Appendix

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Notations

- $\mathbb{R}^*$: The set $\mathbb{R} \setminus \{0\}$ (i.e., real numbers excluding zero).
- $\|\cdot\|$: The Euclidean norm. item $\lambda_{\mathbb{R}^d}$: The Lebesgue measure on $\mathbb{R}^d$.
- **Diameter**: For $\mathcal{C} \subset \mathbb{R}^d$, we define its diameter as:
$$D_{\mathcal{C}} := \sup\{\|x-y\| \mid x, y \in \mathcal{C}\}.$$
- **Indicator function**: For a set $A \subset \mathbb{R}^d$, $\mathbb{1}_A(x)$ is defined as $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise.
- **Component-wise Minimum**: For $v \in \mathbb{R}^d$:
$$v_{\min} := \min_{1 \le j \le d} v_j.$$
- **Special Vectors**:
  - $\mathbf{1}_M = (1, \ldots, 1) \in \mathbb{R}^M$.
  - $\mathbf{0}_M = (0, \ldots, 0) \in \mathbb{R}^M$.
  - $\mathbf{e}_j \in \mathbb{R}^M$, for any $1 \le j \le M$,is the vector with zeros except for the j-th entry, which is equal to 1.
- **Probability Measures**:
  - $\mathcal{P}(\mathbb{R}^d)$: The set of probability measures on $\mathbb{R}^d$.
  - For $\rho \in \mathcal{P}(\mathbb{R}^d)$, $\mathrm{Supp}(\rho)$ denotes its support.
- **Asymptotic Orders**:
  - $\mathcal{O}(\cdot)$ and $o(\cdot)$: Standard approximation orders.
  - $f \lesssim g$ means there exists a constant $C > 0$ such that $f(\cdot) \le Cg(\cdot)$.
  - $a \asymp b$ means both $a \lesssim b$ and $b \lesssim a$.
- **Filtration**: We denote by $\mathcal{F}_n$ the filtration generated by the sample $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mu$, i.e.,
$$\mathcal{F}_n = \sigma\left(X_1, \ldots, X_n\right), \quad n \ge 1.$$
- **Sets**:
  - For any $\varepsilon > 0$, define:
$$K_\varepsilon := \left\{\mathbf{g} \in \mathbb{R}^M \mid \forall i \in [\![1, M]\!], \mu\left(\mathbb{L}_i(\mathbf{g})\right) \ge \varepsilon\right\}.$$
  - Define:
$$K_+ := \left\{\mathbf{g} \in \mathbb{R}^M \mid \forall i \in [\![1, M]\!], \mu\left(\mathbb{L}_i(\mathbf{g})\right) > 0\right\}.$$
- **Density of an Absolutely Continuous Measure**: For an absolutely continuous measure ρ on $\mathbb{R}^d$, we denote its density w.r.t. the Lebesgue measure by f_ρ.
- **Orthogonal of Vect$(\mathbf{1})$**:
  - For any $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^M$, we define:
$$\|\mathbf{g} - \mathbf{g}'\|_v = \|\mathrm{Proj}_{\mathbf{1}^\perp}\left(\mathbf{g} - \mathbf{g}'\right)\|^2.$$
  - Inner product in this space:
$$\langle \mathbf{g}, \mathbf{g}' \rangle_v = \langle \mathrm{Proj}_{\mathbf{1}^\perp}(\mathbf{g}), \mathrm{Proj}_{\mathbf{1}^\perp}(\mathbf{g}')\rangle.$$
- **Strong Convexity**: We discuss the strong convexity of the semi-dual function H when the strong convexity holds on the orthogonal complement of $\mathbf{1}$.

A Further details on our Assumptions

A.1 The Ma-Trundinger-Wang Properties

In the semi-discrete setting, the class of cost functions verifying the Ma–Trudinger–Wang (MTW) properties [24] is defined as the set of cost functions satisfying the following conditions: (Reg), (Twist), and Loeper's condition (QC), detailed below.

$$c\left(\cdot, y_i\right) \in C^2(\text{Supp}(\mu)), \forall i \in \{1, \ldots, M\} \tag{Reg}$$

$$\nabla_x c\left(x, y_i\right) \neq \nabla_x c\left(x, y_k\right), \forall x \in \text{Supp}(\mu), i \neq k \tag{Twist}$$

Definition A.1 (Loeper's condition)**.** We say c satisfies *Loeper's condition* if for each $i \in \{1, \ldots, M\}$ there exists a convex set $X_i \subset \mathbb{R}^d$ and a C^2 diffeomorphism $\exp_i^c(\cdot) : X_i \to \text{Supp}(\mu)$ such that

$$\forall t \in \mathbb{R},\ 1 \leq k, i \leq M,\ \{p \in X_i \mid -c(\exp_i^c(p), y_k) + c(\exp_i^c(p), y_i) \leq t\} \text{ is convex.} \tag{QC}$$

Definition A.2 (c-convexity)**.** We say that $X \subset \mathbb{R}^d$ is c-convex if $(\exp_i^c)^{-1}(X)$ is a convex set for every $i \in \{1, \ldots, N\}$

For a detailed discussion on this class of cost functions and their implications in the semi-discrete optimal transport framework, we refer the reader to Section 1.5 of [20]

A.2 The Poincaré-Wirtinger Inequality

A probability measure $\rho = w(x)\, dx$ on a domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the *weighted Poincaré–Wirtinger inequality* (PW) if

$$\int_\Omega |f - \mathbb{E}_\rho[f]|\, d\rho \ \leq\ C_{\mathrm{PW}} \int_\Omega |\nabla f|\, d\rho, \qquad \forall f \in C^1(\Omega).$$

The existence of a finite constant C_{PW} provides a quantitative connectedness of the source measure and is necessary for H to be locally strongly convex. We provide here two examples of measures satisfying (PW). Notably, Example 1 shows that non-degenerate Gaussians, mixture of non-degenerate Gaussians and Student distributions satisfy (PW), when we take their restrictions on any ball $B(0, R)$, $R > 0$.

Example 2. (bounded support, density bounded above and below) Let Ω be bounded, connected, α-Holder with $\alpha \in (0, 1]$, and assume $0 < m \leq w(x) \leq M < \infty$ almost everywhere.

Note that the assumption of the support being bounded from above and below is classical in the semi-discrete OT literature, as in [12, 29].

Example 3. (Annular support with radial concave profile, ([20], Proposition A.1)) Let $0 < r < R$, and let $\overline{\rho} \in \mathcal{C}^0([0, R])$ be a nonnegative function such that $\overline{\rho}(s) = 0$ for $s \in [0, r]$, and $\overline{\rho}$ is concave on $[r, R]$, with

$$\int_r^R \overline{\rho}(s)\, ds = 1.$$

Define the probability measure ρ on the annulus $X := B(0, R) \subset \mathbb{R}^d$ by

$$\rho(x) = \frac{1}{\|x\|^{d-1}\omega_{d-1}}\, \overline{\rho}(\|x\|),$$

where ω_{d-1} denotes the surface volume of the unit sphere $\mathbb{S}^{d-1}$. Then ρ satisfies the weighted Poincaré–Wirtinger inequality for some positive constant.

B Properties of the semi-discrete OT problem

Regularity properties of H

In the main article, we concisely presented the regularity properties of the function H. In this section, we provide a more detailed breakdown of these properties, organizing them into sub-properties and referring to the corresponding proofs in the quadratic case with unbounded support (assumptions B1–B3).

- **Differentiability:** The function H is differentiable on the entire space $\mathbb{R}^M$, and we denote its gradient by ∇H (see Proposition B.5).
- **Local C^2 regularity:** There exists a radius $r > 0$ such that H is twice continuously differentiable (C^2) on the ball $B(\mathbf{g}^*, r)$ (see Proposition B.5).
- **Local strong convexity:** The function H is strongly convex on the ball $B(\mathbf{g}^*, r)$ (see Proposition B.12).
- **Hölder continuity of the Hessian:** If the density f_μ is α-Hölder continuous for some $\alpha \in (0, 1]$, then the Hessian of H inherits this regularity and is also α-Hölder continuous (see Corollary B.11).

B.1 Known results in the compact case

In this section, we recall known properties of the semi-dual semi-discrete problem when the support of the source measure μ is c-convex and contained within a compact set and c is a cost satisfying the `MTW` properties. We will then extend these results to the non-compact case for the quadratic cost.

Here, we fix $\varepsilon > 0$ and recall that $K_\varepsilon := \left\{\mathbf{g} \in \mathbb{R}^M : \forall i \in [\![1, M]\!], \mu\left(\mathcal{L}_i(\mathbf{g})\right) \geq \varepsilon\right\}$. The two theorems presented below are taken from [20] and have been adapted to our notation. We emphasize that the authors of [20] considered the semi-dual OT problem as a concave problem, studying the objective function $-H$ instead of H under their notation. For a better understanding of the constants in their theorems, we refer the reader to their article.

Proposition B.1 (Theorem 1.1 in [20])**.** *Let μ be an absolutely continuous density with bounded support included in $\mathbb{R}^d$, then the functional H is C^1 smooth, its gradient is given by*

$$\nabla H(\mathbf{g})_i = -\mu(\mathbb{L}_i^c(\mathbf{g})) + w_i \ ,$$

and its Hessian by

$$(i \neq j) \qquad \nabla^2 H(\mathbf{g})_{ij} = -\int_{\mathbb{L}_i(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g})} \frac{f_\mu(x)}{\|y_i - y_j\|} \, \mathrm{d}\mathcal{H}^{d-1}(x),$$
$$\nabla^2 H(\mathbf{g})_{ii} = -\sum_{j \neq i} \nabla^2 H(\mathbf{g})_{ij} \ .$$

Proposition B.2 (Theorem 5.1 in [20])**.** *Under the assumption (A1), that μ satisfies a weighted (1,1)-Poincaré–Wirtinger inequality, there exists a constant λ such that for any $\mathbf{g} \in K_\varepsilon$, the second smallest eigenvalue of $\nabla^2 H(\mathbf{g})$, denoted $\lambda_2(\nabla^2 H(\mathbf{g}))$, satisfies*

$$\lambda_2(\nabla^2 H(\mathbf{g})) > \lambda.$$

That is, H is strongly convex on K_ε, considering the problem on $\mathbf{1}^\perp$.

Theorem B.3 (Theorem 1.3 in [20])**.** *If μ has its density f_μ in $C^{0,\alpha}(Supp(\mu))$. Then, the functional H is $C^{2,\alpha}$ on the set*

$$K_\varepsilon := \left\{\mathbf{g} \in \mathbb{R}^M, \forall i, \mu\left(\mathbb{L}_i(\mathbf{g})\right) > \varepsilon\right\} \ ,$$

Lastly, we state a result concerning the quantitative stability of Laguerre cells, as presented in [5].

Lemma B.4 (Lemma 5.5 in [5])**.** *Under the same assumptions as in Proposition B.2, for $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^M$, we have*

$$\mu(\mathbb{L}_i^c(\mathbf{g}) \setminus \mathbb{L}_i^c(\mathbf{g}')) \lesssim M\|\mathbf{g} - \mathbf{g}'\|_\infty, \qquad \forall i \in [\![1, M]\!] \ .$$

Once again, we refer to [5] for a more detailed understanding of the constant involved in this lemma.

B.2 New properties for the non-compact case with the quadratic Euclidean cost

In this section, we give second order properties of the semi-dual when the source measure is not supported on a compact. In the compact case, this properties are already known as discussed in Appendix B.1.

For the reader's convenience, we recall the Assumptions that we made for the non-compact case.

Assumption (Non-compact case) **(B1)** The cost is quadratic: $c(x, y) = \frac{1}{2}\|x - y\|^2$ and the measure μ has a finite second-order moment.

(B2) There exists a compact set $K \subset \mathbb{R}^d$ with $\mu(K) \geq 1 - \frac{1}{4}w_{\min}$, such that the probability measure μ_K with density $f_{\mu_K}(x) := c_K f_\mu(x)\mathbf{1}_K(x)$ satisfies a Poincaré-Wirtinger inequality.

(B3) The density f_μ satisfies the following integrability and regularity condition: for $R > 1$ and $r \geq 1$, define

$$f_\mu^R := f_\mu \cdot \mathbf{1}_{\|x\| \leq R}, \quad f_\mu^{R+r} := f_\mu \cdot \mathbf{1}_{R+(r-2) \leq \|x\| \leq R+r}.$$

Assume there exist $C > 0$ and a modulus of continuity ω such that for all $\delta > 0$,

$$\sum_{r=0}^{\infty}(R+r)^{d-1}\omega_{f_\mu}^{R+r}(\delta) \leq C\,\omega(\delta), \quad \sum_{r=0}^{\infty}(R+r)^{d-1}C_{f_\mu}^{R+r} < \infty, \tag{6}$$

where $C_{f_\mu}^{R+r} := \sup_{x \in \mathbb{R}^d} f_\mu^{R+r}(x)$ and $\omega_{f_\mu}^{R+r}$ is the modulus of continuity of f_μ^{R+r}.

Additional notation. Since here the cost is fixed, we define the Laguerre cells by $\mathbb{L}_i(\mathbf{g})$ instead of $\mathbb{L}_i^c(\mathbf{g})$.

B.2.1 Definition and regularity of the Hessian

Proposition B.5. *Under Assumptions (B1) and (B3), the semi-dual H is differentiable everywhere, and its gradient is given by*

$$\nabla H(\mathbf{g})_i = \mu(\mathbb{L}_i(\mathbf{g})) - w_i\ .$$

Moreover H is C^2 smooth on $K_{w_{\min}/2} \cap \mathcal{C}$ and its Hessian is given by

$$\begin{aligned}(i \neq j) \qquad \nabla^2 H(\mathbf{g})_{ij} &= -\int_{\mathbb{L}_i(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g})} \frac{f_\mu(x)}{\|y_i - y_j\|}\,\mathrm{d}\mathcal{H}^{d-1}(x),\\ \nabla^2 H(\mathbf{g})_{ii} &= -\sum_{j \neq i} \nabla^2 H(\mathbf{g})_{ij}\ .\end{aligned}$$

Proof. **Definition of the gradient.** The proof follows the lines of the proof of Theorem 4.1 of [20] and extend this result to the non-compact case. If x is in the interior of Laguerre cell $\mathbb{L}_j(\mathbf{g})$, we have

$$\nabla_{\mathbf{g}} h(\mathbf{g}, x) = \mathbb{1}_{i=j} - w_i.$$

where we recall that $h(\mathbf{g}, x) = -\mathbf{g}^c(x) - \sum_{i=1}^M w_j g_j$. Since the boundaries of the Laguerre cells are defined by the intersections of M hyperplanes, they form a negligible set with respect to the measure μ. As a result, the gradient definition of H follows immediately.

Definition of the Hessian. Fix $i \in [\![1, M]\!]$. We aim to prove the differentiability of the measure $\mu(\mathbb{L}_i(\mathbf{g}))$ with respect to g_j and that its differential is defined by:

$$\begin{aligned}\frac{\partial \mu(\mathbb{L}_i(\mathbf{g}))}{\partial g_j} &= -\int_{\mathbb{L}_i(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g})} \frac{f_\mu(x)}{\|y_i - y_j\|}\,\mathrm{d}\mathcal{H}^{d-1}(x), \quad j \neq i,\\ \frac{\partial \mu(\mathbb{L}_j(\mathbf{g}))}{\partial g_j} &= -\sum_{i \neq j} \frac{\partial \mu(\mathbb{L}_i(\mathbf{g}))}{\partial g_j}.\end{aligned}$$

This gives us exactly the line $\left(\nabla^2 H(\mathbf{g})_{1i}, \ldots, \nabla^2 H(\mathbf{g})_{Mi}\right)$ of the Hessian.

Suppose $i \neq j$.

Suppose $\delta \geq 0$. Defining $h_{ij}(x) := \frac{1}{2}\|x - y_i\|^2 - \frac{1}{2}\|x - y_j\|^2 - g_i + g_j$, note that we have $\mathbb{L}_i(\mathbf{g}) = \cap_j h_{ij}^{-1}(] - \infty, 0])$.

We also have $\mathbb{L}_i(\mathbf{g} + \delta\mathbf{e}_j) = \mathbb{L}_i(\mathbf{g}) \setminus \left(\cap_{k \neq j} h_{ik}^{-1}(] - \infty, 0]) \cap h_{ij}^{-1}\left([-\delta, 0]\right)\right)$. Note that h is Lipschitz and for all x, $\|\nabla h(x)\| = \|y_j - y_i\|$. Moreover, under assumption (B3), we can use Lemma F.2, which states that for all hyperplane H, $\int_H f_\mu d\mathcal{H}^{d-1} \lesssim 1$. Therefore, we can apply the coarea formula to pass from the second to the third equality below,

$$\begin{aligned}\mu\left(\mathbb{L}_i(\mathbf{g}+\delta\mathbf{e}_j)\right) &= \mu(\mathbb{L}_i(\mathbf{g})) - \mu\left(\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0]) \cap h_{ij}^{-1}\left([-\delta,0]\right)\right) \\ &= \mu(\mathbb{L}_i(\mathbf{g})) - \int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}([-\delta,0])} f_\mu(x)\mathrm{d}x \\ &= \mu(\mathbb{L}_i(\mathbf{g})) - \int_{-\delta}^{0}\int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{f_\mu(x)}{\|y_j - y_i\|}\mathrm{d}\mathcal{H}^{d-1}(x)\mathrm{d}t \,.\end{aligned}$$

By analogy, for $\delta \leq 0$ we have:

$$\mathbb{L}_i(\mathbf{g}+\delta\mathbf{e}_j) = \mathbb{L}_i(\mathbf{g}) \cup \left(\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0]) \cap h_{ij}^{-1}\left([-\delta,0]\right)\right) \,,$$

Using the coarea formula gives:

$$\mu\left(\mathbb{L}_i(\mathbf{g}+\delta\mathbf{e}_j)\right) = \mu(\mathbb{L}_i(\mathbf{g})) + \int_{0}^{-\delta}\int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{f_\mu(x)}{\|y_j - y_i\|}\mathrm{d}\mathcal{H}^{d-1}(x)\mathrm{d}t \,.$$

Applying Lemma B.6, which is stated and proved later in the appendice, the integrand defined above is continuous on $K_{w_{\min}/2} \cap \mathcal{C}$. As a consequence, we can apply the Fundamental Theorem of Calculus and justify the limit in

$$\lim_{\delta\to 0^-} \frac{\mu(\mathbb{L}_i(g+\delta\mathbf{e}_j)) - \mu(\mathbb{L}_i(\mathbf{g}))}{\delta} = -\int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{0\})} f_\mu(x)\frac{1}{\|y_j - y_i\|}\,\mathrm{d}\mathcal{H}^{d-1}(x) \,.$$

By symmetry

$$\lim_{\delta\to 0^+} \frac{\mu(\mathbb{L}_i(g+\delta\mathbf{e}_j)) - \mu(\mathbb{L}_i(\mathbf{g}))}{\delta} = -\int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{0\})} f_\mu(x)\frac{1}{\|y_j - y_i\|}\,\mathrm{d}\mathcal{H}^{d-1}(x) \,.$$

Since $\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0]) \cap h_{ij}^{-1}(\{0\}) = \mathbb{L}_i(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g})$, we thus obtain

$$F_{ij}(\mathbf{g}) := \frac{\partial\mu(\mathbb{L}_i(\mathbf{g}))}{\partial g_j} = -\int_{\mathbb{L}_i(\mathbf{g})\cap\mathbb{L}_j(\mathbf{g})} \frac{f_\mu(x)}{\|y_i - y_j\|}\,d\mathcal{H}^{d-1}(x), \quad j \neq i \,.$$

Suppose $i = j$**.** No matter $\mathbf{g} \in \mathbb{R}^M$, the Laguerre cells verifies $\mu\left(\cup_i \mathbb{L}_j(\mathbf{g})\right) = \mu(\mathbb{R}^d) = 1$. Therefore,

$$\frac{\partial}{\partial g_j}\sum_{i=1}^{M} \mu\left(\mathbb{L}_i(\mathbf{g})\right) = \frac{\partial}{\partial g_j} 1 = 0.$$

This equality gives $\frac{\partial}{\partial g_j}\mu\left(\mathbb{L}_j(\mathbf{g})\right) = -\sum_{i\neq j}\frac{\partial}{\partial g_j}\mu\left(\mathbb{L}_i(\mathbf{g})\right)$.

□

Lemma B.6. *Under assumption (B1) and (B3), for any* $\mathbf{g} \in K_{w_{\min}/2} \cap \mathcal{C}$ *the function* F_{f_μ} *defined for all* $t \in \mathbb{R}$ *as*

$$F_{f_\mu}(t) = \int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{f_\mu(x)}{\|y_j - y_i\|}\,\mathrm{d}\mathcal{H}^{d-1}(x) \,.$$

admits ω *as modulus of continuity in some neighborhood of* $\{0\}$.

Proof of Lemma B.6. We keep the same notation as in [20]. Restricting ourselves to the quadratic cost, we get

$$\begin{aligned}\varepsilon_{nd} = \varepsilon_{tw} &= \min_{i\neq j}\|y_i - y_j\| \,, \\ C_\nabla^R &= O(R) \,, \\ C_{exp} &= O(1) \,, \\ C_{cond} &= O(1) \,, \\ C_{det} &= O(1) \,.\end{aligned}$$

In order to apply the results in [20], we need to extend Proposition 4.5 to measures with unbounded support in the case of the quadratic cost. We prove the following result:

Proposition B.7. *Consider $\mathcal{K}_\varepsilon := \{\mathbf{g} \in \mathbb{R}^{N-1} \,|\, \mu(\mathbb{L}_i(\mathbf{g})) > \varepsilon\,, \forall i = 1, \dots, M\}$ for some $\varepsilon > 0$ sufficiently small and let $\mathcal{C}$ be the compact set of the projection. Then, there exists a positive constant δ_1, such that for all $\mathbf{g} \in \mathcal{K}_\varepsilon \cap \mathcal{C}$ and all $p \in \mathbb{R}^d$ such that there exist $i \neq j$, for which $c(p, y_i) - c(p, y_0) = g_i - g_0$ and $c(p, y_j) - c(p, y_0) = g_j - g_0$, then*

$$\left(\frac{\langle y_i - y_0, y_j - y_0 \rangle}{\|y_i - y_0\| \|y_j - y_0\|} \right)^2 \leq 1 - \delta_1^2\,. \tag{7}$$

Remark B.8. Note that the transversality *equation* (7) is independent of $p \in \mathbb{R}^d$. However, p is still involved in the transversality *condition* by the value of the difference of the costs.

Proof. Fix an index i. The set of points $p \in \mathbb{R}^d$ such that $c(p, y_i) - c(p, y_0) = g_i - g_0$ defines a hyperplane in $\mathbb{R}^d$. The intersection of two such hyperplanes, denoted $H_{ij}(g)$, is, unless the hyperplanes are parallel, a codimension-2 affine subspace of $\mathbb{R}^d$.

For each such $H_{ij}(g)$, let $d_{ij}(g) \in \mathbb{R}^d$ be the orthogonal projection of the origin 0_d onto $H_{ij}(g)$. Since the vector of potentials $(g_i)_{i=0,\dots,M-1}$ is bounded due to the projection set $\mathcal{C}$, the set

$$\{d_{ij}(g) \,:\, i \neq j \text{ admissible, and } g \in \mathcal{K}_\varepsilon \cap \mathcal{C}\}$$

is contained in a ball $B(0, R-1)$ for some sufficiently large $R > 0$.

Hence, for every $g \in \mathcal{K}_\varepsilon \cap \mathcal{C}$ and every admissible pair (i, j), there exists a point $p \in H_{ij}(g)$ lying in the interior of the ball $B(0, R)$.

We now apply the transversality result from [20] to the measure

$$\mu_R := \mathbf{1}_{B(0,R)} \cdot \frac{\mu}{\mu(B(0, R))},$$

which is the normalized restriction of μ to $B(0, R)$. This result guarantees transversality of the hyperplanes H_{ij} associated with potentials in the set $\mathcal{K}'_{\varepsilon'}$ (defined analogously for μ_R and some $\varepsilon' > 0$) over the whole space $\mathbb{R}^d$.

Choose R such that $\mu(B(0, R)) \geq 1 - \varepsilon/2$. Then it is sufficient to take

$$\varepsilon' = \frac{\varepsilon/2}{1 - \varepsilon/2},$$

since one can check that $\mathcal{K}_\varepsilon \subset \mathcal{K}'_{\varepsilon'}$. This completes the proof. □

We aim to use the transversality result of Kitagawa et al. on a decomposition of $\mathbb{R}^d$ into balls centered at 0. To this goal, we now prove a transversality result at the boundary of $B(0, R)$ uniform for R sufficiently large.

Proposition B.9. *Consider $\mathcal{K}_\varepsilon := \{\mathbf{g} \in \mathbb{R}^{N-1} \,|\, \mu(\mathbb{L}_i(\mathbf{g})) > \varepsilon\,, \forall i = 1, \dots, M\}$ for some $\varepsilon > 0$ sufficiently small and let $\mathcal{C}$ be the compact set of the projection. Then, for any positive constant δ_2 there exists R sufficiently large such that for all $\mathbf{g} \in \mathcal{K}_\varepsilon \cap \mathcal{C}$ and all $p \in \mathbb{R}^d$: if there exists i such that $c(p, y_i) - c(p, y_0) = g_i - g_0$ for some $p \in \partial B(0, R)$, then*

$$\left\langle \frac{p}{\|p\|}, \frac{y_i - y_0}{\|y_i - y_0\|} \right\rangle \leq 1 - \delta_2^2\,. \tag{8}$$

Proof. Rewrite the condition $c(p, y_i) - c(p, y_0) = g_i - g_0$ as $\langle p, y_i - y_0 \rangle = g_i - g_0$, dividing by $\|p\|$ we get

$$\left\langle \frac{p}{\|p\|}, y_i - y_0 \right\rangle = \frac{g_i - g_0}{\|p\|}\,. \tag{9}$$

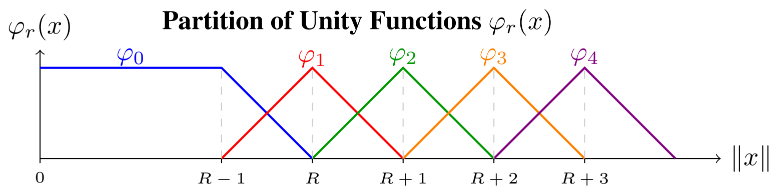
The right hand side tends to 0 uniformly with R since $g_i - g_0$ lies in a compact set. The conclusion follows directly. □

The rest of the proof follows the lines of the proofs Appendix B of [20] with $\varepsilon_{tr} = \delta_1$ as in Proposition B.7. From Proposition B.9 applied to $\delta_2 = \delta_1$ there exists $R > 1$ such as the transversality condition on the boundary (8) holds for every $B(0, R+r)$ and the same δ_1 for every $r \geq 0$. Let us fix such $R > 1$ so that Assumption (5) is also satisfied.

The next step is to decompose the integral using a partition of unity. We define the sequence of functions $(\varphi_r)_{r\geq 0}$ by

$$\varphi_r(x) = \begin{cases} (R - \|x\|)_+ \wedge 1, & \text{if } r = 0, \\ (\|x\| - (R + r - 2))_+ \wedge (R + r - \|x\|)_+, & \text{if } r \geq 1, \end{cases} \qquad x \in \mathbb{R}^d.$$

An illustration of the functions $\varphi_r(x)$ defined above is shown below:



By definition, $\sum_{r\geq 0} \varphi(x) = 1$, for all $x \in \mathbb{R}^d$, and every φ_r is supported on $\{R + r - 2 \leq \|x\| \leq R+r\}$ for every $r \geq 1$, φ_0 being supported by $B(0, R)$. Moreover φ_r is Lipschitz continuous and we denote ω_{φ_r} its modulus of continuity on its support. The moduli of continuity satisfy $\omega_{\varphi_r}(\delta) \leq |\delta|$, $\delta > 0$ $r \geq 0$. Then we decompose the integral

$$\int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{f_\mu(x)}{\|y_j - y_i\|} \mathrm{d}\mathcal{H}^{d-1}(x) = \sum_{r=0}^{\infty} \int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{f_\mu(x)\varphi(x)}{\|y_j - y_i\|} \mathrm{d}\mathcal{H}^{d-1}(x)\,. \tag{10}$$

We apply Proposition B.1 of [20] to each term of the decomposition. We recall below his crucial result, tracking the order of the constants established by [20].

Proposition B.10. *Let σ be a continuous non-negative function on $B(0, R)$ bounded by σ_∞ and with modulus of continuity ω_σ. Let the functions h_{ij} satisfy the transversality conditions* (7) *and* (8) *with the same constant $\varepsilon_{tr} > 0$. Then*

$$F_\sigma(t) := \int_{\cap_{k\neq j} h_{ik}^{-1}(]-\infty,0])\cap h_{ij}^{-1}(\{t\})} \frac{\sigma(x)}{\|y_j - y_i\|} \mathrm{d}\mathcal{H}^{d-1}(x)$$

has modulus of continuity

$$\omega_{h_\sigma}(\delta) = C_1 \omega_\sigma(C_2 \delta) + C_3|\delta|$$

where $C_1 = O(\mathcal{H}^{d-1}(\partial B(0, R)))$, $\mathcal{H}^{d-1}(\partial B(0, R)) = O(R^{d-1})$, $C_2 = O(\varepsilon_{tw}^{-1}) = O(1)$ and $C_3 = O(\sigma_\infty C(d, 2R)\varepsilon_{tr}^{-4} + \mathcal{H}^{d-1}(\partial B(0, R)))$, where $C(d, 2R)$ defined in (3.5) of [20] satisfies $C(d, 2R) = O(R^{d-1})$.

When applying Proposition B.10 to the continuous function $\sigma^{R+r}(x) := f_\mu(x)\varphi(x)$ we easily estimate $\sigma_\infty^{R+r} \leq C_{f_\mu}^{R+r}$ and $w_{\sigma^{R+r}} \leq \omega_{f_\mu}^{R+r} + C_{f_\mu}^{R+r}\omega_{\varphi_r}$. Using that $\omega_{\varphi_r}(\delta) \leq |\delta|$ we obtain:

$$\omega_{h_{\sigma^{R+r}}}(\delta) = O\left((R + r)^{d-1}\right)\left(\omega_{f_\mu}^{R+r}(O(1)\delta) + O(C_{f_\mu}^{R+r})\delta\right)\,,$$

uniformly for every $\mathbf{g} \in \mathcal{K}_\varepsilon \cap \mathcal{C}$ on a neighborhood of $\{0\}$. Noticing that this neighborhood depends solely on the transversality properties that are common to every $r \geq 0$ and since

$$\omega_{F_{f_\mu}} = \sum_{r\geq 0} \omega_{h_{\sigma^{R+r}}}$$

from the decomposition in (10) the desired result follows under the assumption in (5). $\square$

Corollary B.11. *Under Assumptions (B1) and (B3), if moreover f_μ is α-Hölder with $\alpha \in (0, 1]$, then the Hessian of the semi-dual H is also α-Hölder on $K_{w_{\min}/2} \cap \mathcal{C}$.*

Proof. Since f_μ is α-Hölder, the function f_μ^{R+r} with $r \geq 0$ are also α-Hölder and we note $\omega_{f_\mu}^{R+r} = \kappa_{f_\mu}^{R+r}\delta^\alpha$, applying Proposition B.10,

$$\begin{aligned}
&\omega_{h_{\sigma R+r}}(\delta) O\left((R+r)^{d-1}\right)\left(\omega_{f_\mu}^{R+r}(O(1)\delta) + O(C_{f_\mu}^{R+r})\delta\right) \\
&\qquad = O((R+r)^{d-1})\kappa_{f_\mu}^{R+r} O(1)\delta^\alpha + ((R+r)^{d-1}) O(C_{f_\mu}^{R+r})\delta \,.
\end{aligned}$$

By the summability conditions of assumption (B3), for some constants $C_1, C_2 > 0$, we have

$$\begin{aligned}
\omega_{F_{f_\mu}} &= \sum_{r \geq 0} \omega_{h_{\sigma R+r}} \\
&= O(1)\delta^\alpha \sum_{r\geq 0} O((R+r)^{d-1})\kappa_{f_\mu}^{R+r} + \delta \sum_{r \geq 0} O(C_{f_\mu}^{R+r})((R+r)^{d-1}) \\
&\leq C_1 \delta^\alpha + C_2 \delta \,,
\end{aligned}$$

Applying Proposition B.3 in [20] under our hypothesis shows that for $\mathbf{g}, \mathbf{g}' \in K_{w_{\min}/2} \cap \mathcal{C}$, there exists a constant C depending on ε_{tr} and f_μ such that we have

$$\left|\nabla^2 H(\mathbf{g}) - \nabla^2 H(\mathbf{g}')\right| \leq \omega_{F_{f_\mu}}(\|\mathbf{g} - \mathbf{g}'\|_\infty) + C\|\mathbf{g} - \mathbf{g}'\|$$

which gives the α-Hölder regularity since $\omega_{F_{f_\mu}} \leq C_1\delta^\alpha + C_2\delta$. □

B.2.2 Local strong convexity with respect to the mass of Laguerre cells

Proposition B.12. *Under Assumptions (B1-B3) H is strongly convex on $K_{\frac{1}{2}w_{\min}}$.*

Proof. Recall $K \subset \mathbb{R}^d$ the compact set such that $\mu(K) > 1 - \frac{1}{4}w_{\min}$ and μ_K, the probability measure with density defined by $f_{\mu_K}(x) = c_K f_\mu(x)\mathbf{1}_K(x)$, with $c_K \in [1, 2]$, satisfies a weighted Poincaré-Wirtinger inequality.

We thus can use Proposition B.2, which states that the semi-dual between μ_R and ν is strongly-convex on $\mathrm{Vect}_\mathbf{1}^\perp$ on the set

$$K^R_{\frac{1}{4}w_{\min}} := \left\{\mathbf{g} \in \mathbb{R}^M : \forall i \in [\![1, M]\!], \mu_R(\mathbb{L}_i^R(\mathbf{g})) \geq \frac{1}{4}w_{\min}\right\}.$$

This gives us that, for any $\mathbf{g} \in K^R_{\frac{1}{4}w_{\min}}$, there exists $\lambda_K > 0$, lower bounding the second smallest value of the semi-dual function between μ_R and ν. This also gives us that for any $\mathbf{g} \in K^R_{\frac{1}{4}w_{\min}}$, the second smallest eigenvalue of the matrix B defined by

$$\begin{aligned}
(i \neq j) \qquad B_{ij} &= -\int_{\mathbb{L}_i(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g})} \frac{f_\mu(x)\mathbf{1}_{B(0,R)}(x)}{\|y_i - y_j\|}\,\mathrm{d}\mathcal{H}^{d-1}(x), \\
B_{ii} &= -\sum_{j \neq i} B_{ij} \,,
\end{aligned}$$

is lower bounded by $\lambda_K / c_R \geq \lambda_K/2 > 0$.

Since for any $\mathbf{g}$, and $i \neq j$, $\nabla^2 H(\mathbf{g})_{ij} \leq B_{ij}$ and that both $\nabla^2 H(\mathbf{g})$ and B are Laplacian matrices, we apply Lemma F.1 to obtain that the second smallest eigenvalue of the hessian $\nabla^2 H$ is lower bounded by λ_R / c_R on $K^R_{\frac{1}{4}w_{\min}}$.

For any $\mathbf{g} \in K_{w_{\min}/2}$, we thus have for all $i \in [\![1, M]\!], \mu(\mathbb{L}_i(\mathbf{g}) \cap B(0, R)) \geq \frac{1}{4}w_{\min}$. That is, $K_{\frac{1}{2}w_{\min}} \subset K^R_{\frac{1}{4}w_{\min}}$, which completes the proof. □

B.2.3 Quantitative stability of the Laguerre cells

Lemma B.13. *Under Assumptions (B1-B3), we have*

$$\mu(\mathbb{L}_i^c(\mathbf{g}) \setminus \mathbb{L}_i^c(\mathbf{g}')) \lesssim M\|\mathbf{g} - \mathbf{g}'\|_\infty, \qquad \forall i \in [\![1, M]\!] \,.$$

Proof. For all $j \in [\![1, M]\!]$, if x is in the interior of $\mathbb{L}_j(\mathbf{g})$, we have

$$\nabla \mathbf{g}^c(x) = x - y_j. \tag{11}$$

Therefore, given $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^M$, if there exists $j \in [\![1, M]\!]$ such that x is the interior of $\mathbb{L}_j(\mathbf{g}) \cap \mathbb{L}_j(\mathbf{g}')$ we have

$$\nabla \mathbf{g}^c(x) = \nabla(\mathbf{g}')^c(x).$$

Moreover, since the support of ν is finite, we have

$$\sup_{x \in \mathbb{R}^d} \|\nabla \mathbf{g}^c(x) - \nabla(\mathbf{g}')^c(x)\| = \max_{i \neq j} \|y_i - y_j\| \,.$$

Hence, to bound the error of $T(\mathbf{g})(x) = x - \nabla \mathbf{g}^c(x)$ and $T_{\mu,\nu} = x - \nabla(\mathbf{g}^*)^c(x)$, we just need to bound the difference of measure of Laguerre cells made by $\mathbf{g}$ and $\mathbf{g}^*$. More generally, we now proceed here to bound the difference of measure of Laguerre cells between $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^M$ fixed arbitrarily.

Our proof will follow some arguments from [20]. Let us fix $i \in [\![1, M]\!]$ and suppose $x \in \mathbb{L}_i(\mathbf{g}) \setminus \mathbb{L}_i(\mathbf{g}')$. The definition of the Laguerre cells implies that there is a $k \neq i$ such that $c(x, y_k) + \mathbf{g}'_k < c(x, y_i) + \mathbf{g}'_i$ while $c(x, y_i) + \mathbf{g}_i \leq c(x, y_k) + \mathbf{g}_k$. Combining these two inequalities yields to

$$\mathbf{g}'_k - \mathbf{g}'_i < c(x, y_i) - c(x, y_k) \leq \mathbf{g}_k - \mathbf{g}_i.$$

Hence, writing $f_{ik}(x) = c(x, y_i) - c(x, y_k)$, we have

$$\mathbb{L}_i(\mathbf{g}) \setminus \mathbb{L}_i(\mathbf{g}') \subset \bigcup_{k \neq i} f_{ik}^{-1}\left([\mathbf{g}'_k - \mathbf{g}'_i, \mathbf{g}_k - \mathbf{g}_i]\right) \,. \tag{12}$$

We now now bound $\mu\left(f_{ik}^{-1}\left([\mathbf{g}'_k - \mathbf{g}'_i, \mathbf{g}_i - \mathbf{g}_k]\right)\right)$ using the coarea formula, using that for all x, $\|\nabla f_{ik}(x)\| = \|y_i - y_k\|$:

$$\begin{aligned}
\mu\left(f_{ik}^{-1}([a,b])\right) &= \int_{f_{ik}^{-1}([a,b])} \mathrm{d}\mu(x) \\
&= \int_a^b \int_{f_{ik}^{-1}(\{t\})} \frac{1}{\|\nabla f_{ik}(x)\|} \mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \mathrm{d}t \\
&= \int_a^b \int_{f_{ik}^{-1}(\{t\})} \frac{1}{\|y_i - y_k\|} \mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \mathrm{d}t \,.
\end{aligned}$$

Observe that for the quadratic cost, $f_{ik}(x) = \langle x, y_k - y_i \rangle + \|y_i\|^2 - \|y_k\|^2$ and so $f_{ik}^{-1}(\{t\}) = \left\{x \in \mathbb{R}^d, \langle x, y_k - y_i \rangle = t - \frac{1}{2}(\|y_i\|^2 + \|y_k\|^2)\right\}$ and is therefore a hyperplane. Applying Lemma F.2, there exists a constant C such that, for any i, k and t, we have

$$\int_{f_{ik}^{-1}(\{t\})} \frac{1}{\|y_i - y_k\|} \mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \leq C \,.$$

Therefore, we have

$$\mu\left(f_{ik}^{-1}([a,b])\right) \leq \int_a^b \frac{C}{\|y_i - y_k\|} \leq (b-a) \frac{C}{\|y_i - y_k\|} \,.$$

Since $\mathbf{g}_k - \mathbf{g}_i - (\mathbf{g}'_k - \mathbf{g}'_i) \leq 2 \|\mathbf{g} - \mathbf{g}'\|_\infty$, by combining the above with (12) we conclude

$$\mu\left(\mathbb{L}_i(\mathbf{g}) \setminus \mathbb{L}_i(\mathbf{g}')\right) \leq \sum_{k \neq i} \mu\left(f_{ik}^{-1}\left([\mathbf{g}'_k - \mathbf{g}'_i, \mathbf{g}_k - \mathbf{g}_i]\right)\right) \lesssim M \|\mathbf{g} - \mathbf{g}'\|_\infty. \tag{13}$$

$\square$

B.3 New properties in both our compact and non compact settings

Theorem 5.1. Under Assumptions (A1) or (B1-3), the function $\mathbf{g} \mapsto \|T(\mathbf{g}) - T_{\mu,\nu}\|^2_{L^2(\mu)}$ is Lipschitz with respect to the infinity norm $\|\cdot\|_\infty$. Moreover, the Lipschitz constant grows at most quadratically in M.

Proof. Using that no matter $\mathbf{g} \in \mathbb{R}^M$, for any $x \in \mathbb{R}^d$

$$\|T_{\mu,\nu}(\mathbf{g}^*) - T_{\mu,\nu}(\mathbf{g})\| \leq \begin{cases} 0 & \text{if } x \in \mathbb{L}_i(\mathbf{g}^*) \cap \mathbb{L}_i(\mathbf{g}) \text{ for a certain } i \in [\![1, M]\!], \\ \max_{i \neq j} \|y_i - y_j\| & \text{else,} \end{cases}$$

and that for any $\mathbf{g}, \mathbf{g}' \in \mathbb{R}^M$, we have $\mu(\mathbb{L}_i(\mathbf{g}) \setminus \mathbb{L}_i(\mathbf{g}')) \lesssim M\|\mathbf{g} - \mathbf{g}'\|$ using Proposition B.4 in the compact case, or Proposition B.13, we have for any $p \in [1, \infty)$

$$\begin{aligned} \|T_{\mu,\nu}(\mathbf{g}^*) - T_{\mu,\nu}(\mathbf{g})\|^p_{L_p(\mu)} &= \int_{\mathbb{R}^d} \|T_{\mu,\nu}(\mathbf{g}^*)(x) - T_{\mu,\nu}(\mathbf{g})(x)\|^p \mathrm{d}\mu(x) \\ &= \sum_{i=1}^M \int_{\mathbb{L}_i(\mathbf{g}^*)} \|T_{\mu,\nu}(\mathbf{g}^*)(x) - T_{\mu,\nu}(\mathbf{g})(x)\|^p \mathrm{d}\mu(x) \\ &\leq \sum_{i=1}^M \int_{\mathbb{L}_i(\mathbf{g}^*) \setminus \mathbb{L}_i(\mathbf{g})} \max_{i \neq j} \|y_i - y_j\|^p \mathrm{d}\mu(x) \\ &\leq \sum_{i=1}^M \max_{i \neq j} \|y_i - y_j\|^p \mu\left(\mathbb{L}_i(\mathbf{g}^*) \setminus \mathbb{L}_i(\mathbf{g})\right) \\ &\lesssim M^2 \|\mathbf{g}^* - \mathbf{g}\|_\infty, \end{aligned}$$

where we used $\mu(\mathbb{L}_i(\mathbf{g}) \setminus \mathbb{L}_i(\mathbf{g}^*)) \lesssim M\|\mathbf{g} - \mathbf{g}^*\|$ for the last line.

□

Lemma B.14. *For any $\varepsilon > 0$, there exists $\mathbf{g}^* \in K_\varepsilon$ and $d_\varepsilon > 0$ such that $B(\mathbf{g}^*, d_\varepsilon) \subset K_\varepsilon$.*

Proof. This result is a simple application of the results on the quantitative stability of Laguerre cells stated in Proposition B.4 and Proposition B.13. □

Proposition B.15. *(Hessian and Local strong convexity). Under Assumptions (A1) or (B1-3), H is twice differentiable. Moreover, there exists a constant $\lambda > 0$ and a radius $r > 0$, such that for any $\mathbf{g} \in \mathbb{R}^M$, $\|\mathbf{g}^* - \mathbf{g}\|_v \leq r$ implies that the second smallest value of the Hessian $\nabla^2 H(\mathbf{g})$ is lower bounded by λ.*

Proof. The fact that H is twice differentiable was already known in the compact case and is proven in Proposition B.5 for the non compact case under (B1-3).

Under assumptions (A1) and using Proposition B.2 or under assumptions (B1-3) and using Proposition B.12, H is strongly convex on $K_{\frac{1}{2}w_{\min}}$. Applying Lemma B.14 concludes the proof. □

Proposition B.16. *There exists $\eta > 0$, such that uniformly in $\mathbf{g} \in \mathcal{C}$, we have*

$$\langle \nabla H(\mathbf{g}), \mathbf{g} - \mathbf{g}^* \rangle_v \geq \eta \|\mathbf{g} - \mathbf{g}^*\|_v^2 .$$

Proof. Observe that, by Proposition B.12, H is locally strongly convex on the orthogonal of $\mathbf{1}$, on the set

$$K_{\frac{1}{2}w_{\min}} = \left\{\mathbf{g} \,:\, \forall i \in [\![1, M]\!], \mu(\mathbb{L}_i(\mathbf{g})) \geq \frac{1}{2} w_{\min}\right\}.$$

Therefore, for any $\mathbf{g} \in K_{\frac{1}{2}w_{\min}}$, we have $\langle \nabla H(\mathbf{g}), \mathbf{g} - \mathbf{g}^* \rangle_v \geq \lambda \|\mathbf{g} - \mathbf{g}^*\|_v^2$.

Now, suppose $\mathbf{g} \in \mathcal{C} \setminus K_{\frac{1}{2}w_{\min}}$. Using Lemma B.14, we know that there exists $d_0 > 0$ such that $B(\mathbf{g}^*, d_0) \subset K_{\frac{1}{2}w_{\min}}$. Therefore, by the convexity of H, there exists $c > 0$ such that

$$\langle \nabla H(\mathbf{g}), \mathbf{g} - \mathbf{g}^* \rangle_v \geq H(\mathbf{g}) - H(\mathbf{g}^*) > c .$$

Defining

$$\lambda' := \inf_{\mathbf{g}\in\mathcal{C}\setminus K_{\frac{1}{2}w_{\min}}} \left\{ \frac{H(\mathbf{g})-H(\mathbf{g}^*)}{\|\mathbf{g}-\mathbf{g}^*\|_v^2} \right\} > c/\mathrm{Diam}_2(\mathcal{C})^2,$$

we have for any $\mathbf{g}\in\mathcal{C}\setminus K_{\frac{1}{2}w_{\min}}$

$$\langle \nabla H(\mathbf{g}), \mathbf{g}-\mathbf{g}^*\rangle_v \geq \lambda' \|\mathbf{g}-\mathbf{g}^*\|_v^2 \,.$$

Taking $\eta = \min\{\lambda, \lambda'\}$ concludes the proof. □

C Proofs of the convergence rates of PSGD

C.1 Fast convergence rates for MTW costs

C.1.1 Convergence of the non-averaged iterates.

Theorem 4.5 (Non averaged iterates). Under Assumptions (A1) or (B1-3) and for any decay step of the form $\gamma_n = \gamma_1/n^b$ with $\gamma_1 > 0, b\in(1/2,1)$, we have the convergence rate

$$\mathbb{E}[\|\mathbf{g}_n-\mathbf{g}^*\|_v^2] = \mathcal{O}\left(\frac{\gamma_n}{\eta}\right) \,.$$

Proof. By definition of the gradient step at time $n+1$, sampling $X_{n+1}\sim\mu$ and since $\mathbf{g}^*\in\mathcal{C}$, we have $\|\nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1})\|_v \leq 2$ a.s. and

$$\begin{aligned}
\|\mathbf{g}_{n+1}-\mathbf{g}^*\|_v^2 &= \|\mathrm{Proj}_{\mathcal{C}}(\mathbf{g}_n - \gamma_{n+1}\nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1})) - \mathbf{g}^*\|_v^2 \\
&\leq \|\mathbf{g}_n - \gamma_{n+1}\nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}) - \mathbf{g}^*\|_v^2 \\
&\leq \left(\|\mathbf{g}_n-\mathbf{g}^*\|_v^2 - 2\gamma_{n+1}\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n-\mathbf{g}^*\rangle_v + \gamma_{n+1}^2\|\nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1})\|_v^2\right) \\
&\leq \left(\|\mathbf{g}_n-\mathbf{g}^*\|_v^2 - 2\gamma_{n+1}\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n-\mathbf{g}^*\rangle_v + 4\gamma_{n+1}^2\right) \,. \qquad (14)
\end{aligned}$$

Using Proposition B.16, we have for any $\mathbf{g}\in\mathcal{C}$, $\langle \nabla H(\mathbf{g}_n), \mathbf{g}_n-\mathbf{g}^*\rangle_v \geq \eta\|\mathbf{g}_n-\mathbf{g}^*\|_v^2$. Therefore, taking the conditional expectation, we obtain

$$\mathbb{E}\left[\|\mathbf{g}_{n+1}-\mathbf{g}^*\|_v^2 \mid \mathcal{F}_n\right] \leq \|\mathbf{g}_n-\mathbf{g}^*\|_v^2(1-2\eta\gamma_{n+1}) + 4\gamma_{n+1}^2 \,.$$

Taking the expectation and applying Lemma F.3 with $\delta_n = \mathbb{E}[\|\mathbf{g}_n-\mathbf{g}^*\|_v^2]$ and $m_n = 4\gamma_n$ gives

$$\mathbb{E}\left[\|\mathbf{g}_{n+1}-\mathbf{g}^*\|_v^2\right] \leq \exp\left(-2\eta\sum_{k=\lceil n/2\rceil}^{n}\gamma_k\right)\left(\sum_{n=n_0}^{n} 4\gamma_n^2 + \mathbb{E}[\|\mathbf{g}_{n_0}-\mathbf{g}^*\|_v^2]\right) + \frac{4}{\eta}\gamma_{\lceil n/2\rceil-1}$$

where $n_0 = \min\{n\in\mathbb{N}, \eta\gamma_{n+1}\leq 1\}$. Remark that the exponential term converges exponentially fast. Indeed, we have $\sum_{k=\lceil n/2\rceil}^{n}\gamma_k \gtrsim n^{1-b}$ with $1-b>0$. Moreover, $\|\mathbf{g}_{n_0}-\mathbf{g}^*\|_v^2 \leq \mathrm{Diam}_2(\mathcal{C})^2$. Therefore, since for all $n\geq 2$, $\gamma_{\lceil n/2\rceil-1}\leq 2^b\gamma_n$, we have the desired convergence rate

$$\mathbb{E}\left[\|\mathbf{g}_{n+1}-\mathbf{g}^*\|_v^2\right] \lesssim \frac{\gamma_n}{\eta}$$

which concludes the proof.

□

C.1.2 Convergence rate for higher order moments of the non-averaged iterates.

We prove here the convergence rate of higher order moments of the error $\|\mathbf{g}_n-\mathbf{g}^*\|_v$. This convergence will be useful for the convergence rate of the averaged iterates of PSGD. While this proposition directly proves Theorem 4.4 by the use of Jensen's inequality, the proof is slightly more cumbersome so we decided to make a separate case.

Proposition C.1. *Under Assumptions (A1) or (B1-3) and for any decay step of the form $\gamma_n = \gamma_1/n^b$ with $\gamma_1 > 0, b \in (1/2, 1)$ and $p \in \{1, 2, 3\}$, we have the convergence rate*

$$\mathbb{E}\left[\|\mathbf{g}_n - \mathbf{g}^*\|^p\right] \lesssim \frac{\gamma_n^p}{\eta^p}$$

where $(\mathbf{g}_n)$ is the sequence of non-averaged iterates of PSGD.

Proof. By definition of the gradient step at time $n+1$, sampling $X_{n+1} \sim \mu$ and using inequality (14),

$$\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6 \leq \left(\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 - 2\gamma_{n+1}\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v + 4\gamma_{n+1}^2\right)^3 .$$

Using that $(A+B+C)^3 = \sum_{a+b+c=3} \frac{3!}{a!b!c!} A^a B^b C^c$, we obtain

$$\begin{aligned}
\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6 \leq \|\mathbf{g}_n - \mathbf{g}^*\|_v^6 & \\
& - 6\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 \gamma_{n+1}\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v \\
& + 12\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 \gamma_{n+1}^2 \\
& + 12\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 \gamma_{n+1}^2\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^2 \\
& - 48\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 \gamma_{n+1}^3\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v \\
& + 48\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 \gamma_{n+1}^4 \\
& - 8\gamma_{n+1}^3\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^3 \\
& + 48\gamma_{n+1}^4\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^2 \\
& - 96\gamma_{n+1}^5\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v \\
& + 2^6\gamma_{n+1}^6.
\end{aligned}$$

Taking the conditional expectation and already omitting some negative terms thanks to $\langle \nabla H(\mathbf{g}_n), \mathbf{g}_n - \mathbf{g}^*\rangle_v \geq 0$, which follows from the fact that H is convex and $\mathbf{g}^*$ is a minimizer, gives the simplification

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6 \mid \mathcal{F}_n\right] \leq \|\mathbf{g}_n - \mathbf{g}^*\|_v^6 & \\
& - 6\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 \gamma_{n+1}\left\langle \nabla H(\mathbf{g}_n), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v \\
& + 12\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 \gamma_{n+1}^2 \\
& + 12\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 \gamma_{n+1}^2 \mathbb{E}\left[\left\langle \nabla h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^2 |\mathcal{F}_n\right] \\
& + 48\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 \gamma_{n+1}^4 \\
& + 48\gamma_{n+1}^4 \mathbb{E}\left[\left\langle \nabla h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^2 |\mathcal{F}_n\right] \\
& + 2^6\gamma_{n+1}^6 \\
& - 8\gamma_{n+1}^3 \mathbb{E}\left[\left\langle \nabla_{\mathbf{g}} h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v^3 |\mathcal{F}_n\right] .
\end{aligned}$$

Using Proposition B.16, we have for any $\mathbf{g} \in \mathcal{C}$, $\left\langle \nabla H(\mathbf{g}_n), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v \geq \eta\|\mathbf{g}_n - \mathbf{g}^*\|_v^2$. The Cauchy-Schwarz inequality gives $|\left\langle \nabla h(\mathbf{g}_n, X_{n+1}), \mathbf{g}_n - \mathbf{g}^*\right\rangle_v| \leq 2\|\mathbf{g}_n - \mathbf{g}^*\|_v$. Combining these two inequalities we obtain

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6 \mid \mathcal{F}_n\right] \leq \|\mathbf{g}_n - \mathbf{g}^*\|_v^6(1 - 6\eta\gamma_{n+1}) + 60\gamma_{n+1}^2\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 + 240\gamma_{n+1}^4\|\mathbf{g}_n - \mathbf{g}^*\|_v^2 & \\
+ 2^6\gamma_{n+1}^6 + 64\gamma_{n+1}^3\|\mathbf{g}_n - \mathbf{g}^*\|_v^3 & .
\end{aligned}$$

Using Young's (generalized) inequality $ab = ac\frac{b}{c} \leq \frac{(ac)^p}{p} + \frac{b^q}{c^q q}$ for $c \neq 0, \frac{1}{p} + \frac{1}{q} = 1$ and applying it to $60\gamma_{n+1}\|\mathbf{g}_n - \mathbf{g}^*\|_v^4$ with $c = \left(\frac{2}{3\eta}\right)^{2/3}, p = 3, q = \frac{3}{2}$ gives $60\gamma_{n+1}\|\mathbf{g}_n - \mathbf{g}^*\|_v^4 \leq \frac{60^3\gamma_{n+1}^3}{3}$ ·

$\left(\frac{2}{3\eta}\right)^2 + \eta\|\mathbf{g}_n - \mathbf{g}^*\|_v^6$. Analogously, one has $64\gamma_{n+1}^2\|\mathbf{g}_n - \mathbf{g}^*\|_v^3 \le \frac{2^{10}}{\eta}\gamma_{n+1}^4 + \eta\|\mathbf{g}_n - \mathbf{g}^*\|_v^6$. Thus, taking the expectation, we have

$$\mathbb{E}[\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6] \le \mathbb{E}[\|\mathbf{g}_n - \mathbf{g}^*\|_v^6](1 - 4\eta\gamma_{n+1}) + \gamma_{n+1}^4\left(240 \cdot \mathrm{Diam}_2(\mathcal{C})^2 + \frac{32}{\eta^2}\cdot 10^3\right) + \frac{2^{10}}{\eta}\gamma_{n+1}^5 + 64\gamma_{n+1}^6,$$

where the terms involving $\mathrm{Diam}_2(\mathcal{C})$ appears from the crude bound $\|\mathbf{g}_n - \mathbf{g}\|_v \le \mathrm{Diam}_2(\mathcal{C})$.

Applying Lemma F.3 in a similar way as in the proof of Theorem 4.4 gives $\mathbb{E}[\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^6] \lesssim \frac{\gamma_n^3}{\eta^3}$, so by Jensen's inequality, we conclude

$$\mathbb{E}[\|\mathbf{g}_{n+1} - \mathbf{g}^*\|_v^{2p}] \lesssim \frac{\gamma_n^p}{\eta^p} \quad \text{for } p \in \{1,2,3\}\,.$$

$\square$

C.2 Convergence of the averaged iterates.

Theorem 4.4 (Averaged iterates) Under Assumptions (A1) or (B1–B3), and assuming that f_μ is α-Hölder with $\alpha \in (0,1]$, for any decay schedule of the form $\gamma_n = \gamma_1/n^b$ with $\gamma_1 > 0$ and $b \in \left(\frac{1}{1+\alpha}, 1\right)$, we have the convergence rate

$$\mathbb{E}[\|\overline{\mathbf{g}}_n - \mathbf{g}^*\|_v^2] = \mathcal{O}\left(1/n\right).$$

Without assuming f_μ to be α-Hölder, and for $b \in (1/2, 1)$, we still obtain

$$\mathbb{E}[\|\overline{\mathbf{g}}_n - \mathbf{g}^*\|_v^2] = \mathcal{O}\left(1/n^b\right).$$

Proof. For this proof, we introduce the additional following notation:

For any $c > 0$ we define the function $t \mapsto \Psi_c(t)$ such that

$$\sum_{t=1}^{T} t^{-c} \le \Psi_c(T) := \begin{cases} 1 + \ln(T+1) & \text{if } c = 1, \\ \frac{2c-1}{c-1} & \text{if } c > 1, \\ 1 + \frac{1}{1-c}(T+1)^{1-c} & \text{if } c < 1. \end{cases} \tag{15}$$

We start by a decomposition of the gradient step, already present in [18]. We define the differences

$$\begin{aligned} p_k &:= \mathrm{Proj}_{\mathcal{C}}\left(\mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)\right) - \left(\mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)\right), \\ \xi_{k+1} &:= \nabla H\left(\mathbf{g}_k\right) - \nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right), \\ \delta_k &:= \nabla H\left(\mathbf{g}_k\right) - \nabla^2 H(\mathbf{g}^*)\left(\mathbf{g}_k - \mathbf{g}^*\right). \end{aligned}$$

Noting I_M the identity matrix in $\mathcal{M}_M(\mathbb{R})$, we observe that, by incorporating each introduced term sequentially, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{g}_{k+1} - \mathbf{g}^* &= \mathrm{Proj}_{\mathcal{C}}\left(\mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h(\mathbf{g}_k, X_{k+1})\right) - \mathbf{g}^* \\ &= \mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h(\mathbf{g}_k, X_{k+1}) - \mathbf{g}^* - p_k \\ &= \mathbf{g}_k - \gamma_{k+1}\nabla H(\mathbf{g}_k, X_{k+1}) - \mathbf{g}^* + \gamma_{k+1}\xi_{k+1} - p_k \\ &= \left(I_M - \gamma_{k+1}\nabla^2 H(\mathbf{g}^*)\right)\left(\mathbf{g}_k - \mathbf{g}_*\right) - \gamma_{k+1}\delta_k + \gamma_{k+1}\xi_{k+1} + p_k\,. \end{aligned}$$

Thus, we have that

$$\nabla^2 H(\mathbf{g}^*)\left(\mathbf{g}_k - \mathbf{g}^*\right) = \frac{\mathbf{g}_k - \mathbf{g}_{k+1}}{\gamma_{k+1}} - \delta_k + \xi_{k+1} + \frac{p_k}{\gamma_{k+1}}\,.$$

Observing that $\frac{1}{n+1}\sum_{k=0}^n(\mathbf{g}_k - \mathbf{g}^*) = \overline{\mathbf{g}}_n - \mathbf{g}^*$, we have the following decomposition of the averaged iterates

$$\nabla^2 H(\mathbf{g}^*)\left(\overline{\mathbf{g}}_n - \mathbf{g}^*\right) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{\mathbf{g}_k - \mathbf{g}_{k+1}}{\gamma_{k+1}} - \frac{1}{n+1}\sum_{k=0}^{n}\delta_k + \frac{1}{n+1}\sum_{k=0}^{n}\xi_{k+1} + \frac{1}{n+1}\sum_{k=0}^{n}\frac{p_k}{\gamma_{k+1}}\,.$$

We will now give the convergence rate of each sum.

• Convergence rate for $\frac{1}{n+1}\sum_{k=0}^{n}\frac{\mathbf{g}_k-\mathbf{g}_{k+1}}{\gamma_{k+1}}$.

$$\begin{aligned}\sum_{k=0}^{n}\frac{\mathbf{g}_k-\mathbf{g}_{k+1}}{\gamma_{k+1}} &= \sum_{k=0}^{n}\frac{(\mathbf{g}_k-\mathbf{g}^*)-(\mathbf{g}_{k+1}-\mathbf{g}^*)}{\gamma_{k+1}} \\ &= \sum_{k=0}^{n}\frac{\mathbf{g}_k-\mathbf{g}^*}{\gamma_{k+1}} - \sum_{k=0}^{n}\frac{\mathbf{g}_{k+1}-\mathbf{g}^*}{\gamma_{k+1}} \\ &= \sum_{k=1}^{n}\left(\frac{1}{\gamma_{k+1}}-\frac{1}{\gamma_k}\right)(\mathbf{g}_k-\mathbf{g}^*) + \frac{\mathbf{g}_0-\mathbf{g}^*}{\gamma_1} - \frac{\mathbf{g}_{n+1}-\mathbf{g}^*}{\gamma_{n+1}}.\end{aligned}$$

Remark that $\gamma_{n+1}^{-1}-\gamma_n^{-1}\leq 2\gamma_1^{-1}n^{b-1}$. Using Minkowski's inequality and that, by Theorem 4.4 (non-averaged iterates), $\mathbb{E}\left[\|\mathbf{g}_n-\mathbf{g}^*\|_v^2\right]\lesssim\frac{\gamma_1}{\eta}(n+1)^{-b}$,

$$\mathbb{E}\left[\left\|\sum_{k=0}^{n}\frac{\mathbf{g}_k-\mathbf{g}_{k+1}}{\gamma_{k+1}}\right\|_v^2\right]^{\frac{1}{2}} \lesssim \frac{1}{\eta}\Psi_{1-b/2}(n+1) + \mathrm{Diam}_2(\mathcal{C})\gamma_1^{-1} + \frac{1}{\sqrt{\gamma_1\eta}}(n+1)^{b/2}\ .$$

We thus have the convergence rate

$$\frac{1}{n+1}\mathbb{E}\left[\left\|\sum_{k=0}^{n}\frac{\mathbf{g}_k-\mathbf{g}_{k+1}}{\gamma_{k+1}}\right\|_v^2\right]^{\frac{1}{2}} \lesssim \frac{1}{\eta(n+1)^{1-b/2}}\ .$$

• Convergence rate for $\frac{1}{n+1}\sum_{k=0}^{n}\delta_k$.

We recall that $\delta_k=\nabla H(\mathbf{g}_k)-\nabla^2 H(\mathbf{g}^*)(\mathbf{g}_k-\mathbf{g}^*)$ and that the Hessian using either Theorem B.3 or Proposition 4.1, depending on our setting, there exists a ball $B(\mathbf{g}^*,d_1)$ with $d_1>0$ where H is α-Hölder. Therefore, applying Lemma F.4, if $\mathbf{g}_k\in B(\mathbf{g}^*,d_1)$, we have

$$\|\delta_k\|\lesssim\|\mathbf{g}_k-\mathbf{g}^*\|_v^{1+\alpha}\ .$$

Otherwise, since the Hessian, whose expression is provided in Proposition B.5, is uniformly bounded by an application of Lemma F.2, there exists a constant C_δ such that for any $\mathbf{g}\in\mathcal{C}$, $\|\nabla H(\mathbf{g})-\nabla^2 H(\mathbf{g}^*)(\mathbf{g}-\mathbf{g}^*)\|\leq C_\delta$.

Since $\mathbb{P}(\mathbf{g}_k\notin B(\mathbf{g}^*,d_1))=\mathbb{P}(\|\mathbf{g}_k-\mathbf{g}_*\|>d_1)$, using Markov's inequality gives

$$\begin{aligned}\mathbb{E}[\|\delta_k\|_v^2] &= \mathbb{E}[\|\delta_k\|_v^2\mathbf{1}_{\mathbf{g}_k\in B(\mathbf{g}^*,d_1)}] + \mathbb{E}[\|\delta_k\|_v^2\mathbf{1}_{\mathbf{g}_k\notin B(\mathbf{g}^*,d_1)}] \\ &\lesssim \mathbb{E}\left[\|\mathbf{g}_k-\mathbf{g}^*\|_v^{2+2\alpha}\right] + \frac{C_\delta^2}{d_1^{2+2\alpha}}\mathbb{E}[\|\mathbf{g}_k-\mathbf{g}^*\|_v^{2+2\alpha}] \\ &\lesssim \mathbb{E}[\|\mathbf{g}_k-\mathbf{g}^*\|_v^{2+2\alpha}]\ .\end{aligned}$$

Therefore, using Minkowski's inequality, we have

$$\begin{aligned}\frac{1}{n+1}\mathbb{E}\left[\left\|\sum_{k=0}^{n}\delta_k\right\|_v^2\right]^{\frac{1}{2}} &\lesssim \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\eta^{\frac{1+\alpha}{2}}}\gamma_{k+1}^{\frac{1+\alpha}{2}} \\ &\leq \frac{1}{\eta^{\frac{1+\alpha}{2}}(n+1)}\Psi_{\frac{b+\alpha b}{2}} \\ &\lesssim \frac{1}{\eta^{\frac{1+\alpha}{2}}(n+1)^{\frac{b+\alpha b}{2}}}\ .\end{aligned}$$

• Convergence rate for $\frac{1}{n+1}\sum_{k=0}^{n}\xi_{k+1}$.

We recall that $\xi_{k+1} = \nabla H(\mathbf{g}_k) - \nabla_{\mathbf{g}} h(\mathbf{g}_k, X_{k+1})$ and thus $\mathbb{E}[\xi_{k+1}] = 0$.

Observe that

$$\mathbb{E}\left[\left\|\sum_{k=0}^{n} \xi_{k+1}\right\|_v^2\right] = \mathbb{E}\left[\left\|\sum_{k=0}^{n-1} \xi_{k+1}\right\|_v^2 + 2\left\langle \sum_{k=0}^{n-1} \xi_{k+1}, \xi_{n+1}\right\rangle_v + \|\xi_{n+1}\|_v^2\right]$$

with

$$\mathbb{E}\left[\left\langle \sum_{k=0}^{n-1} \xi_{k+1}, \xi_{n+1}\right\rangle_v\right] = \mathbb{E}\left[\left\langle \sum_{k=0}^{n-1} \xi_{k+1}, \mathbb{E}\left[\xi_{n+1}|\mathcal{F}_n\right]\right\rangle_v\right] = 0\,.$$

Thus, since for all k, $\mathbb{E}\left[\|\xi_k\|^2\right] \leq 4$, we have the convergence rate

$$\frac{1}{n+1}\mathbb{E}\left[\left\|\sum_{k=0}^{n} \xi_{k+1}\right\|_v^2\right]^{\frac{1}{2}} \leq \frac{2}{\sqrt{n+1}}\,.$$

• **Convergence rate for $\frac{1}{n+1}\sum_{k=0}^{n} \frac{p_k}{\gamma_{k+1}}$.**

Take d_0 such that $B(\mathbf{g}^*, d_0) \subset \mathcal{C}$. Using the notation $\nabla_k := \nabla_{\mathbf{g}} h(\mathbf{g}_k, X_{k+1})$ for conciseness, we obtain

$$\begin{aligned}
\mathbb{E}\left[\|p_k\|_v^2\right] &= \mathbb{E}\left[\left\|\mathrm{Proj}_{\mathcal{C}}\left(\mathbf{g}_k - \gamma_{k+1}\nabla_k\right) - \left(\mathbf{g}_k - \gamma_{k+1}\nabla_k\right)\right\|_v^2\right] \\
&= \mathbb{E}\left[\left\|\mathrm{Proj}_{\mathcal{C}}\left(\mathbf{g}_k - \gamma_{k+1}\nabla_k\right) - \left(\mathbf{g}_k - \gamma_{k+1}\nabla_k\right)\right\|_v^2 \mathbf{1}_{\mathbf{g}_k - \gamma_{k+1}\nabla_k \notin \mathcal{C}}\right]
\end{aligned}$$

Since for any $y \in \mathcal{C}$, one has $\|x - \mathrm{Proj}_{\mathcal{C}}(x)\|_v \leq \|x - y\|_v$, taking $y = \mathbf{g_k}$, and since $\mathbf{g}_k - \gamma_{k+1}\nabla_k \notin \mathcal{C}$ is satisfied only if $\|\mathbf{g}_k - \gamma_{k+1}\nabla_k - \mathbf{g}^*\|_v > d_0$, we have

$$\begin{aligned}
\mathbb{E}\left[\|p_k\|_v^2\right] &\leq \mathbb{E}\left[\|\gamma_{k+1}\nabla_k\|_v^2 \mathbf{1}_{\|\mathbf{g}_k - \gamma_{k+1}\nabla_k - \mathbf{g}^*\|_v > d_0}\right] \\
&\leq 4\gamma_{k+1}^2 \frac{\mathbb{E}\left[\|\mathbf{g}_k - \gamma_{k+1}\nabla_k - \mathbf{g}^*\|_v^4\right]}{d_0^4} \\
&\leq \frac{\gamma_{k+1}^2}{d_0^4}\left(2^5\mathbb{E}\left[\|\mathbf{g}_k - \mathbf{g}^*\|_v^4\right] + 2^9\gamma_{k+1}^4\right) \\
&\lesssim \frac{1}{\eta^2}\gamma_{k+1}^4
\end{aligned}$$

Where we used Markov's inequality and the inequality $(A+B)^4 \leq 2^3(A^4 + B^4)$, for all $A, B \in \mathbb{R}$ and that by Proposition C.1, $\mathbb{E}\left[\|\mathbf{g}_n - \mathbf{g}^*\|^4\right] \lesssim \frac{\gamma_1^2}{\eta^2}(n+1)^{-2b}$.

We thus have

$$\begin{aligned}
\frac{1}{n+1}\mathbb{E}\left[\left\|\sum_{k=0}^{n} \frac{p_k}{\gamma_{k+1}}\right\|_v^2\right]^{\frac{1}{2}} &\lesssim \frac{1}{n+1}\sum_{k=0}^{n} \frac{\gamma_{k+1}}{\eta} \\
&\lesssim \frac{\gamma_1}{\eta(n+1)^b}\,.
\end{aligned}$$

• **Conclusion.**

Using the convergence rate of all our terms, Cauchy-Schwarz inequality and that $(A+B)^2 \leq 2(A^2 + B^2)$ for all $A, B \in \mathbb{R}$ we conclude

$$\begin{aligned}
\mathbb{E}\left[\left\|\nabla^2 H(\mathbf{g}^*)\left(\overline{\mathbf{g}}_n - \mathbf{g}^*\right)\right\|_v^2\right] &= \mathbb{E}\left[\left\|\frac{1}{n+1}\sum_{k=0}^{n} \frac{\mathbf{g}_k - \mathbf{g}_{k+1}}{\gamma_{k+1}} - \frac{1}{n+1}\sum_{k=0}^{n}\delta_k + \frac{1}{n+1}\sum_{k=0}^{n}\xi_{k+1} + \frac{1}{n+1}\sum_{k=0}^{n}\frac{p_k}{\gamma_{k+1}}\right\|_v^2\right] \\
&\lesssim \frac{1}{\eta^2(n+1)^{2-b}} + \frac{1}{\eta^{\frac{1+\alpha}{\alpha}}(n+1)^{b+\alpha b}} + \frac{1}{n+1} + \frac{\gamma_1^4}{\eta^2(n+1)^{2b}}\,.
\end{aligned}$$

Since $b \in (\frac{1}{2}, 1)$ and $b \in (\frac{1}{1+\alpha}, 1)$ the limiting term is $\frac{1}{n+1}$ and we have

$$\mathbb{E}\left[\left\|\nabla^2 H(\mathbf{g}^*)\left(\overline{\mathbf{g}}_n - \mathbf{g}^*\right)\right\|_v^2\right] \lesssim \frac{1}{n+1}\,.$$

Finally, observe that there is an orthogonal matrix U such that $\nabla^2 H(\mathbf{g}^*) = U \operatorname{diag}(\lambda_1, \ldots, \lambda_{M-1}, 0)\, U^\top$. Therefore, noting by abuse of notation

$$\left(\nabla^2 H(\mathbf{g}^*)\right)^{-1} = U \operatorname{diag}\left(\lambda_1^{-1}, \ldots, \lambda_{M-1}^{-1}, 0\right) U^\top$$

the inverse of ∇_k^2 in the space $\mathrm{Vect}(\mathbf{1}_M)^\perp$ we finally have

$$\mathbb{E}[\|\overline{\mathbf{g}}_n - \mathbf{g}^*\|_v^2] \lesssim \frac{1}{\lambda^2(n+1)}$$

where $\lambda = \min_{j \in [\![1, M-1]\!]} \lambda_j > 0$ by either Theorem B.3 or Proposition B.12. $\square$

C.3 Convergence rate of PSGD in the general setting

Theorem 3.2 (PSGD in the general setting) Assuming that the semi-dual problem (4) admits at least one solution $\mathbf{g}^*$ and that there exists a compact set K such that $\mu(K) \geq 1 - \frac{1}{2} w_{\min}$, choosing the learning rate $\gamma_n = \gamma_1 / n^b$ with $\gamma_1 = \frac{\mathrm{Diam}(\mathcal{C})}{2\sqrt{2}}$ and $b = \frac{1}{2}$, we obtain

$$\mathbb{E}\left[H\left(\overline{\mathbf{g}}_k\right) - H\left(\mathbf{g}^*\right)\right] \leq \frac{4\sqrt{2}\,\mathrm{Diam}(\mathcal{C})}{\sqrt{n}}.$$

Proof. Define $\gamma_k = \frac{\gamma_1}{\sqrt{k}}$ for $\gamma_1 > 0, k \geq 1$ and denote by $\mathbf{g}^* \in \mathcal{C}$ a minimizer of the functional H. Thanks to Jensen's inequality coupled with the fact that no matter $\mathbf{g} \in \mathbb{R}^M$: $H(\mathbf{g}) - H(\mathbf{g}^*) \leq \nabla H(\mathbf{g})^\top (\mathbf{g} - \mathbf{g}^*)$, it comes

$$\mathbb{E}\left[H\left(\overline{g}_n\right) - H\left(\mathbf{g}^*\right)\right] \leq \mathbb{E}\left[\frac{1}{n+1}\sum_{k=0}^{n} H\left(\mathbf{g}_k\right) - H\left(\mathbf{g}^*\right)\right] \leq \frac{1}{n+1}\mathbb{E}\left[\sum_{k=0}^{n} \nabla H\left(\mathbf{g}_k\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right].$$

Then, since no matter $k,\ X_{k+1} \sim \mu$, we have $\mathbb{E}\left[\nabla_{\mathbf{g}} h(\mathbf{g}_k, X_{k+1}) \mid \mathcal{F}_k\right] = \nabla H(\mathbf{g}_k)$, we have

$$\mathbb{E}\left[H\left(\overline{g}_n\right) - H\left(\mathbf{g}^*\right)\right] \leq \frac{1}{n+1}\mathbb{E}\left[\sum_{k=0}^{n} \nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right]. \tag{16}$$

We will proceed to bound the right hand side of this inequality.

By definition of $\mathbf{g}_{k+1}$, and since $\mathbf{g}^* \in \mathcal{C}$ and $\mathrm{Proj}_{\mathcal{C}}$ is 1-Lipschitz, it comes

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{g}_{k+1} - \mathbf{g}^*\|^2\right] &\leq \mathbb{E}\left[\left\|\mathrm{Proj}_{\mathcal{C}}\left(\mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)\right) - \mathbf{g}^*\right\|^2\right] \\
&\leq \mathbb{E}\left[\left\|\mathbf{g}_k - \gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right) - \mathbf{g}^*\right\|^2\right] \\
&\leq \mathbb{E}\left[\|\mathbf{g}_k - \mathbf{g}^*\|^2 + \gamma_{k+1}^2 \left\|\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)\right\|^2 - 2\gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right].
\end{aligned}$$

In addition, since $\|\nabla_{\mathbf{g}} h(\mathbf{g}, X)\| \leq 2$ a.s. no matter $\mathbf{g} \in \mathbb{R}^M$ and $X \in \mathbb{R}^d$,

$$\mathbb{E}\left[2\gamma_{k+1}\nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right] \leq \mathbb{E}\left[\|\mathbf{g}_k - \mathbf{g}^*\|^2 - \|\mathbf{g}_{k+1} - \mathbf{g}^*\|^2 + 4\gamma_{k+1}^2\right].$$

Then, with the help of Abel's summation formula,

$$\begin{aligned}
2\mathbb{E}\left[\sum_{k=0}^{n} \nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right] &\leq \mathbb{E}\left[\sum_{k=0}^{n} \frac{\|\mathbf{g}_k - \mathbf{g}^*\|^2 - \|\mathbf{g}_{k+1} - \mathbf{g}^*\|^2}{\gamma_{k+1}}\right] + 4\sum_{k=0}^{n}\gamma_{k+1} \\
&\leq \mathbb{E}\left[\sum_{k=1}^{n} \|\mathbf{g}_k - \mathbf{g}^*\|^2 \left(\frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k}\right)\right] - \frac{\|\mathbf{g}_{k+1} - \mathbf{g}^*\|^2}{\gamma_{k+1}} + \frac{\|\mathbf{g}_0 - \mathbf{g}^*\|^2}{\gamma_1} + 4\sum_{k=0}^{n}\gamma_{k+1}
\end{aligned}$$

Then, since for all k, $\|\mathbf{g}_k - \mathbf{g}^*\| \leq \mathrm{Diam}_2(\mathcal{C})$, it comes

$$\begin{aligned}
2\mathbb{E}\left[\sum_{k=0}^{n} \nabla_{\mathbf{g}} h\left(\mathbf{g}_k, X_{k+1}\right)^\top \left(\mathbf{g}_k - \mathbf{g}^*\right)\right] &\leq \mathrm{Diam}_2(\mathcal{C})^2 \sum_{k=1}^{n}\left(\frac{1}{\gamma_{k+1}} - \frac{1}{\gamma_k}\right) + \frac{D^2}{\gamma_1} + 4\sum_{k=1}^{n}\gamma_k \\
&\leq \frac{\mathrm{Diam}_2(\mathcal{C})^2}{\gamma_{n+1}} + 4\sum_{k=1}^{n}\gamma_k \\
&\leq \frac{\mathrm{Diam}_2(\mathcal{C})^2}{\gamma_1}\sqrt{n+1} + 8\gamma_1\sqrt{n+1}
\end{aligned}$$

Using the inequality (16) we obtain

$$\mathbb{E}\left[H\left(\overline{g}_n\right) - H\left(\mathbf{g}^*\right)\right] \leq \frac{1}{\sqrt{n+1}}\left(\frac{\mathrm{Diam}_2(\mathcal{C})^2}{\gamma_1} + 8\gamma_1\right).$$

The best constant γ_1 is then $\gamma_1 = \frac{\mathrm{Diam}_2(\mathcal{C})}{2\sqrt{2}}$, but no matter $\gamma_1 > 0$, we have the desired convergence rate

$$\mathbb{E}\left[H\left(\overline{g}_n\right) - H\left(\mathbf{g}^*\right)\right] = \mathcal{O}(1/\sqrt{n}).$$

□

D Proof of Lemma 3.1: Localisation of a projection set

Lemma 3.1 (Existence of a projection set) As soon as the semi-dual problem is well-posed, there exists a minimizer $\mathbf{g}^*$ in the set

$$\mathcal{C} := \left\{\mathbf{g} \in \{0\} \times \mathbb{R}^{M-1} \mid |g_j| \leq \|c\|_{K,\infty}\right\}$$

where $\|c\|_{K,\infty} := \sup_{x\in K, j\in[\![1,M]\!]} |c(x, y_j)|$ for any compact set K satisfying $\mu(K) \geq 1 - \frac{1}{2}\min w_j$.

Proof. By the first order condition, the minimizer of H satisfies $\mu(\mathbb{L}_j(\mathbf{g})) = w_j$ for all $j \in [\![1, M]\!]$. In particular, one can restrict the search set for an optimal potential to the set of potentials defined by

$$\mathcal{L} := \left\{\mathbf{g} \in \mathbb{R}^M : \forall j \in [\![1, M]\!], \mu(\mathbb{L}_j(\mathbf{g})) \geq \frac{2}{3}w_{\min}\right\}.$$

Let us show that this set is contained in an L^∞ ball with an explicit radius. Consider any compact set K such that $\mu(K) \geq 1 - \frac{1}{2}w_{\min}$. For $\mathbf{g} \in \mathcal{L}$ and any $j \in [\![1, M]\!]$ we get that

$$\begin{aligned}
\mu(\mathbb{L}_j(\mathbf{g}) \cap K) &= 1 - \mu((\mathbb{R}^M \setminus \mathbb{L}_j(\mathbf{g})) \cup (\mathbb{R}^M \setminus K)) \\
&\geq 1 - \mu((\mathbb{R}^M \setminus \mathbb{L}_j(\mathbf{g}))) - \mu((\mathbb{R}^M \setminus K)) \\
&= 1 - (1 - \mu(\mathbb{L}_j(\mathbf{g}))) - (1 - \mu(K)) \\
&\geq \frac{w_{\min}}{6} > 0,
\end{aligned}$$

and so $\mathbb{L}_j(\mathbf{g}) \cap K \neq \emptyset$. In particular, for every $j \in [\![1, M]\!]$, there exists $x_j \in \mathbb{L}_j(\mathbf{g}) \cap K$ and so for all $i \in [\![1, M]\!]$

$$c(x_j, y_j) - g_j \leq c(x_j, y_i) - g_i\ .$$

Therefore, using the fact that the cost is non-negative, we have

$$\begin{aligned}
\max_i g_i - \min_j g_j &\leq \max_i \max_j \{c(x_j, y_i) - c(x_j, y_j)\} \\
&\leq \max_{i,j\in[\![1,M]\!]} \sup_{x\in K} c(x, y_i) = \|c\|_{K,\infty}\ .
\end{aligned}$$

Moreover, since $H(\mathbf{g} + \lambda \mathbb{1}_M) = H(\mathbf{g})$ one can fix $g_1 = 0$, which concludes the proof. □

E Minimax estimation of OT quantities

Consider $P, Q \in \mathcal{P}(\mathbb{R}^d)$ with densities f_P and f_Q. We recall that the Hellinger distance is defined by

$$d_{\mathrm{H}}(P,Q) := \left(\int_{\mathbb{R}^d}\left(\sqrt{f_P(x)} - \sqrt{f_Q(x)}\right)^2 \mathrm{d}\lambda_{\mathbb{R}^d}(x)\right)^{\frac{1}{2}}. \tag{17}$$

We also recall the formulation of Le Cam's Lemma. We refer to [32], Chapter 15, for more details.

Lemma E.1. *(Le Cam's Lemma.) Let $\mathcal{P}$ be a set of probability distributions on a measurable space, and consider the problem of estimating a parameter $\theta \in \Theta$ with a loss function ℓ defined, for all $\hat{\theta}, \theta \in \Theta$, as*

$$\ell(\hat{\theta}, \theta) = d(\hat{\theta}, \theta)^p,$$

where $p \geq 1$ is an integer, and d is a distance on Θ. Then, for all $\theta_1, \theta_2 \in \Theta$,

$$R_M = \inf_{T^{(n)}} \sup_{\theta\in\Theta} \mathbb{E}_\theta[\ell(\theta, T^{(n)}(X))] \geq \frac{1}{2^p}\left(1 - \sqrt{n} d_{\mathrm{H}}(P_{\theta_1}, P_{\theta_2})\right) d(\theta_1,\theta_2)^p,$$

where R_M is the minimax risk, and $T^{(n)}$ is an estimator based on n i.i.d samples from P_θ.

In the sequel, we note $\mathcal{P}_H$ the set of absolute continuous measures that verify Assumption A.

E.1 Kantorovich potential

Proof. Consider $\nu = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}$ and fix the cost to transfer mass to be the usual quadratic cost $c(x,y) = \frac{1}{2}\|x-y\|^2$.

For $\delta \geq 0$, we define $\mu_{\theta^\delta} = \mathcal{N}(\delta, 1)$. Since $d = 1$, the optimal transport map is monotone non-decreasing (see, for instance, Chapter 2 in [30]). Thus, we must have the identity

$$T_{\mu_{\theta_\delta},\nu}(x) = \begin{cases} 0, & \text{if } x \leq \delta, \\ 1, & \text{otherwise.}\end{cases}$$

Therefore the vector $\theta^\delta \in \mathbb{R}^2$ solves the semi-dual problem if and only if it satisfies the following inequalities

$$\begin{aligned} c(x,0) - \theta_1^\delta \leq c(x,1) - \theta_2^\delta\,, \quad \forall x \leq \delta\,, \\ c(x,1) - \theta_2^\delta \leq c(x,0) - \theta_1^\delta\,, \quad \forall x > \delta\,. \end{aligned}$$

Since $c(x,y) = \frac{1}{2}\|x-y\|^2$ we can fix $\theta_1^\delta = 0$ and compute

$$\theta^\delta = \left(0, \frac{1}{2} - \delta\right).$$

Therefore, we parameterized the family of probability measures $\mu_\theta \in \mathcal{P}_H$ so that the couple (θ^c, θ) is the unique solution in $\Theta \subset \left\{\theta = (\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 = 0\right\}$ of the dual of $\mathrm{OT}(\mu_\theta, \nu)$. In this class of probabilities, the minimax estimation of the optimal transport potential θ, given $n > 0$ i.i.d samples of the source measure, can be written as

$$R_n^\Theta := \inf_{\hat{\theta}^{(n)}} \sup_{\theta\in\Theta} \mathbb{E}_{\mu_\theta}\left[\|\hat{\theta}^{(n)} - \theta\|^2\right],$$

where $\hat{\theta}^{(n)}$ is based on the n i.i.d samples from the source measure μ. Note that

$$R_n^\Theta \leq \inf_{\mathbf{g}^{(n)}} \sup_{\mu\in\mathcal{P}_H} \mathbb{E}_\mu\left[\|\mathbf{g}^{(n)} - \mathbf{g}^*\|^2\right], \tag{18}$$

where the infimum is taken over all vectors $\mathbf{g}^{(n)}$ based on n i.i.d samples of μ.

Using the closed form of the Hellinger distance between Gaussian distributions, we have

$$d_{\mathrm{H}}(\mu_{\theta^0}, \mu_{\theta^\delta}) = \sqrt{1 - \exp\left(-\frac{\delta^2}{8}\right)}.$$

For $\delta \to 0$, the Taylor expansion gives $d_{\mathrm{H}}(\mu_{\theta^0}, \mu_{\theta^\delta}) = \delta/\sqrt{8} + o(\delta)$. Applying Le Cam's Lemma with $\delta = 1/\sqrt{n}$ gives

$$\begin{aligned} R_n^\Theta &\geq \frac{1}{4}\left(1 - \sqrt{n} d_{\mathrm{H}}\left(\mu_{\theta^0}, \mu_{\theta^\delta}\right)\right) \|\theta^0 - \theta^\delta\|_2^2 \\ &\geq \frac{1}{4}\left(1 - 1/\sqrt{8} + o(1)\right)\frac{1}{n} \\ &\geq \frac{1}{10n} + o\left(\frac{1}{n}\right) \end{aligned}$$

Using the inequality (18) concludes the proof. □

E.2 OT map

Proof. Define for $\delta \in [0,1]$ the set of probability measures μ_δ, with density:

$$f_{\mu_\delta}(x) = \mathbf{1}_{x \in [0,1]}\big(1 + \delta g(x)\big), \qquad x \in [0,1],$$

where

$$g(x) = \begin{cases} 2(1-2x), & x \in [0, 1/2], \\ -2(2x-1), & x \in [1/2, 1]. \end{cases}$$

The squared Hellinger distance between μ_0 (uniform) and μ_δ is:

$$\begin{aligned} d_{\mathrm{H}}(\mu_0, \mu_\delta)^2 &= \frac{1}{2}\int_0^1 \left(\sqrt{1} - \sqrt{1 + \delta g(x)}\right)^2 dx \\ &= \frac{1}{6\delta}\left((1-2\delta)^{3/2} + 6\delta - (1+2\delta)^{3/2}\right) \\ &= \frac{1}{2}\delta^2 + o(\delta^2) \end{aligned}$$

when $\delta \to 0$. Therefore, we obtain

$$d_{\mathrm{H}}(\mu_0, \mu_\delta) = \frac{1}{\sqrt{2}}\delta + o(\delta)\ .$$

Since $\mu_\delta \in \mathcal{P}_H$, $\delta \in [0,1]$, we have

$$\inf_{T^{(n)}} \sup_{\mu \in \mathcal{P}_H} \mathbb{E}_\mu \left[\left\|T^{(n)} - T_{\mu,\nu}\right\|^p_{L^p(\mu)}\right] \geq \inf_{T^{(n)}} \sup_{\delta \in [0,1]} \mathbb{E}_{\mu_\delta}\left[\left\|T^{(n)} - T_{\mu_\delta,\nu}\right\|^p_{L^p(\mu_\delta)}\right]\ . \tag{19}$$

From the relation $1 - 2\delta \leq f_{\mu_\delta}(x) \leq 1 + 2\delta, \forall x \in [0,1]$, we infer that no matter $T^{(n)}$ and δ small enough

$$\left\|T^{(n)} - T_{\mu_\delta,\nu}\right\|^p_{L^p(\mu_\delta)} \geq \frac{1}{2}\left\|T^{(n)} - T_{\mu_\delta,\nu}\right\|^p_{L^p(\mathcal{U}(0,1))}\ . \tag{20}$$

Consider $\nu = \frac{1}{2}\delta_{\{0\}} + \frac{1}{2}\delta_{\{1\}}$ as above. Recall that the optimal transport map is monotone non-decreasing in dimension 1. Moreover by definition $T_{\mu_\delta,\nu}(x) \in \{0,1\}$, $x \in [0,1]$, and $\mu_\delta(T^{-1}_{\mu_\delta,\nu}(0)) = 1/2$. Therefore one identifies $T_{\mu_\delta,\nu} = \mathbf{1}_{x \geq M_\delta}$ where M_δ is the median of μ_δ, $\delta > 0$, satisfying

$$\int_0^{M_\delta} \big(1 + \delta g(x)\big) dx = \frac{1}{2}.$$

Noticing that $M_\delta \in [0, 1/2]$, we solve

$$\int_0^{M_\delta} \big(1 + 2\delta(1-2x)\big)dx = \frac{1}{2} \, ,$$

which gives the solution

$$M_\delta = \frac{1 + 2\delta - \sqrt{1+4\delta^2}}{4\delta} = \frac{1}{2} - \frac{1}{2}\delta + o(\delta) \, .$$

Observe that

$$\|T_{\mu_0,\nu} - T_{\mu_\delta,\nu}\|^p_{L^p(\mathcal{U}(0,1))} = \int_0^1 |\mathbf{1}_{x \geq M_0} - \mathbf{1}_{x\geq M_\delta}|^p \, dx = |M_\delta - M_0| \, .$$

We proved the relation $\|T_{\mu_0,\nu} - T_{\mu_\delta,\nu}\|^p_{L^p(\mathcal{U}(0,1))} = \delta + o(\delta)$. Applying Le Cam's Lemma with $\delta = \frac{1}{\sqrt{n}}$ for n sufficiently large and any $p \in [1, \infty)$, we obtain

$$\begin{aligned} \inf_{T^{(n)}} \sup_{\delta \in [0,1]} \mathbb{E}_{\mu_\delta} \left[\left\| T^{(n)} - T_{\mu_\delta,\nu} \right\|^p_{L^p(\mathcal{U}(0,1))} \right] &\geq \frac{1}{2^p} \left(1 - \sqrt{n} d_{\mathrm{H}}(\mu_0, \mu_\delta)\right) \|T_{\mu_0,\nu} - T_{\mu_\delta,\nu}\|^p_{L^p(\mathcal{U}(0,1))} \\ &\geq \frac{1}{2^p} \left(1 - \frac{1}{\sqrt{2}} + o\left(1\right)\right) \left(\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) \\ &\gtrsim \frac{1}{\sqrt{n}} \, . \end{aligned}$$

Therefore, combining inequalities (19) and (20) we conclude

$$\inf_{T^{(n)}} \sup_{\mu \in \mathcal{P}_H} \mathbb{E}_\mu \left[\left\| T^{(n)} - T_{\mu,\nu} \right\|^p_{L^p(\mu)} \right] \gtrsim \frac{1}{\sqrt{n}} \, .$$

□

F Technical Lemmas

F.1 Technical Lemmas for Appendix B

Lemma F.1. *Perturbation of Laplacian Matrices. Let A and B be symmetric Laplacian matrices of the same size such that:*

$$A_{ij} \leq 0, \quad B_{ij} \leq A_{ij} \quad \textit{for all } i \neq j \, .$$

Suppose $\lambda_2(A) > 0$, where $\lambda_2(A)$ denotes the second smallest eigenvalue of A. Then:

$$\lambda_2(B) \geq \lambda_2(A)$$

where $\lambda_2(B)$ is the second smallest eigenvalue of B.

Proof. We recall the variational characterization of the second smallest eigenvalue of a Laplacian matrix M is

$$\lambda_2(M) = \min_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x} \, .$$

Define the matrix $C = B - A$. For $i \neq j$:

$$C_{ij} = B_{ij} - A_{ij} \leq 0 \, ,$$

and for diagonal elements

$$C_{ii} = B_{ii} - A_{ii} = -\sum_{j \neq i} C_{ij} \geq 0 \, .$$

Thus, C is a Laplacian matrix and so it is positive, semi-definite.

Let y_2 be the eigenvector corresponding to $\lambda_2(B)$, the second smallest eigenvalue of B, that is

$$By_2 = \lambda_2(B)y_2 \quad \text{with } y_2 \perp \mathbf{1}.$$

Since $A = B - C$, we have

$$y_2^T A y_2 = y_2^T (B - C) y_2 = y_2^T B y_2 - y_2^T C y_2.$$

Thus,

$$\frac{y_2^T A y_2}{y_2^T y_2} = \lambda_2(B) - \frac{y_2^T C y_2}{y_2^T y_2}.$$

By the variational principle:

$$\lambda_2(A) = \min_{x \perp \mathbf{1}} \frac{x^T A x}{x^T x} \leq \frac{y_2^T A y_2}{y_2^T y_2} \leq \lambda_2(B) - \frac{y_2^T C y_2}{y_2^T y_2}.$$

Since C is a Laplacian matrix, we have $y_2^T C y_2 \geq 0$, no matter y_2 and thus $\lambda_2(A) \leq \lambda_2(B)$ which completes the proof.

□

Lemma F.2. *Let f_μ satisfy, for all $x \in \mathbb{R}^d$, the decay condition*

$$\sum_{r \geq 1} r^{d-1} \sup_{x \in \mathbb{R}^d \setminus B(0, r-1)} f_\mu(x) < \infty \ .$$

Then, there exists a constant $C > 0$ such that, for any hyperplane $H \subset \mathbb{R}^d$, we have

$$\int_H f_\mu(x) \, \mathrm{d}\mathcal{H}^{d-1}(x) \leq C.$$

Proof. Defining for any $r \in \mathbb{N}^*$, $R(r) := B(0, r) \setminus B(0, r-1)$, we have

$$\begin{aligned}
\int_H f_\mu(x) d\mathcal{H}^{d-1}(x) &= \sum_{r \geq 1} \int_{H \cap R(r)} f_\mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&\leq \sum_{r \geq 1} \int_{H \cap R(r)} \sup_{x \in R(r)} f_\mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&\leq \sum_{r \geq 1} \int_{H \cap R(r)} \sup_{x \in \mathbb{R}^d \setminus B(0, r-1)} f_\mu(x) \mathrm{d}\mathcal{H}^{d-1}(x)
\end{aligned}$$

Then, using the fact that $\mathcal{H}^{d-1}(H \cap R(r)) \leq \mathcal{H}^{d-1}(H \cap B(0, r))$, and noting that $H \cap B(0, r)$ is a $(d-1)$-dimensional ball of radius r, we have

$$\mathcal{H}^{d-1}(H \cap R(r)) \leq \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} r^{d-1}.$$

Therefore, incorporating this bound, we obtain:

$$\int_H f_\mu(x) \mathrm{d}\mathcal{H}^{d-1}(x) \leq \sum_{r \geq 1} \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} r^{d-1} \sup_{x \in \mathbb{R}^d \setminus B(0, r)} f_\mu(x)$$

which is finite by our decay assumption on f_μ.

□

F.2 Technical Lemmas for Appendix C

Lemma F.3. *Let $(\gamma_n)_{n\geq 0}$ and $(m_n)_{n\geq 0}$ be some positive and decreasing sequences and let $(\delta_n)_{n\geq 0}$, satisfying the following:*

- *The sequence δ_n follows the recursive relation:*
$$\delta_{n+1} \leq \left(1-\omega\gamma_{n+1}\right)\delta_n + m_{n+1}\gamma_{n+1}, \tag{21}$$
with $\delta_0 \geq 0$ and $\omega > 0$.
- *γ_n converges to 0.*
- *Let $n_0 = \inf\left\{n \geq 1\, : \omega\gamma_{n+1} \leq 1\right\}$, δ_{n_0} is non-negative.*

Then, for all $n \geq n_0$, we have the upper bound:

$$\delta_n \leq \exp\left(-\omega \sum_{i=n_0+1}^{n} \gamma_i\right)\left(\sum_{k=n_0}^{n}\gamma_k m_k + \delta_{n_0}\right) + \frac{1}{\omega} m_{\lceil \frac{n}{2}\rceil - 1}$$

Proof. For all $n \geq n_0$, one has

$$\delta_n \leq \underbrace{\prod_{i=n_0+1}^{n}\left(1-\omega\gamma_i\right)\delta_{n_0}}_{=:U_{1,n}} + \underbrace{\sum_{k=n_0+1}^{n}\prod_{i=k+1}^{n}\left(1-\omega\gamma_i\right)\gamma_k m_k}_{=:U_{2,n}}$$

One can consider two cases: $\lceil n/2\rceil - 1 \leq n_0$ and $\lceil n/2\rceil - 1 > n_0$.

Case where $\lceil n/2\rceil - 1 \leq n_0 < n$: Since m_k is decreasing,

$$\begin{aligned}
U_{2,n} &\leq m_{n_0+1}\sum_{k=n_0+1}^{n}\prod_{i=k+1}^{n}\left(1-\omega\gamma_i\right)\gamma_k \\
&= \frac{1}{\omega} m_{n_0+1}\sum_{k=n_0+1}^{n}\prod_{i=k+1}^{n}\left(1-\omega\gamma_i\right) - \prod_{i=k}^{n}\left(1-\omega\gamma_i\right) \\
&= \frac{1}{\omega} m_{n_0+1}\left(1-\prod_{i=n_0+1}^{n}\left(1-\omega\gamma_i\right)\right) \\
&\leq \frac{1}{\omega} m_{n_0+1}
\end{aligned}$$

Since m_k is decreasing, it comes $U_{2,n} \leq \frac{1}{\omega} m_{\lceil n/2\rceil}$.

Case where $\lceil n/2\rceil - 1 > n_0$: As in [4], for all $m = n_0+1, \ldots, n$, one has

$$U_{2,n} \leq \exp\left(-\omega\sum_{k=m+1}^{n}\gamma_k\right)\sum_{k=n_0+1}^{m}\gamma_k m_k + \frac{1}{\omega} m_m$$

Then, taking $m = \lceil n/2\rceil - 1$, it comes

$$U_{2,n} \leq \exp\left(-\omega\sum_{k=\lceil n/2\rceil}^{n}\gamma_k\right)\sum_{k=n_0+1}^{\lceil n/2\rceil - 1}\gamma_k m_k + \frac{1}{\omega} m_{\lceil n/2\rceil - 1}$$

$\square$

Lemma F.4. *Linearization of the Hessian If a function $H : \mathbb{R}^M \to \mathbb{R}$ is such that its Hessian is α-Hölder on the ball $B(0,r)$, with $r>0, \alpha \in (0,1)$ and constant L, then for any $\mathbf{g}, \mathbf{g}^* \in B(0,r)$, we have*

$$\left\|\nabla H(\mathbf{g}) - \nabla^2 H(\mathbf{g}^*)(\mathbf{g}-\mathbf{g}^*)\right\| \leq C\left\|\mathbf{g}-\mathbf{g}^*\right\|^{1+\alpha},$$

where $C = \frac{L}{\alpha+1}$.

Proof. Consider the Taylor expansion of $\nabla H(\mathbf{g})$ around $\mathbf{g}^*$:

$$\nabla H(\mathbf{g}) = \nabla H(\mathbf{g}^*) + \nabla^2 H(\mathbf{g}^*)(\mathbf{g} - \mathbf{g}^*) + R(\mathbf{g}),$$

where the remainder term $R(\mathbf{g})$ is given by:

$$R(\mathbf{g}) = \int_0^1 \left[\nabla^2 H(\mathbf{g}^* + t(\mathbf{g} - \mathbf{g}^*)) - \nabla^2 H(\mathbf{g}^*)\right] (\mathbf{g} - \mathbf{g}^*)\mathrm{d}t.$$

By the assumption of α-Hölder continuity of the Hessian, we have

$$\|\nabla^2 H(\mathbf{g}^* + t(\mathbf{g} - \mathbf{g}^*)) - \nabla^2 H(\mathbf{g}^*)\| \leq L\|t(\mathbf{g} - \mathbf{g}^*)\|^\alpha = Lt^\alpha \|\mathbf{g} - \mathbf{g}^*\|^\alpha.$$

Thus,

$$\|R(\mathbf{g})\| \leq \int_0^1 Lt^\alpha \|\mathbf{g} - \mathbf{g}^*\|^\alpha \|\mathbf{g} - \mathbf{g}^*\|\mathrm{d}t = L\|\mathbf{g} - \mathbf{g}^*\|^{1+\alpha} \int_0^1 t^\alpha \mathrm{d}t.$$

Evaluating the integral:

$$\int_0^1 t^\alpha \mathrm{d}t = \frac{1}{\alpha + 1}.$$

Therefore,

$$\|R(\mathbf{g})\| \leq \frac{L}{\alpha + 1} \|(\mathbf{g} - (\mathbf{g}^*\|^{1+\alpha}.$$

This implies the desired inequality:

$$\|\nabla H(\mathbf{g}) - \nabla^2 H(\mathbf{g}^*)(\mathbf{g} - (\mathbf{g}^*)\| \leq C\|(\mathbf{g} - (\mathbf{g}^*\|^{1+\alpha},$$

where $C = \frac{L}{\alpha+1}$. □