A. Algebra definitions

A.1. Formal definitions for Universal Algebra

Universal algebra is the field of mathematics that studies algebraic structures, which are defined as a set A along with its own collection of operations. An n-ary operation on A is a function that takes n elements of A and returns a single element from the set. More formally (Burris & Sankappanavar, 1981b; Jonnson, 1967; Day, 1969):

Definition 5. N-ary function For A non-empty set and n nonnegative integer we define $A^0 = \{\emptyset\}$ and, for $n > 0$, A^n is the set of n-tuples of elements from A. An n-ary operation (or function) on A is any function f from A^n to A; n is the arity (or rank) of f. An operation f on A is called an n-ary operation if its arity is n.

Definition 6. Algebraic Structure An algebra $\mathcal{A}$ is a pair $\langle A, F \rangle$ where A is a non-empty set called universe and F is a set of finitary operations on A.

Apart from the operations on A, an algebra is further defined by axioms, that in the particular case of universal algebras are often of the form of identities. The collection of algebraic structures defined by equational laws are called varieties. (Hyland & Power, 2007)

Definition 7. Variety A nonempty class K of algebras of type $\mathcal{F}$ is called a variety if it is closed under subalgebras, homomorphic images, and direct products.

Definition 8. A *lattice* **L** is an algebraic structure composed by a non-empty set L and two binary operations $\vee$ and $\wedge$ satisfying the following axioms and their duals obtained exchanging $\vee$ and $\wedge$:

$$x \vee y \approx y \vee x \qquad \text{(commutativity)}$$
$$x \vee (y \vee z) \approx (x \vee y) \qquad \text{(associativity)}$$
$$x \vee x \approx x \qquad \text{(idempotency)}$$
$$x \approx x \vee (x \wedge y) \qquad \text{(absorption)}$$

Theorem 3 ((Birkhoff, 1940))**.** *Definition 2 and Definition 8 are equivalent.*

Congruence lattices of algebraic structures are partially ordered sets such that every pair of elements has unique supremum and infimum determined by the underlying algebra. This object is important relatively to algebraic structures' properties, many of which can be described by omission or admission of certain subpatterns in a graph.

Definition 9. Congruence Lattice
For every algebra $\mathcal{A}$ on the set A, the identity relation on A, and $A \times A$ are trivial congruences. An algebra with no other congruences is called simple. Let $\mathrm{Con}(\mathcal{A})$ be the set of congruences on the algebra $\mathcal{A}$. Because congruences are closed under intersection, we can define a meet operation: $\wedge : \mathrm{Con}(\mathcal{A}) \times \mathrm{Con}(\mathcal{A}) \to \mathrm{Con}(\mathcal{A})$ by simply taking the intersection of the congruences $E_1 \wedge E_2 = E_1 \cap E_2$. Congruences are not closed under union, however we can define the closure operator of any binary relation E, with respect to a fixed algebra $\mathcal{A}$, such that it is a congruence, in the following way: $\langle E \rangle_{\mathcal{A}} = \bigcap \{F \in \mathrm{Con}(\mathcal{A}) \mid E \subseteq F\}$. Note that the closure of a binary relation is a congruence and thus depends on the operations in $\mathcal{A}$, not just on the carrier set. Now define $\vee : \mathrm{Con}(\mathcal{A}) \times \mathrm{Con}(\mathcal{A}) \to \mathrm{Con}(\mathcal{A})$ as $E_1 \vee E_2 = \langle E_1 \cup E_2 \rangle_{\mathcal{A}}$. For every algebra $\mathcal{A}$, $(\mathrm{Con}(\mathcal{A}), \wedge, \vee)$ with the two operations defined above forms a lattice, called the congruence lattice of $\mathcal{A}$.

Definition 10. Subalgebra Let **A** and **B** be two algebras of the same type. Then **B** is a *subalgebra* of **A** if $B \subseteq A$ and every fundamental operation of **B** is the restriction of the corresponding operation of **A**, i.e., for each function symbol f, $f^{\mathbf{B}}$ is $f^{\mathbf{A}}$ restricted to **B**.

Definition 11. Homomorphic image Suppose **A** and **B** are two algebras of the same type $\mathcal{F}$. A mapping $\alpha : A \to B$ is called a *homomorphism* from **A** to **B** if

$$\alpha f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(\alpha a_1, \ldots, \alpha a_n)$$

for each n-ary f in $\mathcal{F}$ and each sequence $a_1, \ldots, a_n$ from **A**. If, in addition, the mapping α is onto then **B** is said to be a homomorphic image of **A**.

Definition 12. Direct product Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two algebras of the same type $\mathcal{F}$. We define the direct product $\mathbf{A}_1 \times \mathbf{A}_2$ to be the algebra whose universe is the set $A_1 \times A_2$, and such that for $f \in \mathcal{F}$ and $a_i \in A_1$, $a'_i \in A_2$, $1 \leq i \leq n$,

$$f^{\mathbf{A}_1 \times \mathbf{A}_2}(\langle a_1, a'_1 \rangle, \ldots, \langle a_n, a'_n \rangle) = \langle f^{\mathbf{A}_1}(a_1, \ldots, a_n), f^{\mathbf{A}_2}(a'_1, \ldots, a'_n) \rangle$$