$$\begin{aligned}
\psi_{21} &= \frac{1}{k_2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) \\
&> \frac{1}{k_2}\log\left(\frac{n_1}{k_1}\log\left(\frac{n_1}{k_1}\right)\log\left(\frac{n_2}{k_2}\right)\right) \\
&\geq \frac{1}{k_2}\log\left(\frac{n_1}{k_1}\right) \\
&\geq \nu_{21}.
\end{aligned}$$

This concludes the proof.

□

The lemma below ensure that R is a lower bound on the rate obtained in Lemma 3.

Lemma 12. *Suppose that $k_1^2 \leq \bar{c} n_1 k_2$ for some $\bar{c} \in (0, (2e)^{-4})$ and $k_1 \leq c' n_1$ and $k_2 \leq c' n_2$ for some small enough constant $c' > 0$. Furthermore, suppose that $k_2 < k_1 \log\big((n_1 k_2)/k_1^2\big)$ and $\frac{n_1}{k_1} \geq e \log\left(\frac{n_2}{k_2}\right)$.*

1. *Then for any $\alpha_4 > 0$, there exist constants $c_\mu > 0$ and $C_* \geq 1$ such that if $\mu^2 \leq c_\mu R$, then*

$$\mathbb{E}\Big[\exp(\mu^2 XY)\mathbf{1}\Big(X \geq \frac{k_2}{\log\big(\frac{n_1 k_2}{k_1^2}\big)} \vee C_* \frac{k_1^2}{n_1}\Big]\Big)\Big] < \alpha_4.$$

2. *Moreover, it holds that*

$$R \geq \begin{cases} \frac{1}{4}\big((\psi_{21} + \nu_{12}) + (\psi_{12} + \nu_{21})\big) \wedge \phi_{12} \wedge \phi_{21} & \text{if } k_1 \leq k_2 \log\left(\frac{n_2}{k_2}\right) \\ \frac{1}{2}\big(\psi_{21} + \nu_{12}\big) \wedge \phi_{12} \wedge \phi_{21} & \text{otherwise.} \end{cases}$$

Consequently, we have $R \geq \big(\psi_{21} + \nu_{12}\big) \wedge \big(\psi_{12} + \nu_{21}\big) \wedge \phi_{12} \wedge \phi_{21}$.

Proof of Lemma 12.

1. Note that the relation $k_2 < k_1 \log\big((n_1 k_2)/k_1^2\big)$ implies that

$$\frac{n_1 k_2}{k_1^2} < \frac{n_1}{k_1}\log\left(\frac{n_1 k_2}{k_1^2}\right) \leq \frac{2n_1}{k_1}\log\left(\frac{2n_1}{k_1}\right) \qquad \text{by Lemma 30.(i)}$$

which yields

$$k_2 \le 4k_1 \log(n_1/k_1). \tag{14}$$

Now, we obtain

$$\frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right) = \frac{n_2}{k_2} \cdot \frac{k_1}{k_2} \log\left(\frac{n_1}{k_1}\right) \ge \frac{1}{4c'} \ge 1$$

provided $c' \le 1/4$, and

$$\frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right) \ge \frac{1}{\bar{c}} \log\left(\frac{1}{c'}\right) \ge 1$$

provided c' and $\bar{c}$ are small enough, which ensures that $\nu_{12} = \frac{1}{k_1} \log\left(\frac{n_2}{k_2}\right)$ and $\nu_{21} = \frac{1}{k_2} \log\left(\frac{n_1}{k_1}\right)$.

Assume first that $k_1 < k_2 \log(n_2/k_2)$. Then, we have

$$\begin{aligned}
R &\le \psi_{12} + \psi_{21} \\
&= \frac{1}{k_1} \log\left(1 + \frac{n_2 k_1}{k_2^2} \log\left(\frac{n_1}{k_1}\right)\right) + \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right) \\
&\le \frac{1}{k_1} \log\left(2 \frac{n_2}{k_2} \frac{k_1}{k_2} \log\left(\frac{n_1}{k_1}\right)\right) + \frac{1}{k_2} \log\left(1 + \frac{n_1}{k_1} 4 \log\left(\frac{n_1}{k_1}\right) \log\left(\frac{n_2}{k_2}\right)\right) \\
&\le \frac{1}{k_1} \log\left(\frac{n_2}{k_2}\right) + \frac{1}{k_1} \log\left(\frac{2k_1}{k_2} \log\left(\frac{n_1}{k_1}\right)\right) + \frac{1}{k_2} \log\left(\frac{8}{e} \left(\frac{n_1}{k_1}\right)^2 \log\left(\frac{n_1}{k_1}\right)\right) \\
&\le \frac{1}{k_1} \log\left(\frac{n_2}{k_2}\right) + \frac{1}{k_1} \log\left(\frac{2k_1}{k_2} \log\left(\frac{n_1}{k_1}\right)\right) + \frac{4}{k_2} \log\left(\frac{n_1}{k_1}\right) \\
&\le \frac{1}{k_1} \log\left(\frac{n_2}{k_2}\right) + \frac{6}{k_2} \log\left(\frac{n_1}{k_1}\right) \\
&\le 6\nu_{12} + 6\nu_{21}.
\end{aligned}$$

In the second to last inequality, we used the fact that $\log\left(\frac{2k_1}{k_2} \log\left(\frac{n_1}{k_1}\right)\right) \le \frac{2k_1}{k_2} \log\left(\frac{n_1}{k_1}\right)$ due to the inequality $\log(x) \le x$ that holds true for any $x > 0$. It follows from Lemma 3 that if $\mu^2 \le c_\mu R$ for some sufficiently small constant $c_\mu > 0$, then

$$\mathbb{E}\Big[\exp(\mu^2 XY) \mathbf{1}\Big(X \ge \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \vee C_* \frac{k_1^2}{n_1}\Big)\Big] < \alpha.$$

Assume now that $k_1 > k_2 \log\left(\frac{n_2}{k_2}\right)$. Again, we have $R \le \psi_{12} + \psi_{21}$. We aim to show that

$$\psi_{12} + \psi_{21} \le \frac{4}{k_2} \log\left(\frac{k_2 n_1 \log(n_2/k_2)}{2e k_1^2}\right), \tag{15}$$

which will conclude the proof since it ensures that, if $\mu^2 \le c_\mu R$ for some small enough $c_\mu > 0$, then

$$\mathbb{E}\Big[\exp(\mu^2 XY) \mathbf{1}\Big(X \ge \frac{k_2}{\log\left(\frac{n_1 k_2}{k_1^2}\right)} \vee C_* \frac{k_1^2}{n_1}\Big)\Big] < \alpha.$$

by Lemma 3. We observe that the assumption $k_1^2 \le \bar{c} n_1 k_2$ ensures that $\frac{n_1 k_2}{k_1^2} \log(n_2/k_2) > 2$, which yields

$$\begin{aligned}
\psi_{21} &= \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right) \\
&\le \frac{2}{k_2} \log\left(\frac{n_1 k_2}{k_1^2} \log\left(\frac{n_2}{k_2}\right)\right),
\end{aligned}$$

and further implies that

$$\frac{1}{k_2}\log\left(\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right)\le\psi_{21}\le\frac{2}{k_2}\log\left(\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right). \tag{16}$$

Now, by the relations $k_2\le 4k_1\log(n_1/k_1)$ and $k_j\le c'n_j$ for some small enough constant $c'>0$, we also have $\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)>2$, which yields

$$\psi_{12}=\frac{1}{k_1}\log\left(1+\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right)\le\frac{2}{k_1}\log\left(\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right),$$

so that

$$\frac{1}{k_1}\log\left(\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right)\le\psi_{12}\le\frac{2}{k_1}\log\left(\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) \tag{17}$$

To obtain the desired inequality (15), it therefore suffices to show that

$$\frac{1}{k_1}\log\left(\frac{n_2k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right)\le\frac{1}{k_2}\log\left(\frac{n_1k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) \tag{18}$$

under the assumptions $\frac{k_1}{\bar{c}n_1}\le\frac{k_2}{k_1}\le\frac{1}{\log(n_2/k_2)}$ and $\frac{n_j}{k_j}\ge\frac{1}{c'}$ for $j\in\{1,2\}$, as well as $\frac{n_1}{k_1}\ge e\log\left(\frac{n_2}{k_2}\right)$.

To prove this, we let $x=\frac{k_2}{k_1}$, and $a_j=\frac{n_j}{k_j}$ for $j=1,2$ and introduce the function

$$f(x)=\log\left(a_1x\log(a_2)\right)-x\log\left(\frac{a_2\log(a_1)}{x}\right).$$

We note that the desired inequality (18) is equivalent to showing that $f(x)\ge 0$ for any $x\in\left[\frac{1}{\bar{c}a_1},\frac{1}{\log(a_2)}\right]$ where $a_1\ge e\log(a_2)$ and $a_1,a_2\ge\frac{1}{c'}$. We have, for any $x\in\left[\frac{1}{\bar{c}a_1},\frac{1}{\log(a_2)}\right]$

$$\begin{aligned}
f'(x)&=\frac{1}{x}-\log(a_2\log(a_1))+\log(x)+1\\
&=\log(x)+\frac{1}{x}-\log\left(\frac{a_2\log(a_1)}{e}\right),\\
f''(x)&=\frac{1}{x}-\frac{1}{x^2}\le 0,
\end{aligned}$$

since $x\le\frac{1}{\log(a_2)}\le 1$. Therefore, f is concave, and to prove that $f\ge 0$ over $\left[\frac{1}{\bar{c}a_1},\frac{1}{\log(a_2)}\right]$, it suffice to show that $f\left(\frac{1}{\bar{c}a_1}\right)\ge 0$ and $f\left(\frac{1}{\log(a_2)}\right)\ge 0$. We have

$$\begin{aligned}
f\left(\frac{1}{\bar{c}a_1}\right)&=\log\left(\frac{\log(a_2)}{\bar{c}}\right)-\frac{\log(\bar{c}a_1)}{\bar{c}a_1}-\frac{\log(a_2)}{\bar{c}a_1}-\frac{\log\log(a_1)}{\bar{c}a_1}\\
&\ge\log\left(\frac{\log(a_2)}{\bar{c}}\right)-\frac{\log(a_2)}{\bar{c}a_1}-1\\
&\ge\frac{1}{2}\log\left(\frac{\log(a_2)}{\bar{c}}\right)-1\ge 0.
\end{aligned}$$

In the last step, we used the fact that, since $a_1\ge e\log(a_2)$, we also have $\bar{c}a_1\ge\frac{1}{2}\frac{\log(a_2)}{\log\log(a_2/\bar{c})}$ provided a_2 is large enough, which can be enforced by choosing $c'>0$ small enough. Similarly, we have

$$\begin{aligned}
f\left(\frac{1}{\log(a_2)}\right)&=\log(a_1)-\frac{1}{\log(a_2)}\log\left(a_2\log(a_2)\log(a_1)\right)\\
&=\log(a_1)-1-\frac{\log\log(a_2)}{\log(a_2)}-\frac{\log(a_1)}{\log(a_2)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \log(a_1) - 2 - \frac{\log(a_1)}{\log(a_2)} \\
&\geq 0,
\end{aligned}
$$

provided a_1 and a_2 are larger than suitably large constants, which can be enforced by choosing c' small enough. This concludes the proof of the first claim.

2. Assume first that $k_1 < k_2 \log\left(\frac{n_2}{k_2}\right)$. Then we have

$$
\begin{aligned}
\psi_{12} + \psi_{21} &= \frac{1}{k_1}\log\left(1 + \frac{n_2 k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) + \frac{1}{k_2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) \\
&\geq \frac{1}{k_1}\log\left(1 + \frac{n_2}{4k_2}\right) + \frac{1}{k_2}\log\left(\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) \\
&> \frac{1}{4k_1}\log\left(1 + \frac{n_2}{k_2}\right) + \frac{1}{k_2}\log\left(\frac{n_1}{k_1}\right) \\
&\geq \frac{1}{4}\big(\nu_{12} + \nu_{21}\big),
\end{aligned}
$$

which yields $R \geq \frac{1}{8}\big((\psi_{21} + \nu_{12}) + (\psi_{12} + \nu_{21})\big) \wedge \phi_{12} \wedge \phi_{21}$, as desired.

Assume now that $k_1 \geq k_2 \log\left(\frac{n_2}{k_2}\right)$. Combining (16), (17) and (18) yields $\psi_{12} + \psi_{21} \asymp \psi_{21}$. We now show that $\psi_{21} \asymp \psi_{21} + \nu_{12}$. By assumption, we have

$$
\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right) \geq \frac{1}{c} > 1,
$$

which implies that $\nu_{12} = \frac{1}{k_1}\log\left(\frac{n_2}{k_2}\right)$. Therefore, we obtain

$$
\begin{aligned}
\nu_{12} &= \frac{1}{k_1}\log\left(\frac{n_2}{k_2}\right) \\
&\leq \frac{1}{k_1}\log\left(\frac{n_2}{k_2}\right) + \frac{1}{k_1}\log\left(\frac{4k_1\log\left(\frac{n_1}{k_1}\right)}{k_2}\right) && \text{by equation (14)} \\
&\leq \frac{2}{k_1}\log\left(\frac{n_2 k_1}{k_2^2}\log\left(\frac{n_1}{k_1}\right)\right) && \text{using } \log(4x) \leq 2\log(x),\ \forall x \geq 4 \\
&\leq \frac{2}{k_2}\log\left(\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) && \text{by equation (18)} \\
&\leq 2\psi_{21} && \text{by equation (16).}
\end{aligned}
$$

Therefore, we have $\psi_{12} + \psi_{21} \geq \psi_{21} + \nu_{12}/2$, hence $R \geq \frac{1}{2}\big(\psi_{21} + \nu_{12}\big) \wedge \phi_{12} \wedge \phi_{21}$ and the proof is complete.

$\square$

Lemma 13. *Assume that, for some constant $c > 0$, we have $k_2 \leq c n_2$. If $\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right) \leq 1$, then we have $\psi_{21} \leq \phi_{12}$.*

Proof of Lemma 13. Note that, when $\frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right) \leq 1$, we have $\frac{n_1}{k_1^2} \leq 1$, so that $\phi_{12} = \frac{n_1}{k_1^2}\log\left(1 + \frac{n_2}{k_2^2}\right)$. We obtain

$$
\begin{aligned}
\psi_{21} &= \frac{1}{k_2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\log\left(\frac{n_2}{k_2}\right)\right) \\
&\asymp \frac{n_1}{k_1^2}\log\left(\frac{n_2}{k_2}\right)
\end{aligned}
$$

$$\geq \phi_{12},$$

which completes the proof. $\square$

Lemma 14. *It holds that* $R \gtrsim \big(\psi_{12} + \nu_{21}\big) \wedge \big(\psi_{21} + \nu_{12}\big) \wedge \phi_{12} \wedge \phi_{21}$.

Proof of Lemma 14. Assume first that, for some constant $c > 0$, we have $k_1 \geq cn_1$ or $k_2 \geq cn_2$, and by symmetry, assume we have $k_1 \geq cn_1$. Then

$$\psi_{12} = \frac{1}{k_1} \log\left(1 + \frac{n_2}{k_2^2} \log\left(e\binom{n_1}{k_1}\right)\right) \geq \frac{1}{k_1} \log\left(1 + \frac{n_2}{k_2^2}\right) \asymp \phi_{12},$$

which yields that

$$\begin{aligned} R &= (\psi_{12} + \psi_{21}) \wedge \phi_{12} \wedge \phi_{21} \\ &\asymp \phi_{12} \wedge \phi_{21} \\ &\geq \big(\psi_{12} + \nu_{21}\big) \wedge \big(\psi_{21} + \nu_{12}\big) \wedge \phi_{12} \wedge \phi_{21}. \end{aligned}$$

From now on, assume that $k_1 \leq cn_1$ and $k_2 \leq cn_2$, and assume without loss of generality that $\frac{n_1}{k_1} \geq e \log\left(\frac{n_2}{k_2}\right)$, by lemma 27.

If $k_1^2 \geq \bar{c}n_1k_2$, then the result follows by Lemma 8. Now, assume that $k_1^2 < \bar{c}n_1k_2$. If $\frac{k_2^2}{n_2} \leq 2e\frac{k_1^2}{n_1}$, then the result follows by Lemma 9. Else, we have $k_1^2 \geq \bar{c}n_1k_2$ and the result follows by Lemmas 11 and 12.

$\square$

B Proofs for upper bound

B.1 Analysis of total degree test

Recall that the total degree test is defined as $\Delta_{\text{deg}}^h = \mathbf{1}(t_{\text{deg}} > h)$ for a choice of threshold $h > 0$, where

$$t_{\text{deg}} = \frac{1}{\sqrt{n_1n_2p_0(1-p_0)}} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (A_{ij} - p_0) \tag{19}$$

Lemma 15. *Let* $\alpha \in (0,1)$ *be given and define* $h_\alpha = \sqrt{4\log(2/\alpha)}$. *Suppose that* $n_1n_2p_0 \geq \frac{4}{27}h_\alpha^2$ *and* $p_0 \in (0, \frac{1}{4}]$. *Then there exists a constant* $C_\delta > 0$ *such that if*

$$\delta^2 \geq C_\delta p_0(1-p_0)\frac{n_1n_2}{k_1^2k_2^2},$$

then the linear test with threshold $h = h_\alpha$ *satisfies*

$$\mathcal{R}(\Delta_{deg}^h, \delta) \leq \alpha.$$

Proof of Lemma 15. We begin by an analysis of the Type 1 error of Δ_{deg}^h. Under the null hypothesis, it holds that $\mathbb{E}[t_{\text{deg}}] = 0$ and $\text{var}(t_{\text{deg}}) = 1$. Let $\sigma^2 = p_0(1-p_0)n_1n_2$. Note that under our assumptions $n_1n_2p_0 \geq \frac{4}{27}h_\alpha^2$ and $p_0 \leq \frac{1}{4}$, it holds

$$\begin{aligned} \sigma &= \sqrt{p_0(1-p_0)n_1n_2} \\ &\geq \sqrt{\frac{3}{4}p_0n_1n_2} \\ &\geq \sqrt{\frac{3}{4}\frac{4}{27}h^2} \\ &= \frac{1}{3}h \end{aligned}$$

By direct calculation, we have

$$\begin{aligned}
\mathbb{P}_0(\Delta^h_{\text{deg}} = 1) &= \mathbb{P}_0(t_{\text{deg}} > h) \\
&= \mathbb{P}_0\big(\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(A_{ij} - p_0) > \sigma h\big) \\
&\leq \exp\Big(-\frac{\frac{1}{2}h^2\sigma^2}{\sigma^2 + \frac{1}{3}h\sigma}\Big) \quad \text{(by Bernstein's inequality)} \\
&\leq \exp\Big(-\frac{\frac{1}{2}\sigma^2 h^2}{2\sigma^2}\Big) \quad (\text{since } \sigma \geq \frac{1}{3}h) \\
&= \exp\Big(-\frac{1}{4}h^2\Big) \\
&\leq \exp\big(-\log(2/\alpha)\big) \quad (\text{since } h = \sqrt{4\log(2/\alpha)}) \\
&= \frac{\alpha}{2}.
\end{aligned}$$

Now we turn our attention to the Type 2 error. Under the alternative hypothesis, there exist subsets of indices $K_1 \subset [n_1]$ and $K_1 \subset [n_2]$ of sizes $|K_1| = k_1$ and $|K_2| = k_2$ such that $\mathbb{E}[A_{ij}] = p_0 + \delta$ for $(i, j) \in K_1 \times K_2$ and $\mathbb{E}[A_{ij}] = p_0$ otherwise. Since the $\{A_{ij}\}_{i\in[n_1],j\in[n_2]}$ are independent Bernoulli random variables, we can directly compute

$$\begin{aligned}
\mathbb{E}_{\mathbf{P}}[t_{\text{deg}}] &= \frac{1}{\sigma}k_1k_2\delta, \\
\text{var}_{\mathbf{P}}(t_{\text{deg}}) &= \frac{1}{\sigma^2}\big(k_1k_2p_1(1-p_1) + (n_1-k_1)(n_2-k_2)p_0(1-p_0)\big),
\end{aligned}$$

where we let $p_1 = p_0 + \delta$. Now under the assumption that $\delta^2 \geq C_\delta p_0(1-p_0)\frac{n_1n_2}{k_1^2k_2^2}$, it holds

$$\begin{aligned}
\mathbb{E}_{\mathbf{P}}[t_{\text{deg}}] &= \frac{1}{\sigma}k_1k_2\delta \\
&= \frac{k_1k_2}{\sqrt{n_1n_2p_0(1-p_0)}}\delta \\
&\geq C_\delta.
\end{aligned}$$

Therefore, for C_δ taken large enough, we can ensure that $\mathbb{E}_{\mathbf{P}}[t_{\text{deg}}] \geq 2h = 2\sqrt{4\log(2/\alpha)}$. Therefore, we may perform the following calculation:

$$\begin{aligned}
\mathbb{P}_{\mathbf{P}}(\Delta^h_{\text{deg}} = 0) &= \mathbb{P}_{\mathbf{P}}(t_{\text{deg}} \leq h) \\
&= \mathbb{P}_{\mathbf{P}}(t_{\text{deg}} - \mathbb{E}_{\mathbf{P}}t_{\text{deg}} \leq h - \mathbb{E}_{\mathbf{P}}t_{\text{deg}}) \\
&= \mathbb{P}_{\mathbf{P}}\big((t_{\text{deg}} - \mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2 \geq (h - \mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2\big) \\
&\leq \mathbb{P}_{\mathbf{P}}\big((t_{\text{deg}} - \mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2 \geq \frac{1}{4}(\mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2\big) \\
&\leq 4\frac{\text{var}_{\mathbf{P}}(t_{\text{deg}})}{(\mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2},
\end{aligned}$$

where the final inequality uses Markov's. From here, we apply our closed form expressions of $\text{var}_{\mathbf{P}}(t_{\text{deg}})$ and $\mathbb{E}_{\mathbf{P}}[t_{\text{deg}}]$ derived above to compute:

$$\begin{aligned}
\frac{\text{var}_{\mathbf{P}}(t_{\text{deg}})}{(\mathbb{E}_{\mathbf{P}}t_{\text{deg}})^2} &= \frac{k_1k_2p_1(1-p_1) + (n_1-k_1)(n_2-k_2)p_0(1-p_0)}{k_1^2k_2^2\delta^2} \\
&\leq \frac{k_1k_2(p_0+\delta) + \sigma^2}{k_1^2k_2^2\delta^2} \\
&= \text{I} + \text{II} + \text{III},
\end{aligned}$$

where we define $\text{I} = \frac{p_0}{k_1k_2\delta^2}$, $\text{II} = \frac{1}{k_1k_2\delta}$, and $\text{III} = \frac{\sigma^2}{k_1^2k_2^2\delta^2}$. We control each of these terms separately. First, we have

$$\text{I} = \frac{p_0}{k_1k_2\delta^2}$$

$$
\begin{aligned}
&\leq \frac{k_1 k_2 p_0}{n_1 n_2 p_0(1-p_0)C_\delta} \quad (\text{since } \delta^2 \geq C_\delta \frac{n_1 n_2 p_0(1-p_0)}{k_1^2 k_2^2}\ ) \\
&\leq \frac{4}{3C_\delta} \quad (\text{since } k_i \leq n_i \text{ for } i \in \{1,2\} \text{ and } 1-p_0 \geq \frac{3}{4}) \\
&\leq \frac{\alpha}{24},
\end{aligned}
$$

where the final inequality holds for C_δ taken sufficiently large. Now we consider II. Recall that $n_1 n_2 p_0 \geq \frac{4}{27}h^2 = \frac{8}{27}\log(2/\alpha)$. Therefore, $p_0 \geq \frac{C}{n_1 n_2}$ where $C > 0$ is a constant that depends on α. We thus have

$$
\begin{aligned}
\delta &\geq \sqrt{C_\delta p_0(1-p_0)\frac{n_1 n_2}{k_1^2 k_2^2}} \\
&\geq \sqrt{C_\delta \frac{3}{4}}\sqrt{p_0 \frac{n_1 n_2}{k_1^2 k_2^2}} \quad (\text{since } 1-p_0 \geq \frac{3}{4}) \\
&\geq \frac{C'}{k_1 k_2},
\end{aligned}
$$

where $C' = \sqrt{\frac{3}{4}C_\delta C}$. Using this lower bound on δ, we have

$$
\begin{aligned}
\text{II} &= \frac{1}{k_1 k_2 \delta} \\
&\leq \frac{1}{C'} \\
&\leq \frac{\alpha}{24}.
\end{aligned}
$$

where the final inequality holds for C_δ taken sufficiently large. Finally, to control III we have

$$
\begin{aligned}
\text{III} &= \frac{\sigma^2}{k_1^2 k_2^2 \delta^2} \\
&\leq \frac{n_1 n_2 p_0(1-p_0)k_1^2 k_2^2}{C_\delta n_1 n_2 p_0(1-p_0)k_1^2 k_2^2} \quad (\text{since } \delta^2 \geq C_\delta \frac{n_1 n_2 p_0(1-p_0)}{k_1^2 k_2^2}) \\
&= \frac{1}{C_\delta} \\
&\leq \frac{\alpha}{24}
\end{aligned}
$$

where the final inequality holds for C_δ taken large enough. Combining these bounds on I, II, and III gives us

$$
\begin{aligned}
\mathbb{P}_{\mathbf{P}}(\Delta^h_{\text{deg}} = 0) &\leq 4\frac{\text{var}_{\mathbf{P}}(t_{\text{deg}})}{(\mathbb{E}_{\mathbf{P}} t_{\text{deg}})^2} \\
&\leq 4(\text{I} + \text{II} + \text{III}) \\
&\leq 4(3\frac{\alpha}{24}) \\
&= \frac{\alpha}{2}.
\end{aligned}
$$

Combining this inequality with our bound on the Type I error, we have

$$
\mathcal{R}(\Delta^h_{\text{deg}}, \delta) \leq \alpha,
$$

and the proof is complete. □

B.2 Analysis of the truncated degree test

The following lemma controls the risk of $\Delta^h_{\text{trunc-deg},1}$. The analysis of $\Delta^h_{\text{trunc-deg},2}$ is analogous.

Lemma 16. *Let $\alpha \in (0,1)$ and define $h_\alpha = C^* \left(\sqrt{n_2 \exp\left(-c' \log(1+\frac{n_2}{k_2^2})\right) \log(2/\alpha)} + \log(2/\alpha) \right)$ for constants $c', C^* > 0$ to be determined later. Suppose that $n_2 \geq c k_2^2$ for a constant $c > 0$. Then there exist constants $C_\delta, C_\alpha > 0$ and $C \geq \frac{8}{3}$ such that if $\frac{1}{4} \geq p_0 \geq \frac{C_\alpha}{n_1} \log\left(1 + \frac{n_2}{k_2^2}\right)$ and*

$$\delta^2 \geq C_\delta p_0(1-p_0) \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right),$$

then the truncated degree test with threshold $h = h_\alpha$ and $\tau = \sqrt{C \log\left(1 + \frac{n_2}{k_2^2}\right)}$ satisfies

$$\mathcal{R}(\Delta^{h_\alpha}_{\textit{trunc-deg},1}, \delta) \leq \alpha.$$

Proof of Lemma 16. First, we control the Type 1 error of $\Delta^{h_\alpha}_{\text{trunc-deg},1}$. Let $\sigma = \sqrt{n_1 p_0(1-p_0)}$. By Lemma 19, there exist constants $C', c, c', C^* > 0$ such that if $\tau \in [C', c\sigma]$, then

$$\mathbb{P}_0(\Delta^{h_\alpha}_{\text{trunc-deg},1} > h_\alpha) \leq \frac{\alpha}{2}.$$

We impose that $c' \geq 1$ by taking C in the definition of τ sufficiently large. Therefore we just need to verify that $\tau \in [C', c\sigma]$. Since $n_2 \geq c k_2^2$, it holds

$$\begin{aligned}
\tau &= \sqrt{C \log(1 + \frac{n_2}{k_2^2})} \\
&> \sqrt{C \log(1+c)} \\
&\geq C',
\end{aligned}$$

where the final inequality holds for C taken large enough. On the other hand, since $\frac{1}{4} \geq p_0 \geq \frac{C_\alpha}{n_1} \log\left(1 + \frac{n_2}{k_2^2}\right)$, we have

$$\begin{aligned}
c\sigma &= c\sqrt{n_1 p_0(1-p_0)} \\
&\geq c\sqrt{C_\alpha \frac{3}{4} \log\left(1 + \frac{n_2}{k_2^2}\right)} \\
&\geq \sqrt{C \log\left(1 + \frac{n_2}{k_2^2}\right)} \quad \text{(for } C_\alpha \text{ large enough)} \\
&= \tau.
\end{aligned}$$

Therefore $\tau \in [C', c\sigma]$, and the desired bound on the Type 1 error holds.

Now we aim to control the Type 2 error. Under the alternative hypothesis, there exist sets $K_1 \in \mathcal{P}_{k_1}(n_1)$ and $K_2 \in \mathcal{P}_{k_2}(n_2)$ such that $\mathbb{E}[\mathbf{A}] = \mathbf{P}$ with entries $P_{ij} \geq p_0 + \delta$ if $(i,j) \in K_1 \times K_2$ and $P_{ij} = p_0$ otherwise. Let $\theta = \frac{k_1 \delta}{\sigma}$. Then by Lemma 20, there exist constants $\tilde{c}, c, C' > 0$ such that if $\tau \in [C', c\sigma]$, $k_1(p_0+\delta) \geq C'$, and $\theta \geq C'\tau$, then

$$\mathbb{E}\left[\sum_{j=1}^{n_2} \left(W_j - \nu_\tau^{k_1}\right) \mathbf{1}(\bar{A}_j > \tau)\right] \geq \tilde{c} k_2 \theta \sigma \log\left(1 + \frac{\theta}{\sigma}\right). \tag{20}$$

As when we controlled the Type 1 error, we enforce $\tau \in [C', c\sigma]$ by taking C in the definition of τ and C_α sufficiently large, in that order. To verify $k_1(p_0+\delta) \geq C'$, we calculate

$$\begin{aligned}
k_1(p_0+\delta) &\geq k_1\delta \\
&\geq \sqrt{n_1 p_0(1-p_0) C_\delta \log\left(1 + \frac{n_2}{k_2^2}\right)} \\
&\geq \sqrt{C C_\delta \log\left(1 + \frac{n_2}{k_2^2}\right)} \quad \text{(since } \sigma \gtrsim 1 \text{, as shown above)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sqrt{CC_\delta \log(1+c)} \\
&\geq C',
\end{aligned}
$$

where the finally inequality holds for C_δ taken sufficiently large. Finally, to verify $\theta \geq C'\tau$, we have

$$
\begin{aligned}
\theta &= \frac{k_1\delta}{\sqrt{n_1 p_0(1-p_0)}} \\
&\geq \sqrt{C_\delta \log\left(1+\frac{n_2}{k_2^2}\right)} \\
&\geq C'\sqrt{C\log\left(1+\frac{n_2}{k_2^2}\right)} \quad \text{(for } C_\delta \text{ taken large enough)} \\
&= C'\tau.
\end{aligned}
$$

Therefore, (20) holds by Lemma 20. Additionally, by Lemma 21, there exist constants $C_1, c, C' > 0$ such that if $\tau \in [C', c\sigma]$,$k_1(p_0+\delta) \geq C'$, and $\theta \geq 2\tau$, then

$$
\operatorname{var}\left(\sum_{j=1}^{n_2}\left(W_j - \nu_\tau^{k_1}\right)\mathbf{1}(\bar{A}_j > \tau)\right) \leq C_1 k_2(\sigma^2+\theta\sigma)\log^2\left(1+\frac{\theta}{\sigma}\right) + C_1(n_2-k_2)\left(\sigma\tau\log(1+\frac{\tau}{\sigma})\right)^2 \exp\left(-\frac{3}{8}\tau^2\right). \tag{21}
$$

As before, we enforce $\tau \in [C', c\sigma]$,$k_1(p_0+\delta) \geq C'$, and $\theta \geq 2\tau$ by taking C, C_α, and C_δ sufficiently large in that order. To control the Type 2 error with (20) and (21) and Chebyshev's inequality, we need to verify that the right-hand side of (20) is at least $2h_\alpha$. In what follows, we define

$$
\bar{\delta}^2 := C_\delta p_0(1-p_0)\frac{n_1}{k_1^2}\log\left(1+\frac{n_2}{k_2^2}\right),
$$

so that $\delta \geq \bar{\delta}$. First, we have

$$
\begin{aligned}
\log\left(1+\frac{\theta}{\sigma}\right) &= \log\left(1+\frac{k_1\delta}{\sigma^2}\right) \\
&\geq \log\left(1+\frac{k_1\bar{\delta}}{\sigma^2}\right).
\end{aligned}
$$

We now aim to show that there exists a constant $\bar{C} > 0$ such that $\frac{k_1\bar{\delta}}{\sigma^2} \leq \bar{C}$. By the definitions of $\bar{\delta}$ and σ^2, it suffices to show that

$$
\log\left(1+\frac{n_2}{k_2^2}\right) \leq \bar{C}\sigma^2.
$$

Recall that $\tau^2 = C\log\left(1+\frac{n_2}{k_2^2}\right)$. Since we have already shown that $\tau^2 \lesssim \sigma^2$, it holds that there exists a constant $\bar{C} > 0$ such that $\log\left(1+\frac{n_2}{k_2^2}\right) \leq \bar{C}\sigma^2$ as desired. Therefore, there exists a constant $\bar{c} > 0$ such that $\log\left(1+\frac{\theta}{\sigma}\right) \geq \log\left(1+\frac{k_1\bar{\delta}}{\sigma}\right) \geq \bar{c}\frac{k_1\bar{\delta}}{\sigma}$. Thus we have

$$
\begin{aligned}
\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right) &\geq ck_2\theta k_1\bar{\delta} \quad (\text{for } c = \tilde{c}\bar{c}) \\
&\geq ck_2\frac{(k_1\bar{\delta})^2}{\sigma} \\
&= cC_\delta k_2\sigma\log\left(1+\frac{n_2}{k_2^2}\right) \\
&\geq C'_\delta k_2\log\left(1+\frac{n_2}{k_2^2}\right),
\end{aligned}
$$

where the final inequality uses $\sigma \gtrsim 1$, and $C'_\delta > 0$ is a positive multiple of C_δ. Since we have enforced $c' \geq 1$ in the definition of h_α, it holds

$$
h_\alpha = C^*\left(\sqrt{n_2\exp\left(-c'\log(1+\frac{n_2}{k_2^2})\right)\log\left(\frac{2}{\alpha}\right)}\right)
$$

$$
\begin{aligned}
&\leq C^* \left(k_2 \sqrt{\frac{\frac{n_2}{k_2^2}\log(2/\alpha)}{1+\frac{n_2}{k_2^2}}} \right) \\
&\leq C^* \left(k_2 \sqrt{\log(2/\alpha)} \right) \\
&\leq \frac{C_\delta'}{2} \left(k_2 \log\left(1+\frac{n_2}{k_2^2}\right) \right),
\end{aligned}
$$

where the final inequality holds for C_δ taken sufficiently large, and uses that $\frac{n_2}{k_2^2} \geq c > 0$. Therefore, we have shown that for C_δ taken sufficiently large, it holds

$$
\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{n_2} \big(W_j - \nu_\tau^{k_1}\big)\mathbf{1}(\bar{A}_j > \tau)\right] &\geq \tilde{c} k_2 \theta\sigma \log\left(1+\frac{\theta}{\sigma}\right) \\
&\geq 2h_\alpha.
\end{aligned}
$$

Letting $U := \sum_{j=1}^{n_2} \big(W_j - \nu_\tau^{k_1}\big)\mathbf{1}(\bar{A}_j > \tau)$, we therefore can conclude

$$
\begin{aligned}
\mathbb{P}_{\mathbf{P}}\big(\Delta^{h_\alpha}_{\text{trunc-deg},1} = 0\big) &= \mathbb{P}_{\mathbf{P}}\big(U \leq h_\alpha\big) \\
&= \mathbb{P}_{\mathbf{P}}\big(U - \mathbb{E}[U] \leq h_\alpha - \mathbb{E}[U]\big) \\
&= \mathbb{P}_{\mathbf{P}}\Big(\big(U - \mathbb{E}[U]\big)^2 \geq \big(h_\alpha - \mathbb{E}[U]\big)^2\Big) \quad \text{(since } \mathbb{E}[U] \geq 2h_\alpha\text{)} \\
&\leq \mathbb{P}_{\mathbf{P}}\Big(\big(U - \mathbb{E}[U]\big)^2 \geq \big(\mathbb{E}[U]\big)^2\Big) \\
&\leq \frac{\mathrm{var}\big(U\big)}{\big(\mathbb{E}[U]\big)^2} \quad \text{(by Markov's inequality)} \\
&\leq \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\big(1+\frac{\theta}{\sigma}\big) + C_1(n_2-k_2)\big(\sigma\tau\log(1+\frac{\tau}{\sigma})\big)^2 \exp\big(-\frac{3}{8}\tau^2\big)}{\big(\tilde{c}k_2\theta\sigma\log\big(1+\frac{\theta}{\sigma}\big)\big)^2} \\
&= \mathrm{I} + \mathrm{II},
\end{aligned}
$$

where we define

$$
\mathrm{I} = \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\big(1+\frac{\theta}{\sigma}\big)}{\big(\tilde{c}k_2\theta\sigma\log\big(1+\frac{\theta}{\sigma}\big)\big)^2},
$$

and

$$
\mathrm{II} = \frac{C_1(n_2-k_2)\big(\sigma\tau\log(1+\frac{\tau}{\sigma})\big)^2 \exp\big(-\frac{3}{8}\tau^2\big)}{\big(\tilde{c}k_2\theta\sigma\log\big(1+\frac{\theta}{\sigma}\big)\big)^2}.
$$

We control these terms separately. First we have

$$
\begin{aligned}
\mathrm{I} &= \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\big(1+\frac{\theta}{\sigma}\big)}{\big(\tilde{c}k_2\theta\sigma\log\big(1+\frac{\theta}{\sigma}\big)\big)^2} \\
&= \frac{C_1(\sigma^2+\theta\sigma)}{\tilde{c}^2 k_2\theta^2\sigma^2} \\
&= \frac{C_1}{\tilde{c}^2 k_2\theta^2} + \frac{C_1}{\tilde{c}^2 k_2\theta\sigma} \\
&\leq \frac{C_1}{C_\delta'\tilde{c}^2 k_2\sqrt{\log\left(1+\frac{n_2}{k_2^2}\right)}} \\
&\quad + \frac{C_1}{C_\delta'\tilde{c}^2 k_2\log\left(1+\frac{n_2}{k_2^2}\right)} \\
&\leq \frac{C_1}{C_\delta'\tilde{c}^2 k_2\sqrt{\log(1+c)}} + \frac{C_1}{C_\delta'\tilde{c}^2 k_2\log(1+c)}
\end{aligned}
$$

$$\leq \frac{\alpha}{4},$$

where the final inequality holds for C_δ taken large enough. Recall that $C \geq \frac{8}{3}$ in the definition of τ. Then it holds

$$\begin{aligned}
\mathrm{II} &= \frac{C_1(n_2-k_2)\left(\sigma\tau\log(1+\frac{\tau}{\sigma})\right)^2\exp\left(-\frac{3}{8}\tau^2\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\
&\leq \frac{C_1 n_2}{\tilde{c}^2k_2^2\left(1+\frac{n_2}{k_2^2}\right)}\cdot\frac{\left(\tau\log\left(1+\frac{\tau}{\sigma}\right)\right)^2}{\left(\theta\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\
&\leq \frac{C_1}{\tilde{c}^2}\cdot\frac{\left(C\theta\log(1+c)\right)^2}{\left(\theta\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \quad \text{(since } \tau \leq c\sigma \text{ and } \tau \leq C\theta\text{)} \\
&\leq \frac{C_1}{\tilde{c}^2}\cdot\frac{\left(C\log(1+c)\right)^2}{C_\delta'\left(k_2\log\left(1+\frac{n_2}{k_2^2}\right)\right)} \\
&\leq \frac{C_1}{\tilde{c}^2}\cdot\frac{\left(C\log(1+c)\right)^2}{C_\delta'\left(k_2\log\left(1+c\right)\right)} \\
&\leq \frac{\alpha}{4},
\end{aligned}$$

where again the final inequality holds for C_δ taken sufficiently large. Therefore, we have shown

$$\mathbb{P}_{\mathbf{P}}\left(\Delta^{h_\alpha}_{\text{trunc-deg},1}=0\right) \leq \mathrm{I}+\mathrm{II} \leq \frac{\alpha}{2}.$$

Combining this with our bound on the Type 1 error, it holds

$$\mathcal{R}(\Delta^{h_\alpha}_{\text{trunc-deg},1},\delta) \leq \alpha,$$

and the proof is complete. $\square$

B.3 Analysis of the max truncated degree test

The following lemma controls the risk of $\Delta^h_{\text{max-trunc-deg},1}$. The analysis of $\Delta^h_{\text{max-trunc-deg},2}$ is analogous.

Lemma 17. *Let $\alpha \in (0,1)$ and define*

$$h_\alpha = C^*\left(\sqrt{n_2\exp\left(-c'\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)\right)\log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)}+\log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right)$$

for constants $c', C^* > 0$ to be determined later. Suppose that $\frac{n_2}{k_2^2}\log\binom{n_1}{k_1} > c$ for a constant $c \geq 0$. Then there exist constants $C_\delta, C_\alpha > 0$ and $C \geq \frac{8}{3}$ such that if $\frac{1}{4} \geq p_0 \geq \frac{C_\alpha}{k_1k_2}\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right)$ and

$$\delta^2 \geq C_\delta p_0(1-p_0)\left(\frac{1}{k_1}\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)+\frac{1}{k_2k_1}\log\binom{n_1}{k_1}\right),$$

then the truncated degree test with threshold $h = h_\alpha$ and $\tau = \sqrt{C\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)}$ satisfies

$$\mathcal{R}(\Delta^{h_\alpha}_{\textit{trunc-deg},1},\delta) \leq \alpha.$$

Proof of Lemma 17. Let $\sigma = \sqrt{k_1p_0(1-p_0)}$. We begin by controlling the Type 1 error.

$$\begin{aligned}
\mathbb{P}_0(\Delta^{h_\alpha}_{\text{max-trunc-deg},1}=1) &= \mathbb{P}_0(t_{\text{max-trunc-deg},1} > h_\alpha) \\
&= \mathbb{P}_0\left(\max\left\{\sum_{j=1}^{n_2}(W_{J_1,j}-\nu^{k_1}_\tau)\mathbf{1}(t_{J_1,j}>\tau)\,\Big|\, J_1 \in \mathcal{P}_{k_1}(n_1)\right\} > h_\alpha\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \binom{n_1}{k_1}\mathbb{P}_0\left(\sum_{j=1}^{n_2}(W_{J_1,j}-\nu_\tau^{k_1})\mathbf{1}(t_{J_1,j}>\tau)>h_\alpha\right)\\
&\leq \binom{n_1}{k_1}\exp\left(-\log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right)\\
&=\frac{\alpha}{2},
\end{aligned}
$$

where the second inequality follows from Lemma 19 as long as h_α is defined with the constants C^* and c' obtained from Lemma 19 and $\tau \in [C', c\sigma]$ for some constants $C', c > 0$. We impose that the constant c' is made to be greater than 1 by taking the constant C in the definition of τ sufficiently large. Therefore, we just need to verify that $\tau \in [C', c\sigma]$ to finalize our control of the Type 1 error. First, we have

$$
\begin{aligned}
\tau &= \sqrt{C\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)}\\
&> \sqrt{C\log(1+c)}\\
&\geq C',
\end{aligned}
$$

where the final inequality holds for C taken sufficiently large. Additionally, using $\frac{1}{4} \geq p_0 \geq \frac{C_\alpha}{k_1k_2}\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right)$ we have

$$
\begin{aligned}
c\sigma &= c\sqrt{k_1p_0(1-p_0)}\\
&\geq c\sqrt{\frac{3}{4}C_\alpha\log\left(e\frac{n_2}{k_2}\binom{n_1}{k_1}\right)}\\
&\geq \sqrt{C\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)} \quad \text{(for } C_\alpha \text{ large enough)}\\
&= \tau.
\end{aligned}
$$

Therefore $\tau \in [C', c\sigma]$, and the bound on the Type 1 error holds.

Now we turn our attention to the Type 2 error. Under the alternative hypothesis, there exist sets $K_1 \in \mathcal{P}_{k_1}(n_1)$ and $K_2 \in \mathcal{P}_{k_2}(n_2)$ such that $\mathbb{E}[\mathbf{A}] = \mathbf{P}$ with entries $P_{ij} \geq p_0+\delta$ if $(i,j) \in K_1 \times K_2$ and $P_{ij} = p_0$ otherwise. Let $\theta = \frac{k_1\delta}{\sigma}$. Then by Lemma 20, there exist constants $\tilde{c}, c, C' > 0$ such that if $\tau \in [C', c\sigma]$, $k_1(p_0+\delta) \geq C'$, and $\theta \geq C'\tau$, then

$$
\mathbb{E}\left[\sum_{j=1}^{n_2}(W_{K_1,j}-\nu_\tau^{k_1})\mathbf{1}(t_{K_1,j}>\tau)\right] \geq \tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right). \tag{22}
$$

As when we controlled the Type 1 error, we enforce $\tau \in [C', c\sigma]$ by taking C in the definition of τ and C_α sufficiently large, in that order. To verify $k_1(p_0+\delta) \geq C'$, we calculate

$$
\begin{aligned}
k_1(p_0+\delta) &\geq k_1\delta\\
&\geq \sqrt{k_1p_0(1-p_0)C_\delta\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)}\\
&\geq \sqrt{\frac{3}{4}C_\alpha\log\left(e\frac{n_2}{k_2}\binom{n_1}{k_1}\right)C_\delta\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)}\\
&\geq \sqrt{\frac{3}{4}C_\alpha C_\delta\log(1+c)}\\
&\geq C',
\end{aligned}
$$

where the finally inequality holds for C_δ taken sufficiently large. Finally, to verify $\theta \geq C'\tau$, we have

$$
\theta = \frac{k_1\delta}{\sqrt{k_1p_0(1-p_0)}}
$$

$$
\begin{aligned}
&\geq \sqrt{C_\delta \left(\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \frac{1}{k_2}\log\binom{n_1}{k_1}\right)} \\
&\geq C'\sqrt{C\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)} \quad \text{(for } C_\delta \text{ taken large enough)} \\
&= C'\tau.
\end{aligned}
$$

Therefore, (22) holds due to Lemma 20. Additionally, by Lemma 21, there exist constants $C_1, c, C' > 0$ such that if $\tau \in [C', c\sigma]$, $k_1(p_0 + \delta) \geq C'$, and $\theta \geq 2\tau$, then

$$
\operatorname{var}\left(\sum_{j=1}^{n_2}\left(W_{K_1,j} - \nu_\tau^{k_1}\right)\mathbf{1}(t_{K_1,j} > \tau)\right) \leq C_1 k_2(\sigma^2 + \theta\sigma)\log^2\left(1 + \frac{\theta}{\sigma}\right) + C_1(n_2 - k_2)\left(\sigma\tau\log(1 + \frac{\tau}{\sigma})\right)^2\exp\left(-\frac{3}{8}\tau^2\right). \tag{23}
$$

As before, we enforce $\tau \in [C', c\sigma]$,$k_1(p_0+\delta) \geq C'$, and $\theta \geq 2\tau$ by taking C, C_α, and C_δ sufficiently large in that order. We aim to control the Type 2 error by combining (22) and (23) in an application of Chebyshev's inequality. Before we can do this, we need to verify that the right-hand side of (22) is at least $2h_\alpha$. In what follows, we define

$$
\bar{\delta}^2 := C_\delta p_0(1 - p_0)\left(\frac{1}{k_1}\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \frac{1}{k_2 k_1}\log\binom{n_1}{k_1}\right),
$$

so that $\delta \geq \bar{\delta}$ by assumption. First, we have

$$
\begin{aligned}
\log\left(1 + \frac{\theta}{\sigma}\right) &= \log\left(1 + \frac{k_1\delta}{\sigma^2}\right) \\
&\geq \log\left(1 + \frac{k_1\bar{\delta}}{\sigma^2}\right).
\end{aligned}
$$

We now aim to show that there exists a constant $\bar{C} > 0$ such that $\frac{k_1\bar{\delta}}{\sigma^2} \leq \bar{C}$. By the definitions of $\bar{\delta}$ and σ^2, it suffices to show that

$$
\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \frac{1}{k_2}\log\binom{n_1}{k_1} \leq \bar{C}\sigma^2.
$$

Recall that $\tau^2 = C\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)$. Since we have already shown that $\tau \leq c\sigma$, it is clear that $\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) \leq C'\sigma^2$ for some $C' > 0$. Furthermore, by the assumption $\frac{1}{4} \geq p_0 \geq \frac{C_\alpha}{k_1 k_2}\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right)$, it holds

$$
\begin{aligned}
\sigma^2 &= k_1 p_0(1 - p_0) \\
&\geq \frac{3}{4}\frac{C_\alpha}{k_2}\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right) \\
&\geq \frac{3}{4}\frac{C_\alpha}{k_2}\log\binom{n_1}{k_1},
\end{aligned}
$$

and hence $k_2^{-1}\log\binom{n_1}{k_1} \leq C''\sigma^2$ for some $C'' > 0$. Taking $\bar{C} = (C' + C'')/C_\delta$, we have that $\frac{k_1\bar{\delta}}{\sigma^2} \leq \bar{C}$. Therefore, there exists a constant $\bar{c} > 0$ such that $\log\left(1 + \frac{\theta}{\sigma}\right) \geq \log\left(1 + \frac{k_1\bar{\delta}}{\sigma}\right) \geq \bar{c}\frac{k_1\bar{\delta}}{\sigma}$. With this result in hand, we have

$$
\begin{aligned}
\tilde{c}k_2\theta\sigma\log\left(1 + \frac{\theta}{\sigma}\right) &\geq ck_2\theta k_1\bar{\delta} \quad \text{(for } c = \tilde{c}\bar{c}\text{)} \\
&\geq ck_2\frac{(k_1\bar{\delta})^2}{\sigma} \\
&= cC_\delta k_2\sigma\left(\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \frac{1}{k_2}\log\binom{n_1}{k_1}\right) \\
&\geq C'_\delta\left(k_2\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \log\binom{n_1}{k_1}\right),
\end{aligned}
$$

where the final inequality uses $\sigma \gtrsim 1$, and $C'_\delta > 0$ is a positive multiple of C_δ. Note that, since we have enforced $c' \geq 1$ in the definition of h_α, it holds

$$\begin{aligned} h_\alpha &= C^*\left(\sqrt{n_2 \exp\left(-c' \log(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1})\right)\log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)} + \log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right) \\ &\leq C^*\left(k_2\sqrt{\frac{\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}}{1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}}} + \log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right) \\ &\leq C^*\left(k_2 + \log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right) \\ &\leq \frac{C'_\delta}{2}\left(k_2 \log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \log\binom{n_1}{k_1}\right), \end{aligned}$$

where the final inequality holds for C_δ taken sufficiently large, and uses that $\frac{n_2}{k_2^2}\log\binom{n_1}{k_1} \geq c > 0$. To summarize, we have shown that for C_δ taken large enough, it holds

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^{n_2}\left(W_{K_1,j} - \nu_\tau^{k_1}\right)\mathbf{1}(t_{K_1,j} > \tau)\right] &\geq \tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right) \\ &\geq 2h_\alpha. \end{aligned}$$

Letting $U_{K_1} := \sum_{j=1}^{n_2}\left(W_{K_1,j} - \nu_\tau^{k_1}\right)\mathbf{1}(t_{K_1,j} > \tau)$, we therefore can conclude

$$\begin{aligned} \mathbb{P}_{\mathbf{P}}\left(t_{\text{max-trunc-deg}} \leq h_\alpha\right) &\leq \mathbb{P}_{\mathbf{P}}\left(U_{K_1} \leq h_\alpha\right) \\ &= \mathbb{P}_{\mathbf{P}}\left(U_{K_1} - \mathbb{E}[U_{K_1}] \leq h_\alpha - \mathbb{E}[U_{K_1}]\right) \\ &= \mathbb{P}_{\mathbf{P}}\left(\left(U_{K_1} - \mathbb{E}[U_{K_1}]\right)^2 \geq \left(h_\alpha - \mathbb{E}[U_{K_1}]\right)^2\right) \quad \text{(since } \mathbb{E}[U_{K_1}] \geq 2h_\alpha\text{)} \\ &\leq \mathbb{P}_{\mathbf{P}}\left(\left(U_{K_1} - \mathbb{E}[U_{K_1}]\right)^2 \geq \left(\mathbb{E}[U_{K_1}]\right)^2\right) \\ &\leq \frac{\mathrm{var}\left(U_{K_1}\right)}{\left(\mathbb{E}[U_{K_1}]\right)^2} \quad \text{(by Markov's inequality)} \\ &\leq \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\left(1+\frac{\theta}{\sigma}\right) + C_1(n_2-k_2)\left(\sigma\tau\log(1+\frac{\tau}{\sigma})\right)^2\exp\left(-\frac{3}{8}\tau^2\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\ &= \mathrm{I} + \mathrm{II}, \end{aligned}$$

where we define

$$\mathrm{I} = \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\left(1+\frac{\theta}{\sigma}\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2},$$

and

$$\mathrm{II} = \frac{C_1(n_2-k_2)\left(\sigma\tau\log(1+\frac{\tau}{\sigma})\right)^2\exp\left(-\frac{3}{8}\tau^2\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2}.$$

We control each of these terms separately. First we have

$$\begin{aligned} \mathrm{I} &= \frac{C_1 k_2(\sigma^2+\theta\sigma)\log^2\left(1+\frac{\theta}{\sigma}\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\ &= \frac{C_1(\sigma^2+\theta\sigma)}{\tilde{c}^2k_2\theta^2\sigma^2} \\ &= \frac{C_1}{\tilde{c}^2k_2\theta^2} + \frac{C_1}{\tilde{c}^2k_2\theta\sigma} \\ &\leq \frac{C_1}{C'_\delta\tilde{c}^2k_2\sqrt{\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \log\binom{n_1}{k_1}}} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{C_1}{C'_\delta \tilde{c}^2 k_2 \left(\log\left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \log\binom{n_1}{k_1}\right)} \\
&\leq \frac{C_1}{C'_\delta \tilde{c}^2 k_2 \sqrt{\log(1+c)}} + \frac{C_1}{C'_\delta \tilde{c}^2 k_2 \log(1+c)} \\
&\leq \frac{\alpha}{4},
\end{aligned}
$$

where the final inequality holds for C_δ taken large enough. Now we control II. Recall that $C \geq \frac{8}{3}$ in the definition of τ. Then it holds

$$
\begin{aligned}
\mathrm{II} &= \frac{C_1(n_2 - k_2)\left(\sigma\tau\log(1+\frac{\tau}{\sigma})\right)^2 \exp\left(-\frac{3}{8}\tau^2\right)}{\left(\tilde{c}k_2\theta\sigma\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\
&\leq \frac{C_1 n_2}{\tilde{c}^2 k_2^2 \left(1 + \frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)} \cdot \frac{\left(\tau\log\left(1+\frac{\tau}{\sigma}\right)\right)^2}{\left(\theta\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \\
&\leq \frac{C_1}{\tilde{c}^2} \cdot \frac{\left(C\theta\log(1+c)\right)^2}{\left(\theta\log\left(1+\frac{\theta}{\sigma}\right)\right)^2} \quad \text{(since } \tau \leq c\sigma \text{ and } \tau \leq C\theta\text{)} \\
&\leq \frac{C_1}{\tilde{c}^2} \cdot \frac{\left(C\log(1+c)\right)^2}{C'_\delta\left(k_2\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right) + \log\binom{n_1}{k_1}\right)} \\
&\leq \frac{C_1}{\tilde{c}^2} \cdot \frac{\left(C\log(1+c)\right)^2}{C'_\delta\left(k_2\log\left(1+c\right)\right)} \\
&\leq \frac{\alpha}{4},
\end{aligned}
$$

where again the final inequality holds for C_δ taken sufficiently large. Therefore, we have shown

$$
\mathbb{P}_{\mathbf{P}}\left(t_{\text{max-trunc-deg},1} \leq h_\alpha\right) \leq \mathrm{I} + \mathrm{II} \leq \frac{\alpha}{2}.
$$

Combining this with our bound on the Type 1 error, it holds

$$
\mathcal{R}(\Delta^{h_\alpha}_{\text{max-trunc-deg},1}, \delta) \leq \alpha,
$$

and the proof is complete. $\square$

B.4 Proof of Theorem 2

Here, we provide the proof of Theorem 2, which gives an upper bound on the minimax rate of separation δ^*. We let

$$
\tilde{R} = \left(\psi_{12} + \nu_{21}\right) \wedge \left(\psi_{21} + \nu_{12}\right) \wedge \phi_{12} \wedge \phi_{21}.
$$

We define our optimal test as

$$
\Delta^* = \begin{cases}
\Delta^{h_3}_{\text{max-trunc-deg},1} & \text{if } \tilde{R} = \psi_{12} + \nu_{21} \\
\Delta^{h_3}_{\text{max-trunc-deg},2} & \text{if } \tilde{R} = \psi_{21} + \nu_{12} \\
\Delta^{h_1,h_2}_{a} & \text{if } \tilde{R} = \phi_{12} \\
\Delta^{h'_1,h'_2}_{b} & \text{if } \tilde{R} = \phi_{21},
\end{cases}
$$

where

$$
\Delta^{h_1,h_2}_{a} = \begin{cases}
\Delta^{h_1}_{\text{trunc-deg},1} & \text{if } \frac{n_2}{k_2^2} \geq c_1, \\
\Delta^{h_2}_{\text{deg}} & \text{otherwise}
\end{cases}
\qquad \text{and} \qquad
\Delta^{h'_1,h'_2}_{b} = \begin{cases}
\Delta^{h'_1}_{\text{trunc-deg},2} & \text{if } \frac{n_1}{k_1^2} \geq c_1, \\
\Delta^{h'_2}_{\text{deg}} & \text{otherwise.}
\end{cases}
$$

and, for some small enough $\alpha > 0$, we let

$$
h_3 = C^*\left(\sqrt{n_2\exp\left(-c'\log\left(1+\frac{n_2}{k_2^2}\log\binom{n_1}{k_1}\right)\right)\log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)} + \log\left(\frac{2}{\alpha}\binom{n_1}{k_1}\right)\right)
$$

$$h_4 = C^* \left(\sqrt{n_1 \exp\left(-c' \log\left(1 + \frac{n_1}{k_1^2} \log\binom{n_2}{k_2}\right)\right) \log\left(\frac{2}{\alpha}\binom{n_2}{k_2}\right)} + \log\left(\frac{2}{\alpha}\binom{n_2}{k_2}\right)\right)$$
$$h_1 = C^* \left(\sqrt{n_2 \exp\left(-c' \log(1 + \frac{n_2}{k_2^2})\right) \log(2/\alpha)} + \log(2/\alpha)\right)$$
$$h_1' = C^* \left(\sqrt{n_1 \exp\left(-c' \log(1 + \frac{n_1}{k_1^2})\right) \log(2/\alpha)} + \log(2/\alpha)\right)$$
$$h_2 = h_2' = \sqrt{4 \log(2/\alpha)}.$$

We recall the assumption

$$p_0 \geq \begin{cases} \frac{C_\eta}{k_1 k_2} \log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right) & \text{if } \tilde{R} = \left(\psi_{12} + \nu_{21}\right) \wedge \left(\psi_{21} + \nu_{12}\right) \\ \frac{C_\eta}{n_1} \log\left(1 + \frac{n_2}{k_2^2}\right) & \text{if } \tilde{R} = \phi_{12} \text{ and } n_2 > k_2^2 \\ \frac{C_\eta}{n_2} \log\left(1 + \frac{n_1}{k_1^2}\right) & \text{if } \tilde{R} = \phi_{21} \text{ and } n_1 > k_1^2 \\ \frac{C_\eta}{n_1 n_2} & \text{otherwise.} \end{cases}$$

By Lemma 14, we always have $R \gtrsim \tilde{R}$. Assume first that $\tilde{R} = \left(\psi_{12} + \nu_{21}\right) \wedge \left(\psi_{21} + \nu_{12}\right)$, and, by symmetry, assume that we have $\tilde{R} = \psi_{12} + \nu_{21}$. In this case, we have $\Delta^* = \Delta^{h_3}_{\text{max-trunc-deg},1}$, and the result follows by Lemma 17. We can proceed similarly in the case where $\tilde{R} = \psi_{21} + \nu_{12}$.

Assume now that $\tilde{R} = \phi_{12}$ and $n_2 > k_2^2$. Then the result follows by Lemma 16. Similarly, if $\tilde{R} = \phi_{21}$ and $n_1 > k_1^2$, the result follows by Lemma 16. Finally, if none of the conditions above are satisfied, then we have $\Delta^* = \Delta^{h_2}_{\text{deg}}$ and the result follows by Lemma 15. The proof is complete.

C Additional results

C.1 Technical results for the analysis of the truncated chi-square tests

Lemma 18. *Let $n \in \mathbb{N}$ and $p \in (0, 1/4)$. Let $X \sim \text{Bin}(n, p)$ and, for any $a > 0$, define*

$$Z = (X - np)/\sigma, \qquad \text{where } \sigma = \sqrt{np(1-p)}$$
$$v(x) = \sigma x \log\left(1 + \frac{x}{\sigma}\right), \quad \forall x > 0.$$

For any $\alpha \geq 1$, there exist two constants $C_\alpha, \bar{C} > 0$ such that, for any $a \in \left[C_\alpha, \sigma\right]$, we have $\mathbb{E}\left[v(Z)^\alpha \middle| Z \geq a\right] \leq \bar{C} v(a)^\alpha$.

Proof of Lemma 18. We first note that the event $\{Z \geq a\}$ is equivalent to $\{X \geq k_{\min}\}$ where $k_{\min} = \lceil np + a\sigma \rceil$. Let $c^* > 0$ be a constant whose value will be adjusted later. Let also $\alpha > 0$. We have

$$\begin{aligned} \mathbb{E}\left[v(Z)^\alpha \mathbf{1}\big(Z \geq a\big)\right] &= \mathbb{E}\left[v(Z)^\alpha \mathbf{1}\big(X \geq k_{\min}\big)\right] \\ &= \mathbb{E}\left[v(Z)^\alpha \mathbf{1}\big(k_{\min} \leq X \leq k_{\min} + c^* a\sigma\big)\right] \\ &\quad + \mathbb{E}\left[v(Z)^\alpha \mathbf{1}\big(k_{\min} + c^* a\sigma < X \leq n-2\big)\right] \\ &\quad + \mathbb{E}\left[v(Z)^\alpha \mathbf{1}\big(X \geq n-1\big)\right] \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{24}$$

We control each term separately.

Term III. We use the inequality $\log(ab) \leq 2a\log(b)$ when $a > 1$ and $b \geq 2$. Indeed,

$$\log(b) \geq \frac{1}{2}\log(b+1) \geq \frac{1}{2}\log\left(\frac{ab}{a} + (1-\frac{1}{a})\cdot 1\right) \geq \frac{1}{2a}\log(ab)$$

by concavity of the logarithm. Now, noting that $n(1-p) \geq p$, we have

$$\begin{aligned}
\mathrm{III} &= \mathbb{E}\left[v\left(\frac{X-np}{\sigma}\right)^{\alpha}\mathbf{1}\big(X \geq n-1\big)\right] \\
&= v\left(\frac{n(1-p)-1}{\sigma}\right)^{\alpha} np^{n-1}(1-p) + v\left(\frac{n(1-p)}{\sigma}\right)^{\alpha} p^{n} \\
&\leq 2v\left(\frac{n(1-p)}{\sigma}\right)^{\alpha} np^{n-1}(1-p) \\
&= 2\left(\frac{\sigma}{p}\right)^{\alpha+2}\log^{\alpha}\left(1+\frac{1}{p}\right)p^{n} \\
&= 2\left(\sqrt{\frac{n(1-p)}{p}}\right)^{\alpha+2} p^{n-\alpha} \\
&=\leq 2\exp\left(\frac{\alpha+2}{2}\left(\log(n(1-p)) + \log\left(\frac{1}{p}\right)\right) - (n-\alpha)\log\left(\frac{1}{p}\right)\right).
\end{aligned}$$

Now, recall that we have $\sqrt{n/4} \geq \sigma \geq a \geq C_{\alpha}$. Therefore, we have $\frac{\alpha+2}{2}\log(n(1-p)) \leq \frac{n-\alpha}{4}\log(1/p)$ and $\frac{\alpha+2}{2}\log(1/p) \leq \frac{n-\alpha}{4}\log(1/p)$ provided C_{α} is sufficiently large. Therefore,

$$\mathrm{III} \leq 2\exp(-n/2) \leq 2e^{-\frac{a^2}{2}} \leq 4a\mathbb{P}(Z \geq a). \tag{25}$$

In the last line, we used Theorem 2.1 from [5], which states that whenever $p \leq \frac{1}{4}$, it holds that $\mathbb{P}(Z \geq a) \geq 1 - \Phi(a) \geq \frac{1}{2a}e^{-\frac{a^2}{2}}$ for $a \geq 2$ where $\Phi(a)$ denotes the cumulative distribution function of the standard normal distribution.

Term I. Note that, if $k \in [k_{\min}, k_{\min} + c^* a\sigma]$, then

$$\begin{aligned}
v\left(\frac{k-np}{\sigma}\right)^{\alpha} &\leq v\left(\frac{k_{\min} + c^* a\sigma - np}{\sigma}\right)^{\alpha} \\
&= v\left(\frac{\lceil np + a\sigma\rceil + c^* a\sigma - np}{\sigma}\right)^{\alpha} \\
&\leq v\left(\frac{(1+c^*)a\sigma + 1}{\sigma}\right)^{\alpha} \\
&\leq v\Big((1+\frac{3}{2}c^*)a\Big)^{\alpha} \\
&\leq (1+2c^*)^{\alpha}\, v(a)^{\alpha}
\end{aligned}$$

provided the constant $C_{\alpha} > 0$ is chosen to be large enough. Therefore, we obtain

$$\begin{aligned}
\mathrm{I} &= \mathbb{E}\left[v\left(\frac{X-np}{\sigma}\right)^{\alpha}\mathbf{1}\big(k_{\min} \leq X \leq k_{\min} + c^* a\sigma\big)\right] \\
&\leq (1+2c^*)^{\alpha} v(a)^{\alpha}\mathbb{P}(Z \geq a).
\end{aligned} \tag{26}$$

Term II. We let $j_0 = \lfloor c^* a\sigma \rfloor + 1$ and $k' = k_{\min} + j_0$, so that the event $X > k_{\min} + c^* a\sigma$ is equivalent to $X \geq k'$. Therefore, we obtain

$$\begin{aligned}
\mathrm{II} &= \sum_{k=k'}^{n-2} v\left(\frac{k-np}{\sigma}\right)^{\alpha} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&\leq \sum_{k=k'}^{n-2} v\left(\frac{k-np}{\sigma}\right)^{\alpha} \frac{C_0}{c_0^2\sqrt{2\pi}} \sqrt{\frac{n}{(n-k)k}} \left(\frac{n(1-p)}{n-k}\right)^{n-k} \left(\frac{np}{k}\right)^{k} \qquad \text{by Lemma 25} \\
&\leq \frac{C_0}{c_0^2\sqrt{2\pi}} \sum_{k=k'}^{n-2} v\left(\frac{k-np}{\sigma}\right)^{\alpha} \sqrt{\frac{n}{(n-k)k}} \left(1 - \frac{k-np}{n(1-p)}\right)^{-n+k} \left(1 + \frac{k-np}{np}\right)^{-k} \\
&\leq \frac{C_0}{c_0^2\sqrt{2\pi}} \sum_{j=j_0}^{n-k_{\min}-2} v\left(\frac{a\sigma+j+1}{\sigma}\right)^{\alpha} \sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}} \\
&\qquad \left(1 - \frac{k_{\min}+j-np}{n(1-p)}\right)^{-n+k_{\min}+j} \left(1 + \frac{k_{\min}+j-np}{np}\right)^{-k_{\min}-j} \\
&=: \tilde{C} \sum_{j=j_0}^{n-k_{\min}-2} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha} \sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}} \left(1 - \frac{a\sigma+j+1}{n(1-p)}\right)^{-n+k_{\min}+j} \left(1 + \frac{a\sigma+j}{np}\right)^{-k_{\min}-j},
\end{aligned}$$

where $\tilde{C} = 2\frac{C_0}{c_0^2\sqrt{2\pi}}$ and where we used the fact that $v\left(\frac{a\sigma+j+1}{\sigma}\right)^{\alpha} \leq 2v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}$ provided that $\sigma \geq a \geq C_\alpha$ for some large enough constant C_α. Moreover, we note that

$$\begin{aligned}
&\left(1 - \frac{a\sigma+j+1}{n(1-p)}\right)^{-n+k_{\min}+j} \left(1 + \frac{a\sigma+j}{np}\right)^{-k_{\min}-j} \\
&= \exp\left(-(n-k_{\min}-j)\log\left(1 - \frac{a\sigma+j+1}{n(1-p)}\right) - (k_{\min}+j)\log\left(1 + \frac{a\sigma+j}{np}\right)\right) \\
&\leq \exp\left(-(n-np-a\sigma-j)\log\left(1 - \frac{a\sigma+j+1}{n(1-p)}\right) - (np+a\sigma+j)\log\left(1 + \frac{a\sigma+j}{np}\right)\right) \\
&= \exp\left(-n(1-p)\left(1 - \frac{a\sigma+j}{n(1-p)}\right)\log\left(1 - \frac{a\sigma+j+1}{n(1-p)}\right) - np\left(1 + \frac{a\sigma+j}{np}\right)\log\left(1 + \frac{a\sigma+j}{np}\right)\right).
\end{aligned}$$

We now argue that the latter quantity is at most a constant times the quantity below:

$$\exp\left(-n(1-p)\left(1 - \frac{a\sigma+j}{n(1-p)}\right)\log\left(1 - \frac{a\sigma+j}{n(1-p)}\right) - np\left(1 + \frac{a\sigma+j}{np}\right)\log\left(1 + \frac{a\sigma+j}{np}\right)\right).$$

Indeed, the ratio of the two is

$$\exp\left(-n(1-p)\left(1 - \frac{a\sigma+j}{n(1-p)}\right)\log\left(1 - \frac{\frac{1}{n(1-p)}}{1 - \frac{a\sigma+j}{n(1-p)}}\right)\right),$$

and since $j \leq n - k_{\min} - 2 \leq n(1-p) - a\sigma - 2$, it follows that

$$\frac{\frac{1}{n(1-p)}}{1 - \frac{a\sigma+j}{n(1-p)}} \leq \frac{\frac{1}{n(1-p)}}{\frac{2}{n(1-p)}} = \frac{1}{2}.$$

Using the inequality $\log(1-x) \geq -2x$ for $x \in (0, 1/2)$, the above ratio is controlled as

$$\exp\left(-n(1-p)\left(1 - \frac{a\sigma+j}{n(1-p)}\right)\log\left(1 - \frac{\frac{1}{n(1-p)}}{1 - \frac{a\sigma+j}{n(1-p)}}\right)\right) \leq \exp(2).$$

Combining these calculations, we obtain

$$\mathrm{II} \leq \tilde{C} \sum_{j=j_0}^{n-k_{\min}-2} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha} \sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}}$$

$$\exp\left(-n(1-p)\left(1-\frac{a\sigma+j}{n(1-p)}\right)\log\left(1-\frac{a\sigma+j}{n(1-p)}\right)-np\left(1+\frac{a\sigma+j}{np}\right)\log\left(1+\frac{a\sigma+j}{np}\right)\right)$$

$$=\tilde{C}\sum_{j=j_0}^{n-k_{\min}-2} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}}\exp\left(-n(1-p)h\left(-\frac{a\sigma+j}{n(1-p)}\right)-np\,h\left(\frac{a\sigma+j}{np}\right)\right)$$

where we have defined the Bennett function

$$h(x)=(1+x)\log(1+x)-x,\qquad \forall x>-1.$$

Now, we define the sets of indices $J_1=\{j_0,\ldots,\lfloor 2\sigma^2\rfloor\}$ and $J_2=\{\lfloor 2\sigma^2\rfloor+1,\ldots,n-k_{\min}-2\}$.
We have

$$\begin{aligned}
\mathrm{II} &\le \tilde{C}\sum_{j\in J_1} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}}\exp\left(-n(1-p)h\left(-\frac{a\sigma+j}{n(1-p)}\right)-np\,h\left(\frac{a\sigma+j}{np}\right)\right)\\
&\quad+\tilde{C}\sum_{j\in J_2} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}}\exp\left(-n(1-p)h\left(-\frac{a\sigma+j}{n(1-p)}\right)-np\,h\left(\frac{a\sigma+j}{np}\right)\right)\\
&=: \mathrm{II}_{J_1}+\mathrm{II}_{J_2}
\end{aligned}\tag{27}$$

We now control these two terms separately. Suppose first that $j\in J_1$. Then we have

$$\frac{a\sigma+j}{np}\wedge\frac{a\sigma+j}{n(1-p)}\le\frac{a\sigma+2\sigma^2}{\sigma^2}\le 3,$$

and, since $j\le\sigma^2\le n/4$, we also have

$$\begin{aligned}
\sqrt{\frac{n}{(n-k_{\min}-j)(k_{\min}+j)}} &\le \sqrt{\frac{n}{(n-np-a\sigma-1-n/4)(k_{\min}+j)}}\\
&\le \sqrt{\frac{n}{(n/4-1)(k_{\min}+j)}}\\
&\le 3\sqrt{\frac{1}{k_{\min}+j}}.
\end{aligned}$$

In the last line, we used the fact that, since $\sqrt{n/4}\ge\sigma\ge a\ge C_\alpha$, we have $n\ge 4C_\alpha^2$, so that $\frac{n}{n/4-1}\le 9$ for C_α large enough. Moreover, we have $h(x)\ge x^2/5$ for any $x\in(-1,3]$. Therefore,

$$\begin{aligned}
\frac{1}{3\tilde{C}}\mathrm{II}_{J_1} &\le \sum_{j\in J_1} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{1}{k_{\min}+j}}\exp\left(-n(1-p)h\left(-\frac{a\sigma+j}{n(1-p)}\right)-np\,h\left(\frac{a\sigma+j}{np}\right)\right)\\
&\le \sum_{j\in J_1} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{1}{k_{\min}}}\exp\left(-\frac{1}{5}\frac{(a\sigma+j)^2}{n(1-p)}-\frac{1}{5}\frac{(a\sigma+j)^2}{np}\right)\\
&= \sum_{j\in J_1} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{1}{k_{\min}}}\exp\left(-\frac{1}{5}\frac{(a\sigma+j)^2}{\sigma^2}\right).
\end{aligned}$$

Note that the function $x\mapsto x^{\alpha}e^{-\frac{x^2}{5}}$ is decreasing over $[\sqrt{5\alpha/2},\infty)$. Therefore, choosing C_α large enough again, and letting $\tilde{C}$ denote a constant whose value may change in each appearance, we obtain

$$\begin{aligned}
&\frac{1}{3\tilde{C}}\mathrm{II}_{J_1}\sum_{j\in J_1} v\left(\frac{a\sigma+j}{\sigma}\right)^{\alpha}\sqrt{\frac{1}{k_{\min}}}\exp\left(-\frac{1}{5}\frac{(a\sigma+j)^2}{\sigma^2}\right)\\
&\qquad\le \frac{1}{\sqrt{k_{\min}}}\int_{j_0}^{\infty} v\left(\frac{a\sigma+x}{\sigma}\right)^{\alpha}e^{-\frac{(a\sigma+x)^2}{5\sigma^2}}dx \qquad \text{where } j_0-1=\lfloor c^*a\sigma\rfloor\\
&\qquad\le \frac{\sigma}{\sqrt{k_{\min}}}\int_{\sqrt{\frac{2}{5}}\left(a+\frac{1}{\sigma}\lfloor c^*a\sigma\rfloor\right)}^{\infty}\left(\frac{5}{2}\right)^{1/2} v\left(\sqrt{\frac{5}{2}}y\right)^{\alpha}e^{-\frac{y^2}{2}}dy\\
&\qquad\le \tilde{C}\frac{\sigma}{\sqrt{k_{\min}}}\int_a^{\infty} v(y)^{\alpha}e^{-\frac{y^2}{2}}dy \qquad \text{by taking } c^* \text{ large enough}
\end{aligned}$$

$$\leq \tilde{C}\frac{\sigma}{\sqrt{k_{\min}}}a^{\alpha-1}\log(1+a)^{\alpha}\exp\left(-\frac{a^2}{2}\right) \qquad \text{by Lemma 22}$$

so that

$$\mathrm{II}_{J_1} \leq C_{J_1} v(a)^{\alpha}\mathbb{P}\left(Z \geq a\right), \tag{28}$$

for some constant $C_{J_1} > 0$, where we used the fact that $\sigma \leq \sqrt{k_{\min}}$ and Theorem 2.1 from [5].

Now, we turn to the term II_{J_2}. Note that, when $j \in J_2$, we have

$$\frac{a\sigma + j}{np} \geq \frac{a\sigma + 2np(1-p)}{np} \geq 2(1-p) \geq \frac{3}{2}.$$

Moreover, since $j \geq \sigma^2$, we also have

$$\sqrt{\frac{n}{n-k_{\min}-j}}e^{-j/4} = \left(1 - \frac{k_{\min}+j}{n}\right)^{-1/2} e^{-j/4} = (1-x)^{-1/2}\, e^{(-nx+k_{\min})/4}$$

where $x = (k_{\min}+j)/n \in [2p, 1-\frac{2}{n}]$. We now prove that the function $g : x \mapsto (1-x)^{-1/2}\, e^{-nx+k_{\min}}$ is bounded over the interval $[2p, 1-\frac{2}{n}]$ independently of n. Indeed, the derivative of $\log g$ satisfies

$$(\log g)'(x) = \frac{1}{2(1-x)} - n/4,$$

so that g is decreasing over $[2p, 1-\frac{2}{n}]$. It now suffices to evaluate $g(2p)$

$$\begin{aligned} g(2p) &= (1-2p)^{-1/2}\exp\left(-2np + k_{\min}\right) \leq (1/2)^{-1/2}\exp\left(-2np + np + a\sigma + 1\right) \\ &\leq (1/2)^{-1/2}\exp\left(-\sigma^2/2\right) \leq (1/2)^{-1/2}\exp\left(-C_{\alpha}^2/2\right). \end{aligned}$$

Therefore, there exists a constant $\bar{c}$ such that, for any n and any $j \in J_2$, we have

$$\sqrt{\frac{n}{n-k_{\min}-j}}e^{-j} \leq \bar{c}.$$

By the inequality $h(x) \geq \frac{x}{2}$ that holds for any $x \geq 3/2$, we have

$$\begin{aligned} \frac{1}{\tilde{C}}\mathrm{II}_{J_2} &= \sum_{j\in J_2} v\left(a + \frac{j}{\sigma}\right)^{\alpha}\sqrt{\frac{n}{(k_{\min}+j)(n-k_{\min}-j)}}\exp\left(-n(1-p)h\left(-\frac{a\sigma+j}{n(1-p)}\right) - np\,h\left(\frac{a\sigma+j}{np}\right)\right) \\ &\leq \sum_{j\in J_2} v\left(a + \frac{j}{\sigma}\right)^{\alpha}\frac{1}{\sqrt{j}}\sqrt{\frac{n}{n-k_{\min}-j}}\exp\left(-np\,h\left(\frac{a\sigma+j}{np}\right)\right) \\ &\leq \sum_{j\in J_2} v\left(\frac{2j}{\sigma}\right)^{\alpha}\frac{1}{\sqrt{j}}\sqrt{\frac{n}{n-k_{\min}-j}}\exp\left(-\frac{a\sigma+j}{2}\right) \\ &\leq \tilde{C}\bar{c}\sum_{j\in J_2} v\left(\frac{j}{\sigma}\right)^{\alpha}\frac{1}{\sqrt{j}}\exp\left(-\frac{j}{4}\right) \\ &\leq \frac{\tilde{C}}{\sigma^{\alpha}}\int_{\lfloor 2\sigma^2\rfloor}^{\infty} x^{\alpha-1/2}\log(1+x)^{\alpha}\exp\left(-x/4\right)dx \\ &= \frac{\tilde{C}}{\sigma^{\alpha}}\int_{\lfloor 2\sigma^2\rfloor/4} (4y)^{\alpha-1/2}\log(1+4y)^{\alpha}\exp\left(-y\right)4dy \\ &= \frac{\tilde{C}}{\sigma^{\alpha}}\int_{\lfloor 2\sigma^2\rfloor/4} y^{\alpha-1/2}\log(1+y)^{\alpha}\exp\left(-y\right)4dy \\ &\leq \frac{\tilde{C}}{\sigma^{\alpha}}\left(\frac{\lfloor 2\sigma^2\rfloor}{4}\right)^{\alpha-1/2}\log\left(1+\frac{\lfloor 2\sigma^2\rfloor}{4}\right)^{\alpha}\exp\left(-\frac{\lfloor 2\sigma^2\rfloor}{4}\right) \qquad \text{by Lemma 23} \\ &\leq \frac{\tilde{C}}{\sigma^{\alpha}}\left(\frac{2\sigma^2}{4}\right)^{\alpha-1/2}\log\left(1+\frac{2\sigma^2}{4}\right)^{\alpha}\exp\left(-\frac{2\sigma^2}{4}+\frac{1}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq C_{J_2} \frac{v(\sigma)^\alpha}{\sigma} \exp\left(-\frac{\sigma^2}{2}\right) \qquad \text{for some constant } C_{J_2} \\
&\leq C_{J_2} \frac{v(a)^\alpha}{a} \exp\left(-\frac{a^2}{2}\right) \\
&\leq C_{J_2} v(a)^\alpha \mathbb{P}\left(Z \geq a\right) \qquad (29)
\end{aligned}
$$

where the last inequality follows from Theorem 2.1 from [5].

To conclude the proof of the lemma, it remains to combine equations (24), (25), (26), (27), (28) and (29) to obtain

$$
\begin{aligned}
\mathbb{E}\left[Z^\alpha | Z \geq a\right] &= \frac{\mathbb{E}\left[Z^\alpha \mathbf{1}\big(Z \geq a\big)\right]}{\mathbb{P}(Z \geq a)} \\
&= \frac{\mathrm{I} + \mathrm{II} + \mathrm{III}}{\mathbb{P}(Z \geq A)} \\
&\leq \frac{\mathrm{I} + \mathrm{II}_{J_1} + \mathrm{II}_{J_2} + \mathrm{III}}{\mathbb{P}(Z \geq A)} \\
&\leq \frac{(1+2c^*)^\alpha v(a)^\alpha \mathbb{P}\left(Z \geq v(a)\right) + C_{J_1} v(a)^\alpha \mathbb{P}\left(Z \geq a\right) + C_{J_2} v(a)^\alpha \mathbb{P}\left(Z \geq a\right) + 4v(a)\mathbb{P}(Z \geq a)}{\mathbb{P}(Z \geq a)} \\
&\leq \bar{C} v(a)^\alpha
\end{aligned}
$$

for some constant $\bar{C}$ and where we used $v(a) \leq a^\alpha$ since $a, \alpha \geq 1$. This completes the proof.

□

We recall that the Bennett function is defined as

$$
\begin{aligned}
h_B(x) &= (1+x)\log(1+x) - x, \qquad \forall x > -1. &(30)\\
h_B(-1) &= 1. &(31)
\end{aligned}
$$

Now, we let $Y \sim \mathrm{Bin}(n, p_0)$ for some $n \in \mathbb{N}$ and define

$$
\begin{aligned}
w(x) &= n(1-p_0)\, h_B\left(-\frac{x - np_0}{n(1-p_0)}\right) + np_0\, h_B\left(\frac{x - np_0}{np_0}\right), \forall x \in [0, n] &(32)\\
Z &= (Y - np_0)/\sigma, \qquad \text{where } \sigma = \sqrt{np_0(1-p_0)} \\
W &= w(Y) \\
\nu_a &= \mathbb{E}\left[W | Z \geq a\right] &(33)\\
\gamma_a &= \mathbb{E}\left[W^2 | Z \geq a\right]. &(34)
\end{aligned}
$$

We deduce the following corollary.

Corollary 1. *Recall the definitions of the functions ν and γ from (33) and (34). There exist sufficiently large constants $C, \bar{C}$ such that, for any $a \in [C, \sigma]$, it holds that $\nu_a \leq \bar{C}\sigma a \log\left(1 + \frac{a}{\sigma}\right) \leq \bar{C} a^2$ and $\gamma_a \leq \bar{C}(\sigma a)^2 \log^2\left(1 + \frac{a}{\sigma}\right) \leq \bar{C} a^4$.*

Lemma 19. *Let $m, n \in \mathbb{N}$ and $p_0 \in (0, 1/4)$. Let $Y_1, \ldots, Y_m \overset{iid}{\sim} \mathrm{Bin}(n, p_0)$ and for any $j \in \{1, \ldots, m\}$ define $Z_j = (Y_j - np_0)/\sigma$, where $\sigma = \sqrt{np_0(1-p_0)}$. For any $j \in \{1, \ldots, m\}$, define $W_j = w(Y_j)$ where w is defined in (32) and recall the definition of ν_a from (33) for $a > 0$. Then there exist universal constants $C, C^*, c, c' > 0$ such that for any $a \in [C, c\sigma]$ and $x > 0$, we have*

$$
\mathbb{P}\left(\sum_{j=1}^m \left(W_j - \nu_a\right) \mathbf{1}_{\{Z_j \geq a\}} \geq C^*\left(\sqrt{me^{-c'a^2}x} + x\right)\right) \leq e^{-x}.
$$

Proof of Lemma 19. In this proof, we will write p rather than p_0 to alleviate the notation. Let $Y \sim \mathrm{Bin}(n, p)$ and $Z = (Y - np)/\sigma$, and consider $X = (W - \nu_a)\mathbf{1}_{\{Z \geq a\}}$ where

$$
W = n(1-p)\, h_B\left(-\frac{Y - np}{n(1-p)}\right) + np\, h_B\left(\frac{Y - np}{np}\right).
$$

We first derive a bound for the moment-generating function $\mathbb{E}\left(e^{\lambda X}\right)$. Since $\mathbb{E}(X)=0$, we have

$$\mathbb{E}\left(e^{\lambda X}\right)=1+\mathbb{E}\left(e^{\lambda X}-1-\lambda X\right).$$

By the deterministic bound

$$e^{x}-1-x \leq \begin{cases}(-x)\wedge x^{2}, & \text{if } x<0\\ x^{2}, & \text{if } 0\leq x\leq 1\\ e^{x}, & \text{if } x\geq 1\end{cases}$$

we have

$$\mathbb{E}\left(e^{\lambda X}-1-\lambda X\right)\leq \lambda^{2}\mathbb{E}\left(X^{2}\mathbf{1}_{\{X<0\}}\right)+\lambda^{2}\mathbb{E}\left(X^{2}\mathbf{1}\{0\leq X\leq 1/\lambda\}\right)+\mathbb{E}\left(e^{\lambda X}\mathbf{1}_{\{X\geq 1/\lambda\}}\right)$$

and we will bound the three terms separately. By Corollary 1, the first term is bounded as

$$\begin{aligned}
\mathbb{E}\left(\left(|X|\wedge X^{2}\right)\mathbf{1}_{\{X<0\}}\right) &\leq \mathbb{P}(X<0)\left[\left|\nu_a-w(np+a\sigma)\right|\wedge\left(\nu_a-w(np+a\sigma)\right)^{2}\right]\\
&\leq \mathbb{P}(X<0)\left[\left|\nu_a-\frac{a^{2}}{8}\right|\wedge\left(\nu_a-\frac{a^{2}}{8}\right)^{2}\right] \qquad \text{by Lemma 32.2}\\
&\leq \mathbb{P}(Z\geq a)\left(\nu_a-\frac{a^{2}}{8}\right)^{2}\\
&\leq (\bar{C}-1/8)^{2}a^{4}\,\mathbb{P}\left(Y-np\geq a\sigma\right) \qquad \text{by Corollary 1}\\
&\leq (\bar{C}-1/8)^{2}a^{4}\exp\left(-\frac{a^{2}/2}{1+\frac{a/\sigma}{3}}\right) \qquad \text{by Lemma 24}\\
&= (\bar{C}-1/8)^{2}a^{4}\exp\left(-\frac{3a^{2}}{8}\right)\\
&\leq \exp\left(-\frac{a^{2}}{4}\right)
\end{aligned}$$

provided the universal constant C such that $a\geq C$ is chosen large enough depending on $\bar{C}$. We now bound the second term. Recalling the definition of the function γ from (34), we have

$$\begin{aligned}
\mathbb{E}\left(X^{2}\mathbf{1}_{\{0<X\leq 1/\lambda\}}\right) &\leq \mathbb{E}\left(X^{2}\mathbf{1}_{\{X>0\}}\right)=\mathbb{E}\left[\left(W-\nu_a\right)^{2}\mathbf{1}_{\{Z>\sqrt{\nu_a}\}}\right]\\
&\leq 2\mathbb{E}\left[W^{2}\mathbf{1}_{\{Z>\sqrt{\nu_a}\}}\right]+2\nu_a^{2}\,\mathbb{P}\left(Z>\sqrt{\nu_a}\right)\\
&= 2\left(\gamma_{\sqrt{\nu_a}}+\nu_a^{2}\right)\mathbb{P}\left(Z>\sqrt{\nu_a}\right)\\
&\leq 2\left(\bar{C}\nu_a^{2}+\nu_a^{2}\right)\exp\left(-\frac{\nu_a/2}{1+\frac{\sqrt{\nu_a}}{3\sigma}}\right) \qquad \text{by Lemmas 18 and 24}\\
&\leq 2\left(\bar{C}^{3}+\bar{C}^{2}\right)a^{4}\exp\left(-\frac{\nu_a/2}{1+\sqrt{\bar{C}}/3}\right) \qquad \text{by Lemma 18 and } a\leq\sigma\\
&\leq \exp\left(-\frac{\nu_a}{2\bar{C}}\right)\\
&\leq \exp\left(-\frac{a^{2}}{2\bar{C}}\right)
\end{aligned}$$

provided the universal constant C such that $a\geq C$ is chosen large enough depending on $\bar{C}$.

Finally we control the third term. We note that the event $X\geq\frac{1}{\lambda}$ is equivalent to $W\geq\nu_a+\frac{1}{\lambda}$. We let $k_0\in\mathbb{N}$ be the unique integer larger than or equal to np such that the event $\left\{W\geq\nu_a+\frac{1}{\lambda}\right.$ and $\left.Z\geq 0\right\}$ is equivalent to $\{Y\geq k_0\}$. This integer exists by Lemma 26 since w is increasing over $[np,n]$ by Lemma 26, $w(np+a\sigma)=0$, $\lim_{x\to n}w(x)=2n\log(1/p)>\nu_a+\frac{1}{\lambda}$ for $\lambda\leq\frac{1}{2}$

and $\nu_a \le \bar{C}a^2 \le \bar{C}c^2\sigma^2 \le \bar{C}cn < 2n\log(1/p)$ for c chosen sufficiently small depending on $\bar{C}$. Moreover, we also have $k_0 \ge np + a\sigma$ since $w(np+a\sigma) \le a^2 < \nu_a + \frac{1}{\lambda}$. Now, we have

$$
\begin{aligned}
\mathbb{E}\left[e^{\lambda X}\mathbf{1}\big(X \ge 1/\lambda\big)\right] &= \mathbb{E}\left[\exp\left(\lambda(W-\nu_a)\right)\mathbf{1}\big(W \ge \nu_a + \frac{1}{\lambda}\big)\right] \\
&= e^{-\lambda\nu_a}\sum_{k=k_0}^{n} e^{\lambda w(k)}\mathbb{P}(Y=k) \\
&\le e^{-\lambda\nu_a}e^{\lambda w(n)}\mathbb{P}(Y=n) + e^{-\lambda\nu_a}e^{\lambda w(k_0)}\mathbb{P}(Y=k_0) \\
&\quad + \frac{C_0}{\sqrt{2\pi}c_0^2}e^{-\lambda\nu_a}\sum_{k=k_0}^{n-1}\sqrt{\frac{n}{(n-k)k}} \\
&\quad \times \exp\left((2\lambda-1)\left\{n(1-p)h_B\left(-\frac{k-np}{n(1-p)}\right) + np\, h_B\left(\frac{k-np}{np}\right)\right\}\right) \\
&\qquad \text{(by Lemma 25)} \\
&= e^{-\lambda\nu_a}e^{\lambda w(n)}\mathbb{P}(Y=n) + e^{-\lambda\nu_a}e^{\lambda w(k_0)}\mathbb{P}(Y=k_0) \\
&\quad + \frac{C_0}{\sqrt{2\pi}c_0^2}e^{-\lambda\nu_a}\sum_{I_1\cup I_2}\sqrt{\frac{n}{(n-k)k}}e^{\left(\lambda-\frac{1}{2}\right)w(k)} \qquad (35)
\end{aligned}
$$

where we have defined $I_1 = \{k_0+1,\ldots,\lfloor C^*\sigma^2\rfloor\}$ and $I_2 = \{\lfloor C^*\sigma^2\rfloor+1,\ldots,n-1\}$ for some constant C^* that will be adjusted later. We first compute the sum over the set of indices I_1. Below, we use the notation C' to denote a constant whose value may change in each appearance. Using the change of variables

$$
\begin{aligned}
&y = w(x) \\
&dx = \frac{dy}{w' \circ w^{-1}(y)} = \left(\log\left(\frac{w^{-1}(y)}{np}\frac{n(1-p)}{n-w^{-1}(y)}\right)\right)^{-1}dy
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{I_1}\sqrt{\frac{n}{(n-k)k}}e^{\left(\lambda-\frac{1}{2}\right)w(k)} \\
&\le C'\int_{k_0}^{\lfloor C^*\sigma\rfloor+1}\sqrt{\frac{n}{(n-x)x}}e^{(\lambda-\frac{1}{2})w(x)}dx \\
&\le \frac{C'}{\sigma}\int_{k_0}^{\lfloor C^*\sigma\rfloor+1}e^{(\lambda-\frac{1}{2})w(x)}dx \\
&= \frac{C'}{\sigma}\int_{w(k_0)}^{w(\lfloor C^*\sigma^2\rfloor+1)}e^{(\lambda-\frac{1}{2})y}\left(\log\left(\frac{w^{-1}(y)}{np}\frac{n(1-p)}{n-w^{-1}(y)}\right)\right)^{-1}dy \\
&= \frac{C'}{\sigma}\int_{w(k_0)}^{w(\lfloor C^*\sigma^2\rfloor+1)}e^{(\lambda-\frac{1}{2})y}\left(\log\left(\frac{w^{-1}(y)}{np}\frac{1}{1-\frac{w^{-1}(y)-np}{n(1-p)}}\right)\right)^{-1}dy \\
&\le \frac{C'}{\sigma}\int_{w(k_0)}^{w(\lfloor C^*\sigma^2\rfloor+1)}e^{(\lambda-\frac{1}{2})y}\left(\log\left(\frac{w^{-1}(y)}{np}\left(1+\frac{w^{-1}(y)-np}{n(1-p)}\right)\right)\right)^{-1}dy.
\end{aligned}
$$

Moreover, we have $w(x) \le \frac{(x-np)^2}{2\sigma^2}$ for any $x \in [np,n]$. Therefore, $w^{-1}(y) \ge np + \sigma\sqrt{2y}$. Therefore, we can further control the display above as

$$
\begin{aligned}
&\sum_{I_1}\sqrt{\frac{n}{(n-k)k}}e^{\left(\lambda-\frac{1}{2}\right)w(k)} \\
&\le \frac{C'}{\sigma}\int_{w(k_0)}^{w(\lfloor C^*\sigma^2\rfloor+1)}e^{(\lambda-\frac{1}{2})y}\left(\log\left(\left(1+\frac{\sigma\sqrt{2y}}{np}\right)\left(1+\frac{\sigma\sqrt{2y}}{n(1-p)}\right)\right)\right)^{-1}dy
\end{aligned}
$$

$$\leq \frac{C'}{\sigma} \int_{w(k_0)}^{w(\lfloor C^*\sigma^2 \rfloor + 1)} e^{(\lambda - \frac{1}{2})y} \left(\log \left(1 + \frac{\sqrt{2y}}{\sigma} \right) \right)^{-1} dy.$$

Moreover, we have $w(x) \leq \frac{(x-np)^2}{\sigma^2}$ for any $x > 0$ by Lemma 32 so that $w(\lfloor C^*\sigma^2 \rfloor + 1) \leq C'\sigma^2$, which yields that, for some constant c' that depends on C^*, we have

$$\log \left(1 + \frac{\sqrt{2y}}{\sigma} \right) \geq c' \frac{\sqrt{y}}{\sigma}, \qquad \forall y \in \left[w(k_0), w(\lfloor C^*\sigma^2 \rfloor + 1) \right].$$

We obtain

$$\begin{aligned}
\sum_{I_1} \sqrt{\frac{n}{(n-k)k}} e^{\left(\lambda - \frac{1}{2}\right) w(k)} &\leq \frac{C'}{\sigma} \int_{w(k_0)}^{w(\lfloor C^*\sigma^2 \rfloor + 1)} e^{(\lambda - \frac{1}{2})y} \frac{\sigma}{\sqrt{y}} dy \\
&\leq C' \int_{(\frac{1}{2} - \lambda) w(k_0)}^{\infty} e^{-z} \frac{1}{\sqrt{z}} \frac{1}{\sqrt{1/2 - \lambda}} dz \\
&\leq C' \int_{(\frac{1}{2} - \lambda)(\nu_a + \frac{1}{\lambda})}^{\infty} e^{-z} \frac{1}{\sqrt{z}} \frac{1}{\sqrt{1/2 - \lambda}} dz \\
&\leq \frac{C' \exp\left(-(\frac{1}{2} - \lambda)(\nu_a + \frac{1}{\lambda})\right)}{\sqrt{(\frac{1}{2} - \lambda)(\nu_a + \frac{1}{\lambda})}},
\end{aligned}$$

which yields

$$\frac{C_0}{\sqrt{2\pi c_0^2}} e^{-\lambda \nu_a} \sum_{I_1} \sqrt{\frac{n}{(n-k)k}} e^{\left(\lambda - \frac{1}{2}\right) w(k)} \leq \frac{C_0}{\sqrt{2\pi c_0^2}} e^{-\lambda \nu_a} \frac{C' \exp\left(-(\frac{1}{2} - \lambda)(\nu_a + \frac{1}{\lambda})\right)}{\sqrt{(\frac{1}{2} - \lambda)(\nu_a + \frac{1}{\lambda})}} \tag{36}$$

We now show that, for any $k \in I_2 \setminus \{n-1\}$, we have

$$\sqrt{\frac{n}{(n-k-1)(k+1)}} e^{\left(\lambda - \frac{1}{2}\right) w(k+1)} \leq \sqrt{2} e^{-2} \sqrt{\frac{n}{(n-k)k}} e^{\left(\lambda - \frac{1}{2}\right) w(k)}.$$

Indeed, we have

$$\sqrt{\frac{n}{(n-k-1)(k+1)}} \leq \sqrt{2} \sqrt{\frac{n}{(n-k)k}}.$$

Moreover, since the function w is increasing and convex over $[np, n)$ by Lemma 26, we have

$$\begin{aligned}
w(k+1) - w(k) \geq w'(k) &\geq 2 \log \left(\frac{k}{np} \frac{n(1-p)}{n-k} \right) \\
&\geq 2 \log \left(\frac{k_0}{np} \frac{n(1-p)}{n-k_0} \right) \qquad \text{since } w' \text{ is increasing over } [np, n] \\
&\geq \log \left(\frac{np + \lfloor C^*\sigma^2 \rfloor}{np} \frac{n(1-p)}{n - np - \lfloor C^*\sigma^2 \rfloor} \right) \\
&= \log \left(\left(1 + \frac{\lfloor C^*\sigma^2 \rfloor}{np} \right) \frac{1}{1 - \frac{\lfloor C^*\sigma^2 \rfloor}{n(1-p)}} \right) \\
&\geq \log \left(\left(1 + \frac{\lfloor C^*\sigma^2 \rfloor}{np} \right) \left(1 + \frac{\lfloor C^*\sigma^2 \rfloor}{n(1-p)} \right) \right) \\
&\geq \log \left(1 + \frac{\lfloor C^*\sigma^2 \rfloor}{\sigma^2} \right) \\
&\geq 8
\end{aligned}$$

for C^* larger than an absolute constant. For $\lambda \leq 1/4$, we therefore obtain

$$\sqrt{\frac{n}{(n-k-1)(k+1)}} e^{\left(\lambda-\frac{1}{2}\right)w(k+1)} \leq \sqrt{2}e^{-2}\sqrt{\frac{n}{(n-k)k}} e^{\left(\lambda-\frac{1}{2}\right)w(k)}.$$

Therefore, the sum over I_2 can be controlled as by a geometric series, and we obtain

$$\begin{aligned}\frac{C_0}{\sqrt{2\pi}c_0^2} e^{-\lambda\nu_a} \sum_{k\in I_2} \sqrt{\frac{n}{(n-k)k}} e^{\left(\lambda-\frac{1}{2}\right)w(k)} &\leq C' e^{-\lambda\nu_a} \sqrt{\frac{n}{(n-\lfloor C^*\sigma^2\rfloor)\lfloor C^*\sigma^2\rfloor}} e^{\left(\lambda-\frac{1}{2}\right)w(\lfloor C^*\sigma^2\rfloor)} \\ &\leq C'' e^{-\lambda\nu_a} \frac{1}{\sigma} e^{\left(\lambda-\frac{1}{2}\right)(\nu_a+1/\lambda)} \end{aligned} \tag{37}$$

provided C^* is large enough.

To conclude, it remains to control the terms $e^{-\lambda\nu_a}e^{\lambda w(n)}\mathbb{P}(Y=n)$ and $e^{-\lambda\nu_a}e^{\lambda w(k_0)}\mathbb{P}(Y=k_0)$. For $k \in \{k_0, n\}$, we have

$$\begin{aligned} & e^{-\lambda\nu_a}e^{\lambda w(k)}\mathbb{P}(Y=k) \\ &\leq e^{-\lambda\nu_a}e^{\lambda w(k)} \frac{C_0}{c_0^2\sqrt{2\pi}} \sqrt{\frac{n}{(n-k)k}} \exp\left(-n(1-p)\, h_B\left(-\frac{k-np}{n(1-p)}\right) - np\ h_B\left(\frac{k-np}{np}\right)\right) \text{ by Lemma 25} \\ &\leq \frac{C_0}{c_0^2\sqrt{2\pi}} e^{-\lambda\nu_a} \sqrt{\frac{2}{k}} e^{(\lambda-\frac{1}{2})w(k)} \\ &\leq \frac{C_0}{c_0^2\sqrt{2\pi}} e^{-\lambda\nu_a} \sqrt{\frac{2}{\sigma^2}} e^{(\lambda-\frac{1}{2})(\nu_a+\frac{1}{\lambda})} \end{aligned} \tag{38}$$

by definition of k_0. Combining equations (35), (36), (37) and (38), we obtain

$$\begin{aligned}\mathbb{E}\left[e^{\lambda X}\mathbf{1}\big(X \geq 1/\lambda\big)\right] &\leq Ce^{-\lambda\nu_a+(\lambda-\frac{1}{2})(\nu_a+\frac{1}{\lambda})} \\ &\leq Ce^{-\frac{\nu_a}{2}-\frac{1}{2\lambda}} \\ &\leq C\lambda^2 e^{-\frac{\nu_a}{2}}.\end{aligned}$$

Assembling these calculations, we have for $\lambda \in [0, M]$

$$\mathbb{E}e^{\lambda X} \leq 1 + C'\lambda^2 e^{-ca^2} \leq \exp\left(C'\lambda^2 e^{-ca^2}\right)$$

where $C' > 0$ is a constant that depends on $\bar{C}$. Then, for any $t > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^m (W_j - \nu_a)\,\mathbf{1}_{\{Z_j\geq a\}} > t\right) &\leq \inf_{\lambda\leq M} e^{-\lambda t}\left(\mathbb{E}e^{\lambda X}\right)^m \leq \inf_{\lambda\leq M} \exp\left(-\lambda t + C'm\lambda^2 e^{-ca^2}\right) \\ &= \exp\left\{-\left(\frac{Mt^2e^{ca^2}}{C'm} \wedge \frac{t}{2M}\right)\right\}.\end{aligned}$$

Setting $t = C^*\left(\sqrt{me^{-ca^2}x} + x\right)$ for C^* sufficiently large, we obtain the desired result. $\square$

Lemma 20. *Let $n, k \in \mathbb{N}$ such that $k \leq n/2$, $p_0 \in (0, 1/4)$ and $p_1 \in [0, 1]$. Let $Y = A + B$ where $A \sim \text{Bin}(k, p_1)$ and $B \sim \text{Bin}(n-k, p_0)$ are independent and define $Z = (Y - np_0)/\sigma$, where $\sigma = \sqrt{np_0(1-p_0)}$. Let w denote the function defined in (32) and let $W = w(Y)$. Let $\theta = \frac{k(p_1-p_0)}{\sigma}$ and recall the definition of ν_a from (33) for $a > 0$. Then there exist universal constants $\tilde{c}, c, C > 0$ such that, if $kp_1 \geq C$, then for any $a \in [C, c\sigma]$,*

$$\mathbb{E}\left\{(W-\nu_a)\,\mathbf{1}\{Z \geq a\}\right\} \begin{cases} = 0, & \text{if } \theta = 0 \\ \geq \tilde{c}\,\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right), & \text{if } \theta \geq Ca. \end{cases}$$

Proof of Lemma 20. The fact that $\mathbb{E}\left\{(W-\nu_a)\,\mathbf{1}\{Z\geq a\}\right\}=0$ when $\theta=0$ follows by definition of ν_a.

Suppose now that $\theta\geq Ca$. Recalling that the function w is increasing over $[np_0,n]$ by Lemma 26, we have

$$
\begin{aligned}
&\mathbb{E}\left[(W-\nu_a)\mathbf{1}\big(Z\geq a\big)\right]\\
&\geq \mathbb{E}\left[W\mathbf{1}\big(Z\geq \theta\big)\right]-\nu_a\\
&= E\left[w(Y)\mathbf{1}\big(Y\geq kp_1+(n-k)p_0\big)\right]-\nu_a\\
&\geq w(kp_1+(n-k)p_0)\mathbb{P}\left(Y\geq kp_1+(n-k)p_0\right)-\bar{C}a^2 \qquad \text{by Corollary 1}\\
&= c'\mathbb{P}\left(Y\geq kp_1+(n-k)p_0\right)\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)-\bar{C}a^2 \qquad \text{for some } c'>0 \text{ by Corollary 2.}
\end{aligned}
$$

Moreover, the median $m_{n,p}$ of a binomial distribution with parameters n and p satisfies $|m_{n,p}-np|<\log(2)$ (see [2], Theorem 2). Therefore, we have

$$
\mathbb{P}\left(Y\geq np_1\right)\geq \mathbb{P}\left(A\geq kp_1\right)\mathbb{P}\left(B\geq (n-k)p_0\right)
$$

We show that $P\left(A\geq kp_1\right)\geq\frac{1}{4}$ provided $kp_1\geq C$ for some large enough constant $C>0$. We have

$$
\begin{aligned}
P\left(A\geq kp_1\right)&\geq \mathbb{P}\left(A\geq m_{k,p_1}\right)-\mathbb{P}\left(A=\lfloor kp_1\rfloor-1\right)\\
&\geq \frac{1}{2}-\mathbb{P}\left(A=\lfloor kp_1\rfloor-1\right).
\end{aligned}
$$

Moreover, letting $j=\lfloor kp_1\rfloor-1$, we have

$$
\begin{aligned}
\mathbb{P}\left(A=j\right)&=\binom{k}{j}p_1^j(1-p_1)^{k-j}\\
&\leq C_0\sqrt{\frac{k}{(k-j)j}}\left(\frac{k(1-p_1)}{k-j}\right)^{k-j}\left(\frac{kp_1}{j}\right)^j \qquad \text{by Lemma 25}\\
&\leq C_0\frac{2}{\sqrt{kp_1}}\left(1+\frac{kp_1-j}{j}\right)^j\\
&\leq C_0\frac{2}{\sqrt{kp_1}}\exp\left(kp_1-j\right)\\
&\leq 2C_0e^2\frac{1}{\sqrt{kp_1}}\\
&\leq\frac{1}{4}
\end{aligned}
$$

provided $kp_1\geq C$, as claimed. Similarly, we can show that $P\left(B\geq (n-k)p_0\right)\geq\frac{1}{4}$ provided $(n-k)p_0\geq C$ for some large enough constant $C>0$. Therefore, we have proved $\mathbb{P}(Y\geq np_1)\geq\frac{1}{16}$, which yields

$$
\mathbb{E}\left[(W-\nu_a)\mathbf{1}\big(Z\geq a\big)\right]\geq\frac{c'}{16}\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)-\bar{C}a^2.
$$

Moreover, since $\theta\geq Ca$, we have

$$
\begin{aligned}
\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)&\geq\sigma Ca\log\left(1+\frac{Ca}{\sigma}\right)\\
&\geq\sigma Ca\log(2)\cdot\frac{Ca}{\sigma} \qquad \text{provided } c\leq 1/C\\
&\geq C^2\log(2)a^2.
\end{aligned}
$$

We may now choose C large enough that $\bar{C}a^2\leq\frac{c'}{32}\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)$, which yields

$$
\mathbb{E}\left[(W-\nu_a)\mathbf{1}\big(Z\geq a\big)\right]\geq\frac{c'}{32}\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)
$$

and concludes the proof.

□

Lemma 21. *Let $n, k \in \mathbb{N}$ such that $k \leq n/2$, $p_0 \in (0, 1/4)$ and $p_1 \in [0,1]$. Let $Y = A + B$ where $A \sim \mathrm{Bin}(k, p_1)$ and $B \sim \mathrm{Bin}(n-k, p_0)$ are independent and define $Z = (Y - np_0)/\sigma$, where $\sigma = \sqrt{np_0(1-p_0)}$. Let w denote the function defined in (32) and let $W = w(Y)$. Let $\theta = \frac{k(p_1 - p_0)}{\sigma}$ and recall the definition of ν_a from (33) for $a > 0$. Then there exist universal constants $C_1, c, C > 0$ such that, if $kp_1 \geq C$, then for any $a \in [C, c\sigma]$,*

$$\mathrm{Var}\left\{(W - \nu_a)\mathbf{1}\{Z \geq a\}\right\} \leq \begin{cases} C_1\left(\sigma a \log\left(1 + \frac{a}{\sigma}\right)\right)^2 \exp\left(-\frac{3}{8}a^2\right) & \text{if } \theta = 0 \\ C_1(\sigma^2 + \sigma\theta)\log^2\left(1 + \frac{\theta}{\sigma}\right) & \text{if } \theta \geq 2a. \end{cases}$$

Proof of Lemma 21. Let $a \geq C$. In this proof, we use the notation $\tilde{C}$ to denote a constant whose value may change in each appearance. When $\theta = 0$, we have

$$\begin{aligned}
\mathrm{Var}\left\{(W - \nu_a)\mathbf{1}\{Z \geq a\}\right\} &\leq \mathbb{E}\left\{(W - \nu_a)^2 \mathbf{1}\{Z \geq a\}\right\} \\
&= \mathbb{E}\left[W^2 \mathbf{1}\{Z \geq a\}\right] - 2\nu_a \mathbb{E}\left[W \mathbf{1}\{Z \geq a\}\right] + \nu_a^2 \mathbb{P}(Z \geq a) \\
&= \left(\gamma_a - \nu_a^2\right)\mathbb{P}(Z \geq a) \\
&\leq \bar{C}\left(\sigma a \log\left(1 + \frac{a}{\sigma}\right)\right)^2 \exp\left(-\frac{\frac{1}{2}a^2}{1 + \frac{a/\sigma}{3}}\right) \qquad \text{by Corollary 1 and Lemma 24} \\
&\leq \bar{C}\left(\sigma a \log\left(1 + \frac{a}{\sigma}\right)\right)^2 \exp\left(-\frac{3}{8}a^2\right).
\end{aligned}$$

For $\theta \neq 0$, we have

$$\begin{aligned}
&\mathrm{Var}\left\{(W - \nu_a)\mathbf{1}\{Z \geq a\}\right\} \\
&= \mathbb{E}\left[\mathrm{Var}\left\{(W - \nu_a)\mathbf{1}\{Z \geq a\} \mid \mathbf{1}\{Z \geq a\}\right\}\right] \\
&\quad + \mathrm{Var}\left[\mathbb{E}\left\{(W - \nu_a)\mathbf{1}\{Z \geq a\} \mid \mathbf{1}\{Z \geq a\}\right\}\right] \\
&= \mathbb{P}(Z \geq a)\,\mathrm{Var}\left(W | Z \geq a\right) \\
&\quad + \mathbb{P}(Z < a)\mathbb{P}(Z \geq a)\left\{\mathbb{E}\left(W - \nu_a | Z \geq a\right)\right\}^2. \qquad (39)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathbb{P}(Z \geq a)\,\mathrm{Var}\left(W | Z \geq a\right) &\leq \mathbb{E}\left\{\mathrm{Var}\left(W \mid \mathbf{1}\{Z \geq a\}\right)\right\} \\
&\leq \mathrm{Var}\left(W\right) = \mathbb{E}[W^2] - \mathbb{E}^2[W]. \qquad (40)
\end{aligned}$$

Now, we recall that $\mathbb{E}Y = kp_1 + (n-k)p_0$. We observe that, for any $x \in [0, n]$

$$\begin{aligned}
w(x) &= (n - x)\log\left(\frac{n - x}{n(1 - p_0)}\right) + x\log\left(\frac{x}{np_0}\right) \\
&= (n - x)\log\left(\frac{n - \mathbb{E}Y}{n(1 - p_0)}\right) + x\log\left(\frac{\mathbb{E}Y}{np_0}\right) + (n - x)\log\left(\frac{n - x}{n - \mathbb{E}Y}\right) + x\log\left(\frac{x}{\mathbb{E}Y}\right) \\
&=: nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right) + (x - \mathbb{E}Y)\log\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1 - p_0)}{n - \mathbb{E}Y}\right) + w_{p_1}(x),
\end{aligned}$$

where $D_{KL}(p||q) = (1-p)\log\left(\frac{1-p}{1-q}\right) + p\log\left(\frac{p}{q}\right)$ for any $p, q \in (0,1)$. Therefore, we obtain

$$\begin{aligned}
\mathbb{E}(W) &= nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right) + \mathbb{E}(w_{p_1}(Y)) \\
\implies \mathbb{E}^2(W) &\geq \left(nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right)\right)^2 + 2nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right)\mathbb{E}(w_{p_1}(Y)).
\end{aligned}$$

Moreover, using the inequality $ab \leq a^2 + b^2$ that holds true for any $a, b \in \mathbb{R}$, we obtain

$$\mathbb{E}[W^2] = \left(nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right)\right)^2 + \mathbb{V}(Y)\log^2\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1 - p_0)}{n - \mathbb{E}Y}\right) + \mathbb{E}(w_{p_1}(Y)^2) + 2nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\, p_0\right)\mathbb{E}(w_{p_1}(Y))$$

$$
\begin{aligned}
&+\mathbb{E}\left[(Y-\mathbb{E}Y)\log\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1-p_0)}{n-\mathbb{E}Y}\right)w_{p_1}(Y)\right]\\
&\le\left(nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\,p_0\right)\right)^2+2\mathbb{V}(Y)\log^2\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1-p_0)}{n-\mathbb{E}Y}\right)+2\mathbb{E}(w_{p_1}(Y)^2)+2nD_{KL}\left(\frac{\mathbb{E}Y}{n}\,\Big|\Big|\,p_0\right)\mathbb{E}(w_{p_1}(Y)).
\end{aligned}
$$

Recalling $k\le n/2$, we have $\frac{k(p_1-p_0)}{n}\le 1/2$. We obtain

$$
\begin{aligned}
\mathbb{V}(W)&\le 2\mathbb{V}(Y)\log^2\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1-p_0)}{n-\mathbb{E}Y}\right)+2\mathbb{E}(w_{p_1}(Y)^2)\\
&\le 2\mathbb{V}(Y)\log^2\left(\frac{\mathbb{E}Y}{np_0}\frac{n(1-p_0)}{n-\mathbb{E}Y}\right)+2\mathbb{E}\left(4\left(\frac{Y-\mathbb{E}Y}{\sigma(Y)}\right)^4\right)\qquad\text{by Lemma 32.1}\\
&\le 2\left(kp_1+(n-k)p_0\right)\log^2\left(\left(1+\frac{k(p_1-p_0)}{np_0}\right)\frac{n(1-p_0)}{n(1-p_0)-k(p_1-p_0))}\right)+C\\
&\le Cnp_0\left(1+\frac{k(p_1-p_0)}{np_0}\right)\log\left(1+\frac{k(p_1-p_0)}{np_0(1-p_0)}\right)+C\qquad\text{since }\frac{k(p_1-p_0)}{n(1-p_0)}\le 1/2\\
&\le C\left(\sigma^2+\theta\sigma\right)\log^2\left(1+\frac{\theta}{\sigma}\right).
\end{aligned}
$$

Finally, writing $Z=\theta+X$ where $X=\frac{Y-kp_1-(n-k)p_0}{\sigma}$, we have by Lemma 18

$$
\begin{aligned}
&\mathbb{E}\left(W-\nu_a|Z\ge a\right)\\
&=\mathbb{E}\Big[w(kp_1+(n-k)p_0+\sigma X)\,\big|\,\theta+X\ge a\Big]\\
&\le\mathbb{E}\Big[w(kp_1+(n-k)p_0+\sigma X)\,\big|\,X\ge a\Big]\\
&\le\tilde{C}\,\mathbb{E}\left[(k(p_1-p_0)+\sigma X)\log\left(1+\frac{k(p_1-p_0)+\sigma X}{\sigma^2}\right)\,\big|\,X\ge a\right]\qquad\text{by Corollary 2}\\
&\le\tilde{C}\left(\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)+\mathbb{E}\left[\sigma X\log\left(1+\frac{X}{\sigma}\right)\,\big|\,X\ge a\right]\right)\\
&\le\tilde{C}\left(\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)+\sigma a\log\left(1+\frac{a}{\sigma}\right)\right)\\
&\le\tilde{C}\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right).
\end{aligned}\tag{41}
$$

If $|\theta|\ge 2a$, notice that, by Bernstein's inequality, we have

$$
\begin{aligned}
\mathbb{P}(Z<a)=\mathbb{P}\left(X<-(\theta-a)\right)&\le\exp\left(-\frac{\frac{1}{2}(a-\theta)^2}{1+\frac{(\theta-a)/\sigma}{3}}\right)\\
&\le\exp\left(-\frac{\frac{1}{2}\left(1-\bar{C}^{-1}\right)^2\theta^2}{1+\frac{(1-\bar{C}^{-1})\theta/\sigma}{3}}\right)\\
&\le\tilde{C}/\theta^2\\
&\le\frac{\tilde{C}}{\sigma\theta\log\left(1+\frac{\theta}{\sigma}\right)}
\end{aligned}\tag{42}
$$

if $\theta\ge C$ for some large enough $C>0$. The result follows by combining (39), (40), (41) and (42).

$\square$

Lemma 22. *Let $Y\sim\mathcal{N}(0,1)$ and let $\alpha,\beta\ge 1$ and $x>0$ such that $x^2\ge 2(\alpha-1)$. Then we have*

$$
\begin{aligned}
\mathbb{E}\left[Y^\alpha\mathbf{1}_{Y\ge x}\right]&\le 2x^{\alpha-1}e^{-x^2/2}&&\textit{provided }x^2\ge 2(\alpha-1)\\
\mathbb{E}\left[Y^\alpha\log(1+Y)^\beta\mathbf{1}_{Y\ge x}\right]&\le 2x^{\alpha-1}\log(1+x)^\beta e^{-x^2/2}&&\textit{provided }x^2\ge 2(\alpha+\beta-1).
\end{aligned}
$$

Proof of Lemma 22. By integration by parts, we have

$$\int_x^{\infty} y^{\alpha} e^{-y^2/2} dy = x^{\alpha-1} e^{-x^2/2} + (\alpha-1) \int_x^{\infty} y^{\alpha-2} e^{-y^2/2} dy$$
$$\leq x^{\alpha-1} e^{-x^2/2} + \frac{1}{2} \int_x^{\infty} y^{\alpha} e^{-y^2/2} dy \text{ using that } y^2 \geq 2(\alpha-1) \text{ over } [x, +\infty),$$

so that $\quad \mathbb{E}\big[Y^{\alpha} \mathbf{1}_{Y \geq x}\big] = \int_x^{\infty} y^{\alpha} e^{-y^2/2} dy \leq 2x^{\alpha-1} e^{-x^2/2}.$

Similarly,

$$\begin{aligned}
\int_x^{\infty} y^{\alpha} \log(1+y)^{\beta} e^{-y^2/2} dy &= x^{\alpha-1} \log(1+y)^{\beta} e^{-x^2/2} \\
&\quad + \int_x^{\infty} \left((\alpha-1) y^{\alpha-2} \log(1+y)^{\beta} + \beta \frac{y^{\alpha-1}}{1+y} \log(1+y)^{\beta-1} \right) e^{-y^2/2} dy \\
&\leq x^{\alpha-1} \log(1+y)^{\beta} e^{-x^2/2} \\
&\quad + \int_x^{\infty} (\alpha+\beta-1) y^{\alpha-2} \log(1+y)^{\beta} e^{-y^2/2} dy \\
&\leq x^{\alpha-1} \log(1+y)^{\beta} e^{-x^2/2} \\
&\quad + \frac{1}{2} \int_x^{\infty} y^{\alpha} \log(1+y)^{\beta} e^{-y^2/2} dy \text{ using that } y^2 \geq 2(\alpha+\beta-1) \text{ over } [x, +\infty),
\end{aligned}$$

so that $\quad \mathbb{E}\big[v(Y)^{\alpha} \mathbf{1}_{Y \geq x}\big] = \int_x^{\infty} y^{\alpha} \log(1+y)^{\beta} e^{-y^2/2} dy \leq 2x^{\alpha-1} \log(1+y)^{\beta} e^{-x^2/2}.$

□

Lemma 23. *For any $\alpha > 0$ and $x \geq 2\alpha$, it holds that*

$$\int_x^{\infty} y^{\alpha} e^{-y} dy \leq 2x^{\alpha} e^{-x}.$$

Proof of Lemma 23. By integration by parts, we have

$$\int_x^{\infty} y^{\alpha} e^{-y} dy \leq 2x^{\alpha} e^{-x} = x^{\alpha} e^{-x} + \int_x^{\infty} \alpha x^{\alpha-1} e^{-x} dx$$
$$\leq x^{\alpha} e^{-x} + \frac{1}{2} \int_x^{\infty} x^{\alpha} e^{-x} dx \qquad \text{since } y \geq 2\alpha \text{ over } [x, \infty),$$

from which it follows that

$$\int_x^{\infty} y^{\alpha} e^{-y} dy \leq 2x^{\alpha} e^{-x},$$

as claimed. □

Lemma 24. *Let $n \in \mathbb{N}$ and $p \in (0, 1/2)$. Let $X \sim \mathrm{Bin}(n, p)$ and $\sigma = \sqrt{np(1-p)}$. For any $t > 0$, we have*

$$\mathbb{P}\left(\frac{X - np}{\sigma} \geq t \right) \leq \exp\left(-\frac{t^2/2}{1 + \frac{t}{3\sigma}} \right)$$
$$\mathbb{P}\left(\frac{X - np}{\sigma} \leq -t \right) \leq \exp\left(-\frac{t^2/2}{1 + \frac{t}{3\sigma}} \right)$$

Proof. The result follows by a direct application of the Bernstein inequality for binomial distributions.
□

Lemma 25. *There exist two absolute constants C_0, c_0 such that, for any $n, k \in \mathbb{N}$ such that $k < n$ and $p \in (0,1)$, we have*

$$\begin{aligned}\binom{n}{k} p^k (1-p)^{n-k} &\leq \frac{C_0}{c_0^2\sqrt{2\pi}} \sqrt{\frac{n}{(n-k)k}} \left(\frac{n(1-p)}{n-k}\right)^{n-k} \left(\frac{np}{k}\right)^k \\ &= \frac{C_0}{c_0^2\sqrt{2\pi}} \sqrt{\frac{n}{(n-k)k}} \exp\left(-n(1-p)h_B\left(-\frac{k-np}{n(1-p)}\right) - np\, h_B\left(\frac{k-np}{np}\right)\right).\end{aligned}$$

Proof. We recall that, by the Stirling formula, we have

$$m! \sim \sqrt{2\pi m}\left(\frac{m}{e}\right)^m, \qquad \text{as } m \to \infty.$$

Since the function $m \in \mathbb{N} \mapsto \frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}$ is positive and converges to 1, there exist two absolute constants $c_0, C_0 > 0$ such that

$$c_0\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \leq m! \leq C_0\sqrt{2\pi m}\left(\frac{m}{e}\right)^m, \qquad \forall m \in \mathbb{N}. \tag{43}$$

The result follows. □

Lemma 26. *The function w defined in (32) is convex and increasing over $[np_0, n)$.*

Proof. The derivative of w is given by

$$w'(x) = 2\log\left(\frac{x}{np_0}\frac{n(1-p_0)}{n-x}\right), \quad \forall x \in [np_0, n]$$

which is positive if $x > np_0$. The second derivative of w is given by

$$\frac{2}{n-x} + \frac{2}{x}$$

which is positive for any $x \in [np_0, n)$. □

C.2 Proofs of propositions in the main text

We begin by restating and proving Proposition 1.

Proposition 1. *Let $c > 0$. There exists a constant $C > 0$ such that if*

$$\frac{C}{k_1k_2}\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right) \leq p_0 \leq \frac{1}{4},$$

then it holds $\mathbb{P}_0\left(\mathbf{G} \text{ has an empty } k_1 \times k_2 \text{ bipartite subgraph}\right) \leq c$.

Here, we say that a subgraph $\mathbf{G}_1$ of $\mathbf{G}$ is empty if all the vertices in $\mathbf{G}_1$ has degree equal to 0.

Proof.

$$\begin{aligned}&\mathbb{P}_0\left(\mathbf{G} \text{ has an empty } k_1 \times k_2 \text{ bipartite subgraph}\right)\\ &= \mathbb{P}_0\left(\exists (K_1, K_2) \in \mathcal{P}_{k_1}(n_1) \times \mathcal{P}_{k_2}(n_2), \forall (i,j) \in K_1 \times K_2 : A_{ij} = 0\right)\\ &\leq \binom{n_1}{k_1}\binom{n_2}{k_2}\mathbb{P}\left(\text{Bin}(k_1k_2, p_0) = 0\right)\\ &= \binom{n_1}{k_1}\binom{n_2}{k_2}(1-p_0)^{k_1k_2}\\ &\leq \binom{n_1}{k_1}\binom{n_2}{k_2}e^{-k_1k_2p_0}\\ &\leq \binom{n_1}{k_1}\binom{n_2}{k_2}\exp\left(-C\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \exp\left(-(C-1)\log\left(e\binom{n_1}{k_1}\binom{n_2}{k_2}\right)\right) \quad \text{(for } C>1\text{)} \\
&\leq \exp\left(-(C-1)\right) \\
&\leq c,
\end{aligned}
$$

where the final inequality holds for C taken sufficiently large. $\square$

Now we state and prove Proposition 2.

Proposition 2. *Let $c > 0$. There exists a constant $C > 0$ such that if*

$$
\frac{C}{n_2 k_1}\log\left(e\binom{n_1}{k_1}\right) \leq p_0 \leq \frac{1}{4},
$$

then it holds $\mathbb{P}_0\Big(\exists I \in \mathcal{P}_{k_1}(n_1) : \sum_{i\in I}\sum_{j=1}^{n_2} A_{ij} = 0\Big) \leq c$.

Proof.

$$
\begin{aligned}
&\mathbb{P}_0\Big(\exists I \in \mathcal{P}_{k_1}(n_1) : \sum_{i\in I}\sum_{j=1}^{n_2} A_{ij} = 0\Big) \\
&\leq \binom{n_1}{k_1}\mathbb{P}\left(\text{Bin}(k_1 n_2, p_0) = 0\right) \\
&= \binom{n_1}{k_1}(1-p_0)^{k_1 n_2} \\
&\leq \binom{n_1}{k_1} e^{-k_1 n_2 p_0} \\
&\leq \binom{n_1}{k_1}\exp\left(-C\log\left(e\binom{n_1}{k_1}\right)\right) \\
&\leq \exp\left(-(C-1)\log\left(e\binom{n_1}{k_1}\right)\right) \quad \text{(for } C>1\text{)} \\
&\leq \exp\left(-(C-1)\right) \\
&\leq c,
\end{aligned}
$$

where the final inequality holds for C taken sufficiently large. $\square$

Finally, we state and prove Proposition 3.

Proposition 3. *Suppose that $k_1^2 \geq \bar{c} n_1 k_2$ for a constant $\bar{c} > 0$ and $k_j \leq c_j n_j$ for $j \in \{1,2\}$ where $c_1, c_2 > 0$ are sufficiently small constants. Additionally, suppose that there exists a constant $\alpha > 0$ such that $n_2 \geq k_2^{2+\alpha}$ and that $\frac{n_1}{k_1} \geq e\log(\frac{n_2}{k_2})$. Then it holds*

$$
\frac{(\delta^*)^2}{p_0(1-p_0)} \asymp \frac{1}{k_2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\log(n_2)\right). \tag{44}
$$

In particular, this reveals a phase transition at $\frac{n_1 k_2}{k_1^2}\log(n_2) \asymp 1$.

Proof. Notice that the assumption $\frac{n_1}{k_1} \geq e\log(\frac{n_2}{k_2})$ can be made without loss of generality by Lemma 27. Theorems 1 and 2 together imply

$$
\frac{(\delta^*)^2}{p_0(1-p_0)} \asymp R.
$$

Therefore, we need to show

$$
R \asymp \frac{1}{k_2}\log\left(1 + \frac{n_1 k_2}{k_1^2}\log(n_2)\right).
$$

Note that the assumption $n_2 \geq k_2^{2+\alpha}$ implies that $\phi_{21} = \infty$ if c_2 is large enough. Then by Lemma 8, it holds

$$R \asymp \psi_{21} \wedge \phi_{12}.$$

Furthermore, $n_2 \geq k_2^{2+\alpha}$ implies that $\log(1 + \frac{n_2}{k_2^2}) \asymp \log(n_2)$. Using the inequality $\log(1+x) \leq x$ which holds for any $x \geq 0$, we have

$$\begin{aligned}
\psi_{21} &= \frac{1}{k_2} \log\left(1 + \frac{n_1}{k_1^2} \log\left(e\binom{n_2}{k_2}\right)\right) \\
&\asymp \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log(n_2/k_2)\right) \\
&\asymp \frac{1}{k_2} \log\left(1 + \frac{n_1 k_2}{k_1^2} \log(n_2)\right) \\
&\leq \frac{1}{k_2} \frac{n_1 k_2}{k_1^2} \log(n_2) \\
&\asymp \frac{n_1}{k_1^2} \log\left(1 + \frac{n_2}{k_2^2}\right) \\
&= \phi_{12}.
\end{aligned}$$

This implies $R \asymp \psi_{21} \wedge \phi_{12} = \psi_{21} \asymp \log\left(1 + \frac{n_1 k_2}{k_1^2} \log(n_2)\right)$. The proof is complete. $\square$

C.3 Miscellaneous analytic lemmas

Lemma 27. *For any two real numbers $x, y > 1$, at least one of the two inequalities $x \geq e\log(y)$ or $y \geq e\log(x)$ holds.*

Proof of Lemma 27. We first prove a preliminary result: For any $x > 1$, it holds that $x \geq e\log(e\log(x))$. To see this, define the function $f : (1, \infty) \to \mathbb{R};\ x \mapsto x - e\log(e\log(x))$. We have for any $x > 1$

$$f'(x) = 1 - \frac{e}{x\log(x)}.$$

The only value x^* for which $f(x^*) = 0$ is $x^* = e$, which implies that f is decreasing over $(1, e)$ and increasing over (e, ∞). Hence, f is minimized at e and its corresponding minimum value is

$$f(e) = e - e\log(e\log(e)) = 0.$$

This fact being established, assume now for the sake of contradiction that there exist two real numbers x, y such that $x < e\log(y)$ and $y < e\log(x)$. Then we obtain $x < e\log(e\log(x))$ which is a contradiction. This concludes the proof. $\square$

Lemma 28. *Let $X \sim Bin(n, p)$ with $p \leq \frac{1}{2}$. Then for any $k \in \{1, ..., n\}$, we have*

$$\Pr(X = k) \leq \left(\frac{2enp}{k}\right)^k \exp\left(-np\right)$$

Proof. Using the bound $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$ (Appendix A in [4]), we have

$$\begin{aligned}
\Pr(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&\leq \left(\frac{npe}{k}\right)^k (1-p)^{n-k} \\
&= \left(\frac{npe}{k}\right)^k (1-p)^n (1-p)^{-k}
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{npe}{k}\Big)^k (1-p)^n (1/2)^{-k} \\
&= \Big(\frac{2enp}{k}\Big)^k \Big(1-p\Big)^n \\
&\le \Big(\frac{2enp}{k}\Big)^k \exp\big(-np\big)
\end{aligned}
$$

where the final inequality uses $(1-x)^b \le \exp(-xb)$ for any $x, b \ge 0$. $\square$

Lemma 29. *Let $f : S \to (0,\infty)$ be differentiable, where $S \subseteq (0,\infty)$ is an interval. For $x \in S$, define $g(x) = \frac{1}{x}\log\big(f(x)\big)$ and $g^{(1)}(x) = \frac{1}{x}\log\big(1+f(x)\big)$. If g is decreasing on S, then $g^{(1)}$ is decreasing on S as well.*

Proof. Let $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$ for any $x \in S$. By direct calculation, we have

$$\frac{\mathrm{d}g}{\mathrm{d}x}(x) = \frac{f'(x)}{xf(x)} - \frac{\log\big(f(x)\big)}{x^2}.$$

If g is decreasing on S, then for each $x \in S$ it holds

$$\frac{f'(x)}{xf(x)} < \frac{\log\big(f(x)\big)}{x^2}.$$

Now calculating the derivative of $g^{(1)}$ at any $x \in S$, we have

$$
\begin{aligned}
\frac{\mathrm{d}g^{(1)}}{\mathrm{d}x}(x) &= \frac{f'(x)}{x\big(1+f(x)\big)} - \frac{\log\big(1+f(x)\big)}{x^2} \\
&= \frac{f'(x)}{x\big(1+f(x)\big)}\frac{f(x)}{f(x)} - \frac{\log\big(1+f(x)\big)}{x^2} \\
&< \frac{\log\big(f(x)\big)}{x^2}\frac{f(x)}{\big(1+f(x)\big)} - \frac{\log\big(1+f(x)\big)}{x^2} \\
&\le 0
\end{aligned}
$$

where the final inequality uses $f(x) \le 1 + f(x)$ and $\log\big(f(x)\big) \le \log\big(1+f(x)\big)$. This completes the proof. $\square$

Lemma 30. *Let $x, y > e$.*

(i) If $x/\log(x) \le y/2$, then we have $x \le y\log(y)$.

(ii) If $x/\log(x) \ge y$, then we have $x \ge y\log(y)$.

Proof. (i) We prove the statement by contrapositive. Suppose that $x > y\log(y)$. Then since $t \mapsto \frac{t}{\log(t)}$ is increasing over (e,∞), we have

$$\frac{x}{\log(x)} \ge \frac{y\log(y)}{\log(y\log(y))} = \frac{y}{1+\frac{\log\log(y)}{\log(y)}} > \frac{y}{2}.$$

(ii) Again, we prove the statement by contrapositive. Suppose that $x/\log(x) < y$. Then since $t \mapsto \frac{t}{\log(t)}$ is increasing over (e,∞), we have

$$\frac{x}{\log(x)} < \frac{y\log(y)}{\log(y\log(y))} = \frac{y}{1+\frac{\log\log(y)}{\log(y)}} < y.$$

This completes the proof.

$\square$

Lemma 31. *The following two properties hold.*

(i) For any $x \geq 0$ and $c \in [0,1]$, we have $\log(1+cx) \geq c\log(1+x)$.

(ii) For any $x \geq 0$ and $C \geq 1$, we have $C\log(1+x) \geq \log(1+Cx)$.

Proof of Lemma 31. (i) By concavity of the logarithm, we have $\log(1+cx) = \log\left((1-c)\cdot 1 + c(1+x)\right) \geq (1-c)\log(1) + c\log(1+x) = c\log(1+x)$ for any $c \in [0,1]$.

(ii) We have seen above that $\log(1+cy) \geq c\log(1+y)$ for any $y \geq 0$ and $c \in [0,1]$. Applying this with $y = Cx$ and $c = 1/C$ yields the result.

□

Lemma 32. *Let $n \in \mathbb{N}$ and $p_0 \in (0,1)$ and w denote the function defined in (32). For any $x \in [0,n]$, the following properties hold.*

1. *We have $w(x) \leq \frac{(x-np_0)^2}{\sigma^2}$ where $\sigma^2 = np_0(1-p_0)$*
2. *There exists a universal constant $c_0 > 0$ such that, for any $x \in [np_0, np_0 + c_0\sigma^2]$, we have $w(x) \geq \frac{(x-np_0)^2}{8\sigma^2}$.*
3. *For any $x \in [c_0\sigma^2, n]$, we have $w(x) \asymp \sigma\frac{x-np_0}{\sigma}\log\left(1+\frac{x-np_0}{\sigma^2}\right)$.*

Proof of Lemma 32.

1. Using the inequality $\log(1+x) \leq x$ that holds for any $x \in [-1,\infty)$ with the convention $\log(0) = -\infty$, we have, for any $x \in [0,n]$

$$
\begin{aligned}
w(x) &= n(1-p_0)\,h_B\left(-\frac{x-np_0}{n(1-p_0)}\right) + np_0\,h_B\left(\frac{x-np_0}{np_0}\right)\\
&= n(1-p_0)\left\{\left(1-\frac{x-np_0}{n(1-p_0)}\right)\log\left(1-\frac{x-np_0}{n(1-p_0)}\right)+\frac{x-np_0}{n(1-p_0)}\right\}\\
&\quad + np_0\left\{\left(1+\frac{x-np_0}{np_0}\right)\log\left(1+\frac{x-np_0}{np_0}\right)-\frac{x-np_0}{np_0}\right\}\\
&\leq n(1-p_0)\left\{\left(1-\frac{x-np_0}{n(1-p_0)}\right)\left(-\frac{x-np_0}{n(1-p_0)}\right)+\frac{x-np_0}{n(1-p_0)}\right\}\\
&\quad + np_0\left\{\left(1+\frac{x-np_0}{np_0}\right)\left(\frac{x-np_0}{np_0}\right)-\frac{x-np_0}{np_0}\right\}\\
&= \frac{(x-np_0)^2}{n(1-p_0)} + \frac{2(x-np_0)^2}{np_0}\\
&= \frac{(x-np_0)^2}{\sigma^2}.
\end{aligned}
$$

2. For any $x \in [np_0, np_0 + c\sigma^2]$, we have $\frac{x-np_0}{np_0} \vee \frac{x-np_0}{n(1-p_0)} \in [0,c]$. Choosing c small enough that $\log(1+t) \geq t - \frac{3t^2}{4}$ for any $t \in [-c,c]$, we obtain that,

$$
\begin{aligned}
w(x) &= n(1-p_0)\,h_B\left(-\frac{x-np_0}{n(1-p_0)}\right) + np_0\,h_B\left(\frac{x-np_0}{np_0}\right)\\
&= n(1-p_0)\left\{\left(1-\frac{x-np_0}{n(1-p_0)}\right)\log\left(1-\frac{x-np_0}{n(1-p_0)}\right)+\frac{x-np_0}{n(1-p_0)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ np_0 \left\{ \left(1 + \frac{x - np_0}{np_0}\right) \log\left(1 + \frac{x - np_0}{np_0}\right) - \frac{x - np_0}{np_0} \right\} \\
&\geq n(1-p_0) \left\{ \left(1 - \frac{x - np_0}{n(1-p_0)}\right) \left(-\frac{x - np_0}{n(1-p_0)} - \frac{3}{4}\frac{(x - np_0)^2}{(n(1-p_0))^2}\right) + \frac{x - np_0}{n(1-p)} \right\} \\
&\quad + np_0 \left\{ \left(1 + \frac{x - np_0}{np_0}\right) \left(\frac{x - np_0}{np_0} - \frac{3}{4}\frac{(x - np_0)^2}{(n(1-p_0))^2}\right) - \frac{x - np_0}{np_0} \right\} \\
&\geq n(1-p_0) \frac{1}{4}\frac{(x - np_0)^2}{(n(1-p_0))^2} + np_0 \left[\frac{1}{4}\frac{(x - np_0)^2}{(np_0)^2} - \frac{3}{4}\frac{(x - np_0)^3}{(np_0)^3}\right] \\
&\geq \frac{1}{8}\frac{(x - np_0)^2}{\sigma^2}
\end{aligned}
$$

provided c is small enough.

3. Assume now that $x \in [np_0 + c_0\sigma^2, n]$. We have

$$
\begin{aligned}
n(1-p_0)h_B\left(-\frac{x - np_0}{n(1-p_0)}\right) &= n(1-p_0) \left\{ \left(1 - \frac{x - np_0}{n(1-p_0)}\right) \log\left(1 - \frac{x - np_0}{n(1-p_0)}\right) + \frac{x - np_0}{n(1-p_0)} \right\} \\
&\in \big[0, x - np_0\big].
\end{aligned}
$$

Moreover, using the relation $h_B(y) \asymp y \log(1 + y)$ that holds for any $y > 0$, we have

$$
np_0\, h_B\left(\frac{x - np_0}{np_0}\right) \asymp (x - np_0) \log\left(\frac{x}{np_0}\right) \geq \log\left(1 + \frac{3c_0}{4}\right)(x - np_0).
$$

Therefore, we obtain

$$
\begin{aligned}
w(x) \asymp np_0\, h_B\left(\frac{x - np_0}{np_0}\right) &\asymp (x - np_0) \log\left(\frac{x}{np_0}\right) \\
&\asymp \sigma \frac{x - np_0}{\sigma} \log\left(1 + \frac{x - np_0}{\sigma^2}\right).
\end{aligned}
$$

□

The following corollary follows by combining the properties from Lemma 32.

Corollary 2. *For any $x \in [np_0, n]$, it holds that $w(x) \asymp \sigma z \log(1 + z/\sigma)$ where $z = \frac{x - np_0}{\sigma}$ and $\sigma^2 = np_0(1 - p_0)$.*