SUPPLEMENTARY MATERIAL FOR SUBMISSION #8904

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Paper under double-blind review

1 PROOF OF THEOREM 3.1

Lemma 3.1. *Given an ODE* $\frac{dx(t)}{dt} = g(x(t), t)$ *with a Lipschitz continuous drift function* $g(x(t), t)$*, the probability density* $p_t(x)$ *satisfies the continuity equation:*

$$\frac{\partial p_t(x(t))}{\partial t} + \nabla \cdot p_t(x(t))g(x(t), t) = 0. \tag{1}$$

Theorem 3.1. *Suppose the evolving of* $x(t) \in \mathbb{R}^d$ *satisfies an ODE* $\frac{dx(t)}{dt} = -D_t \nabla_x \log p(x(t))$*, the conditional probability density of* $x(t)$ *given* $x(0)$ *is:*

$$p(x(t)|x(0)) = \mathcal{N}(x(t); x(0), \sigma_t^2 I) = \frac{1}{(2\pi\sigma_t^2)^{d/2}} \exp(-\frac{||x(t) - x(0)||^2}{2\sigma_t^2}). \tag{2}$$

Given two data distributions $x_1 \sim p_1(x_1)$ *and* $x_2 \sim p_2(x_2)$*, based on Eq.(2), the ScoreFlow mapping data from* x_1 *to* x_2 *can be derived as:*

$$\frac{dx(t)}{dt} = \dot{\sigma}_t \sigma_t \nabla_x \log \frac{p_2(x(t))}{p_1(x(t))} = \frac{\dot{\sigma}_t}{\sigma_t}(x_2 - x_1), \tag{3}$$

where $\sigma_t \geq 0$ *denotes a monotonic increasing function with* $\sigma_t \gg 1$ *as* $t \to \infty$*,* d *denotes the number of dimensions.*

Proof. Without loss of generality, we first consider the case of dimension 1. According to Lemma 3.1, we first substitute the given ODE into Equation (1) and obtain:

$$\frac{\partial p(x(t))}{\partial t} = D_t \nabla^2 p(x(t)), \tag{4}$$

To solve this equation, we perform a spatial Fourier transform on x:

$$\begin{aligned} \mathcal{F}\left[\frac{\partial p(x(t))}{\partial t}\right] &= D_t \mathcal{F}\left[\nabla^2 p(x(t))\right] \\ \Longrightarrow \qquad \frac{\partial}{\partial t}\mathcal{F}_t(\omega) &= (-i\omega)^2 D_t \mathcal{F}_t(\omega). \end{aligned} \tag{5}$$

By solving this ODE, we have:

$$\begin{aligned} \mathcal{F}_t(\omega) &= \mathcal{F}_0(\omega) e^{-\omega^2 \int_0^t D_s ds} \\ \overset{\int_0^t D_s ds = \frac{1}{2}\sigma_t^2}{\Longrightarrow} \quad \mathcal{F}_t(\omega) &= \mathcal{F}_0(\omega) e^{-\frac{1}{2}\omega^2 \sigma_t^2}. \end{aligned} \tag{6}$$

Using the convolution theorem, we perform the inverse Fourier transform and obtain:

$$\begin{aligned} p(x(t)) &= \int p(x_0) \frac{1}{(2\pi\sigma_t^2)^{1/2}} \exp(-\frac{|x(t) - x(0)|^2}{2\sigma_t^2}) dx \\ \Longrightarrow \qquad p(x(t)|x(0)) &= \frac{1}{(2\pi\sigma_t^2)^{1/2}} \exp(-\frac{||x(t) - x(0)||^2}{2\sigma_t^2}). \end{aligned} \tag{7}$$

In multi-dimensional situations, we have:

$$\begin{aligned} p(x(t)|x(0)) &= \frac{1}{(2\pi\sigma_t^2)^{d/2}} \exp(-\frac{||x(t) - x(0)||^2}{2\sigma_t^2}) \\ &= \mathcal{N}(x(t); x(0), \sigma_t^2 I), \end{aligned} \tag{8}$$

and the ODE can be written as:

$$dx(t) = -\dot{\sigma}_t \sigma_t \nabla_x \log p(x(t))dt. \tag{9}$$

Therefore, given $x(0) \in \mathbb{R}^d$ as the initial state of ODE (9), according to Equation (8), we can obtain $x(t)$ by computing:

$$x(t) = x(0) + \sigma_t \epsilon, \epsilon \sim \mathcal{N}(0, I). \tag{10}$$

If $\sigma_t \geq 0$ denotes a monotonic increasing function with $\sigma_t \gg 1$ as $t \to \infty$, when t is sufficiently large, we have:

$$x(t) = \sigma_t \epsilon \tag{11}$$

Therefore, given $x_1 \sim p_1$ and $x_2 \sim p_2$ as the inital states of the ODE (9), when T is sufficiently large, we have:

$$\begin{aligned} x_1(T) &= x_1 + \int_0^T -\dot{\sigma}_t \sigma_t \nabla_{x_1} \log p_1(x(t))dt = \sigma_t \epsilon, \\ x_2(T) &= x_2 + \int_0^T -\dot{\sigma}_t \sigma_t \nabla_{x_2} \log p_2(x(t))dt = \sigma_t \epsilon, \end{aligned} \tag{12}$$

Therefore, we have:

$$\begin{aligned} x_2 &= x_1 + \int_0^T -\dot{\sigma}_t \sigma_t \nabla_{x_1} \log p_1(x(t))dt - \int_0^T -\dot{\sigma}_t \sigma_t \nabla_{x_2} \log p_2(x(t))dt \\ &= x_1 + \int_0^T \dot{\sigma}_t \sigma_t \nabla_x \log \frac{p_2(x(t))}{p_1(x(t))} dt \\ \Longrightarrow \quad \frac{dx(t)}{dt} &= \dot{\sigma}_t \sigma_t \nabla_x \log \frac{p_2(x(t))}{p_1(x(t))} \end{aligned} \tag{13}$$

Utilizing the Equation (8), we obtain:

$$\begin{aligned} \frac{dx(t)}{dt} &= \dot{\sigma}_t \sigma_t \nabla_x \log \frac{p_2(x(t))}{p_1(x(t))} \\ &= \dot{\sigma}_t \sigma_t \nabla_x \log \frac{\mathcal{N}(x(t); x_2, \sigma_t^2 I)}{\mathcal{N}(x(t); x_1, \sigma_t^2 I)} \\ &= \dot{\sigma}_t \sigma_t \frac{x_2 - x_1}{\sigma_t^2} = \frac{\dot{\sigma}_t}{\sigma_t}(x_2 - x_1) \end{aligned} \tag{14}$$

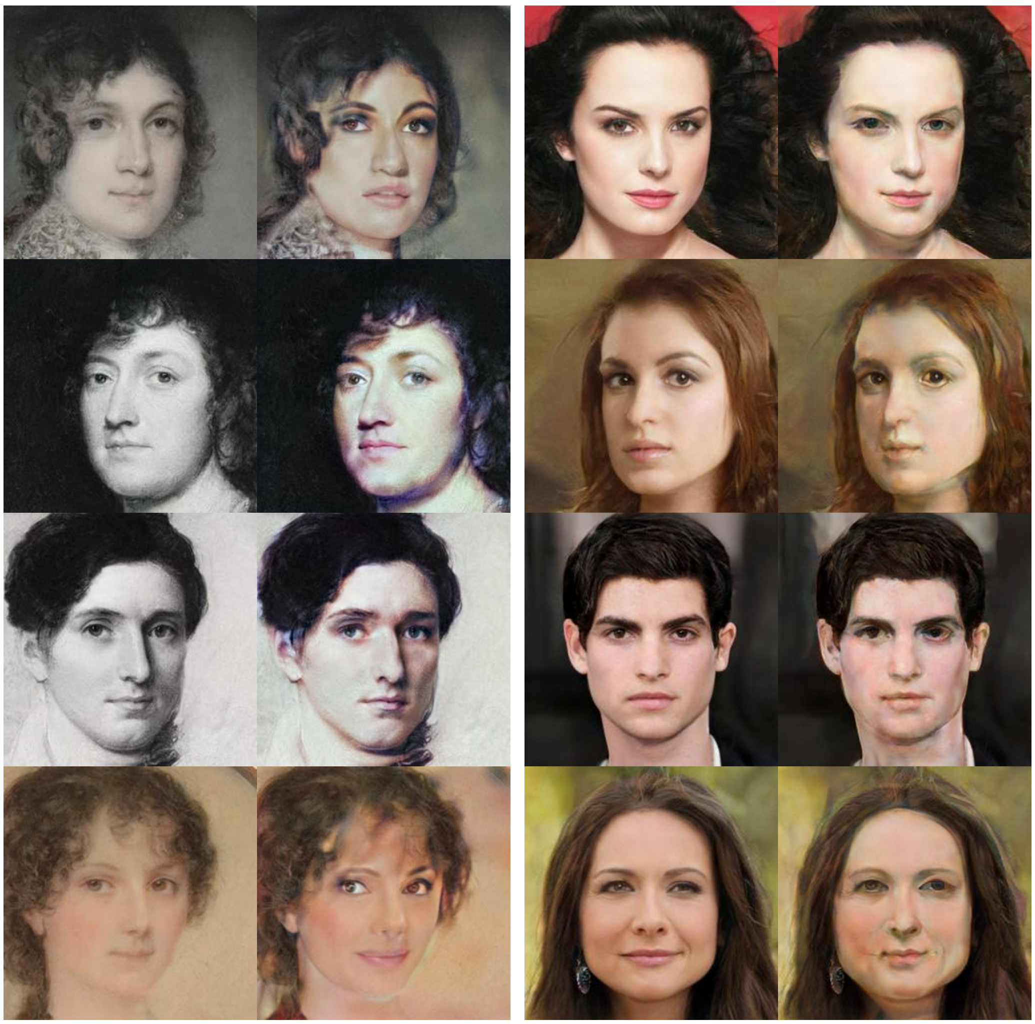
This completes the proof. □

2 More Experimental Results

2.1 256×256 Images Generated by a Unified ScoreFlow Trained on CelebA and MetFace

Figure 1: Generated Images on CelebA

Figure 2: Generated Images on MetFace



(a) CelebA→MetFace (b) MetFace→CelebA

Figure 3: (a) The image translation from CelebA to MetFace, via solving $dx = [f_\theta(x(t), t, c = 1) - f_\theta(x(t), t, c = 0)]dt$. (b) The reverse translation from MetFace to CelebA.

2.2 256×256 Images Generated by a Conditional ScoreFlow Trained on AFHQ

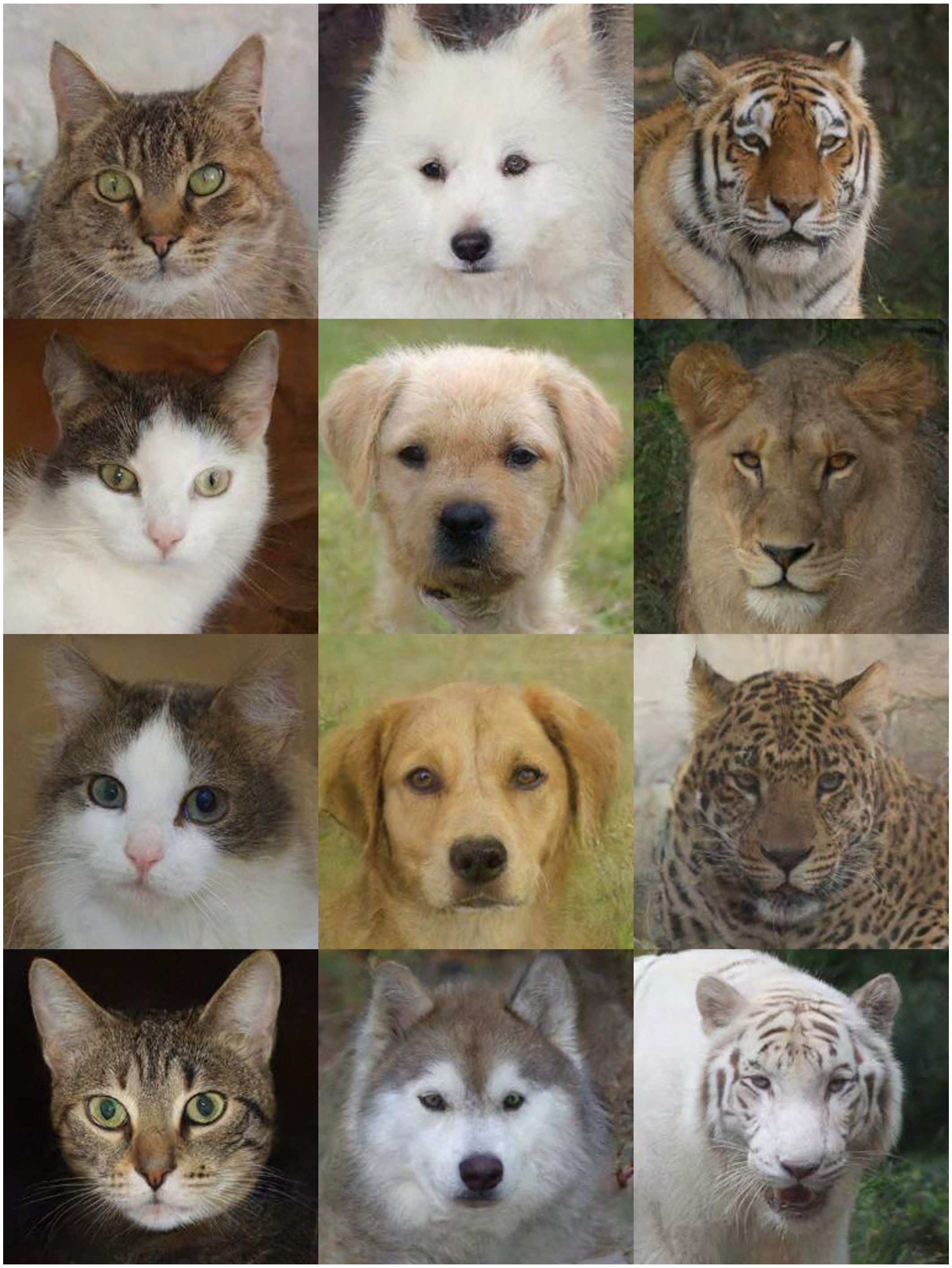


Figure 4: Generated Images on AFHQ