Best Arm Identification in Rare Events
(Supplementary Material)

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A THE $\mathcal{K}_{inf}$ PROBLEM AND RELATED REFORMULATIONS

A.1 DUAL FORM OF $\mathcal{K}_{inf}$

The following well-known Lemma gives the dual representations of $\mathcal{K}^U_{inf}(.,.)$ and $\mathcal{K}^L_{inf}(.,.)$. We follow the approach used in Honda and Takemura [2010], Agrawal et al. [2020].

Lemma 2. *Consider any discrete distribution η with a finite support $\{y_j\}_{j\in[n]}$ and an upper bound B. We assume $y_j \geq 0, \forall j$ and $0 < x < B$.*

a) The dual representation of $\mathcal{K}^U_{inf}(\eta, x)$ is

$$\mathcal{K}^U_{inf}(\eta, x) = \max_{\lambda_U \in \left[0, \frac{1}{B-x}\right]} \sum_{j=0}^{n} \eta_j \log(1 + \lambda_U(x - y_j)).$$

The optimal λ^*_U in the dual maximization above is characterised by:

$$\begin{cases} \lambda^*_U = 0, & \text{if } x < \mu_\eta, \\ \lambda^*_U = \frac{1}{B-x}, & \text{if } x > \mu_\eta \text{ and } \sum_{j=0}^{n_i} \eta_j \left(\frac{B-x}{B-y_j}\right) < 1, \\ \sum_j \frac{y_j \eta_j}{1+\lambda^*_U(x-y_j)} = x, & \text{If } x > \mu_\eta \text{, and } \sum_{j=0}^{n} \eta_j \left(\frac{B-x}{B-y_j}\right) \geq 1. \end{cases}$$

The support of the primal optimizer κ^* satisfies $supp(\eta) \subseteq supp(\kappa^*) \subseteq supp(\eta) \cup \{B\}$. The constraint is tight at optimality:

$$\mu_{\kappa^*} = x.$$

Further for $y_j \in supp(\eta)$:

$$\kappa^*(y_j) = \frac{n_j}{1 + \lambda^*_U(x - y_j)}.$$

b) The dual representation of $\mathcal{K}^L_{inf}(\eta, x)$ is

$$\mathcal{K}^L_{inf}(\eta, x) = \max_{\lambda_L \in \left[0, \frac{1}{x}\right]} \sum_{j=0}^{n} \eta_j \log(1 - \lambda_L(x - y_j)).$$

The optimal λ^*_L in the dual maximization above is characterised by:

$$\begin{cases} \lambda^*_L = 0, & \text{if } x \geq \mu_\eta, \\ \sum_j \frac{(y_j - x)\eta_j}{1-\lambda^*_L(x-y_j)} = 0, & \text{If } x < \mu_\eta. \end{cases}$$

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The support of the primal optimizer κ^* satisfies supp$(\eta) =$ supp(κ^*). The constraint is tight at optimality:

$$\mu_{\kappa^*} = x.$$

Further for $y_j \in$ supp(η):

$$\kappa^*(y_j) = \frac{n_j}{1 - \lambda_L^*(x - y_j)}.$$

Proof. See sections A.2 and A.3. ☐

A.2 PROOF OF LEMMA 2A

Define the set $\mathcal{D} := \{0\} \cup [b, B]$. Suppose a probability distribution η has finite support (say $\{0, y_1, ..., y_n\}$ for some n) from $\mathcal{D}$. Let $\mathcal{M}^+(\mathcal{D})$ denote the set of positive finite measures on $\mathcal{D}$. We want to find $\mathcal{K}_{inf}^U(\eta, x)$, which is defined as

$$\mathcal{K}_{inf}^U(\eta, x) = \min_{\substack{\text{supp}(\kappa)\subseteq\mathcal{D} \\ \mathbb{E}[\kappa]\geq x}} KL(\eta, \kappa).$$

We shall develop a Lagrangian duality for the above quantity in the space $\mathcal{M}^+(\mathcal{D})$. The Lagrangian with multiplier $\lambda = (\lambda_1, \lambda_2)$ and $\kappa \in \mathcal{M}^+(\mathcal{D})$ is:

$$\mathcal{L}(\kappa, \lambda) := KL(\eta, \kappa) + \lambda_1(x - \int_{\mathcal{D}} y d\kappa(y)) + \lambda_2(1 - \int_{\mathcal{D}} d\kappa(y)).$$

Then the dual objective becomes

$$\mathcal{L}(\lambda) := \inf_{\kappa\in\mathcal{M}^+(\mathcal{D})} \mathcal{L}(\kappa, \lambda).$$

Let us define two quantities useful in the analysis:

$$h(y, \lambda) := -\lambda_2 - \lambda_1 y,$$

$$Z(\lambda) := \{y \in \mathcal{D} : h(y, \lambda) = 0\}.$$

We define the set

$$\begin{aligned}\mathcal{R}_2 &:= \{\lambda \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}, \lambda \neq 0, \inf_{y\in\mathcal{D}} h(y, \lambda) \geq 0\} \\ &= \{\lambda \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}, \lambda \neq 0, -\lambda_2 \geq \lambda_1 B \geq 0\}.\end{aligned}$$

The lemma below shows that in maximising the dual objective $\mathcal{L}(\lambda)$, it is enough to restrict ourselves to the set $\mathcal{R}_2$.

Lemma A.1.a.

$$\max_{\substack{\lambda_1\geq 0, \\ \lambda_2\in\mathbb{R}}} \mathcal{L}(\lambda) = \max_{\lambda\in\mathcal{R}_2} \mathcal{L}(\lambda)$$

Proof. Suppose $\lambda \notin \mathcal{R}_2$. Then, there is a $y_0 \in \mathcal{D}$ such that $h(y_0, \lambda) < 0$. We know that for any $M > 0$, we have a measure $\kappa_M \in \mathcal{M}^+(\mathcal{D})$ such that

$$\kappa_M(y_0) = M, \ \frac{d\kappa_M}{d\eta}(y) = 1, \forall y \in \text{supp}(\eta)\backslash\{y_0\}$$

So, we must have that supp$(\kappa_M) = \{y_0\} \cup$ supp(η).

$$\begin{aligned}\mathcal{L}(\kappa_M, \lambda) &= \int_{\mathcal{D}} \log\left(\frac{d\eta}{d\kappa_M}(y)\right) d\eta(y) + \int_{\mathcal{D}} h(y, \lambda) d\kappa_M(y) + \lambda_1 x + \lambda_2 \\ &= \eta(y_0) \log\left(\frac{\eta(y_0)}{M}\right) + M h(y_0, \lambda) + \int_{\text{supp}(\eta)} h(y, \lambda) d\kappa_M(y) + \lambda_1 x + \lambda_2.\end{aligned}$$

Now as $M \to \infty$ the first two terms tend to $-\infty$ while the other terms remain bounded and gives the result. ☐

The next lemma characterises the minimizer κ^* in the dual objective $\mathcal{L}(\lambda)$. The support of κ^* is contained in $\text{supp}(\eta) \cup Z(\lambda)$ and its density wrt η (wherever it is well-defined) is $1/h(y,\lambda)$.

Lemma A.1.b. *For $\lambda \in \mathcal{R}_2$, $\kappa^* \in \mathcal{M}^+(\mathcal{D})$ that minimizes $\mathcal{L}(\kappa,\lambda)$ satisfies $\text{supp}(\eta) \subseteq \kappa^* \subseteq \text{supp}(\eta) \cup Z(\lambda)$. Also, for $y \in \text{supp}(\eta)$, $h(y,\lambda) > 0$, and*

$$\frac{d\kappa^*}{d\eta} = \frac{1}{-\lambda_1 - \lambda_2 y}.$$

Proof. Given $\lambda \in \mathcal{R}_2$, the inner optimization problem is strictly convex in κ. This means that a unique minimizer κ^* must exist. This κ^* must satisfy for any arbitrary $\kappa_1, \kappa_t := (1-t)\kappa^* + t\kappa_1$, $\left.\frac{\partial \mathcal{L}(\kappa_t,\lambda)}{\partial t}\right|_{t=0} \geq 0$.
Let us define $\mathcal{L}(t) := \mathcal{L}(\kappa_t, \lambda)$ which is

$$\int_{\text{supp}(\eta)} \log\left(\frac{d\eta}{d\kappa_t}(y)\right) d\eta(y) + \int_{\mathcal{D}} h(y,\lambda) d\kappa_t(y) + \lambda_1 x + \lambda_2.$$

Then,

$$\frac{d\mathcal{L}(t)}{dt} = \int_{\text{supp}(\eta)} \frac{d\eta}{d\kappa^*}(y)(d\kappa^*(y) - d\kappa_1(y)) + \int_{\mathcal{D}} h(y,\lambda)(d\kappa_1(y) - d\kappa^*(y)).$$

So,

$$\left.\frac{d\mathcal{L}(t)}{dt}\right|_{t=0} = -\int_{\mathcal{D}\setminus\text{supp}(\eta)} h(y,\lambda) d\kappa^*(y)) + \int_{\mathcal{D}\setminus\text{supp}(\eta)} h(y,\lambda)(d\kappa_1(y)).$$

Now, $\lambda \in \mathcal{R}^2$ guarantees that $\mathcal{L}^{'}(0) \geq 0$. This completes our proof. □

Remark A.1.1. *If $y \in Z(\lambda)$, then y can only be $-\frac{\lambda_2}{\lambda_1}$. Therefore, we get that $Z(\lambda) = \left\{-\frac{\lambda_2}{\lambda_1}\right\}$, if $\lambda_1 \geq 0, -\frac{\lambda_2}{\lambda_1} \in \mathcal{D}$ and $Z(\lambda) = \emptyset$, otherwise.*

It now remains to find $\max_{\lambda \in \mathcal{R}_2} \mathcal{L}(\lambda)$ in order to characterise the Lagrangian dual of $\mathcal{K}^U_{inf}(\eta, x)$.

If $Z(\lambda) = \Phi$, $\text{supp}(\kappa^*) = \text{supp}(\eta)$. We can then say from the characterization of κ^* that

$$\mathcal{K}^U_{inf}(\eta, x) = \max_{\lambda \in \mathcal{R}_2} \sum_{j=0}^{n} \eta_j \log(-\lambda_2 - \lambda_1 y_j)$$

The first order conditions tell us that $\sum_j \frac{\eta_j}{\lambda_2 - \lambda_1 y_j} = 1$ and $\sum_j \frac{y_j \eta_j}{\lambda_2 - \lambda_1 y_j} = x$. Multiplying the first equation by $-\lambda_2$ and the second by $-\lambda_1$ and then adding the two would give us that $\lambda_2 - \lambda_1 x = 1$. And $\lambda_2 \geq \lambda_1 B \Rightarrow 1 + \lambda_1 x \geq \lambda_1 B \Rightarrow \lambda_1 \in \left[0, \frac{1}{B-x}\right]$. We can therefore conclude that

$$\mathcal{K}^U_{inf}(\eta, x) = \max_{\lambda_1 \in \left[0, \frac{1}{B-x}\right]} \sum_{j=0}^{n} \eta_j \log(1 + \lambda_1(x - y_j))$$

If $Z(\lambda) \neq \Phi$, then $-\frac{\lambda_2}{\lambda_1} \leq B$. But $\lambda \in \mathcal{R}_2$ implies that $-\frac{\lambda_2}{\lambda_1} \geq B$. Hence, $-\frac{\lambda_2}{\lambda_1} = B$. Then, we can say that

$$\mathcal{K}^U_{inf}(\eta, x) = \max_{\lambda_1 \geq 0} \sum_{j=0}^{n} \eta_j \log(\lambda_1(B - y_j)).$$

Let λ^*_U denote the maximizing λ_1, $\kappa^*(B)$ denote the mass that κ^* puts at B. Then, we get from the first order conditions that $\sum_j \frac{\eta_j}{\lambda^*_U(B-y_j)} + \kappa^*(B) = 1$ and $\sum_j \frac{y_j \eta_j}{\lambda^*_U(B-y_j)} + B\kappa^*(B) = x$. Multiplying the first equation by B and adding to the second gives us that $B - x = \frac{1}{\lambda^*_U} \Rightarrow \lambda^*_U = \frac{1}{B-x}$. Therefore, in this case,

$$\mathcal{K}^U_{inf}(\eta, x) = \sum_{j=0}^{n} \eta_j \log\left(\frac{B - y_j}{B - x}\right).$$

Note that this can happen iff $\sum_{j=0}^{n} \eta_j \log\left(\frac{B-x}{B-y_j}\right) \leq 1$.

Irrespective of whether or not $Z(\lambda) = \Phi$, we can say that

$$\mathcal{K}_{inf}^{U}(\eta, x) = \max_{\lambda_1 \in \left[0, \frac{1}{B-x}\right]} \sum_{j=0}^{n} \eta_j \log(1 + \lambda_1(x - y_j))$$

. Let us define $p(\lambda_1) := \sum_{j=0}^{n} \eta_j \log(1 + \lambda_1(x - y_j))$, $\lambda_1 \in \left[0, \frac{1}{B-x}\right]$. Then, $p^{'}(\lambda_1) = \sum_{j=0}^{n} \frac{\eta_j(x-y_j)}{1+\lambda_1(x-y_j)}$ and $p^{''}(\lambda_1) = -\sum_{j=0}^{n} \frac{\eta_j(x-y_j)^2}{(1+\lambda_1(x-y_j))^2}$. The expression for $p^{''}$ leads us to conclude that p is always concave in λ_1 and hence, must have a unique maximizer.

If $x \leq \mathbb{E}_\eta$, note that $p^{'}(0) = x - \sum_{j=0}^{n} \eta_j y_j \leq 0$, i.e., p decreases in $\left[0, \frac{1}{B-x}\right]$. Hence, we must have $\mathcal{K}_{inf}^{U}(\eta, x) = \max_{\lambda_1 \in \left[0, \frac{1}{B-x}\right]} p(\lambda_1) = p(0) = 0$. Since the maximizer is $\lambda_U^* = 0$, we know from the definition of $Z(\lambda)$ that $Z(\lambda) = \Phi$, and therefore, $\text{supp}(\kappa^*) = \text{supp}(\eta)$.

If $x > \mathbb{E}_\eta$, then we have that $p^{'}(0) > 0$, meaning that p is increasing at $\lambda_1 = 0$ and therefore, may take the maximum value at either $\lambda_U^* = \frac{1}{B-x}$ or $\lambda_U^* \in \left(0, \frac{1}{B-x}\right)$. Let us first compute $p^{'}\left(\frac{1}{B-x}\right)$.

$$\begin{aligned}
p^{'}\left(\frac{1}{B-x}\right) &= \sum_{j=0}^{n} \eta_j \frac{(x-y_j)(B-x)}{(B-y_j)} \\
&= (B-x) \sum_{j=0}^{n} \frac{\eta_j x - \eta_j B + \eta_j B - \eta_j y_j}{B-y_j} \\
&= -(B-x)^2 \sum_{j=0}^{n} \frac{\eta_j}{B-y_j} + (B-x) \\
&= (B-x)\left[1 - \sum_{j=0}^{n} \eta_j \left(\frac{B-x}{B-y_j}\right)\right]
\end{aligned}$$

If $p^{'}\left(\frac{1}{B-x}\right) \leq 0$, then p must reach its maximum in $\left(0, \frac{1}{B-x}\right)$. This happens iff $\sum_{j=0}^{n} \eta_j \left(\frac{B-x}{B-y_j}\right) \geq 1$.

If $p^{'}\left(\frac{1}{B-x}\right) > 0$, then p must reach its maximum at $\frac{1}{B-x}$. This happens iff $\sum_{j=0}^{n} \eta_j \left(\frac{B-x}{B-y_j}\right) < 1$.

Remark A.1.2. *For the rare event setup, it is now easy to check that mass will be put at $B_i \gamma^{-\alpha_i}$ in $\mathcal{K}_{inf}^{U}(p_i, x)$ iff $x > F_0(\gamma)$, where $F_0(\gamma) := \frac{B_i}{\left(\sum_{j=1}^{n} \frac{a_{ij} p_{ij}}{B_i - a_{ij}}\right)^{-1} + \gamma^{\alpha_i}}$.*

A.3 PROOF OF LEMMA 2B

We want to find

$$\mathcal{K}_{inf}^{L}(\eta, x) = \min_{\substack{\text{supp}(\kappa) \subseteq \mathcal{D} \\ \mathbb{E}[\kappa] \leq x}} KL(\eta, \kappa)$$

Just as in section A.2, we shall develop a Lagrangian dual for $\mathcal{K}_{inf}^{L}(\eta, x)$. The Lagrangian with multiplier $\lambda = (\lambda_1, \lambda_2)$ is:

$$\mathcal{L}(\kappa, \lambda) := KL(\eta, \kappa) - \lambda_1(x - \int_{\mathcal{D}} y d\kappa(y)) - \lambda_2(1 - \int_{\mathcal{D}} d\kappa(y))$$

Similar to section A.2, define the quantities

$$\mathcal{L}(\lambda) := \inf_{\kappa \in \mathcal{M}^+(\mathcal{D})} \mathcal{L}(\kappa, \lambda),$$

$$h(y,\lambda) := \lambda_2 + \lambda_1 y,$$
$$Z(\lambda) := \{y \in \mathcal{D} : h(y,\lambda) = 0\}$$

and the set

$$\begin{aligned}\mathcal{R}_2 &:= \{\lambda \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 \in \mathbb{R}, \lambda \neq 0, \inf_{y\in\mathcal{D}} h(y,\lambda) \geq 0\} \\ &= \{\lambda \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda \neq 0\}.\end{aligned}$$

As in section A.2 we have the following lemmas:

Lemma A.2.a.

$$\max_{\substack{\lambda_1\geq 0,\\ \lambda_2\in\mathbb{R}}} \mathcal{L}(\lambda) = \max_{\lambda\in\mathcal{R}_2} \mathcal{L}(\lambda)$$

Proof. Suppose $\lambda \notin \mathcal{R}_2$. Then, there is a $y_0 \in \mathcal{D}$ such that $h(y_0,\lambda) < 0$. We know that for any $M > 0$, we have a measure $\kappa_M \in \mathcal{M}^+(\mathcal{D})$ such that

$$\kappa_M(y_0) = M,\ \frac{d\kappa_M}{d\eta}(y) = 1, \forall y \in \text{supp}(\eta)\backslash\{y_0\}$$

So, we must have that $\text{supp}(\kappa_M) = \{y_0\} \cup \text{supp}(\eta)$.

$$\begin{aligned}\mathcal{L}(\kappa,\lambda) &= \int_{\mathcal{D}} \log\left(\frac{d\eta}{d\kappa_M}(y)\right) d\eta(y) + \int_{\mathcal{D}} h(y,\lambda) d\kappa_M(y) - \lambda_1 x - \lambda_2 \\ &= \eta(y_0)\log\left(\frac{\eta(y_0)}{M}\right) + Mh(y_0,\lambda) + \int_{\text{supp}(\eta)} h(y,\lambda) d\kappa_M(y) - \lambda_1 x - \lambda_2\end{aligned}$$

Now as $M \to \infty$ the first two terms tend to $-\infty$ while the other terms remain bounded and we obtain the desired result. □

Lemma A.2.b. *For $\lambda \in \mathcal{R}_2$, $\kappa^* \in \mathcal{M}^+(\mathcal{D})$ that minimizes $\mathcal{L}(\kappa,\lambda)$ satisfies* $\text{supp}(\eta) \subseteq \kappa^* \subseteq \text{supp}(\eta) \cup Z(\lambda)$. *Also, for $y \in \text{supp}(\eta)$, $h(y,\lambda) > 0$, and*

$$\frac{d\kappa^*}{d\eta} = \frac{1}{\lambda_1 + \lambda_2 y}.$$

Proof. Given $\lambda \in \mathcal{R}_2$, the inner optimization problem is strictly convex in κ. This means that a unique minimizer κ^* must exist. This κ^* must satisfy for any arbitrary $\kappa_1, \kappa_t := (1-t)\kappa^* + t\kappa_1$, $\left.\frac{\partial \mathcal{L}(\kappa_t,\lambda)}{\partial t}\right|_{t=0} \geq 0$.

Let us define $\mathcal{L}(t) := \mathcal{L}(\kappa_t,\lambda)$ which is

$$\mathcal{L}(t) = \int_{\text{supp}(\eta)} \log\left(\frac{d\eta}{d\kappa_M}(y)\right) d\eta(y) + \int_{\mathcal{D}} h(y,\lambda) d\kappa_t(y) - \lambda_1 x - \lambda_2.$$

Then,

$$\frac{d\mathcal{L}(t)}{dt} = \int_{\text{supp}(\eta)} \frac{d\eta}{d\kappa^*}(y)(d\kappa^*(y) - d\kappa_1(y)) + \int_{\mathcal{D}} h(y,\lambda)(d\kappa_1(y) - d\kappa^*(y)).$$

So,

$$\left.\frac{d\mathcal{L}(t)}{dt}\right|_{t=0} = -\int_{\mathcal{D}\backslash\text{supp}(\eta)} h(y,\lambda) d\kappa^*(y)) + \int_{\mathcal{D}\backslash\text{supp}(\eta)} h(y,\lambda)(d\kappa_1(y)).$$

Now, $\lambda \in \mathcal{R}^2$ guarantees that $\mathcal{L}^{'}(0) \geq 0$. This completes our proof. □

Note that if $y \in Z(\lambda)$ then $y = -\frac{\lambda_2}{\lambda_1}$ if $-\frac{\lambda_2}{\lambda_1} \in \mathcal{D}$. But because $\lambda \in \mathcal{R}_2$ we have $-\frac{\lambda_2}{\lambda_1} < 0$ and hence $Z(\lambda) = \phi$. This implies $\text{supp}(\kappa^*) = \text{supp}(\eta)$ with the mean and probability conditions

$$1 = \sum_j \frac{\eta_j}{(\lambda_2 + \lambda_1 y_j)}$$
$$x = \sum_j \frac{y_j \eta_j}{(\lambda_2 + \lambda_1 y_j)}$$

These imply $1 = \lambda_2 + \lambda_1 x$. As $\lambda_2 \geq 0$, we have $\lambda_1 \leq \frac{1}{x}$. Thus, denoting the optinal λ_1 by λ_L^*, we get that

$$\mathcal{K}_{inf}^L(\eta, x) = \sum \eta_j \log(1 - \lambda_L^*(x - y_j))$$

with $0 \leq \lambda_L^* \leq 1/x$ and the mean equation

$$x = \sum_j \frac{y_j \eta_j}{(1 - \lambda_L^*(x - y_j))}.$$

A.4 REFORMULATION OF THE LOWER BOUND

We can now use lemma 1 to simplify $\mathcal{P}_i$ (see 7 of the main body) in the rare event setting. We observe that the objective in $\mathcal{P}_i$ is a smooth and strictly convex function. The optimizer, $x_{i,e}^*$, is therefore given by first-order stationarity conditions. Using the dual representation, we can write this as

$$w_1 \lambda_{L_{1i}}^*(x_{i,e}^*) - w_i \lambda_{U_i}^*(x_{i,e}^*) = 0$$

where $\lambda_{U_i}^*, \lambda_{L_{1i}}^*$ are as in lemma 1 and are functions of $x_{i,e}^*$. Now let us define quantities that are useful in reformulating $\mathcal{P}$ to a form suitable for further analysis. Define

$$\begin{aligned}
K_{1i} &:= 1 - x_{i,e}^* \lambda_{L_{1i}}^*(x_{i,e}^*), \\
C_{1i} &:= \lambda_{L_{1i}}^*(x_{i,e}^*) \gamma^{-\alpha_1}, \\
K_i &:= 1 + x_{i,e}^* \lambda_{U_i}^*(x_{i,e}^*), \\
C_i &:= \lambda_{U_i}^*(x_{i,e}^*) \gamma^{-\alpha_i}.
\end{aligned}$$

These quantities will turn out to have bounded limits as $\gamma \to 0$. The stationarity condition may now be rewritten as

$$C_{1i} w_1 \gamma^{\alpha_1} = C_i w_i \gamma^{\alpha_i}. \tag{1}$$

In the rare event setup, the tightness of the constraint in lemma 1 gives us that

$$x_{i,e}^* = \sum_{j=1}^{n_1} \frac{a_{1j} p_{1j}}{K_{1i} + C_{1i} a_{1j}} = \sum_{j=1}^{n_i} \frac{a_{ij} p_{ij}}{K_i - C_i a_{ij}} + B_i \gamma^{-\alpha_i} \left[1 - \sum_{j=1}^{n} \frac{p_{ij}}{K_i - C_i a_{ij}} \gamma^{\alpha_i} - \frac{1 - \sum_{j=1}^{n} p_{ij} \gamma^{\alpha_i}}{K_i} \right]. \tag{2}$$

Since the primal optimizer has the same support as the underlying distribution in part (b) of lemma 1, we must have

$$\sum_{j=1}^{n} \frac{p_{1j}}{K_{1i} + C_{1i} a_{1j}} \gamma^{\alpha_1} + \frac{1 - \sum_{j=1}^{n} p_{1j} \gamma^{\alpha_1}}{K_{1i}} = 1. \tag{3}$$

From their definitions and from the stationarity condition, we have the following relationship between K_{1i} and K_i:

$$w_1(1 - K_{1i}) = w_i(K_i - 1). \tag{4}$$

Let $\mathcal{P}_i = \inf_{x \in [\mu_i, \mu_1]} \mathcal{K}_i(w_1, w_i, x)$ (see (7) from the main body). We know from the Envelope Theorem that

$$\frac{d\mathcal{K}_i(w_1, w_i, x)}{dx} = -w_1 \lambda_{L_{1i}^*} + w_i \lambda_{U_i^*}.$$

The first order stationarity condition $\frac{d\mathcal{K}_i(w_1, w_i, x)}{dx} = 0$ implies that $w_1 \lambda_{L_{1i}^*} = w_i \lambda_{U_i^*} = \phi_i$, (say). Let us define $x_i^* := \arg\min_{x \in [\mu_i, \mu_1]} \mathcal{K}_i(w_1, w_i, x)$. It is easy to infer from our derivations of the $\mathcal{K}_{inf}^L$ and $\mathcal{K}_{inf}^U$ expressions that

$$\begin{aligned}
\mathcal{K}_{inf}^L(p_1, x_i^*) &= KL(p_1, \tilde{p}_1^{(i)}) \\
\mathcal{K}_{inf}^U(p_i, x_i^*) &= KL(p_i, \tilde{p}_i)
\end{aligned} \tag{5}$$

where

$$\tilde{p}_{1j}^{(i)} = \frac{p_{1j}}{1 - \lambda_{L_{1i}^*}(x_i^* - a_{1j}\gamma^{-\alpha_1})} = \frac{p_{1j}}{\left(1 - \frac{\phi_i}{w_1}x_i^*\right) + \frac{\phi_i a_{1j}}{w_1 \gamma^{\alpha_1}}}$$
$$\tilde{p}_{ij} = \frac{p_{ij}}{1 + \lambda_{U_i^*}(x_i^* - a_{ij}\gamma^{-\alpha_i})} = \frac{p_{ij}}{\left(1 + \frac{\phi_i}{w_i}x_i^*\right) - \frac{\phi_i a_{ij}}{w_i \gamma^{\alpha_i}}} \tag{6}$$

We note that $\mathbb{E}_{\tilde{p}_1^{(i)}} = \mathbb{E}_{\tilde{p}_i} = x_i^*$.

We can now express $K_{1i} = 1 - \frac{\phi_i}{w_1}x_i^* - i$, $K_i = 1 + \frac{\phi_i}{w_i}x_i^*$, $C_{1i} = \frac{\phi_i}{w_1\gamma^{\alpha_1}}$, $C_i = \frac{\phi_i}{w_i\gamma^{\alpha_i}}$. The following obvious equations will be helpful.

$$K_{1i} = \frac{1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1}}{1 - \sum_{j=1}^n \tilde{p}_{1j}^{(i)}\gamma^{\alpha_1}}$$
$$K_i = \frac{1 - \sum_{j=1}^n p_{ij}\gamma^{\alpha_i}}{1 - \sum_{j=1}^n \tilde{p}_{ij}\gamma^{\alpha_i}}$$
$$w_1(1 - K_{1i}) = w_i(K_i - 1) = \phi_i x_i^*$$

We also claim that

$$1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1} \le K_{1i} \le 1,$$
$$1 \le K_i \le \left[\frac{1}{1 - \frac{\gamma^{\alpha_1}\mu_1}{\max_j a_{ij}(1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1})}}\right]. \tag{7}$$

For the proof of the first claim, we see that $K_{1i} = 1 - \lambda_{L_{1i}^*}x \le 1$ because $0 \le \lambda_{L_{1i}^*} \le \frac{1}{x} \Rightarrow 0 \le \lambda_{L_{1i}^*}x \le 1$. The lower bound on K_{1i} is trivial.

For the proof of the second claim, we see that $K_i = 1 + \frac{\phi_i}{w_i}x^* \ge 1$. We also have that $w_i(K_i - 1) = \phi_i x^* \le \frac{\phi_i x^*}{K_{1i}} \le \frac{\phi_i x_i^*}{1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1}}$. This implies that $K_i - 1 \le \frac{\phi_i}{w_i\gamma^{\alpha_i}} . \frac{\gamma^{\alpha_i \mu_1}}{1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1}} \le \frac{K_i}{max_j a_{ij}} . \frac{\gamma^{\alpha_i \mu_1}}{1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1}}$. As the final step, we can conclude from the above chain of inequalities that $K_i\left(1 - \frac{1}{max_j a_{ij}} . \frac{\gamma^{\alpha_i \mu_1}}{1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1}}\right) \le 1$

These bounds tell us that $K_{1i}, K_i \to 1$ as $\gamma \to 0$. Now, we can write $\mathcal{P}_i$ in terms of K_{1i}, K_i, C_{1i}, C_i as

$$\mathcal{P}_i = w_1\gamma^{\alpha_1}\left[\sum_j p_{1j}\log(K_{1i} + C_{1i}a_{1j}) + \frac{(1 - \sum_{j=1}^n p_{1j}\gamma^{\alpha_1})}{\gamma^{\alpha_1}}\log(K_{1i})\right]$$
$$+ w_i\gamma^{\alpha_i}\left[\sum_j p_{ij}\log(K_i - C_i a_{ij}) + \frac{(1 - \sum_{j=1}^n p_{ij}\gamma^{\alpha_i})}{\gamma^{\alpha_i}}\log(K_i)\right]. \tag{8}$$

The advantage of re-writing $\mathcal{P}_i$ in terms of K_{1i}, K_i, C_{1i}, C_i is that these quantities have bounded well-defined limits and using equations (1),(2),(3),(4), we can eliminate the dependence on x_i^* (whose behaviour is not as easy to analyze when $\gamma \to 0$). The bounds on K_{1i} and K_i will also help us to define the approximate version $\mathcal{P}_{i,a}$ of $\mathcal{P}_i$ (see 9 of main body).

A.5 PROOF OF PROPOSITION 1

Consider i.i.d. draws of the ith arm. Define

$$\tau_{ij}^{(1)} := \text{the first time } a_{ij}\gamma^{-\alpha_i} \text{ is seen in arm } i.$$
$$\tau_{ij}^{(k)} := \text{the } k\text{th inter-arrival time of } a_{ij}\gamma^{-\alpha_i} \text{ in arm } i.$$

Then, we have that

$$\mathbb{P}(\tau_{ij}^{(1)} > n) = (1 - \gamma^{\alpha_i}p_{ij})^n$$

Clearly, the kth inter-arrival time is independent of all the previous inter-arrival times. Hence

$$\mathbb{P}(\tau_{ij}^{(k)} > n_k) = (1 - \gamma^{\alpha_i} p_{ij})^{n_k}$$

Now setting $n_k = t\gamma^{-\alpha_i}$ and taking the limit $\gamma \to 0$ we have

$$\begin{aligned}\lim_{\gamma\to 0} \mathbb{P}(\tau_{ij}^{(k)} > t\gamma^{-\alpha_i}) &= \lim_{\gamma\to 0} (1 - \gamma^{\alpha_i} p_{ij})^{t\gamma^{-\alpha_i}} \\ &= e^{p_{ij}t}\end{aligned}$$

Now as the inter-arrival times are asymptotically independent exponentially distributed, it follows by the standard argument that $N_{ij}(t)$ is asymptotically distributed as Poisson$(p_{ij}t)$. Note that the same argument could have been repeated while assuming two or more support points as a set. We would then get that the count process for the set are asymptotically distributed as sum of the individual Poisson distributions. From computing the Poisson mgf this implies asymptotic independence of these Poisson variables. We omit the arguments as they are standard.

B PROOF OF THEOREM 1

In this section alone, we add the superscript e to C_i, C_{1i} to prevent any confusion, since exact and approximate versions are used simultaneously. Let $C_{1i}^e, C_i^e, x_{i,e}^*$ denote solutions inner minimization problem $\mathcal{P}_i(w)$, and $C_{1i}^a, C_i^a, x_{i,a}^*$ denote solutions to the approximate inner minimization problem $\mathcal{P}_{i,a}(w)$. We have already established bounds on K_{1i} and K_i in A.4. It is straightforward to see from equation 2 of the supplementary material and equations 10 of the main body, that $0 \le C_{1i}^e, C_{1i}^a \le \frac{\sum_j p_{1j}}{\mu_i}, 0 \le C_i^e \le \frac{K_i}{B_i}, C_i^a \le \frac{1}{B_i}$. Using these bounds, one can easily use the definitions of $mathcal P_i$, $\mathcal{P}_{i,a}$ to conclude that $\mathcal{P}_i, \mathcal{P}_{i,a} = \mathcal{O}(\min(w_1\gamma^{\alpha_1}, w_i\gamma^{\alpha_i}))$. $\lim_{\gamma\to 0} \frac{\mathcal{P}_i}{\mathcal{P}_{i,a}} = 1$. becomes an immediate conclusion.

To establish the bound on $|\mathcal{P}_i - \mathcal{P}_{i,a}|$, we'll follow three broad steps: showing that the solutions to $\mathcal{P}_i$ also approximately solve $\mathcal{P}_{i,a}$; showing that solutions to $\mathcal{P}_i$ and solutions to $\mathcal{P}_{i,a}$ are close; using the Lipschitz property of $\tilde{\mathcal{K}}_{inf}^L$ and $\tilde{\mathcal{K}}_{inf}^U$ along with the triangle inequality to connect the bounds derived in the earlier steps and arrive at the proof. $\tilde{\mathcal{K}}_{inf}^L$ and $\tilde{\mathcal{K}}_{inf}^U$ are defined as follows:

$$\tilde{\mathcal{K}}_{inf}^L(z) = \gamma^{\alpha_1}\left(\sum_j p_{1j}\log(1 + za_{1j}) - z\sum_j \frac{a_{1j}p_{1j}}{1 - za_{1j}}\right)$$
$$\tilde{\mathcal{K}}_{inf}^U(m, z) = \gamma^{\alpha_i}\left(\sum_j p_{ij}\log(1 - za_{ij}) + zm\right)$$

Step 1: Solutions to exact problem approximately solve approximate problem

Bounds on K_{1i} (see 7) imply that given any $\epsilon > 0$, we have γ small enough that $K_{1i} \ge 1 - \epsilon$. Then

$$\log\left(\frac{1 - \epsilon + C_{1i}^e a_{1j}}{1 + C_{1i}^e a_{1j}}\right) \le \log\left(\frac{K_{1i} + C_{1i}^e a_{1j}}{1 + C_{1i}^e a_{1j}}\right) \le 0.$$

By Mean Value Theorem (MVT), we have that

$$\log\left(\frac{1 - \epsilon + C_{1i}^e a_{1j}}{1 + C_{1i}^e a_{1j}}\right) \ge -\frac{\epsilon}{1 - \epsilon}$$

and hence,

$$-\frac{\epsilon}{1 - \epsilon} \le \log(K_{1i} + C_{1i}^e a_{1j}) - \log(1 + C_{1i}^e a_{1j}) \le 0.$$

Thus, for small enough γ, $\log(1 + C_{1i}^e a_{1j}) \approx \log(K_{1i} + C_{1i}^e a_{1j})$.

Using the fact that $K_{1i} = 1 - C_{1i}^e x_{i,e}^* \gamma^{\alpha_1}$, we get

$$(1 - \gamma^{\alpha_1}\sum_j p_{1j})\frac{\log(K_{1i})}{\gamma^{\alpha_1}} \le -(1 - \epsilon)C_{1i}^e x_{i,e}^*$$

when $\gamma^{\alpha_1} \sum_j p_{ij} \le \epsilon$. Similarly, we have

$$(1-\gamma^{\alpha_1}\sum_j p_{1j})\frac{\log(K_{1i})}{\gamma^{\alpha_1}} \ge \frac{-C^e_{1i}x^*_{i,e}}{1-C^e_{1i}x^*_{i,e}\gamma^{\alpha_1}} = -C^e_{1i}x^*_{i,e} + \frac{-(C^e_{1i}x^*_{i,e})^2\gamma^{\alpha_1}}{1-C^e_{1i}x^*_{i,e}\gamma^{\alpha_1}}$$

Thus, for γ small enough, we have $(1-\gamma^{\alpha_1}\sum_j p_{1j})\frac{\log(K_{1i})}{\gamma^{\alpha_1}} \approx -C^e_{1i}x^*_{i,e}$. In $\mathcal{K}^L_{inf}$ (from Lemma 1b), $\tilde{p}$ has no probability mass on the upper bound B_i and hence

$$x^*_{i,e} = \sum_j \frac{a_{1j}p_{1j}}{1-C^e_{1i}a_{1j}}.$$

This gives us

$$|\tilde{\mathcal{K}}^L_{inf}(C^e_{1i}) - \mathcal{K}^L_{inf}(K_{1i}, C^e_{1i})| \le 2\gamma^{2\alpha_1}\frac{(\sum_j p_{1j})^2}{1-\sum_j p_{1j}\gamma^{\alpha_1}}$$

Bounds on K_i, imply that for any $\epsilon > 0$, we can choose γ (again independently of w) so that $K_i \le 1+\epsilon$. Then,

$$0 \le \log(K_i + C^e_i a_{ij}) - \log(1+C^e_i a_{ij}) \le \log\left(\frac{1+\epsilon+C^e_i a_{ij}}{1+C^e_i a_{ij}}\right).$$

Now, from MVT we have

$$\log(1+\epsilon+C^e_i a_{ij}) - \log(1+C^e_i a_{ij}) \le \frac{\epsilon}{1+C^e_i a_{ij}} \le \epsilon.$$

Thus, $\log(K_i + C^e_i a_{ij}) \approx \log(1+C^e_i a_{ij})$ when γ is small. From $K_i = 1 + C^e_i x^*_{i,e}\gamma^{\alpha_i}$, we have

$$(1-\epsilon)\frac{C^e_i x^*_{i,e}}{1+C^e_i x^*_{i,e}\gamma^{\alpha_i}} \le (1-\gamma^{\alpha_i}\sum_j p_{ij})\frac{\log(K_i)}{\gamma^{\alpha_i}} \le C^e_i x^*_{i,e}$$

when $\gamma^{\alpha_i} \le \epsilon$. Thus when γ small, $(1-\gamma^{\alpha_i}\sum_j p_{ij})\frac{\log(K_i)}{\gamma^{\alpha_i}} \approx C^e_i x^*_{i,e}$.

We thus have the following bound:

$$|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^{(e)}_i)| \le \frac{\frac{\mu_1}{\max_j a_{ij}}\gamma^{2\alpha_i}}{1-\frac{\mu_1}{\max_j a_{ij}}\gamma^{\alpha_i}}\left(\sum_j p_{ij} + \frac{\mu_1}{\max_j a_{ij}}\right)$$

It may be noted that the bound does not depend on w, which give uniform bounds independent of w.

Step 2: Solutions to exact problem are close to solutions of approximate problem

So far, we have shown that the C^e_{1i}, C^e_i and $x^*_{i,e}$ that solve the exact problem are also good solutions for the approximate problem. However, the solution to our new approximate problem will be C^a_{1i}, C^a_i and $x^*_{i,a}$. We'll now show that this set of solutions to the approximate problem indeed approaches the set of solutions to the actual problem at the rate of $\gamma^{\min(2\alpha_i, \alpha_i+\alpha_1)}$ as $\gamma \to 0$.

We have that

$$x^*_{i,e} = \sum_{j=1}^{n} \frac{a_{1j}p_{1j}}{1-C^e_{1i}x^*_{i,e}\gamma^{\alpha_1}+C^e_{1i}a_{1j}},$$

$$x^*_{i,a} = \sum_{j=1}^{n} \frac{a_{1j}p_{1j}}{1+C^a_{1i}a_{1j}},$$

Note that the above two statements imply that C^e_{1i} and C^a_{1i} are bounded above by $\frac{\sum_j p_{1j}}{\mu_i}$. We collect the following established results:

$$\frac{C^e_{1i}}{C^e_i} = \frac{C^a_{1i}}{C^a_i} = \frac{w_i \gamma^{\alpha_i}}{w_1 \gamma^{\alpha_1}},$$
$$x^*_{i,e} > F_0(\gamma) \Rightarrow C^e_i = \frac{1}{B_i - x^*_{i,e}\gamma^{\alpha_i}},$$
$$x^*_{i,a} > F_0(0) \Rightarrow C^a_i = \frac{1}{B_i},$$
$$x^*_{i,e} \leq F_0(\gamma) \Rightarrow x^*_{i,e} = \sum_{j=1}^{n} \frac{a_{ij}p_{ij}}{1 + C^e_i x^*_{i,e}\gamma^{\alpha_i} - C^e_i a_{1j}}$$
$$x^*_{i,a} \leq F_0(0) \Rightarrow x^*_{i,a} = \sum_{j=1}^{n} \frac{a_{ij}p_{ij}}{1 - C^e_i a_{1j}}$$

where $F_0(\gamma)$ is defined in RemarkA.1.2. In what follows, we shall let $b_i = \min_j a_{ij}$. We shall now establish that, for all w, the solution to the exact and approximate inner optimisations are close when γ is small. We break the analysis into the following four cases.

Case 1. $x^*_{i,e} \leq F_0(\gamma), x^*_{i,a} \leq F_0(0)$.

We have that

$$\begin{aligned} x^*_{i,e} - x^*_{i,a} &= \sum_{j=1}^{n} \frac{a_{1j}p_{1j}(1 - K_{1i} + a_{1j}(C^a_{1i} - C^e_{1i}))}{(1 + C^a_{1i}a_{1j})(K_{1i} + C^e_{1i}a_{1j})} \\ &= \sum_{j=1}^{n} \frac{a_{ij}p_{ij}(1 - K_i - a_{ij}(C^a_i - C^e_{1i}))}{(1 - C^a_i a_{ij})(K_i - C^{(e)}_i a_{1j})} \end{aligned}$$

Splitting terms from the numerator and using $\frac{C^e_{1i}}{C^e_i} = \frac{C^a_{1i}}{C^a_i} = \frac{w_i\gamma^{\alpha_i}}{w_1\gamma^{\alpha_1}}$, we get the following:

$$A(1 - K_{1i}) + B(1 - K_i) = \tilde{A}(C^e_{1i} - C^a_{1i}) + \tilde{B}\frac{w_1\gamma^{\alpha_1}}{w_i\gamma^{\alpha_i}}(C^e_{1i} - C^a_{1i})$$

where

$$A := \sum_{j=1}^{n} \frac{a_{1j}p_{1j}}{(1 + C^a_{1i}a_{1j})(K_{1i} + C^e_{1i}a_{1j})}$$
$$\tilde{A} := \sum_{j=1}^{n} \frac{a^2_{1j}p_{1j}}{(1 + C^a_{1i}a_{1j})(K_{1i} + C^e_{1i}a_{1j})} \geq b_1 A$$
$$B := \sum_{j=1}^{n} \frac{a_{ij}p_{ij}}{(1 - C^a_i a_{ij})(K_i - C^{(e)}_i a_{1j})}$$
$$\tilde{B} := \sum_{j=1}^{n} \frac{a_{ij^2}p_{ij}}{(1 - C^a_i a_{ij})(K_i - C^{(e)}_i a_{1j})} \geq b_i B$$

Therefore,

$$C^e_{1i} - C^a_{1i} = \gamma^{\alpha_i}\frac{Aw_i(1 - K_{1i}) + Bw_i(K_i - 1)}{\tilde{A}w_i\gamma^{\alpha_i} + \tilde{B}w_1\gamma^{\alpha_1}}$$

Using equation (4), we can write that

$$C^e_{1i} - C^a_{1i} = \left(\frac{Aw_i + Bw_1}{\tilde{A}w_i\gamma^{\alpha_i} + \tilde{B}w_1\gamma^{\alpha_1}}\right)\gamma^{\alpha_i}(1 - K_{1i}).$$

Following this, we can use the lower bounds on $\tilde{A}$, $\tilde{B}$ and K_{1i} to conclude that

$$|C_{1i}^e - C_{1i}^a| \leq \left(\frac{\sum_j p_{1j}}{\min(b_1, b_i)}\right)\gamma^{\min(\alpha_1,\alpha_i)}.$$

This also tells us that

$$|x_{i,e}^* - x_{i,a}^*| \leq \mu_1\left(\sum_{j=1}^n p_{1j}\gamma^{\alpha_1} + \frac{B_1\sum_j p_{1j}}{b_1 \wedge b_i}\gamma^{\alpha_1\wedge\alpha_i}\right).$$

And using a similar computation, we can also prove that

$$|C_i^e - C_i^a| \leq \frac{\mu_1\gamma^{\min\alpha_1,\alpha_i}}{min(b_1, b_i)(b_i - \mu_1\gamma^{\alpha_i})}.$$

Case 2. $x_{i,e}^* \geq F_0(\gamma), x_{i,a}^* \geq F_0(0)$**.**

In this case, we can say that

$$|C_i^{(e)} - C_i^a| = \frac{x_{i,e}^*}{B_i(B_i - x_{i,e}^*\gamma^{\alpha_i})}\gamma^{\alpha_i}$$

We also have that

$$x_{i,e}^* = \sum_{j=1}^n \frac{a_{1j}p_{1j}}{1 + \frac{w_i\gamma^{\alpha_i}}{w_1\gamma^{\alpha_1}}C_i^{(e)}(a_{1j} - x_{i,e}^*\gamma^{\alpha_1})}$$
$$x_{i,a}^* = \sum_{j=1}^n \frac{a_{1j}p_{1j}}{1 + \frac{w_i\gamma^{\alpha_I}}{w_1\gamma^{\alpha_1}}C_i^a a_{1j}}.$$

Subtracting the two gives us that

$$|x_{i,e}^* - x_{i,a}^*| \leq \sum_{j=1}^n \frac{a_{1j}p_{1j}\mu_i}{a_{1j} - \mu_1\gamma^{\alpha_i}}\gamma^{\alpha_1} + \sum_{j=1}^n \frac{a_{1j}^2 p_{1j}\mu_i}{B_i(a_{1j} - \mu_1\gamma^{\alpha_i})}\gamma^{\alpha_i}.$$

The above relation, along with the relation between $|C_{1i}^e - C_{1i}^a|$ and $|x_{i,e}^* - x_{i,a}^*|$ as outlined under Case I, may be used to prove that

$$|C_{1i}^e - C_{1i}^a| \leq D_i\gamma^{\min(\alpha_1,\alpha_i)}$$

where D_i is constant depending on arm p_i.

Case 3. $F_0(\gamma) \leq x_{i,e}^*, x_{i,a}^* \leq F_0(0)$**.**

A direct conclusion here would be

$$|x_{i,e}^* - x_{i,a}^*| \leq |F_0(0) - F_0(\gamma)| \leq \frac{B_i}{1 + \gamma^{\alpha_i}\sum_j \frac{a_{ij}p_{ij}}{B_i - a_{ij}}}\Big(\sum_{j=1}^n \frac{a_{ij}p_{ij}}{B_i - a_{ij}}\Big)^2\gamma^{\alpha_i}$$

We have that

$$x_{i,e}^* - x_{i,a}^* = \sum_{j=1}^n \frac{a_{1j}p_{1j}(1 - K_{1i} + a_{1j}(C_{1i}^a - C_{1i}^e))}{(1 + C_{1i}^a a_{1j})(K_{1i} + C_{1i}^e a_{1j})}$$

whence we can conclude that

$$|C_{1i}^e - C_{1i}^a| \leq \frac{(|x_{i,e}^* - x_{i,a}^*| + C^{(e)}x_{i,e}^*\sum_{j=1}^n a_{1j}p_{1j}\gamma^{\alpha_1})}{\frac{b_1\mu_i}{1+B_1C^{(a)}}}$$
$$\Rightarrow |C_{1i}^e - C_{1i}^a| \leq D_i\gamma^{\min(\alpha_1,\alpha_i)}$$

where D_i is again a constant depending on arm p_i. Lastly, we can show that

$$|C_i^e - \frac{1}{B_i}| \leq \frac{(1 - b_i/B_i)}{b_i\mu_i}B_i\left(\sum_j \frac{a_{ij}p_{ij}}{B_i - a_{ij}}\right)^2\gamma^{\alpha_i}$$
$$|C_i^a - \frac{1}{B_i}| \leq \frac{\mu_1}{B_i(B_i - \mu_1\gamma^{\alpha_i})}.\gamma^{\alpha_i}$$

to conclude that

$$|C_i^e - C_i^a| \leq \frac{(1-b_i/B_i)}{b_i\mu_i} B_i \left(\sum_j \frac{a_{ij}p_{ij}}{B_i - a_{ij}}\right)^2 \gamma^{\alpha_i} + \frac{\mu_1}{B_i(B_i - \mu_1\gamma^{\alpha_i})}.\gamma^{\alpha_i}$$

Case 4. $x_{i,e}^* \leq F_0(\gamma) < F_0(0) \leq x_{i,a}^*$.

We first show that $1/B_i < C_i^e$. Suppose this is false. Then, $C_i^a = 1/B_i \geq C_i^e$. From equation (1) for fixed w_1, w_i and γ, we have:

$$C_{1i}^a \geq C_{1i}^e \Rightarrow x_{i,e}^* > \sum_j \frac{a_{1j}p_{1j}}{1 + C_{1i}^e a_{1j}} > \sum_j \frac{a_{1j}p_{1j}}{1 + C_{1i}^a a_{1j}} = x_{i,a}^*$$

But this contradicts the hypothesis of this case. Hence we must have have:

$$\frac{1}{B_i} < C_i^e < \frac{1}{B_i - x_{i,e}^*\gamma^{\alpha_i}}$$

As $C_i^a = \frac{1}{B_i}$, from above we have

$$1 < \frac{C_i^e}{C_i^a} = \frac{C_{1i}^e}{C_{1i}^a} \leq 1 + \frac{x_{i,e}^*\gamma^{\alpha_i}}{B_i - x_{i,e}^*\gamma^{\alpha_i}}$$

And we can conclude that

$$|C_i^a - C_{1i}^a| \leq \frac{\mu_1}{B_i - \mu_1\gamma^{\alpha_1}}\gamma^{\alpha_i}$$
$$|C_{1i}^a - C_{1i}^e| \leq \frac{(\sum_j p_{1j})\mu_1}{\mu_i(B_i - \mu_1\gamma^{\alpha_i})}\gamma^{\alpha_i}$$
$$|x_{i,a}^* - x_{i,e}^*| \leq \frac{\mu_i^2 B_i^2}{B_i - \mu_i}\gamma^{\min\{\alpha_1,\alpha_i\}}$$

This completes the analysis of the four cases and shows that $C_{1i}^a, C_i^a, x_{i,a}^*$ are close to $C_{1i}^e, C_i^e, x_{i,e}^*$ when γ is small.

Step 3: Connecting solutions to exact problem and solutions to approximate problem

We concluded in Step 1 that

$$|\tilde{\mathcal{K}}_{inf}^L(C_{1i}^e) - \mathcal{K}_{inf}^L(K_{1i}, C_{1i}^e)| \leq 2\gamma^{2\alpha_1}\frac{(\sum_j p_{1j})^2}{1 - \sum_j p_{1j}\gamma^{\alpha_1}}$$

and in Step 2 that $|C_{1i}^e - C_{1i}^a|$ is related to $|x_{i,e}^* - x_{i,a}^*|$ by the equation

$$|C_{1i}^e - C_{1i}^a| \leq \frac{|x_{i,e}^* - x_{i,a}^*| + \sum_j a_{1j}p_{1j}C_{1i}^e x_{i,e}^*\gamma^{\alpha_1}}{\sum_j \frac{a_{1j}^2 p_{1j}}{(1+C_{1i}^a a_{1j})(1+C_{1i}^e(a_{1j} - x_{i,e}^*\gamma^{\alpha_1}))}} \leq \frac{|x_{i,e}^* - x_{i,a}^*| + \mu_1\sum_j p_{1j}\gamma^{\alpha_1}}{\mu_2\left(\frac{b_1}{1+B_1\sum_j p_{1j}/\mu_2}\right)}$$

We have:

$$\frac{d}{dz}\tilde{\mathcal{K}}_{inf}^L(z) = \gamma^{\alpha_i}\left(\sum_j \frac{a_{1j}p_{1j}}{1+za_{1j}} - \sum_j \frac{a_{1j}p_{1j}}{1-za_{1j}} - z\sum_j \frac{a_{1j}^2 p_{1j}}{1-za_{1j}}\right)$$

Now, the derivative of $\tilde{\mathcal{K}}_{inf}^L$ can easily be bounded above by $\mu_1\gamma^{\alpha_1}$. This leads us to the following conclusion.

$$|\tilde{\mathcal{K}}_{inf}^L(C_{1i}^e) - \tilde{\mathcal{K}}_{inf}^L(C_{1i}^a)| \leq \frac{\mu_1^2 B_1}{\mu_i b_1}\left[\frac{\frac{\mu_1^3}{\mu_i b_1}\gamma^{\alpha_1} + \mu_1^2(1 + \frac{B_1 \vee B_i}{b_1 \wedge b_i})\frac{1}{(b_i - \mu_1\gamma^{\alpha_i})}\gamma^{\alpha_1\wedge\alpha_i}}{\mu_i\left(\frac{b_i}{1+\frac{\mu_1 B_1}{\mu_i b_1}}\right)}\right]\gamma^{\alpha_1} = \mathcal{O}(\gamma^{(2\alpha_1)\wedge(\alpha_1+\alpha_i)})$$

where we have used the inequalities $C_{1i}^e, C_{1i}^a \leq \frac{\sum_j p_{1j}}{\mu_i}$ and $b_1\sum_j p_{1j} \leq \mu_1$.

We thus have,

$$|\mathcal{K}_{inf}^L(K_{1i}, C_{1i}^e) - \tilde{\mathcal{K}}_{inf}^L(C_{1i}^a)| \leq |\mathcal{K}_{inf}^L(K_{1i}, C_{1i}^e) - \tilde{\mathcal{K}}_{inf}^L(C_{1i}^e)| + |\tilde{\mathcal{K}}_{inf}^L(C_{1i}^e) - \tilde{\mathcal{K}}_{inf}^L(C_{1i}^a)| \leq L_{1i}\gamma^{(2\alpha_1)\wedge(\alpha_1+\alpha_i)}$$

where L_{1i} is a computable constant, and $L_{1i}\gamma^{(2\alpha_1)\wedge(\alpha_1+\alpha_i)}$ can be computed by adding the bounds on $|\mathcal{K}^L_{inf}(K_{1i}, C^e_{1i}) - \tilde{\mathcal{K}}^L_{inf}(C^e_{1i})|$ and $|\tilde{\mathcal{K}}^L_{inf}(C^e_{1i}) - \tilde{\mathcal{K}}^L_{inf}(C^a_{1i})|$.

Similarly from Step 1 we have:

$$|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^e_i)| \leq \frac{\frac{\mu_1}{\max_j a_{ij}}\gamma^{2\alpha_i}}{1 - \frac{\mu_1}{\max_j a_{ij}}\gamma^{\alpha_i}}\left(\sum_j p_{ij} + \frac{\mu_1}{\max_j a_{ij}}\right)$$

To upper bound $|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)|$, we can follow a procedure similar to how $|\mathcal{K}^L_{inf}(K_{1i}, C^e_{1i}) - \tilde{\mathcal{K}}^L_{inf}(C^a_{1i})|$ was bounded. We first use the triangle inequality to make the following split.

$$\begin{aligned}|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)| \leq |\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^e_i)| + |\tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^a_i)| \\ + |\tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^a_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)|\end{aligned}$$

In the right hand side of the above inequality, the bound to the first summand was already obtained. The second and third summands can be bounded above by showing that $\tilde{\mathcal{K}}^U_{inf}$ is Lipschitz in both its arguments, the Lipschitz constants being computable ones. Thus, we have

$$\begin{aligned}|\tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^a_i)| &\leq \gamma^{\alpha_i}(\mu_1 - \mu_i)|C^e_i - C^a_i| \\ &\leq \frac{\mu_1(\mu_1 - \mu_2)}{(b_1 \wedge b_i)(b_i - \mu_1\gamma^{\alpha_i})}\gamma^{(\alpha_1+\alpha_i)\wedge(2\alpha_i)} \\ &+ \frac{(B_i - b_i)(\mu_1 - \mu_2)}{b_i\mu_i}\left(\sum_{j=1}^n \frac{a_{1j}p_{1j}}{B_i - a_{ij}}\right)^2\gamma^{2\alpha_i}..\end{aligned}$$

The bound in the first step was derived by bounding the partial derivative wrt z of $\tilde{\mathcal{K}}^U_{inf}(m, z)$. Similarly bounding the partial derivative wrt m gives

$$|\tilde{\mathcal{K}}^U_{inf}(x^*_{i,e}, C^a_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)| \leq \gamma^{\alpha_i}\frac{|x^*_{i,e} - x^*_{i,a}|}{b_i}$$

$|x^*_{i,e} - x^*_{i,a}|$ is bounded above by the maximum of the upper bounds derived in the four cases of Step 2. We can therefore conclude that,

$$|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)| \leq L_i\gamma^{(\alpha_1+\alpha_i)\wedge(2\alpha_i)}$$

where L_i can be computed as described above. The upper bounds on $|\mathcal{K}^L_{inf}(K_{1i}, C^e_{1i}) - \tilde{\mathcal{K}}^L_{inf}(C^a_{1i})|$ and $|\mathcal{K}^U_{inf}(K_i, C^e_i) - \tilde{\mathcal{K}}^U_{inf}(x^*_{i,a}, C^a_i)|$ give us the proof of Theorem 3. The upper bound on $|V^*(p) - V^*_a(p)|$ can be inferred immediately.

C PROOF OF THEOREM 2

The proof goes through the following steps: first we analyse the behavior of equation (12) and derive some constraints it imposes on the asymptotic behavior of C^a_{1i}, C^a_i; utilising this, we then analyse the behaviour of equation (11) and finally get the five asymptotic regimes noted in the Theorem.

Step 1: Constraint imposed by equation (12) in the asymptotic behaviours of C^a_{1i}, C^a_i.

We first observe that $C^a_{1i} \to 0, C^a_i \to 0$ as $\gamma \to 0$ cannot happen for any $i \in [K]\backslash\{1\}$, because then equation 10 would imply that $\mu_1 = \sum_{j=1}^n a_{1j}p_{1j} = \sum_{j=1}^n a_{ij}p_{ij} = \mu_i$.

Equation (12) from the main body can be re-written (using envelope theorem) as

$$\begin{aligned}&w_1\gamma^{\alpha_1}\left(\sum_j p_{1j}\log(1 + C^a_{1i}a_{1j}) - C^a_{1i}x^*_{i,a}\right) + w_i\gamma^{\alpha_i}\left(\sum_j p_{ij}\log(1 - C^a_i a_{ij}) + C^a_i x^*_{i,a}\right) \\ =&w_1\gamma^{\alpha_1}\left(\sum_j p_{1j}\log(1 + C^a_{1k}a_{1j}) + C^a_{1i}x^*_{k,a}\right) + w_k\gamma^{\alpha_i}\left(\sum_j p_{kj}\log(1 - C^a_i a_{kj}) - C^a_i x^*_{k,a}\right)\end{aligned}$$

for all $i \neq k, i, k \neq 1$. Using equation $w_1 C_{1i}^a \gamma^{\alpha_1} = w_i C_i^a \gamma^{\alpha_i}$, we can simplify this equation to

$$\frac{\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) + \frac{C_{1i}^a}{C_i^a} \sum_j p_{ij} \log(1 - C_i^a a_{ij})}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) + \frac{C_{1k}^a}{C_k^a} \sum_j p_{kj} \log(1 - C_k^a a_{kj})} = 1 \tag{9}$$

for all $i \neq k$. We also re-write (10) from the main body as

$$\sum_j \frac{a_{1j} p_{1j}}{1 + C_{1i}^a a_{1j}} = \sum_j \frac{a_{ij} p_{ij}}{1 - C_i^a a_{ij}}. \tag{10}$$

Now, we analyze the asymptotic behavior of equation (9) as $\gamma \to 0$ on a case-by-case basis.

Case 1: $C_{1i}^a \to A_1^a (> 0), C_i \to 0; C_{1k}^a \to A_{1k}^a (> 0), C_k^a \to 0.$

Taking the limit in equation (9) we get

$$\begin{aligned} 1 &= \lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) + \frac{C_{1i}^a}{C_i^a} \sum_j p_{ij} \log(1 - C_i^a a_{ij})}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) + \frac{C_{1k}^a}{C_k^a} \sum_j p_{kj} \log(1 - C_k^a a_{kj})} \\ &= \frac{\sum_j p_{1j} \log(1 + A_{1i}^a a_{1j}) - A_{1i}^a \sum_j a_{ij} p_{ij}}{\sum_j p_{1j} \log(1 + A_{1k}^a a_{1j}) - A_{1k}^a \sum_j a_{kj} p_{kj}} \end{aligned}$$

Taking $\gamma \to 0$ in (2), we have that

$$\sum_j \frac{a_{1j} p_{1j}}{1 + A_{1i} a_{1j}} = \sum_j a_{ij} p_{ij}$$
$$\sum_j \frac{a_{1j} p_{1j}}{1 + A_{1k} a_{1j}} = \sum_j a_{kj} p_{kj}$$

Hence,

$$\frac{\sum_j f_j(A_{1i})}{\sum_j f_j(A_{1k})} = 1$$

where $f_j(x) := p_{1j}[\log(1 + a_{1j} x) - \frac{x a_{1j}}{1 + x a_{1j}}]$. It is easy to check that f is a monotonically increasing function, and therefore the above equation must imply $A_{1i} = A_{1k}$. But this also means that $\mu_i = \mu_k$, which is against our assumption of all means being distinct.

Case 2: $C_{1i}^a \to A_{1i} (> 0), C_i^a \to 0, C_{1k}^a \to 0, C_k^a \to A_k (> 0)$

As in Case 1 we take the asymptotic limit on 9 to get

$$\begin{aligned} 1 &= \lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) + \frac{C_{1i}^a}{C_i^a} \sum_j p_{ij} \log(1 - C_i^a a_{ij})}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) + \frac{C_{1k}^a}{C_k^a} \sum_j p_{kj} \log(1 - C_k^a a_{kj})} \\ &= \lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + A_{1i}^a a_{1j}) - A_{1i}^a \sum_j a_{ij} p_{ij}}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) - \frac{C_{1k}^a}{A_k} \sum_j p_{kj} \log(1 - A_k^a a_{kj})} \end{aligned}$$

which is impossible, because the denominator of the right hand side approaches 0 as $\gamma \to 0$.

Case 3: $C_{1i}^a \to A_{1i} (> 0), C_i^a \to A_i (> 0), C_{1k}^a \to 0, C_k^a \to A_k (> 0)$

We have that

$$\begin{aligned} 1 &= \lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) + \frac{C_{1i}^a}{C_i^a} \sum_j p_{ij} \log(1 - C_i^a a_{ij})}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) + \frac{C_{1k}^a}{C_k^a} \sum_j p_{kj} \log(1 - C_k^a a_{kj})} \\ &= \lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + A_{1i}^a a_{1j}) + \frac{A_{1i}^a}{A_i^a} \sum_j p_{ij} \log(1 - A_i^a a_{ij})}{\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) - \frac{C_{1k}^a}{A_k} \sum_j p_{kj} \log(1 - A_k^a a_{kj})} \end{aligned}$$

which is impossible, because the denominator of the left hand side approaches 0 as $\gamma \to 0$. That only leaves us with only the following three possibilities.

Case 4: $C_{1i}^a \to A_{1i}(\neq 0), C_i^a \to A_i(\neq 0), C_{1k}^a \to A_{1k}(\neq 0), C_k^a \to A_k(\neq 0)$

From 9, we know

$$\lim_{\gamma\to 0} \frac{\sum_j p_{1j}\log(1+C_{1i}^a a_{1j}) + \frac{w_i\gamma^{\alpha_i}}{w_1\gamma^{\alpha_1}}\sum_j p_{ij}\log(1-C_i^a a_{ij})}{\sum_j p_{1j}\log(1+C_{1k}^a a_{1j}) + \frac{w_k\gamma^{\alpha_k}}{w_1\gamma^{\alpha_1}}\sum_j p_{kj}\log(1-C_k^a a_{kj})}$$

which cannot be ruled out as an impossibility.

Case 5: $C_{1i}^a \to 0, C_i a \to A_i(\neq 0), C_{1k}^a \to 0, C_k^a \to A_k(\neq 0)$

Using $C_{1i}^a w_1\gamma^{\alpha_1} = C_i^a w_i \gamma^{\alpha_i} = \lambda_i \ \forall i \neq 1$ on 9 gives us that

$$\begin{aligned}
&\lim_{\gamma\to 0}\frac{C_{1i}^a}{C_{1k}^a}\frac{\sum_j p_{1j}\frac{\log(1+C_{1i}^a a_{1j})}{C_{1i}^a} + \sum_j p_{ij}\frac{\log(1-C_i a a_{ij})}{C_i^a}}{\sum_j p_{1j}\frac{\log(1+C_{1k}^a a_{1j})}{C_{1k}^a} + \sum_j p_{kj}\frac{\log(1-C_k^a a_{kj})}{C_k^a}} \\
&= \lim_{\gamma\to 0}\frac{C_{1i}^a}{C_{1k}^a}\left(\frac{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{ij}}{A_i}\log(1-A_i a_{ij})}{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{kj}}{A_k}\log(1-A_k a_{kj})}\right) = 1 \\
&\Rightarrow \lim_{\gamma\to 0}\frac{C_{1i}^a}{C_{1k}^a} = \frac{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{kj}}{A_k}\log(1-A_k a_{kj})}{\sum_j a_{1j}p_{1j} + \frac{p_{ij}}{A_i}\log(1-A_i a_{ij})} \\
&\Rightarrow \lim_{\gamma\to 0}\frac{C_i^a w_i\gamma^{\alpha_i}}{C_k^a w_k \gamma^{\alpha_k}} = \left(\frac{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{kj}}{A_k}\log(1-A_k a_{kj})}{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{ij}}{A_i}\log(1-A_i a_{ij})}\right)
\end{aligned}$$

Case 6: $C_{1i}^a \to A_{1i}(\neq 0), C_i^a \to 0, C_{1k}^a \to A_{1k}(\neq 0), C_k^a \to A_k(\neq 0)$

Using $C_{1i}^a w_1\gamma^{\alpha_1} = C_i^a w_i \gamma^{\alpha_i} = \lambda_i \ \forall i \neq 1$ on 9 gives us that

$$\begin{aligned}
&\lim_{\gamma\to 0}\frac{C_{1i}^a}{C_{1k}^a}\frac{\sum_j p_{1j}\frac{\log(1+C_{1i}^a a_{1j})}{C_{1s}^a} + \sum_j p_{ij}\frac{\log(1-C_i^a a_{ij})}{C_i^a}}{\sum_j p_{1j}\frac{\log(1+C_{1k}^a a_{1j})}{C_{1k}^a} + \sum_j p_{kj}\frac{\log(1-C_k^a a_{kj})}{C_k^a}} \\
=&\frac{\sum_j p_{1j}\log(1+A_{1i}a_{1j}) - A_{1i}\mu_i}{\sum_j p_{1j}\log(1+A_{1k}a_{1j}) + \frac{A_{1k}}{A_k}\sum_j p_{kj}\log(1-A_k a_{kj})} = 1
\end{aligned}$$

Step 2: Analysis of equation 11 of the main body.

The Envelope Theorem guarantees that equation 11 of the main body can be rewritten as

$$\sum_{i=2}^{K}\frac{KL(p_1,\tilde{p}_1^{(i)})}{KL(p_i,\tilde{p}_i)} = \sum_{i=2}^{K}\frac{\gamma^{\alpha_1}(\sum_j p_{1j}\log(1+C_{1i}^a a_{1j}) - C_{1i}^a\sum_j a_{1j}\tilde{p}_{1j}^{(i)})}{\gamma^{\alpha_i}(\sum_j p_{ij}\log(1-C_i^a a_{ij}) + C_i^a\sum_j a_{ij}\tilde{p}_{ij})} = 1 \tag{11}$$

because $\frac{\partial\mathcal{P}_{i,a}(w^*)}{\partial w_1} = KL(p_1,\tilde{p}_1^i)$ and $\frac{\partial\mathcal{P}_{i,a}(w^*)}{\partial w_i} = KL(p_i,\tilde{p}_i)$. We shall use this form of equation 11 to derive expressions for $w_i,\ i\in[K]\backslash\{1\}$ under the following cases:

Case 1: $\alpha_1 \neq \alpha_{max}$,
Case 2: $\alpha_1 = \alpha_{max} > \alpha_i,\ \forall i\neq 1$,
Case 3: $\alpha_1 = \alpha_2 = \alpha_{max} > \alpha_i,\ \forall i \neq 1,2$,
Case 4: $\alpha_1 = \alpha_k = \alpha_{max} \geq \alpha_i,\ i\notin\{1,2,k\}$**,** $\alpha_{max} > \alpha_2$ **and** $\zeta > 1$
Case 5: $\alpha_1 = \alpha_k = \alpha_{max} \geq \alpha_i,\ i\notin\{1,2,k\}$**,** $\alpha_{max} > \alpha_2$ **and** $\zeta \leq 1$

where $\alpha_{max} := \max_i \alpha_i$. We shall first show that ***Case 1*** is equivalent to $C_{1i}^a \to 0, C_i^a \to A_i(\neq 0)\forall i\neq 1$

For the “if" direction, let us assume that $\alpha_1 \geq \alpha_i$ for all $i \in [K]\backslash\{1\}$. In the limit as $\gamma \to 0$, we then get that

$$\sum_{i=2}^{K} \frac{KL(p_1, \tilde{p}_1^{(i)})}{KL(p_i, \tilde{p}_i)} = \sum_{i=2}^{K} \frac{\gamma^{\alpha_1}(\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) - C_{1i}^a \sum_j a_{1j} \tilde{p}_{1j}^{(i)})}{\gamma^{\alpha_i}(\sum_j p_{ij} \log(1 - C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})} = 1 \Rightarrow 0 = 1$$

which is an absurdity.

For the “only if" direction, let us suppose that for some $k \in [K]\backslash\{1\}$, $\alpha_1 < \alpha_k$. If $C_k^a \to 0$, from our analysis in Step 1, we can conclude that $C_{1k}^a \to A_{1k} (\neq 0)$. Therefore,

$$\gamma^{\alpha_1 - \alpha_k} \frac{(\sum_j p_{1j} \log(1 + C_{1k}^a a_{1j}) - C_{1k}^a \sum_j a_{1j} \tilde{p}_{1j}^{(k)})}{(\sum_j p_{kj} \log(1 - C_k^a a_{kj}) + C_k^a \sum_j a_{kj} \tilde{p}_{kj})} \to \infty \text{ as } \gamma \to 0$$

contradicting $\sum_{i=2}^{K} \frac{\gamma^{\alpha_1}(\sum_j p_{1j} \log(1+C_{1i}^a a_{1j}) - C_{1i}^a \sum_j a_{1j} \tilde{p}_{1j}^{(i)})}{\gamma^{\alpha_i}(\sum_j p_{ij} \log(1+C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})} = 1$.

From our analysis in Step 1, we can conclude that $C_k^a \to A_k (\neq 0)$ implies that $C_{1k}^a \to 0$ and consequently, $C_{1i}^a \to 0, C_i^a \to A_i (\neq 0)\ \forall i \neq 1$.

Let $\alpha_{max} = \alpha_k$. Since $C_{1i}^a \to 0, C_i^a \to A_i (\neq 0)\ \forall i \neq 1$, we can use Taylor series expansions to write

$$\lim_{\gamma \to 0} \sum_{i=2}^{K} \frac{\gamma^{\alpha_1}(\sum_j p_{1j} \log(1 + C_{1i}^a a_{1j}) - C_{1i}^a \sum_j a_{1j} \tilde{p}_{1j}^{(i)})}{\gamma^{\alpha_i}(\sum_j p_{ij} \log(1 + C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})} = 1$$

$$\Rightarrow \lim_{\gamma \to 0} \sum_{i=2}^{K} \frac{\frac{(C_{1i}^a)^2 \sum_j a_{1j}^2 p_{1j}}{2} \gamma^{\alpha_1 - \alpha_i}}{(\sum_j p_{ij} \log(1 + C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})} = 1$$

We know that $C_{1i}^a = C_i^a \frac{w_i \gamma^{\alpha_i}}{w_1 \gamma^{\alpha_1}}$. This substitution will give us

$$\lim_{\gamma \to 0} \sum_{i=2}^{K} \frac{\frac{(C_i^a)^2 \sum_j a_{1j}^2 p_{1j}}{2}}{(\sum_j p_{ij} \log(1 + C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})} \left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} = 1$$

$$\Rightarrow \sum_{i=2}^{K} \lim_{\gamma \to 0} M_i \left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} = 1; \text{where } M_i := \frac{\frac{(C_i^a)^2 \sum_j a_{1j}^2 p_{1j}}{2}}{(\sum_j p_{ij} \log(1 + C_i^a a_{ij}) + C_i^a \sum_j a_{ij} \tilde{p}_{ij})}$$

If $\alpha_i < \alpha_1$, then $\gamma^{\alpha_i - \alpha_1}$ must go to ∞ as $\gamma \to 0$. But M_i being bounded and $M_i\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} \leq 1$ implies that $\frac{w_i}{w_1} \leq \frac{1}{M_i} \gamma^{\frac{\alpha_1 - \alpha_i}{2}}$. Therefore, $M_i\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} = M_i\left(\frac{C_{1i}^a}{C_i^a}\right)\left(\frac{w_i}{w_1}\right) \to 0$ as $\gamma \to 0$.

If $\alpha_1 < \alpha_i < \alpha_{max}$, let us suppose $M_i\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} = M_i . \frac{C_k^a}{C_i^a} . \frac{w_k \gamma^{\alpha_k}}{w_i \gamma^{\alpha_i}} . \frac{w_i}{w_1} \to L_i \neq 0$ as $\gamma \to 0$. Let us choose an $\epsilon > 0$ such that $L_i - \epsilon > 0$. Then for sufficiently small γ, we get $w_k \gamma^{\alpha_k} > (L_i - \epsilon) \frac{w_1 \gamma^{\alpha_1}}{w_i}$. But due to $M_k\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_k - \alpha_1} \leq 1$, we must have $(L_i - \epsilon)^2 \frac{M_k}{w_i^2} \gamma^{\alpha_1 - \alpha_k} < M_k\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_k - \alpha_1} \leq 1$. This implies that $w_i > (L_i - \epsilon)\sqrt{M_k} \gamma^{\frac{\alpha_1 - \alpha_k}{2}}$. But we cannot have $w_i \to \infty$ as $\gamma \to 0$.

We are thus forced to conclude that only those values of i for which $\alpha_i = \alpha_{max}$ will contribute positively to the sum $\sum_{i=2}^{K} \lim_{\gamma \to 0} M_i\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1}$.

For i such that $\alpha_i = \alpha_{max}$, as $\gamma \to 0$, let $M_i\left(\frac{w_i}{w_1}\right)^2 \gamma^{\alpha_i - \alpha_1} \to L_i \neq 0$. Therefore, in the limit, $w_1 = \sqrt{\frac{M_i}{L_i}} \gamma^{\frac{\alpha_{max} - \alpha_1}{2}} w_i$. This also gives us that as $\gamma \to 0$, for all s, t such that $\alpha_s = \alpha_t = \alpha_{max}$, $\frac{w_s}{w_t} = \sqrt{\frac{M_t L_s}{M_s L_t}} = \sqrt{\frac{L_s}{L_t}} \sqrt{\frac{\sum_j p_{sj} \log(1+A_s a_{sj}) + A_s \sum_j a_{sj} \tilde{p}_{sj}}{\sum_j p_{tj} \log(1+A_t a_{tj}) + A_t \sum_j a_{tj} \tilde{p}_{tj}}}$.

To approximately solve our maxmin problem, we do the following:

Let us fix a k with $\alpha_k = \alpha_{max}$ and set $w_k = 1$. Then, $w_1 = \sqrt{\frac{M_k}{L_k}}\gamma^{\frac{\alpha_{max}-\alpha_1}{2}}$. For the other i such that $\alpha_i < \alpha_{max}$, using $C_i^a w_i \gamma^{\alpha_i} = \frac{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{kj}}{A_k}\log(1-A_k a_{kj})}{\sum_j a_{1j}p_{1j} + \sum_j \frac{p_{ij}}{A_i}\log(1-A_i a_{ij})} C_k^a w_k \gamma^{\alpha_k}$, we get that $w_i = \frac{A_k \sum_j a_{1j}p_{1j} + \sum_j p_{kj}\log(1-A_k a_{kj})}{A_i \sum_j a_{1j}p_{1j} + \sum_j p_{ij}\log(1-A_i a_{ij})}\gamma^{\alpha_k - \alpha_i}$. Note that A_i may be obtained by solving $\mu_1 = \sum_j \frac{a_{ij}p_{ij}}{1-A_i a_{ij}}$. For any other s with $\alpha_s = \alpha_{max}$, we have $w_s = \sqrt{\frac{L_s}{L_k}}\sqrt{\frac{\sum_j p_{sj}\log(1+A_s a_{sj}) + A_s \sum_j a_{sj}\tilde{p}_{sj}}{\sum_j p_{kj}\log(1+A_k a_{kj}) + A_k \sum_j a_{kj}\tilde{p}_{kj}}}$. We use this to evaluate L_k for each "rarest arm" and finally normalize the weights obtained to lie within [0,1].

Special case: *If there is a unique k with $\alpha_k = \alpha_{max}$, then our analysis tells us that $L_k = 1$. Our approximate solution then becomes the normalized form of $w_1 = \sqrt{M_k}\gamma^{\frac{\alpha_{max}-\alpha_1}{2}}$, $w_i = \frac{A_k \sum_j a_{1j}p_{1j} + \sum_j p_{kj}\log(1-A_k a_{kj})}{A_i \sum_j a_{1j}p_{1j} + \sum_j p_{ij}\log(1-A_i a_{ij})}\gamma^{\alpha_k - \alpha_i}$ for $i \neq k, 1$, and $w_k = 1$.*

Before starting on rest of the cases, we'll introduce some additional notation that will be of importance. Let us revisit the following function introduced in section 3.1.

$$g_i(x) = \left\{ y : \sum_j \frac{a_{1j}p_{1j}}{1 + y a_{1j}} = \sum_j \frac{a_{ij}p_{ij}}{1 - x a_{ij}} \right\}$$

Clearly, g_i is decreasing in x, and $g_k(A_k) = A_{1k}$. We now define $f_i(x)$ as

$$f_i(x) := \sum_j p_{1j}\log(1 + g_i(x)a_{1j}) + \frac{g_i(x)}{x}\sum_j p_{ij}\log(1 - x a_{ij})$$

$$f_i(0) := \lim_{x \to 0^+} f_i(x)$$

f_i can also be shown to be decreasing in x and increasing in $g_i(x)$. Further, we define h_i as follows.

$$h_i(x) := \frac{\sum_j p_{1j}\log(1 + g_i(x)a_{1j}) - g_i(x)\sum_j a_{1j}\tilde{p}_{1j}^{(i)}}{\sum_j p_{ij}\log(1 - x a_{ij}) + x a_{ij}\tilde{p}_{ij}}$$

It can be showed that h_i is a decreasing function of x.

We can now turn our attention to ***Case 2***.

Since $\alpha_1 = \alpha_{max}$ uniquely, in the sum

$$\sum_{i=2}^{K} \lim_{\gamma \to 0} \frac{\gamma^{\alpha_1}(\sum_j p_{1j}\log(1 + C_{1i}^a a_{1j}) - C_{1i}^a \sum_j a_{1j}\tilde{p}_{1j}^{(i)})}{\gamma^{\alpha_i}(\sum_j p_{ij}\log(1 - C_i^a a_{ij}) + C_i^a \sum_j a_{ij}\tilde{p}_{ij})} = 1,$$

if we do not have $C_k^a \to 0$ as $\gamma \to 0$ for some k, then the sum on the left becomes equal to 0, which would be a contradiction. We also note that there will be exactly one arm k where $C_k^a \to 0$ as $\gamma \to 0$. Let us separately examine this k^{th} summand.

$$\lim_{\gamma \to 0} \frac{(\sum_j p_{1j}\log(1 + C_{1k}^a a_{1j}) - C_{1k}^a \sum_j a_{1j}\tilde{p}_{1j}^{(i)})}{(\sum_j p_{kj}\log(1 - C_k^a a_{kj}) + C_k^a \sum_j a_{kj}\tilde{p}_{kj})}\gamma^{\alpha_1 - \alpha_k} = \lim_{\gamma \to 0} \frac{2(\sum_j p_{1j}\log(1 + C_{1k}^a a_{1j}) - C_{1k}^a \sum_j a_1\tilde{p}_{1j}^{(k)})}{(C_k^a)^2 \sum_j a_{kj}^2 p_{kj}}\gamma^{\alpha_1 - \alpha_k}$$

Since this term needs to be equal to 1, we must have

$$\lim_{\gamma \to 0} \frac{(C_k^a)^2}{\gamma^{\alpha_k - \alpha_1}} = \lim_{\gamma \to 0} \frac{(C_{1k}^a)^2 w_k^2 \gamma^{\alpha_k - \alpha_1}}{w_1^2} = \frac{\sum_j a_{kj}^2 p_{kj}}{2(\sum_j p_{1j}\log(1 + A_{1k}a_{1j}) - A_{1k}\sum_j a_{1j}\tilde{p}_{1j}^{(k)})}$$

This suggests the following form for w_k.

$$w_k = \frac{1}{A_{1k}}\sqrt{\frac{\sum_j a_{kj}^2 p_{kj}}{2(\sum_j p_{1j}\log(1 + A_{1k}a_{1j}) - A_{1k}\sum_j a_{1j}\tilde{p}_{1j}^{(k)})}}\, w_1 \gamma^{\frac{\alpha_1 - \alpha_k}{2}} \left(=: M_k w_1 \gamma^{\frac{\alpha_1 - \alpha_k}{2}}\right)$$

We shall now establish that $k = 2$.

It can be understood that $g_i(x)$ is the factor by which the mean of arm 1 is reduced to $\frac{a_{ij}p_i}{1-xa_i}$. Hence, we conclude that $g_2(0) < ... < g_K(0)$, implying that $f_2(0) < ... < f_K(0)$.

Observe that (8) can be expressed as (as $A_k = 0$)

$$f_i(A_i) = f_k(A_k) = f_k(0)$$

If $k > 2$, we have $f_2(A_2) < f_2(0) < f_k(0)$, giving us a contradiction. Hence, $k = 2$.

Since for every other arm i, $C^a_{1i} \to A_{1i}(\neq 0)$ and $C^a_i \to A_i(\neq 0)$ as $\gamma \to 0$,

$$w_i = \frac{A_{1i}}{A_i} w_1 \gamma^{\alpha_1 - \alpha_i}$$

where A_{1i} and A_i can be obtained by finding the unique solution to

$$\frac{\sum_j p_{1j} \log(1 + A_{12}a_{1j}) - A_{12} \sum_j a_{2j}p_{2j}}{\sum_j p_{1j} \log(1 + A_{1i}a_{1j}) + \frac{A_{1i}}{A_i} \sum_j p_{ij} \log(1 - A_i a_{ij})} = 1$$

and

$$\sum_j \frac{a_{1j}p_{1j}}{1 + A_{1i}a_{1j}} = \sum_j \frac{a_{ij}p_{ij}}{1 - A_i a_{ij}}$$

the latter equality following from the limit form of the mean equation. We can then use the same normalization technique as in case 1 to find the optimal weights.

For ***Case 3***, if $C^a_{12} \to A_{12}(\neq 0), C^a_2 \to 0$ as $\gamma \to 0$, we have

$$\lim_{\gamma \to 0} \frac{(\sum_j p_{1j} \log(1 + C^a_{12}a_{1j}) - C^a_{12} \sum_j a_{1j}\tilde{p}^{(i)}_{1j})}{(\sum_j p_{2j} \log(1 - C^a_2 a_{2j}) + C^a_2 \sum_j a_{2j}\tilde{p}_{2j})} \gamma^{\alpha_1 - \alpha_2} = \lim_{\gamma \to 0} \frac{2(\sum_j p_{1j} \log(1 + C^a_{12}a_{1j}) - C^a_{12} \sum_j a_{1j}\tilde{p}^{(2)}_{1j})}{(C^a_2)^2 \sum_j a^2_{2j} p_2} = \infty$$

which is impossible, thereby guaranteeing $C^a_{12} \to A_{12}(\neq 0), C^a_2 \to A_2(\neq 0)$ as $\gamma \to 0$, and $w_2 = \frac{A_{12}}{A_2} w_1$. This will enable us to find w_2 as described under case 2.

As already argued in case 2, $C^a_2 \to A_2(\neq 0)$ as $\gamma \to 0$ means that $C^a_i \to A_i(\neq 0)$ as $\gamma \to 0$ for all $i \neq 2$. Therefore, we must have

$$\lim_{\gamma \to 0} \frac{\sum_j p_{1j} \log(1 + C^a_{12}a_{1j}) - C^a_{12} \sum_j a_{1j}\tilde{p}^{(i)}_{1j}}{\sum_j p_{2j} \log(1 - C^a_2 a_{2j}) + C^a_2 \sum_j a_{2j}\tilde{p}_{2j}} = 1$$

where A_{1i} and A_i can be related by

$$\frac{\sum_j p_{1j} \log(1 + A_{12}a_{1j}) + \frac{A_{12}}{A_2} \sum_j p_{2j} \log(1 - A_2 a_{2j})}{\sum_j p_{1j} \log(1 + A_{1i}a_{1j}) + \frac{A_{1i}}{A_i} \sum_j p_{ij} \log(1 - A_i a_{ij})} = 1 \tag{12}$$

and using the mean equation,

$$\sum_j \frac{a_{1j}p_{1j}}{1 + A_{1i}a_{1j}} = \sum_j \frac{a_{ij}p_{ij}}{1 - A_i a_{ij}} \ \forall i$$

Let us denote these by $A_2(A_{12})$ and $A_i(A_{1i})$. Substituting them in 12 and using the defintions of f_i, we have $f_2(A_{12}) = f_i(A_{1i})$.

Each of these f_i's is increasing in A_{1i}. Thus we have $A_{1i} = f_i^{-1} \circ (f_2(A_{12}))$.

Using this, we can solve for A_{12} from equation 11. We observe that each summand in 11 is an increasing function of A_{1i} and hence A_{12}. So a simple efficient scheme to find the solution is to first guess an A_{12} and then use a simple bisection method to numerically get A_{1i}'s for this guess. The mean equations can be used to get the A_i's. Finally, we check if 11 is satisfied (upto tolerance). If LHS of 11 is greater than 1, then we halve our initial guess, and double the guess if lesser than 1. And repeat the earlier procedure till error tolerance is breached.

It only remains to consider ***Cases 4 and 5***. We have already argued under case 3 that $C_j^a \to A_j(\neq 0)$ as $\gamma \to 0$ whenever $\alpha_j = \alpha_{max}$. Corresponding to any such A_j, we can write all other A_i's in terms of A_j. Let us define $\xi_{ij}(x)$ as follows.

$$\xi_{ij}(x) := \left\{ y : \frac{p_{1j}\log(1+g_i(y)a_1) + p_i\frac{g_i(y)}{y}\log(1-ya_i)}{p_{1j}\log(1+g_j(x)a_1) + p_j\frac{g_j(x)}{x}\log(1-ya_i)} = 1 \right\}$$

Let us now define ζ as

$$\zeta := \sum_{\substack{\{k:k\neq 1,\\ \alpha_k=\alpha_{max}\}}} h_k(\xi_{k2}(0)).$$

Equation 11 can now be re-written after taking the limit $\gamma \to 0$ as

$$\sum_{\substack{\{k:k\neq 1,\\ \alpha_k=\alpha_{max}\}}} h_k(A_k) + \lim_{\gamma\to 0}(\gamma^{\alpha_1-\alpha_2}h_2(C_2^a)) = 1$$

The issue now is to determine if $C_2^a \to 0$ as $\gamma \to 0$. We have observed earlier that $h_i(A_i)$ is a decreasing function of A_i and the bijective map ξ_{i2} implies $h_i(A_i)$ is also a decreasing function of A_2. Thus, we have

$$\zeta \geq \sum_{\substack{\{k:k\neq 1,\\ \alpha_k=\alpha_{max}\}}} h_k(A_k).$$

If $\zeta > 1$, then equation 11 can be satisfied only when $C_2^a \to A_2\ (>0)$. Because otherwise, the first term itself would contribute more than 1 and we'd have a contradiction. Similarly, when $\zeta \leq 1$, we must necessarily have $C_2^a \to 0$.

In the case when $\zeta > 1$, the A_i, A_{1i}'s are determined exactly as in 3. If $\zeta \leq 1$ then A_i, A_{1i}'s are determined exactly as in Case 2. This completes our proof.

D THE MEETING POINT OF THE MEANS IN THE APPROXIMATE PROBLEM

Equation (12) in the main body and the Mean Value Theorem together give us the following chain of equalities/inequalities.

$$\begin{aligned}
&\sum_{j=1}^{n} p_{1j}\log(1+C_{1s}a_{1j}) - C_{1s}\tilde{\mu}_s \\
\leq &\sum_{j=1}^{n} p_{1j}\log(1+C_{1s}a_{1j}) - C_{1s}\sum_{j=1}^{n}\frac{a_{sj}p_{sj}}{1-C_s a_{sj}} \\
\leq &\sum_{j=1}^{n} p_{1j}\log(1+C_{1s}a_{1j}) + \frac{C_{1s}}{C_s}\sum_{j=1}^{n} p_{sj}\log(1-C_s a_{sj}) \\
= &\sum_{j=1}^{n} p_{1j}\log(1+C_{1t}a_{1j}) + \frac{C_{1t}}{C_t}\sum_{j=1}^{n} p_{tj}\log(1-C_t a_{tj}) \\
\leq &\sum_{j=1}^{n} p_{1j}\log(1+C_{1t}a_{1j}) - C_{1t}\mu_t
\end{aligned}$$

Regrouping terms among the first and last quantities of the above chain gives us that

$$\frac{C_{1t}}{C_{1s}}\mu_t \leq \frac{1}{C_{1s}}\sum_{j=1}^{n} p_{1j}\log\left(\frac{1+C_{1t}a_{1j}}{1+C_{1s}a_{1j}}\right) + \tilde{\mu}_s$$

Note that $\log\left(\frac{1+C_{1t}a_{1j}}{1+C_{1s}a_{1j}}\right) = \log\left(1 + \frac{(C_{1t}-C_{1s})a_{1j}}{1+C_{1s}a_{1j}}\right) \leq (C_{1t} - C_{1s})\tilde{\mu}_s$, and hence, $\frac{C_{1t}}{C_{1s}}\mu_t \leq \frac{C_{1t}}{C_{1s}}\tilde{\mu}_s$, i.e., $\mu_t \leq \tilde{\mu}_s$.

We conclude from the above analysis that $\forall s, t \neq 2$, $\tilde{\mu}_s \geq \mu_t \Rightarrow \forall s \neq 2$, $\tilde{\mu}_s \geq \mu_2$.

E PROOF OF δ-CORRECTNESS OF TS(A).

Let the set of all possible bandit hypotheses be $\mathcal{H}$. We have $\mathcal{H} = \cup_i \mathcal{H}_i$, where $\mathcal{H}_i$ denotes all bandit instances with arm i having the highest mean. Let $\hat{i}(\tau_\delta)$ denote the recommendation of TS(A) at the stopping time. The error probability for a bandit instance p with arm 1 having the highest mean is given by:

$$\begin{aligned}\mathbb{P}_p(\tau_\delta < \infty, \hat{i}(\tau_\delta) \neq 1) &\leq \mathbb{P}_p(\exists t \in \mathbb{N} : \hat{i}(t) \neq 1, Z_{\hat{i}(t)}(t) > \beta(t, \delta)) \\ &= \mathbb{P}_p(\exists t \in \mathbb{N} : \exists i \neq 1 A(\hat{p}) \subseteq \mathcal{H}_i)\end{aligned}$$

where $A(\hat{p}) := \{p' \in \mathcal{H} | \min_{b \neq \hat{i}(t)} N_{\hat{i}(t)}(t)\mathcal{K}^L_{inf}(\hat{p}_{\hat{i}(t)}(t), \mu'_{\hat{i}(t)}) + N_b(t)\mathcal{K}^U_{inf}(\hat{p}_b(t), \mu'_b) \leq \beta(t, \delta)\}$. This implies:

$$\begin{aligned}\mathbb{P}_p(\tau_\delta < \infty, \hat{i}(\tau_\delta) \neq 1) &\leq \mathbb{P}_p(\exists t \in \mathbb{N} : p \notin A(\hat{p})) \\ &= \mathbb{P}_p(\exists t \in \mathbb{N} : \min_{b \neq \hat{i}(t)} N_{\hat{i}(t)}(t)\mathcal{K}^L_{inf}(\hat{p}_{\hat{i}(t)}(t), \mu_{\hat{i}(t)}) + N_b(t)\mathcal{K}^U_{inf}(\hat{p}_b(t), \mu_b) \geq \beta(t, \delta)) \\ &\leq \sum_{b \neq 1} \mathbb{P}_p(\exists t \in \mathbb{N} : N_{\hat{i}(t)}(t)\mathcal{K}^L_{inf}(\hat{p}_{\hat{i}(t)}(t), \mu_{\hat{i}(t)}) + N_b(t)\mathcal{K}^U_{inf}(\hat{p}_b(t), \mu_b) \geq \beta(t, \delta))\end{aligned} \tag{13}$$

Now a concentration inequality for the above quantity was shown in Agrawal et al. [2021].

Proposition 4.2 in Agrawal et al. [2021].

$$\mathbb{P}\left(\exists n \in \mathbb{N} : N_i(n)\mathcal{K}^U_{inf}(\hat{p}_i(t), \mu_i) + \mathcal{K}^L_{inf}(\hat{p}_j(t), \mu_j) \geq x + 5\log(n+1) + 2\right) \leq e^{-x}.$$

Substituting this in (13) finishes the proof.

F SAMPLE COMPLEXITY GUARANTEE FOR TS(A).

We follow closely the section C.6.2 in Agrawal et al. [2020]. Let $\hat{w}^*(p)$ denote the optimal weights obtained as solutions to the approximate problem described at the beginning of section 3.1 in the main paper. Lemma 14 in Agrawal et al. [2020] then tells us that TS(A) ensures that for all arms $i \in [K]$, $\frac{N_i(lm)}{lm} \overset{a.s.}{\to} \hat{w}^*(p)$ as $l \to \infty$. Recall from section 4 of the main paper that l is the batch index and m is the batch size.

Define the following set

$$\mathcal{I}_\epsilon(p) := B_\zeta(p_1) \times ... \times B_\zeta(p_K)$$

where

$$B_\zeta(p_i) := \{\tilde{p}_i : d_W(p_i, \tilde{p}_i) \leq \zeta, |\tilde{\mu}_i - \mu_i| \leq \zeta\}.$$

Here, d_W is the Wasserstein-1 metric on probability measures and $\tilde{\mu}_i$ is the mean of $\tilde{p}_i$.
Whenever the empirical bandit $\hat{p}(lm) \in \mathcal{I}_\epsilon(p)$, arm1 becomes empirically best. For $\epsilon > 0$, choose $\zeta := \zeta(\epsilon)(< \frac{\mu_1 - \mu_2}{4})$ such that

$$\max_{i \in [K]} |\hat{w}^*_i(p') - \hat{w}^*_i(p)| \leq \epsilon$$

for all $p' \in \mathcal{I}_\epsilon(p)$. For $T \in \mathbb{N}, T \geq m$, define $\ell_0(T) := \max\{1, \frac{T^{1/4}}{m}\}$, $\ell_1(T) := \max\{1, \frac{T^{3/4}}{m}\}$ and $\ell_2(T) := \lfloor \frac{T}{m} \rfloor$. Define the following set

$$\mathcal{G}_T(\epsilon) := \bigcap_{l=\ell_0(T)}^{\ell_2(T)} \{\hat{p}(lm) \in \mathcal{I}_\epsilon(p)\} \bigcap_{l=\ell_1(T)}^{\ell_2(T)} \left\{\max_{i \in [K]} \left| \frac{N_i(lm)}{lm} - \hat{w}^*_i(p) \right| \leq \epsilon \right\}$$

Define the quantities:

$$\tilde{g}(p, w) := \min_{b \neq 1} \mathcal{P}_b(w)$$

$$\tilde{C}_\epsilon(p) := \inf_{\substack{p' \in \mathcal{I}_\epsilon(p) \\ \{w' : ||w' - \hat{w}^*(p)|| \leq \epsilon\}}} \tilde{g}(p', w').$$

where $\mathcal{P}_b$ was defined in equation 7 of the main paper. Now the stopping rule (see section 4 in the main paper) is given by:

$$Z_{k^*}(l) > \beta(lm, \delta)$$

where

$$Z_{k^*}(l) := \min_{b \neq k^*} \inf_{x \leq y} N_{k^*}(lm) \mathcal{K}_{inf}^L(\hat{p}_{k^*}(lm), x) + N_b(lm) \mathcal{K}_{inf}^U(\hat{p}_b(lm), y).$$

where k^* is the empirical best arm and $\beta(t, \delta)$ is the stopping threshold defined as

$$\beta(t, \delta) := \log\left(\frac{K-1}{\delta}\right) + 5\log(t+1) + 2.$$

Note that in $\mathcal{G}_T(\epsilon)$ we have $Z_{k^*}(l) > lm \times \tilde{C}_\epsilon(p)$. Hence, in $\mathcal{G}_T(\epsilon)$,

$$\begin{aligned}
\min\{\tau_\delta, T\} &\leq m.l_1(T) + m \sum_{l=l_1(T)+1}^{l_2(T)} \mathbb{I}\{lm < \tau_\delta\} \\
&\leq m.l_1(T) + m \sum_{l=l_1(T)+1}^{l_2(T)} \mathbb{I}\{Z_{k^*}(l) < \beta(lm, \delta)\} \\
&= m.l_1(T) + m \sum_{l=l_1(T)+1}^{l_2(T)} \mathbb{I}\left\{l < \frac{\beta(lm, \delta)}{m\tilde{C}_\epsilon(p)}\right\} \\
&= m.l_1(T) + \frac{\beta(T, \delta)}{\tilde{C}_\epsilon(p)}
\end{aligned}$$

Define $T_0(\delta, \epsilon) := \inf\left\{t : m.l_1(T) + \frac{\beta(t,\delta)}{\tilde{C}_\epsilon(p)} \leq t\right\}$.

On $\mathcal{G}_T(\epsilon)$, for $T \geq \max\{m, T_0(\delta, \epsilon)\}$, $\min\{\tau_\delta, T\} \leq T$, meaning that for such T, $\tau_\delta \leq T$. Hence, choosing $T_1(\delta, \epsilon) := \max\{m, T_0(\delta, \epsilon) + 1$, we get that $\mathcal{G}_{T_1(\delta,\epsilon)}(\epsilon) \subseteq \{\tau_\delta \leq T_1(\delta, \epsilon)\}$. Then, $\min\{\tau_\delta, T_1(\delta, \epsilon)\} \leq T_1(\delta, \epsilon) \Rightarrow \tau_\delta \leq T_1(\delta, \epsilon)$. This allows us to conclude that

$$\begin{aligned}
\mathbb{E}(\tau_\delta) &= \sum_{t=1}^{\infty} \mathbb{P}(\tau_\delta \geq t) \\
&= \sum_{t=1}^{T_1(\delta,\epsilon)} \mathbb{P}(\tau_\delta \geq t) + \sum_{t=T_1(\delta,\epsilon)+1}^{\infty} \mathbb{P}(\tau_\delta \geq t) \\
&\leq T_0(\delta, \epsilon) + m + \sum_{t=m+1}^{\infty} \mathbb{P}(\mathcal{G}_T^C(\epsilon))
\end{aligned}$$

Now in the same manner as in Agrawal et al. [2020] we can show that $\frac{T_0(\delta,\epsilon)}{\log(1/\delta)} \to \frac{1}{\tilde{C}_\epsilon(p)}$ as $\delta \to 0$. We invoke Lemma 32 in Agrawal et al. [2020] to observe that $\frac{\sum_{t=m+1}^{\infty} \mathbb{P}(\mathcal{G}_T^C(\epsilon))}{\log(1/\delta)} \to 0$. Thus we have for small enough $\epsilon > 0$

$$\limsup_{\delta \to 0} \frac{\mathbb{E}(\tau_\delta)}{\log(1/\delta)} \leq \frac{1}{\tilde{C}_\epsilon(p)}$$

But we observe that by continuity in ϵ, when $\epsilon \to 0$

$$\tilde{C}_\epsilon(p) \to \min_{b \neq 1} \mathcal{P}_b(\hat{w}^*).$$

Note by definition $\min_{b \neq 1} \mathcal{P}_b(\hat{w}^*) \leq V^*(p)$. This inequality shows that TS(A) suffers an increase in sample complexity but this is expected to be small when γ is close to zero since then $\hat{w}^*(p) \approx w^*(p)$.

G ALGORITHMS IN LITERATURE

The following algorithm as per Even-Dar, Mannor & Mansour (2006) provides a simplistic approach towards solving our problem, despite being highly expensive in terms of sampling complexity.

Algorithm 1 Succesive elimination (δ)

Input: Confidence level δ, Upper bounds $[B_i \gamma^{-\alpha_i}]_{i \in [K]}$.
Output: Arm recommendation k^*.

Set $t = 1, S = [K]$.
For all $i \in [K]$, set the empirical means $\hat{\mu}_i^t = 0$.
while |S|>1 **do**
  Sample every arm once, update $\hat{\mu}_i^t$.
  Define $\hat{\mu}_{max}^t := \max_{i \in S} \hat{\mu}_i^t$, $\xi_t := \sqrt{\frac{\log(4Kt^2/\delta)}{t}}$.
  For all $i \in S$ such that $\hat{\mu}_{max}^t - \hat{\mu}_i^t \geq 2\xi_t$, set $S = S \backslash i$.
  $t = t + 1$
end while
Declare the surviving arm as the best arm.

The successive elimination algorithm performs poorly in the rare event setting because a less rare arm which does not have the largest mean becomes likely to survive the elimination and be declared the winner. This is because the less rare arm is likely to produce a nonzero sample, thereby raising its empirical mean, while the more rare arms are yet to turn out any non-zero samples.

Agrawal et al. [2019] describes the following algorithm to meet the lower bound on sampling complexity.

Algorithm 2 Track and Stop

Input: Confidence level δ, Upper bounds $[B_i \gamma^{-\alpha_i}]_{i \in [K]}$.
Output: Arm recommendation k^*.

Generate $\lfloor \frac{m}{k} \rfloor$ samples for each arm.
Set $l = 1$. lm denotes the number of samples.
Compute the empirical bandit $\hat{\mu} = (\hat{\mu})_{\{a \in [K]}$.
Compute the approximate weights $\hat{w}(\hat{\mu})$.
Let $k^* = \arg\max_{a \in [K]} \mathbb{E}[\hat{\mu}_a]$.
Compute $Z(k^*, l, \hat{\mu})$, $\beta(lm, \delta)$.
while $l \leq 2$ or $Z(k^*, l, \hat{\mu}) \geq \beta(lm, \delta)$ **do**
  Compute $s_a = (\sqrt{(l+1)m} - N_a(lm))^+$.
  if $m \geq \sum_a s_a$ **then**
    Generate s_a many samples for each arm a.
    Generate $(m - \sum_a s_a)^+$ independent samples from $\hat{w}(\hat{\mu})$. Let $Count(a)$ be occurrence of a in these samples.
    Generate $Count(a)$ samples from each arm a.
  else
    Solve the load balancing problem minimize $\max_a (s_a - \hat{s}_a)$, where $s_a \geq \hat{s}_a \geq 0$.
    Generate $\hat{s}_a$ samples from each arm a.
  end if
  $l = l + 1$
  Update empirical bandit $\hat{\mu}$ with new samples.
  Update $Z(k^*, l, \hat{\mu})$, $\beta(lm, \delta)$ and $\hat{w}(\hat{\mu})$.
end while
Declare k^* arm as the best arm.

H LIL-UCB, LUCB DEPENDENCE ON SUB-GAUSSIANITY PARAMETER σ

H.1 LIL-UCB

UCB index in this case is given by

$$(1+\beta)(1+\sqrt{\epsilon})\sigma\sqrt{\frac{2(1+\epsilon)\log\left(\frac{(1+\epsilon)N_i(t)}{\delta}\right)}{N_i(t)}}.$$

We have $\sigma = \max_{i\in[K]} B_i\gamma^{-\alpha_i}$ in our setting. Refer to Figure 1 and the discussion following Theorem 2 in Jamieson et al. [2014] for algorithm and the choice of β and ϵ. Refer to Lemma 1 in Jamieson et al. [2014] for choice of σ.

H.2 LUCB

The UCB index is given by

$$\sigma\sqrt{\frac{1}{2N_i(t)}\log\left(\frac{5Kt^4}{4\delta}\right)}$$

$\sigma = \max_{i\in[K]} B_i\gamma^{-\alpha_i}$ here. Refer to section 3.3 and Theorem 1 in Kalyanakrishnan et al. [2012] for choice of UCB index.

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