Supplementary Material for Symmetric Perceptron with Random Labels

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1 Main Result

In this supplementary material, we prove Theorem II.6 from the main text. We reproduce it below for convenience.

Theorem 1.1 (Theorem II.6 from Main Text)**.** *For any $\kappa > 0$, there exists a $p_\kappa^* < 1$ such that the following holds. Fix any $p \in [p_\kappa^*, 1]$ and any $\alpha < \widetilde{\alpha}_c(\kappa, p)$. Then,*

$$\liminf_{n\to\infty} \mathbb{P}\left[\widetilde{S}_\alpha(\kappa, p) \neq \varnothing\right] > 0.$$

Moreover, for any $\kappa \in (0, 0.817)$, there exists a $p_\kappa^{**} > 0$ such that the following holds. Fix any $p \in [0, p_\kappa^{**}]$ and any $\alpha < \widetilde{\alpha}_c(\kappa, p)$. Then,

$$\liminf_{n\to\infty} \mathbb{P}\left[\widetilde{S}_\alpha(\kappa, p) \neq \varnothing\right] > 0.$$

We prove Theorem 1.1 contingent on an assumption regarding a certain real-valued function. It is worth noting that various related results in the field mentioned earlier were also established contingent on an analogous assumption, see e.g. [APZ19, Hypothesis 3], [PX21, Assumption 1], and [DS19, Condition 1.2].

Assumption 1.2. *Following the notation in [APZ19], let*

$$F_{r,\kappa,\alpha}(\beta) \triangleq h(\beta) + \alpha \log_2 \mathbb{P}[|Z_1| \le \kappa, |Z_\beta| \le \kappa]$$
$$F_{u,\kappa,\alpha}(\beta) \triangleq h(\beta) + \alpha \log_2 \mathbb{P}[|Z_1| > \kappa, |Z_\beta| > \kappa],$$

where $Z_1, Z_\beta \sim \mathcal{N}(0,1)$ with correlation $2\beta - 1$ and $h(\beta)$ is the binary entropy function:

$$h(\beta) = -\beta \log_2 \beta - (1-\beta)\log_2(1-\beta).$$

Fix any $p \in [0,1]$ and set $F_{\kappa,\alpha,p}(\beta) = pF_{r,\kappa,\alpha}(\beta) + (1-p)F_{u,\kappa,\alpha}(\beta)$. For any $\kappa > 0$ and $\alpha > 0$ with $F''_{\kappa,\alpha,p}(1/2) < 0$, there is at most one $\beta \in (1/2, 1)$ such that $F'_{\kappa,\alpha,p}(\beta) = 0$.

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Several remarks are in order. Assumption 1.2 is analogous to [APZ19, Hypothesis 3], adopted both for the SBP (corresponding to $p = 1$ in our model) and for the UBP (corresponding to $p = 0$ in our model) therein. Furthermore, for the SBP, [APZ19, Hypothesis 3] is verified by Abbe, Li, and Sly [ALS21]. It is likely that their techniques adapt also to the UBP for a range of κ values, e.g. when $\kappa < \kappa^* \approx 0.817$[1]. In light of these facts, as well as numerical studies reported in Section 3, Assumption 1.2 is indeed plausible. A rigorous verification is left for future work.

2 Proof of Theorem 1.1

Our proof is very similar to that of [APZ19, Proposition 6], and we use the identical notation whenever appropriate. Furthermore, we only prove the first part as the second part is identical.

The proof is based on the *second moment method*.

Lemma 2.1. *Let Z be an integer-valued random variable with $\mathbb{P}[Z \geq 0] = 1$. Then*

$$\mathbb{P}[Z > 0] \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}.$$

Lemma 2.1 is known as the Paley-Zygmund inequality, we provide a proof for completeness.

Proof of Lemma 2.1. Let $I = \mathbb{1}\{Z > 0\}$, thus $\mathbb{P}[Z > 0] = \mathbb{E}[I] = \mathbb{E}[I^2]$. We then conclude by applying Cauchy-Schwarz inequality:

$$\mathbb{P}[Z > 0]\mathbb{E}[Z^2] = \mathbb{E}[I^2]\mathbb{E}[Z^2] \geq \mathbb{E}[Z\mathbb{1}\{Z > 0\}]^2 = \mathbb{E}[Z]^2. \qquad \square$$

We next provide an auxiliary lemma, originally due to Achlioptas and Moore [AM02, Lemma 2]. The version below is reproduced from [APZ19, Lemma 8].

Lemma 2.2. *Let $g(\beta)$ be a real analytic function on $[0,1]$ and let*

$$G(\beta) = \frac{g(\beta)}{\beta^\beta(1-\beta)^{1-\beta}}.$$

Suppose that (a) $G(1/2) > G(\beta)$ for every $\beta \neq 1/2$ and (b) $G''(1/2) < 0$. Then, there exists constants $c_2 > c_1 > 0$ such that

$$c_2 G(1/2)^n \geq \sum_{0\leq \ell \leq n} \binom{n}{\ell} g(\ell/n)^n \geq c_1 G(1/2)^n.$$

In the remainder of the proof, we let $q(\kappa) \triangleq \mathbb{P}[|Z| \leq \kappa]$ where $Z \sim \mathcal{N}(0,1)$.

Equipped with Lemmas 2.1 and 2.2, we let

$$Z = \left|\widetilde{S}_\alpha(\kappa,p)\right| = \sum_{\boldsymbol{\sigma}\in\Sigma_n} \mathbb{1}\left\{\boldsymbol{\sigma} \in \widetilde{S}_\alpha(\kappa,p)\right\}.$$

Theorem II.3 from the main text yields

$$\mathbb{E}[Z] = 2^n q(\kappa)^{p\alpha n}(1-q(\kappa))^{(1-p)\alpha n}. \tag{1}$$

[1] Above κ^*, the model exhibits *replica symmetry breaking* behaviour, see [APZ19] for details.

Second Moment Calculation Next, fix any $\beta \in [0,1]$, let $Z \sim \mathcal{N}(0,1)$ and $Z_\beta \sim \mathcal{N}(0,1)$ with $\mathbb{E}[Z_\beta Z] = 2\beta - 1$. Define

$$q_{r,\kappa}(\beta) = \mathbb{P}[|Z| \le \kappa, |Z_\beta| \le \kappa] \tag{2}$$

$$q_{u,\kappa}(\beta) = \mathbb{P}[|Z| > \kappa, |Z_\beta| > \kappa]. \tag{3}$$

These are precisely the same quantities appearing in [APZ19, Equation 6]. Note that

$$Z^2 = \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}' \in \Sigma_n} \mathbb{1}\left\{\boldsymbol{\sigma} \in \widetilde{S}_\alpha(\kappa,p), \boldsymbol{\sigma}' \in \widetilde{S}_\alpha(\kappa,p)\right\}.$$

Taking expectations of both sides, we obtain

$$\mathbb{E}[Z^2] = 2^n \sum_{0 \le \ell \le n} \binom{n}{\ell} q_{r,\kappa}(\beta)^{p\alpha n} q_{u,\kappa}(\beta)^{(1-p)\alpha n}.$$

Soon, we will apply Lemma 2.2 to $G_{\kappa,\alpha,p}(\beta)$ where

$$G_{\kappa,\alpha,p}(\beta) = \frac{q_{r,\kappa}(\beta)^{p\alpha} q_{u,\kappa}(\beta)^{(1-p)\alpha}}{\beta^\beta (1-\beta)^{1-\beta}}. \tag{4}$$

Suppose first that $G_{\kappa,\alpha,p}(\cdot)$ satisfies the conditions of Lemma 2.2. Then, we immediately obtain

$$\mathbb{E}[Z^2] \le c_2 \cdot 2^n \cdot G(1/2)^n = c_2 \cdot 4^n \cdot q(\kappa)^{2p\alpha n} \cdot (1 - q(\kappa))^{2(1-p)\alpha n},$$

for some $c_2 > 0$. Observe that

$$q_{r,\kappa}(1/2) = q(\kappa)^2 \quad \text{and} \quad q_{u,\kappa}(1/2) = \big(1 - q(\kappa)\big)^2.$$

Now recalling (1) and applying Lemma 2.1, we establish the desired result:

$$\liminf_{n\to\infty} \mathbb{P}\big[\widetilde{S}_\alpha(\kappa,p) \ne \varnothing\big] = \liminf_{n\to\infty} \mathbb{P}[Z > 0] \ge \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} \ge \frac{1}{c_2} > 0.$$

Verifying Conditions of Lemma 2.2 Hence, it suffices to verify that $G_{\kappa,\alpha}(\beta)$ defined in (4) satisfies the conditions of Lemma 2.2. We proceed analogously to [APZ19]. To that end, we let

$$G_{r,\kappa,\alpha}(\beta) = \frac{q_{r,\kappa}(\beta)^\alpha}{\beta^\beta(1-\beta)^{1-\beta}} \quad \text{and} \quad G_{u,\kappa,\alpha}(\beta) = \frac{q_{u,\kappa}(\beta)^\alpha}{\beta^\beta(1-\beta)^{1-\beta}},$$

and obtain

$$G_{\kappa,\alpha,p}(\beta) = G_{r,\kappa,\alpha}(\beta)^p G_{u,\kappa,\alpha}(\beta)^{1-p}. \tag{5}$$

We then set $G_{\kappa,\alpha,p}(\beta) = \exp\big(F_{\kappa,\alpha,p}(\beta)\big)$ as in the proof of [APZ19, Proposition 6] and observe, using (5), that

$$F_{\kappa,\alpha,p}(\beta) = pF_{r,\kappa,\alpha}(\beta) + (1-p)F_{u,\kappa,\alpha}(\beta), \tag{6}$$

where $F_{r,\kappa,\alpha}(\beta)$ is precisely the term arising in [APZ19, Equation 9] and $F_{u,\kappa,\alpha}(\beta)$ is the term defined in [APZ19, Section 2.2.2]. Note that a necessary condition is $F_{\kappa,\alpha,p}(1/2) > F_{\kappa,\alpha,p}(1)$ for all p, which boils down to the condition

$$\alpha < -\frac{1}{p\log_2 q(\kappa) + (1-p)\log_2\big(1 - q(\kappa)\big)} = \widetilde{\alpha}_c(\kappa,p). \tag{7}$$

Next, we have $F''_{\kappa,\alpha,p}(1/2) = pF''_{r,\kappa,\alpha}(1/2)+(1-p)F''_{u,\kappa,\alpha}(1/2)$. Using the expressions for $F''_{r,\kappa,\alpha}(1/2)$ and $F''_{u,\kappa,\alpha}(1/2)$ derived in [APZ19], we get

$$F''_{\kappa,\alpha,p}(1/2) = 4p\left(-1+\frac{2}{\pi}\frac{\alpha\kappa^2e^{-\kappa^2}}{q(\kappa)^2}\right)+4(1-p)\left(-1+\frac{2}{\pi}\frac{\alpha\kappa^2e^{-\kappa^2}}{\big(1-q(\kappa)\big)^2}\right)$$
$$= -4+\alpha\cdot\frac{8}{\pi}\kappa^2e^{-\kappa^2}\left(\frac{p}{q(\kappa)^2}+\frac{1-p}{\big(1-q(\kappa)\big)^2}\right).$$

So, it suffices to verify that

$$\alpha < \frac{\pi}{2\kappa^2e^{-\kappa^2}}\left(\frac{p}{q(\kappa)^2}+\frac{1-p}{\big(1-q(\kappa)\big)^2}\right)^{-1} \tag{8}$$

to ensure $F''_{\kappa,\alpha,p}(1/2) < 0$. We now establish our claim. Fix any $\kappa > 0$. Note that the argument of [APZ19] shows

$$-\frac{1}{\log_2 q(\kappa)} < \frac{\pi}{2\kappa^2e^{-\kappa^2}}q(\kappa)^2. \tag{9}$$

Define

$$\zeta(p,\kappa) = -\frac{1}{p\log_2 q(\kappa)+(1-p)\log_2\big(1-q(\kappa)\big)} - \frac{\pi}{2\kappa^2e^{-\kappa^2}}\left(\frac{p}{q(\kappa)^2}+\frac{1-p}{\big(1-q(\kappa)\big)^2}\right)^{-1}. \tag{10}$$

Note that for any fixed $\kappa > 0$, $p \mapsto \zeta(p,\kappa)$ is continuous. Furthermore, (9) yields $\zeta(1,\kappa) < 0$. So, for any fixed κ, there is a p^*_κ for which $\zeta(p,\kappa) < 0$ for every $p \in [p^*_\kappa, 1]$. Now if $\zeta(p,\kappa) < 0$, then we have

$$\zeta(p,\kappa) = \widetilde{\alpha}_c(\kappa,p) - \frac{\pi}{2\kappa^2e^{-\kappa^2}}\left(\frac{p}{q(\kappa)^2}+\frac{1-p}{\big(1-q(\kappa)\big)^2}\right)^{-1} < 0,$$

so that for any $\alpha < \widetilde{\alpha}_c(\kappa,p)$, (8) holds. We now verify that $F_{\kappa,\alpha,p}(\beta)$ is maximized at $\beta = 1/2$, under Assumption 1.2. As $F_{\kappa,\alpha,p}$ is symmetric around $\beta = \frac{1}{2}$, it suffices to consider $\beta \in [1/2, 1]$. Since $F'_{\kappa,\alpha,p}(1/2) = 0$ and $F''_{\kappa,\alpha,p}(1/2) < 0$, and $F_{\kappa,\alpha,p}$ has at most one critical point in $(1/2, 1)$, it must attain its maxima either at $\beta = 1/2$ or at $\beta = 1$. As $F_{\kappa,\alpha,p}(1/2) > F_{\kappa,\alpha,p}(1)$ verified in (7), the conditions of Lemma 2.2 are satisfied.

The second part of the Theorem 1.1 is established similarly. In this case, an inequality analogous to (9) holds only when $\kappa < \kappa^* = 0.817$, marking the onset of replica symmetric breaking, see [APZ19] for details.

3 Numerical Experiments

3.1 The Function $\zeta(\kappa, p)$

See Figure 1 for a plot of $\zeta(\kappa, p)$ appearing in (10).

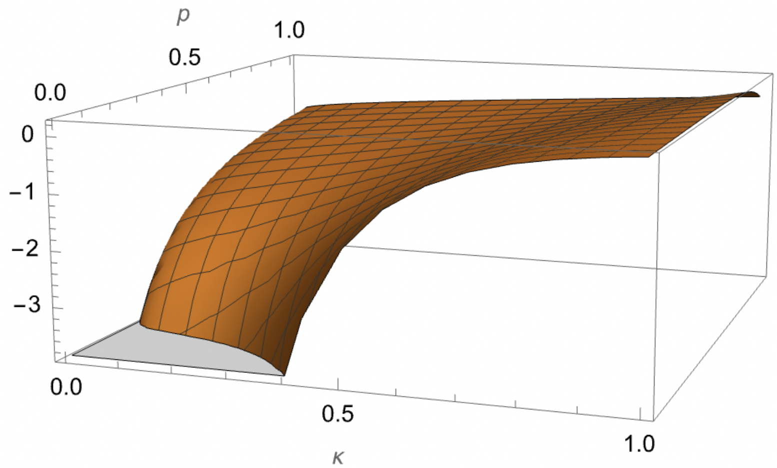

Figure 1: Plot of $\zeta(\kappa, p)$, truncated as $\zeta(0,0) \to -\infty$.

Furthermore, Figure 2 shows the region of (κ, p) pairs for which $\zeta(\kappa, p) < 0$. Recall that for any given $\kappa > 0$, we establish Theorem 1.1 for a range of p values, i.e. $p \in [p^*_\kappa, 1]$ for a suitable p^*_κ. For any fixed $\kappa > 0$, the corresponding p^*_κ can be read off directly from Figure 2.

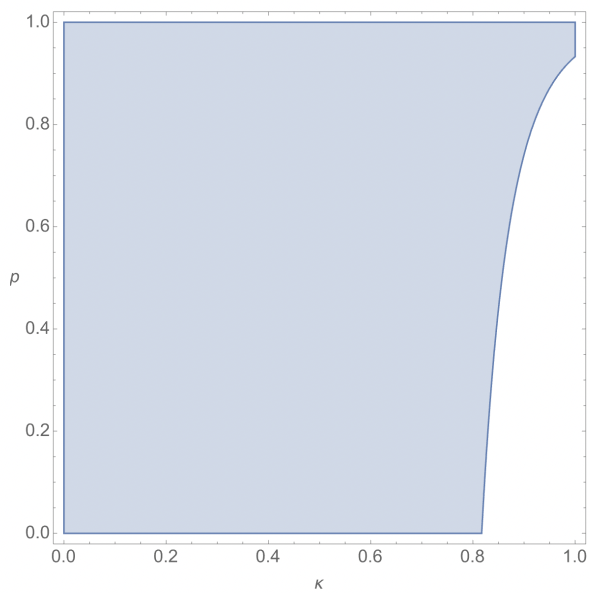

Figure 2: Region of (κ, p) pairs with $\zeta(\kappa, p) < 0$.

3.2 The Function $F_{\kappa,\alpha,p}(\beta)$

We now plot $F_{\kappa,\alpha,p}(\beta)$ appearing in Assumption 1.2, where the axes correspond to p and β. We plotted $F_{\kappa,\alpha,p}$ across p for a broad range of (κ, α) pairs, see Figure 3 for $(\kappa, \alpha) = (0.6, 1)$. This demonstrates typical behavior: Assumption 1.2 is satisfied for all values of p. (6)

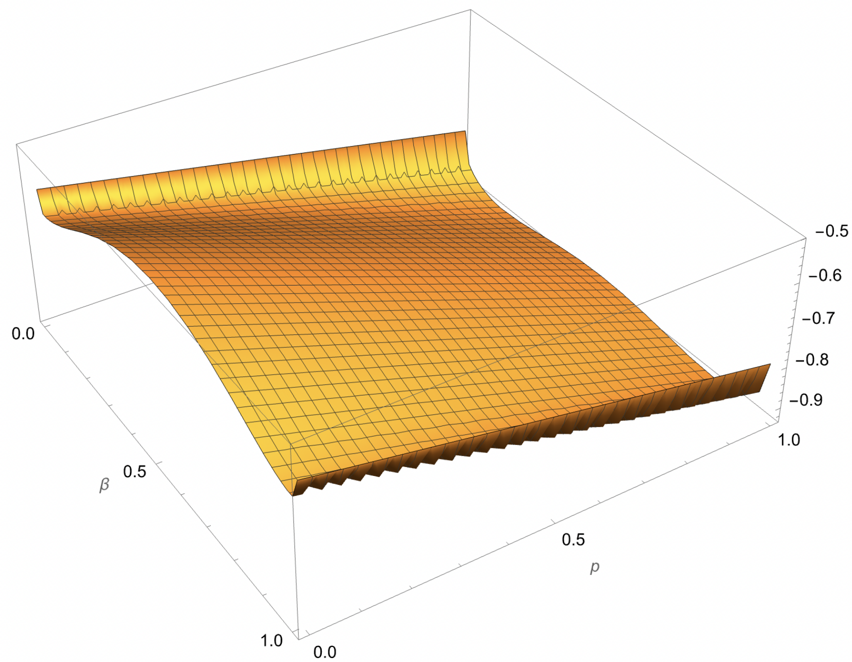

Figure 3: $F_{\kappa,\alpha,p}(\beta)$ for $\kappa = 0.6, \alpha = 1$.

See Figure 4 for a plot of $F_{\kappa,\alpha,p}(\beta)$ for $\kappa = 1.8, \alpha = 0.5$, where the axes correspond to p and β. This demonstrates a phase transition, where Assumption 1.2 is only satisfied for $p \in [p^*_\kappa, 1]$ for a suitable p^*_κ. At $p = 0$, corresponding to the `UBP`, $F_{\kappa,\alpha,p}(1/2)$ is not a local maximum. However, at $p = 1$, corresponding to the `SBP`, $F_{\kappa,\alpha,p}(1/2)$ is a local maximum.

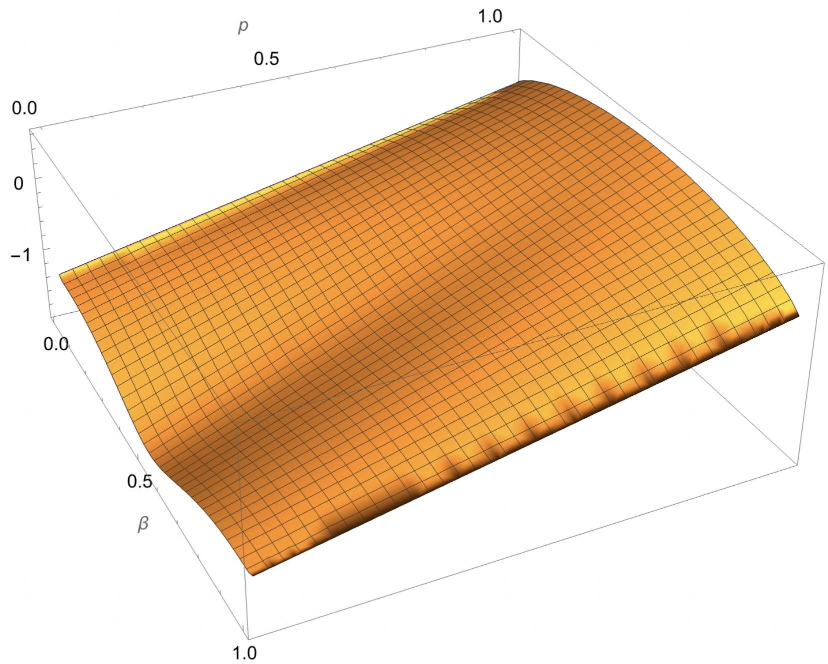

Figure 4: $F_{\kappa,\alpha,p}(\beta)$ for $\kappa = 1.8, \alpha = 0.5$.

References

[ALS21] Emmanuel Abbe, Shuangping Li, and Allan Sly, *Proof of the contiguity conjecture and lognormal limit for the symmetric perceptron*, arXiv preprint arXiv:2102.13069 (2021).

[AM02] Dimitris Achlioptas and Cristopher Moore, *The asymptotic order of the random k-sat threshold*, The 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002. Proceedings., IEEE, 2002, pp. 779–788.

[APZ19] Benjamin Aubin, Will Perkins, and Lenka Zdeborová, *Storage capacity in symmetric binary perceptrons*, Journal of Physics A: Mathematical and Theoretical **52** (2019), no. 29, 294003.

[DS19] Jian Ding and Nike Sun, *Capacity lower bound for the Ising perceptron*, Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, 2019, pp. 816–827.

[PX21] Will Perkins and Changji Xu, *Frozen 1-RSB structure of the symmetric Ising perceptron*, Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, 2021, pp. 1579–1588.