Appendix

A Missing Proofs in Section 3

Proposition 1 (β-Pareto Efficiency, Upper Bound). *There is no neutral rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ that satisfies ϵ-DP and β-Pareto efficiency with $\beta > \mathrm{e}^{\frac{n\epsilon}{m-1}}$.*

Proof. Let $f\colon \mathcal{L}(A)^n \to A$ be a voting rule satisfying ϵ-DP and β-Pareto efficiency. Let P be a profile, where $a_1 \succ a_2 \succ \cdots \succ a_m$, for all $i \in N$. Since a_1 Pareto dominates a_2, a_2 Pareto dominates a_3, etc., we have

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \beta \cdot \mathbb{P}[f(P) = a_2] \\ &\geqslant \beta^2 \cdot \mathbb{P}[f(P) = a_3] \\ &\cdots \\ &\geqslant \beta^{m-1} \cdot \mathbb{P}[f(P) = a_m].\end{aligned}$$

Then we claim that for profile P,

$$\mathbb{P}[f(P) = a_1] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = a_m].$$

Theorefore, we have $\beta^{m-1} \leqslant \mathrm{e}^{n\epsilon}$, i.e., $\beta \leqslant \mathrm{e}^{\frac{n\epsilon}{m-1}}$, as desired.

Finally, we prove the claim above. In fact, for any voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP and neutrality, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = b], \quad \text{for all } a, b \in A. \tag{1}$$

Now, we prove Equation (1). For any profile $P, P' \in \mathcal{L}(A)^n$, let $\ell_0(\cdot,\cdot)$ represents the ℓ_0-distance between them, i.e., $\ell_0(P, P') = \{j \in N : \succ_j \neq \succ'_j\}$. Then, by considering the following operation **Op**, we can see that P can be transferred to P' through $k = \ell_0(P, P')$ times of operations.

- **Op**: Choose a voter $i \in N$ that $\succ_i \neq \succ'_i$, let $\succ_i = \succ'_i$.

Letting $P_0, P_1, \ldots, P_k$ denote all of the profiles, we have the following diagram.

$$P = P_0 \xrightarrow{\mathbf{Op}} P_1 \xrightarrow{\mathbf{Op}} P_2 \xrightarrow{\mathbf{Op}} \cdots \xrightarrow{\mathbf{Op}} P_k = P'.$$

Notice that in each step, only one voter's preference is changed. Consequently, for each i, P_i and P_{i+1} are neighboring profiles. Since f satisfies ϵ-DP, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{\epsilon} \cdot \mathbb{P}[f(P_1) = a] \leqslant \mathrm{e}^{2\epsilon} \cdot \mathbb{P}[f(P_2) = a] \leqslant \ldots \leqslant \mathrm{e}^{k\cdot\epsilon} \cdot \mathbb{P}[f(P') = a].$$

Besides, for any given $P, P' \in \mathcal{L}(A)^n$, there are at most n distinct voters $j \in N$ that $\succ_j \neq \succ'_j$. Therefore, for any profile $P, P' \in \mathcal{L}(A)^n$ and any $a \in A$, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P') = a].$$

Now, for any profile $P \in \mathcal{L}(A)^n$ and an arbitrarily chosen paif of alternatives $a, b \in A$, let P' be the profile transferred from P by swapping a and b in each voter's preference. By the neutrality of f, we have

$$\mathbb{P}[f(P) = a] = \mathbb{P}[f(P') = b] \text{ and } \mathbb{P}[f(P') = a] = \mathbb{P}[f(P) = b].$$

Then it follows that

$$\begin{aligned}\mathbb{P}[f(P)=a] \leqslant & \mathrm{e}^{n\epsilon}\cdot\mathbb{P}[f(P')=a] \\ = & \mathrm{e}^{n\epsilon}\cdot\mathbb{P}[f(P)=b],\end{aligned}$$

which completes the proof. □

Proposition 2 (β-Pareto Efficiency, Lower Bound). *Given $\epsilon\in\mathbb{R}_+$, Mechanism 1 satisfies ϵ-DP and $\mathrm{e}^{\frac{n\epsilon}{2m-2}}$-Pareto efficiency.*

Proof. Let $\mathfrak{E}_{\mathsf{Borda}}\colon\mathcal{L}(A)^*\to\mathcal{R}(A)$ denote the mapping introduced by BordaEXP. Then for any profile $P\in\mathcal{L}(A)^*$ and alternative $a\in A$, we have

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]=\frac{\mathrm{e}^{\mathsf{Borda}_P(a)\epsilon/(2m-2)}}{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_P(c)\epsilon/(2m-2)}}.$$

First, we establish the bound for Pareto efficiency. Given profile $P\in\mathcal{L}(A)^n$, suppose $a,b\in A$ are a pair of alternatives that $a\succ_j b$ for all $j\in N$. It follows that, for each voter $j\in N$, the number of alternatives that are considered worse than b according to her preference order $\succ_j$ is strictly less than the number of alternatives considered worse than a. Formally, we have

$$|\{c\in A: a\succ_j c\}|-|\{c\in A: b\succ_j c\}|\geqslant 1,\quad\text{for all } j\in N.$$

By the definition of Borda score, we have

$$\mathsf{Borda}_P(a)-\mathsf{Borda}_P(b)\geqslant n.$$

Then it follows that

$$\begin{aligned}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a] &= \frac{\mathrm{e}^{\mathsf{Borda}_P(a)\cdot\epsilon/(2m-2)}}{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_P(c)\cdot\epsilon/(2m-2)}} \\ &\geqslant \frac{\mathrm{e}^{\mathsf{Borda}_P(b)}\cdot\mathrm{e}^{n\epsilon/(2m-2)}}{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_P(c)\cdot\epsilon/(2m-2)}} \\ &= \mathrm{e}^{n\epsilon/(2m-2)}\cdot\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=b],\end{aligned}$$

which indicates that $\mathfrak{E}_{\mathsf{Borda}}$ satisfies $\mathrm{e}^{\frac{n\epsilon}{2m-2}}$-Pareto efficiency.

Then we prove the DP-bound. For all neighboring profiles $P,P'\in\mathcal{L}(A)^n$,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P')=a]} &= \frac{\mathrm{e}^{\mathsf{Borda}_P(a)\cdot\frac{\epsilon}{2m-2}}}{\mathrm{e}^{\mathsf{Borda}_{P'}(a)\cdot\frac{\epsilon}{2m-2}}}\cdot\frac{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_{P'}(c)\cdot\frac{\epsilon}{2m-2}}}{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_P(c)\cdot\frac{\epsilon}{2m-2}}} \\ &\leqslant \mathrm{e}^{\epsilon/2}\cdot\sup_{P\in\mathcal{L}(A)^n}\frac{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_{P'}(c)\cdot\frac{\epsilon}{2m-2}}}{\sum\limits_{c\in A}\mathrm{e}^{\mathsf{Borda}_P(c)\cdot\frac{\epsilon}{2m-2}}} \\ &\leqslant \mathrm{e}^{\epsilon},\end{aligned}$$

which completes the proof. □

Lemma 1. *Given $\gamma > 0$, a voting rule f satisfies γ-SD-efficiency if and only if*

$$\frac{1}{\gamma} \geqslant \sup_{P \in \mathcal{L}(A)^n, \xi \in \mathcal{R}(A)} \inf_{j \in N, y \in A} \frac{\sum\limits_{x : x \succ_j y} \mathbb{P}[\xi = x]}{\sum\limits_{x : x \succ_j y} \mathbb{P}[f(P) = x]}.$$

Proof. Suppose that f is not γ-SD-efficient. Then there must be some profile $P \in \mathcal{L}(A)^n$ that $f(P)$ is γ-SD-dominated by some $\xi \in \mathcal{R}(A)$, i.e.,

$$\sum_{x : x \succ_j y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x : x \succ_j y} \mathbb{P}[f(P) = x], \quad \text{for all } y \in A \text{ and } \succ_j \in P,$$

which is equivalent to

$$\frac{1}{\gamma} \leqslant \inf_{j \in N, y \in A} \frac{\sum\limits_{x : x \succ_j y} \mathbb{P}[\xi = x]}{\sum\limits_{x : x \succ_j y} \mathbb{P}[f(P) = x]}.$$

Therefore, f is γ-SD-efficient if and only if for each $P \in \mathcal{L}(A)^n$, there does not exist such ξ, i.e.,

$$\frac{1}{\gamma} \geqslant \inf_{j \in N, y \in A} \frac{\sum\limits_{x : x \succ_j y} \mathbb{P}[\xi = x]}{\sum\limits_{x : x \succ_j y} \mathbb{P}[f(P) = x]}, \quad \text{for all } y \in A \text{ and } P \in \mathcal{L}(A)^n,$$

which is equivalent to

$$\frac{1}{\gamma} \geqslant \sup_{P \in \mathcal{L}(A)^n, \xi \in \mathcal{R}(A)} \inf_{j \in A, y \in A} \frac{\sum\limits_{x : x \succ_j y} \mathbb{P}[\xi = x]}{\sum\limits_{x : x \succ_j y} \mathbb{P}[f(P) = x]}.$$

That completes the proof. □

Proposition 3 (γ-SD-Efficiency, Upper Bound). *Given $\gamma \in \mathbb{R}_+$, there is no neutral voting rule $f \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP and γ-SD-efficiency with $\gamma > \frac{(m-1)e^{n\epsilon}}{(m-1)e^{n\epsilon}+1}$.*

Proof. Consider two profiles, P_1 and P_2, where all voters in P_1 share the same preference order $a_1 \succ a_2 \succ \cdots \succ a_m$. In contrast, in P_2, the voters' preferences are $a_m \succ' a_2 \succ' a_3 \succ' \cdots \succ' a_{m-1} \succ' a_1$. Then the unique SD-efficient lottery for P_1 and P_2 should be $\mathbb{1}_{a_1}$ and $\mathbb{1}_{a_m}$, respectively. Here, $\mathbb{1}_{a_1}$ and $\mathbb{1}_{a_m}$ represent indicator functions defined as follows.

$$\mathbb{P}[\mathbb{1}_{a_1} = a] = \begin{cases} 1 & a = a_1 \\ 0 & \text{otherwise} \end{cases}, \quad \mathbb{P}[\mathbb{1}_{a_m} = a] = \begin{cases} 1 & a = a_m \\ 0 & \text{otherwise} \end{cases}.$$

Let f be any neutral voting rule satisfying ϵ-DP. By Equation (1), we have

$$\begin{aligned} \mathbb{P}[f(P_1) = a_1] &\leqslant e^{n\epsilon} \cdot \mathbb{P}[f(P_2) = a_1] && \text{(by } \epsilon\text{-DP)} \\ &= e^{n\epsilon} \cdot \mathbb{P}[f(P_1) = a_m]. && \text{(by neutrality)} \end{aligned}$$

By symmetry, for any $a \neq a_m$, $\mathbb{P}[f(P_1) = a] \leqslant e^{n\epsilon} \cdot \mathbb{P}[f(P_1) = a_m]$. Therefore,

$$\sum_{a \in A} \mathbb{P}[f(P_1) = a] \leqslant ((m-1)e^{n\epsilon} + 1) \cdot \mathbb{P}[f(P_1) = a_m] \leqslant 1,$$

i.e., $\mathbb{P}[f(P_1)=a_m] \leqslant \frac{1}{(m-1)\mathrm{e}^{n\epsilon}+1}$. If there exists some γ that f satisfies γ-SD-efficiency, there does not exist any $\xi \in \mathcal{R}(A)$, such that

$$\sum_{x:x\succ y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x:x\succ y} \mathbb{P}[f(P_1)=x], \quad \text{for all } y \in A.$$

Therefore, we have

$$\begin{aligned}
\frac{1}{\gamma} &\geqslant \sup_{P\in\mathcal{L}(A)^n} \sup_{\xi\in\mathcal{R}(A)} \inf_{y\in A} \frac{\sum\limits_{x:x\succ y} \mathbb{P}[\xi = x]}{\sum\limits_{x:x\succ y} \mathbb{P}[f(P)=x]} \\
&\geqslant \sup_{\xi\in\mathcal{R}(A)} \inf_{y\in A} \frac{\sum\limits_{x:x\succ y} \mathbb{P}[\xi = x]}{\sum\limits_{x:x\succ y} \mathbb{P}[f(P_1)=x]} \\
&\geqslant \inf_{y\in A} \frac{\sum\limits_{x:x\succ y} \mathbb{P}[\mathbb{1}_{a_1} = x]}{\sum\limits_{x:x\succ y} \mathbb{P}[f(P_1)=x]} \\
&= \frac{1}{\max\limits_{y\in A} \sum\limits_{x:x\succ y} \mathbb{P}[f(P_1)=x]} \\
&= \frac{1}{1-\mathbb{P}[f(P_1)=a_n]} \\
&\geqslant \frac{(m-1)\mathrm{e}^{n\epsilon}+1}{(m-1)\mathrm{e}^{n\epsilon}}.
\end{aligned}$$

In other words, we have $\gamma \leqslant \frac{(m-1)\mathrm{e}^{n\epsilon}}{(m-1)\mathrm{e}^{n\epsilon}+1}$, as desired. $\square$

Proposition 4 (γ-SD-Efficiency, Lower Bound). *Mechanism 2 satisfies ϵ-DP and $\frac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}$-SD-efficiency.*

Proof. Letting $\mathfrak{E}_{\mathsf{Anti}} \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ denote the mapping introduced by Mechanism 2.

For any neighboring profiles $P, P' \in \mathcal{L}(A)^n$ that $P_{-j} = P'_{-j}$ and $\succ_j \neq \succ'_j$, suppose that the chosen ballot in the mechanism is $\succ_i$. Then

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a \mid i \neq j] = \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i \neq j] \qquad \text{(use } C \text{ to denote them)}$$

Further, for any $a \in A$,

$$\begin{aligned}
\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a]} &= \frac{\mathbb{P}[i=j \wedge \mathfrak{E}_{\mathsf{Anti}}(P)=a] + \mathbb{P}[i\neq j \wedge \mathfrak{E}_{\mathsf{Anti}}(P)=a]}{\mathbb{P}[i=j \wedge \mathfrak{E}_{\mathsf{Anti}}(P')=a] + \mathbb{P}[i\neq j \wedge \mathfrak{E}_{\mathsf{Anti}}(P')=a]} \\
&= \frac{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a \mid i=j] + \frac{n-1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a \mid i\neq j]}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a \mid i=j] + \frac{n-1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a \mid i\neq j]} \\
&= \frac{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a \mid i=j] + \frac{n-1}{n}\cdot C}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a \mid i=j] + \frac{n-1}{n}\cdot C} \qquad (P_{-j} = P'_{-j}) \\
&\leqslant \frac{\mathrm{e}^{\epsilon}\cdot\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a \mid i=j] + \frac{n-1}{n}\cdot C}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P')=a \mid i=j] + \frac{n-1}{n}\cdot C} \\
&\leqslant \mathrm{e}^{\epsilon},
\end{aligned}$$

which indicates that $\mathfrak{E}_{\mathsf{Anti}}$ satisfies ϵ-DP. On the other hand, given profile P, suppose the top-ranked and the last-ranked alternative of $\succ_i$ are a_{top} and a_{last}, respectively. Then, for any $\xi \in \mathcal{R}(A)$, we have

$$\frac{\sum\limits_{x:x\succ_i y} \mathbb{P}[\xi = x]}{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} \leqslant \frac{1}{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} = \frac{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathbb{1}_{a_{top}} = x]}{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]}.$$

Theorefore,

$$\begin{aligned}
\sup_{\xi\in\mathcal{R}(A)} \inf_{y\in A} \frac{\sum\limits_{x:x\succ_i y} \mathbb{P}[\xi = x]}{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} &= \inf_{y\in A} \frac{1}{\sum\limits_{x:x\succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} \\
&= \frac{1}{1 - \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a_{last}]} \\
&= \frac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon} + 1}.
\end{aligned}$$

By Lemma 1, $\mathfrak{E}_{\mathsf{Anti}}$ satisfies $\frac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}$-SD-efficiency, which completes the proof. □

Lemma 2. *Given $\gamma \leqslant 1$, γ-PC-efficiency implies γ-SD-efficiency.*

Proof. In fact, we only need to prove that for any $\xi, \zeta \in \mathcal{R}(A)$, $\xi \succeq^{\gamma-SD} \zeta$ implies $\xi \succeq^{\gamma-PC} \zeta$. Let ξ and ζ be two lotteries satisfying $\xi \succeq^{\gamma-SD} \zeta$, i.e.,

$$\sum_{x\succ y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x\succ y} \mathbb{P}[\zeta = y], \qquad \text{for any } y \in A.$$

Then, on the one hand, we have

$$\begin{aligned}
\sum_{x,y\in A\wedge x\succ y} \mathbb{P}[\xi = x] \cdot \mathbb{P}[\zeta = y] &= \sum_{y\in A} \mathbb{P}[\zeta = y] \cdot \sum_{x\succ y} \mathbb{P}[\xi = x] \\
&\geqslant \sum_{y\in A} \mathbb{P}[\zeta = y] \cdot \frac{1}{\gamma} \sum_{x\succ y} \mathbb{P}[\zeta = y] \\
&= \frac{1}{\gamma} \cdot \sum_{x,y\in A\wedge x\succ y} \mathbb{P}[\zeta = y] \cdot \mathbb{P}[\zeta = y].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{\gamma} \cdot \sum_{x,y\in A\wedge x\succ y} \mathbb{P}[\zeta = y]\mathbb{P}[\xi = y] &= \sum_{x\in A} \mathbb{P}[\zeta = y] \cdot \sum_{y\prec x} \mathbb{P}[\xi = y] \\
&= \frac{1}{\gamma} \cdot \sum_{x\in A} \mathbb{P}[\zeta = y] \cdot \sum_{y\prec x} \left(1 - \sum_{x\succeq y} \mathbb{P}[\xi = x]\right) \\
&\leqslant \frac{1}{\gamma} \cdot \sum_{x\in A} \mathbb{P}[\zeta = y] \cdot \sum_{y\prec x} \left(1 - \frac{1}{\gamma} \sum_{x\succeq y} \mathbb{P}[\zeta = y]\right) \\
&\leqslant \frac{1}{\gamma} \cdot \sum_{x\in A} \mathbb{P}[\zeta = y] \cdot \sum_{y\prec x} \left(1 - \sum_{x\succeq y} \mathbb{P}[\zeta = y]\right) \\
&= \frac{1}{\gamma} \cdot \sum_{x,y\in A\wedge x\succ y} \mathbb{P}[\zeta = y] \cdot \mathbb{P}[\zeta = y].
\end{aligned}$$

Then it follows that

$$\sum_{x,y\in A\wedge x\succ y}\mathbb{P}[\xi=x]\cdot\mathbb{P}[\zeta=y]\geqslant\frac{1}{\gamma}\cdot\sum_{x,y\in A\wedge x\succ y}\mathbb{P}[\zeta=x]\cdot\mathbb{P}[\xi=y],$$

which completes the proof. □

Proposition 5 (κ-PC-Efficiency, Upper Bound). *Given any $\kappa,\epsilon\in\mathbb{R}_+$, there is no voting rule $f\colon\mathcal{L}(A)^n\to\mathcal{R}(A)$ satisfying ϵ-DP and κ-PC-efficiency.*

Proof. Consider the profile P, where all voters share the same prefereoce

$$a_1\succ a_2\succ\cdots\succ a_m.$$

Then the unique PC-efficient distribution on A is $\mathbb{1}_{a_1}$. Further, we have

$$\sum_{x,y\colon\ x\succ_i y}\mathbb{P}[\mathbb{1}_{a_1}=x]\cdot\mathbb{P}[f(P)=y]=\sum_{y\colon\ a_1\succ_i y}\mathbb{P}[f(P)=y]=1-\mathbb{P}[f(P)=a_1].$$

However,

$$\sum_{x,y\colon\ x\succ_i y}\mathbb{P}[f(P)=x]\cdot\mathbb{P}[\mathbb{1}_{a_1}=y]=\sum_{x\in A}\mathbb{P}[f(P)=x]\cdot\sum_{y\colon\ x\succ_i y}\mathbb{P}[\mathbb{1}_{a_1}=y]=0.$$

In other words, for all $\kappa\in\mathbb{R}_+$, the lottery $\mathbb{1}_{a_1}$ can κ-PC-dominate any $f(P)$, which completes the proof. □

Proposition 6 (α-Condorcet Criterion, Lower Bound). *Mechanism 3 satisfies* e^ϵ*-Condorcet criterion and ϵ-DP.*

Proof. Let $\mathfrak{R}_{\mathrm{CW}}\colon\mathcal{L}(A)^*\to\mathcal{R}(A)$ denote the mapping introduced by CWRR. Then for any profile $P\in\mathcal{L}(A)^*$ and alternative $a\in A$, we have

$$\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]=\begin{cases}\frac{\mathrm{e}^\epsilon}{\mathrm{e}^\epsilon+m-1}, & a=\mathrm{CW}(P)\\ \frac{1}{\mathrm{e}^\epsilon+m-1}, & \text{otherwise}\end{cases},\quad\text{for all }P\text{ that }\ \mathrm{CW}(P)\text{ exists}.$$

By definition, it is not hard to see that CWRR satisfies e^ϵ-Condorcet criterion. Thus, we only need to prove that CWRR satisfies ϵ-DP. In fact, for any neighboring profiles $P,P'\in\mathcal{L}(A)^n$ and $a\in A$,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]}{\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P')=a]}&\leqslant\frac{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]}{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P')=a]}\\&\leqslant\frac{\mathrm{e}^\epsilon}{\mathrm{e}^\epsilon+m-1}\Big/\frac{1}{\mathrm{e}^\epsilon+m-1}\\&=\mathrm{e}^\epsilon,\end{aligned}$$

which completes the proof. □

Proposition 7 (η-Condorcet Loser Criterion, Upper Bound). *There is no voting rule satisfying ϵ-DP and η-Condorcet loser criterion with $\eta>\mathrm{e}^\epsilon$.*

Proof. Suppose $f\colon\mathcal{L}(A)^n\to A$ be a voting rule satisfying ϵ-DP and η-Condorcet loser criterion. Consider the profile P ($n=2k+1$):

- $k+1$ voters: $a_1\succ a_2\succ\cdots\succ a_m$,

– k voters: $a_m \succ a_{m-1} \succ \cdots \succ a_1$.

By definition, we have $w_P[a_m, a_i] = -1$, for all $a_i \in A\backslash\{a_m\}$, i.e., a_m is the Condorcet loser. Now, letting one voter change her preferece from $a_1 \succ a_2 \succ \cdots \succ a_m$ to $a_m \succ a_{m-1} \succ \cdots \succ a_1$, we can obtain another profile P':

– k voters: $a_1 \succ' a_2 \succ' \cdots \succ' a_m$,
– $k+1$ voters: $a_m \succ' a_{m-1} \succ \cdots \succ' a_1$.

Now we have $w_{P'}[a_1, a_i] = -1$, for all $a_i \in A\backslash\{a\}$, i.e., a_1 is the Condorcet loser for P'. Then

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \eta \cdot \mathbb{P}[f(P) = a_m] && \text{(By } \eta\text{-Condorcet loser)}\\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \eta \cdot \mathbb{P}[f(P') = a_m] && \text{(By } \epsilon\text{-DP)}\\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \eta^2 \cdot \mathbb{P}[f(P') = a_1] && \text{(By } \eta\text{-Condorcet loser)}\\
&\geqslant \mathrm{e}^{-2\epsilon} \cdot \eta^2 \cdot \mathbb{P}[f(P) = a_1], && (\epsilon\text{-DP})
\end{aligned}$$

which indicates that $\mathrm{e}^{-2\epsilon} \cdot \eta^2 \leqslant 1$, i.e., $\eta \leqslant \mathrm{e}^{\epsilon}$. That completes the proof. □

Proposition 8 (η-Condorcet Loser Criterion, Lower Bound). *Mechanism 4 satisfies* e^{ϵ}-*Condorcet loser criterion and* ϵ-*DP.*

Proof. Let $\mathfrak{R}_{\mathrm{CL}} \colon \mathcal{L}(A)^* \to \mathcal{R}(A)$ denote the mapping introduced by CLRR, we have

$$\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a] = \begin{cases} \frac{1}{(m-1)\mathrm{e}^{\epsilon}+1}, & a = \mathrm{CL}(P) \\ \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}, & \text{otherwise} \end{cases}, \quad \text{for all } P \text{ that } \mathrm{CL}(P) \text{ exists.}$$

By definition, it is not hard to see that CLRR satisfies e^{ϵ}-Condorcet criterion. Thus, we only need to prove that CLRR satisfies ϵ-DP. In fact, for any neighboring profiles $P, P' \in \mathcal{L}(A)^n$ and $a \in A$,

$$\begin{aligned}
\frac{\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a]}{\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P') = a]} &\leqslant \frac{\max_{a\in A} \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a]}{\max_{a\in A} \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P') = a]}\\
&\leqslant \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1} \Big/ \frac{1}{(m-1)\mathrm{e}^{\epsilon}+1}\\
&= \mathrm{e}^{\epsilon},
\end{aligned}$$

which completes the proof. □

B Missing Proofs in Section 4

B.1 Results in Table 3 and their proofs

Proposition 9. *Given $\epsilon \in \mathbb{R}_+$, BordaEXP satisfies*

(1) $\frac{e^{\frac{n}{2}}+(m-2)\cdot e^{\frac{n(m-2)}{4m-4}}}{e^{\frac{n}{2}}+(m-1)\cdot e^{\frac{n(m-2)}{4m-4}}}$*-SD-efficiency,*

(2) $e^{\left(\lfloor\frac{n}{2}\rfloor+1\right)\cdot\frac{m}{2m-2}-\frac{n}{2}}$*-Condorcet criterion,*

(3) $e^{\frac{n}{2m-2}-\left(\lceil\frac{n}{2}\rceil-1\right)\frac{m}{2m-2}}$*-Condorcet loser criterion.*

Proof. Let $\mathfrak{E}_{\mathsf{Borda}}$ denote the voting rule introduced by BordaEXP. First, we prove (1). In fact,

$$\begin{aligned}\sup_{P,\xi}\inf_{j,y}\frac{\sum\limits_{x\succ_j y}\mathbb{P}[\xi=x]}{\sum\limits_{x\succ_j y}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=x]}\leqslant&\sup_{P}\inf_{j,y}\frac{1}{\sum\limits_{x\succ_j y}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=x]}\\=&\sup_{P}\inf_{j}\frac{1}{1-\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a^j_{last}]}\\\leqslant&\frac{1}{1-\sup\limits_{P}\inf\limits_{j}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a^j_{last}]}.\end{aligned}$$

where a^j_{last} denote the last-ranked alternative in $\succ_j$. By syemmetry, we have

$$\sup_{P}\inf_{j}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a^j_{last}]=\frac{e^{\frac{n(m-2)}{4m-4}}}{e^{\frac{n}{2}}+(m-1)\cdot e^{\frac{n(m-2)}{4m-4}}}.$$

Then BordaEXP satisfies $\frac{e^{\frac{n}{2}}+(m-2)\cdot e^{\frac{n(m-2)}{4m-4}}}{e^{\frac{n}{2}}+(m-1)\cdot e^{\frac{n(m-2)}{4m-4}}}$. Second, we prove (2). By definition, for any profile P that $\mathrm{CW}(P)$ exists, $\mathrm{CW}(P)$ must defeat each alternative $a \neq \{\mathrm{CW}(P)\}$ in at least half of the votes, i.e., $\mathsf{Borda}_P(\mathrm{CW}(P)) \geqslant (m-1)\left(\lfloor\frac{n}{2}\rfloor+1\right)$. And for each $a \neq \mathrm{CW}(P)$, $\mathsf{Borda}_P(a) \leqslant (m-1)n-(\lfloor\frac{n}{2}\rfloor+1)$. Therefore,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=\mathrm{CW}(P)]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}\geqslant&e^{\frac{(m-1)\left(\lfloor\frac{n}{2}\rfloor+1\right)}{2m-2}-\frac{(m-1)n-(\lfloor\frac{n}{2}\rfloor+1)}{2m-2}}\\=&e^{\left(\lfloor\frac{n}{2}\rfloor+1\right)\cdot\frac{m}{2m-2}-\frac{n}{2}},\end{aligned}$$

which indicates that BordaEXP satisfies $e^{\left(\lfloor\frac{n}{2}\rfloor+1\right)\cdot\frac{m}{2m-2}-\frac{n}{2}}$-Condorcet criterion. Finally, we prove (3). By definition, for any profile P that $\mathrm{CL}(P)$ exists, $a \neq \mathrm{CL}(P)$ must be ranked than $\mathrm{CL}(P)$ in at least a half of votes, i.e., $\mathsf{Borda}_P(\mathrm{CL}(P)) \leqslant (m-1)\left(\lfloor\frac{n}{2}\rfloor-1\right)$. And for each $a \neq \mathrm{CL}(P)$, $\mathsf{Borda}_P(a) \geqslant n-\lceil\frac{n}{2}\rceil+1$. Therefore,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=\mathrm{CL}(P)]}\geqslant&e^{\frac{n-\lceil\frac{n}{2}\rceil+1}{2m-2}-\frac{(m-1)\left(\lfloor\frac{n}{2}\rfloor-1\right)}{2m-2}}\\=&e^{\frac{n}{2m-2}-\left(\lceil\frac{n}{2}\rceil-1\right)\frac{m}{2m-2}},\end{aligned}$$

which indicates that BordaEXP satisfies $e^{\frac{n}{2m-2}-\left(\lceil\frac{n}{2}\rceil-1\right)\frac{m}{2m-2}}$-Condorcet loser criterion. □

Proposition 10. *Given $\epsilon \in \mathbb{R}_+$, RD-Anti satisfies*

(1) 1-*Pareto efficiency,*
(2) $\frac{\left(\lfloor \frac{n}{2}\rfloor-1\right)\mathrm{e}^{\epsilon}+\lceil \frac{n}{2}\rceil+1}{n\mathrm{e}^{\epsilon}}$-*Condorcet criterion,*
(3) $\frac{\left(\lfloor \frac{n}{2}\rfloor-1\right)\mathrm{e}^{\epsilon}+\lceil \frac{n}{2}\rceil+1}{n\mathrm{e}^{\epsilon}}$-*Condorcet loser criterion.*

Proof. First, given profile P, for any $a, b \in A$, a Pareto dominates b means that $a \succ_j b$ for all $j \in N$. Then a, the Pareto dominator, is never ranked last in any $\succ_j$. Therefore, $\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a] \geqslant \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = b]$, which completes the proof of (1). Second, we prove (2). For any profile $P \in \mathcal{L}(A)^n$,

$$|\{j \in N : a^j_{last} = \mathrm{CW}(P)\}| \leqslant \lceil \frac{n}{2} \rceil - 1,$$

otherwise $\mathrm{CW}(P)$ will be the Condorcet loser. Therefore,

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = \mathrm{CW}(P)] \geqslant \frac{\lceil \frac{n}{2} \rceil - 1}{n} \cdot \frac{1}{(m-1)\mathrm{e}^{\epsilon}+1} + \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} \cdot \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}.$$

For any $a \neq \mathrm{CW}(P)$,

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a] \leqslant \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}.$$

Hence, we have

$$\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = \mathrm{CW}(P)]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a]} \geqslant \frac{\left(\lfloor \frac{n}{2} \rfloor - 1\right)\mathrm{e}^{\epsilon} + \lceil \frac{n}{2} \rceil + 1}{n\mathrm{e}^{\epsilon}},$$

which completes the proof. Finally, we prove (3). Given a profile P that $\mathrm{CL}(P)$ exists,

$$|\{j \in N : a^j_{last} = a\}| \leqslant \lceil \frac{n}{2} \rceil - 1,$$

otherwise a will be the Condorcet loser. Therefore,

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a] \geqslant \frac{\lceil \frac{n}{2} \rceil - 1}{n} \cdot \frac{1}{(m-1)\mathrm{e}^{\epsilon}+1} + \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} \cdot \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}.$$

Hoewver,

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = \mathrm{CL}(P)] \leqslant \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}.$$

Hence we have

$$\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = \mathrm{CL}(P)]} \geqslant \frac{\left(\lfloor \frac{n}{2} \rfloor - 1\right)\mathrm{e}^{\epsilon} + \lceil \frac{n}{2} \rceil + 1}{n\mathrm{e}^{\epsilon}},$$

which completes the proof of (3). □

Proposition 11. *Given $\epsilon \in \mathbb{R}_+$, CWRR satisfies* 1-*Pareto efficiency,* $\frac{m-1}{m}$-*SD-efficiency, and* 1-*Condorcet loser criterion.*

Proof. The bounds of Pareto efficiency and Condorcet loser criterion are evident, since for any profile P, neither a Pareto dominated alternative nor the Condorcet loser can be the Condorcet winner. Then we only need to prove the bound of SD-efficiency. Given profile P, we have

$$\sup_{P,\xi} \inf_{j,y} \frac{\sum_{x \succ_j y} \mathbb{P}[\xi = x]}{\sum_{x \succ_j y} \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = x]} \leqslant \frac{1}{1 - \sup_{P} \inf_{j} \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a^j_{last}]}.$$

Then there are two possible cases for the profile, discussed as follows

1. If CW(P) exists, then there must exist some j that $a^j_{last} \neq \mathrm{CW}(P)$. Therefore

$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a^j_{last}] = \frac{1}{\mathrm{e}^\epsilon + m - 1}.$$

2. If CW(P) does not exist, then

$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a^j_{last}] = \frac{1}{m} \geqslant \frac{1}{\mathrm{e}^{\epsilon + m - 1}}.$$

In other words, we have

$$\sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a^j_{last}] = \frac{1}{m},$$

which indicates that CWRR satisfies $\frac{m-1}{m}$-SD-efficiency. That completes the proof. □

Proposition 12. *Given $\epsilon \in \mathbb{R}_+$, CLRR satisfies* 1*-Pareto efficiency,* $\frac{(m-2)\mathrm{e}^\epsilon+1}{(m-1)\mathrm{e}^\epsilon+1}$*-SD-efficiency, and* 1*-Condorcet criterion.*

Proof. The bounds of Pareto efficiency and Condorcet criterion are evident, since for any profile P, neither a Pareto dominator nor the Condorcet winner can be the Condorcet loser. Then we only need to prove the bound of SD-efficiency. Given profile P, we have

$$\sup_{P,\xi} \inf_{j,y} \frac{\sum_{x \succ_j y} \mathbb{P}[\xi = x]}{\sum_{x \succ_j y} \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = x]} \leqslant \frac{1}{1 - \sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a^j_{last}]}.$$

Then there are two possible cases for the profile, discussed as follows

1. If CL(P) exists, considering the profile P, where each $a^j_{last} \neq \mathrm{CL}(P)$ for each $j \in N$, we have

$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a^j_{last}] = \frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon + 1}.$$

2. If CL(P) does not exist, then

$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a^j_{last}] = \frac{1}{m} \leqslant \frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon + 1}.$$

In other words, we have

$$\sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a^j_{last}] = \frac{1}{m},$$

which indicates that CWRR satisfies $\frac{(m-2)\mathrm{e}^\epsilon+1}{(m-1)\mathrm{e}^\epsilon+1}$-SD-efficiency. That completes the proof. □

B.2 Proofs of Theorems 1-6

Theorem 1. *There is no voting rule satisfying ϵ-DP, α-Condorcet criterion and η-Condorcet loser criterion with $\alpha \cdot \eta > \mathrm{e}^\epsilon$.*

Proof. Consider the profile P ($n = 2k + 1$):

– $k + 1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$,

– k voters: $a_m \succ a_{m-1} \succ \cdots \succ a_1$.

By definition, we have $\mathrm{CW}(P) = a_1$, since $w_P[a_1, a_i] = 1$, for all $a_i \neq a_1$. Now consider another profile P' with the same number of voters:

– k voters: $a_1 \succ' a_2 \succ' \cdots \succ' a_m$,
– $k+1$ voters: $a_m \succ a_{m-1} \succ \cdots \succ a_1$.

Then $w_P[a_1, a_i] = -1$, for all $a_i \in A \backslash \{a\}$, i.e., a_1 is a Condorcet loser. Since there is only one voter changes her preference from P to P', we have

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \alpha \cdot \mathbb{P}[f(P) = a_2] && (\alpha\text{-Condorcet criterion})\\
&\geqslant \alpha\eta \cdot \mathbb{P}[f(P) = a_m] && (\eta\text{-Condorcet loser criterion})\\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha\eta \cdot \mathbb{P}[f(P') = a_m] && (\epsilon\text{-DP})\\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha^2 \cdot \eta \cdot \mathbb{P}[f(P') = a_2] && (\alpha\text{-Condorcet criterion})\\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha^2 \cdot \eta^2 \cdot \mathbb{P}[f(P') = a_1] && (\eta\text{-Condorcet loser criterion})\\
&\geqslant \mathrm{e}^{-2\epsilon} \cdot \alpha^2 \cdot \eta^2 \cdot \mathbb{P}[f(P') = a_1], && (\epsilon\text{-DP})
\end{aligned}$$

which indicates that $\mathrm{e}^{-2\epsilon}\alpha^2\eta^2 \leqslant 1$, i.e., $\alpha\eta \leqslant \mathrm{e}^{\epsilon}$. That completes the proof. □

Theorem 2. *If a neutral voting rule $f \colon \mathcal{L}(A)^n \to A$ satisfies ϵ-DP, β-Pareto efficiency, and α-Condorcet criterion, then $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$.*

Proof. Consider the following profile P ($n = 2k+1$):

– $k+1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$;
– k voters: $a_2 \succ \cdots \succ a_m \succ a_1$.

By definition, we have $w_P[a_1, a_i] = 1$, for all $a_i \in A \backslash \{a_1\}$, which indicates that $\mathrm{CW}(P) = a_1$. Also notice that $w_P[a_i, a_j] = n$ for all $i < j$. Thus, a_i Pareto dominates a_j for all $i < j$. The relations among all alternatives are shown in the following graph.

$$a_1 \xrightarrow{\textbf{Condorcet Winner}} a_2 \xrightarrow{\textbf{Pareto}} a_3 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_m. \tag{2}$$

Since f satisfies α-Condorcet criterion and β-Pareto efficiency, we have

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \alpha \cdot \mathbb{P}[f(P) = a_2]\\
&\geqslant \alpha\beta \cdot \mathbb{P}[f(P) = a_3]\\
&\geqslant \cdots\\
&\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m].
\end{aligned}$$

Now, consider another profile P', where all voters' preferences are exactly the same:

$$a_m \succ a_{m-1} \succ \cdots \succ a_1.$$

Then we have the following graph.

$$a_m \xrightarrow{\textbf{Condorcet Winner}} a_{m-1} \xrightarrow{\textbf{Pareto}} a_{m-2} \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_1.$$

Similarly, we have

$$\mathbb{P}[f(P') = a_m] \geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_1].$$

Notice that $|\{j \in N : \succ_j \neq \succ'_j\}| = n$. Therefore,

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m] \\
&\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_m] && (\epsilon\text{-DP}) \\
&\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P') = a_1] \\
&\geqslant \mathrm{e}^{-2n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P) = a_1]. && (\epsilon\text{-DP})
\end{aligned}$$

Then $\mathrm{e}^{-2n\epsilon}\alpha^2\beta^{2m-4} \leqslant 1$, i.e., $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof. □

Theorem 3. *If a neutral voting rule $f \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies ϵ-DP, β-Pareto efficiency, and α-Condorcet loser criterion, then $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$.*

Proof. Consider the following profile P ($n = 2k+1$):

- $k+1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$;
- k voters: $a_2 \succ \cdots \succ a_m \succ a_1$.

By definition, we have $w_P[a_1, a_i] = 1$, for all $a_i \in A \backslash \{a_1\}$, which indicates that $\mathrm{CL}(P) = a_1$. Also notice that $w_P[a_i, a_j] = n$ for all $i < j$. Thus, a_i Pareto dominates a_j for all $i < j$. The relations among all alternatives are shown in the following graph.

$$a_1 \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_{m-1} \xrightarrow{\textbf{Condorcet Loser}} a_m.$$

Since f satisfies α-Condorcet loser criterion and β-Pareto efficiency, we have

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \beta \cdot \mathbb{P}[f(P) = a_2] \\
&\geqslant \cdots \\
&\geqslant \beta^{m-2} \cdot \mathbb{P}[f(P) = a_{m-1}] \\
&\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m].
\end{aligned}$$

Now, consider another profile P', where all voters' preferences are exactly the same:

$$a_m \succ a_{m-1} \succ \cdots \succ a_1.$$

Then we have the following graph.

$$a_m \xrightarrow{\textbf{Pareto}} a_{m-1} \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{Condorcet Loser}} a_1.$$

Similarly, we have

$$\mathbb{P}[f(P') = a_m] \geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_1].$$

Notice that $|\{j \in N : \succ_j \neq \succ'_j\}| = n$. Therefore,

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m] \\
&\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_m] && (\epsilon\text{-DP}) \\
&\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P') = a_1] \\
&\geqslant \mathrm{e}^{-2n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P) = a_1]. && (\epsilon\text{-DP})
\end{aligned}$$

Then $\mathrm{e}^{-2n\epsilon}\alpha^2\beta^{2m-4} \leqslant 1$, i.e., $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof. □

Proposition 13. *Condorcet method satisfies SD-efficiency on $\mathcal{D}_C$.*

Proof. Let P be an arbitrarily chosen profile in $\mathcal{D}_C$. Then we only need to proof that there does not exist $\xi \in \mathcal{R}(A)$ that SD-dominates $\mathsf{CM}(P)$.

In fact, if there exists such a ξ, we can obtain by definition that

$$\sum_{b \succ_j a} \mathbb{P}[\xi = b] \geqslant \sum_{b \succ_j a} \mathbb{P}[\mathsf{CM}(P) = b], \quad \text{for all } j \in N \text{ and } a \in A.$$

Since for any $a \in A$ that $\mathrm{CW}(P) \succ_j a$, we have

$$\sum_{b \succ_j a} \mathbb{P}[\mathsf{CM}(P) = b] = \mathbb{P}[\mathsf{CM}(P) = \mathrm{CW}(P)] = 1,$$

which indicates that

$$\sum_{b \succ_j a} \mathbb{P}[\xi = b] \geqslant 1.$$

Therefore, for any $a \in A$ that $\mathrm{CW}(P) \succ_j a$, $\mathbb{P}[\xi = a] = 0$. However, according to the definition of $\mathrm{CW}(P)$, each $a \in A$ must be ranked behind $\mathrm{CW}(P)$ in some $\succ_j$. Hence we have $\xi = \mathsf{CM}(P)$, a contradiction. □

Theorem 4. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, α-Condorcet criterion, and γ-SD efficiency with $\gamma > \frac{\alpha+m-1-\alpha \mathrm{e}^{-n\epsilon}}{\alpha+m-1}$.*

Proof. Consider the profile P, where all voters' vote are exactly the same, i.e.,

$$a_1 \succ_j a_2 \succ_j \cdots \succ_j a_m, \quad \text{for all } j \in N.$$

It is not hard to see that $\mathrm{CW}(P) = a_1$. Since f satisfies α-Condorcet criterion, we have $\mathbb{P}[f(P) = a] \leqslant \mathbb{P}[f(P) = a_1]/\alpha$, for all $a \in A \backslash \{a_1\}$. Therefore,

$$1 = \mathbb{P}[f(P) = a_1] + \sum_{a \in A \backslash \{a_1\}} p[f(P) = a] \leqslant \left(1 + \frac{m-1}{\alpha}\right) \mathbb{P}[f(P) = a_1],$$

i.e., $\mathbb{P}[f(P) = a_1] \geqslant \frac{\alpha}{\alpha+m-1}$. Further, by Equation (1), we have

$$\mathbb{P}[f(P) = a_m] \geqslant \mathrm{e}^{-n\epsilon} \cdot \mathbb{P}[f(P) = a_1] \geqslant \frac{\alpha \mathrm{e}^{-n\epsilon}}{\alpha + m - 1}.$$

However, for profile P, the unique SD-efficient lottery is $\mathbb{1}_{a_1}$. In other words, all lotteries $\xi \in \mathcal{R}(A)$ that $\xi \neq \mathbb{1}_{a_1}$ are γ-SD-dominated by $\mathbb{1}_{a_1}$ with $\gamma > 1$. Further,

$$\begin{aligned}
\frac{\sum_{x \succ y} \mathbb{P}[\mathbb{1}_{a_1} = x]}{\sum_{x \succ y} \mathbb{P}[f(P) = x]} \geqslant & \frac{\inf_{y \in A} \sum_{x \succ y} \mathbb{P}[\mathbb{1}_{a_1} = x]}{\sup_{y \in A} \sum_{x \succ y} \mathbb{P}[f(P) = x]} \\
= & \frac{1}{1 - \mathbb{P}[f(P) = a_m]} \\
\geqslant & \frac{1}{1 - \frac{\alpha \mathrm{e}^{-n\epsilon}}{\alpha+m-1}} \\
= & \frac{\alpha + m - 1 - \alpha \mathrm{e}^{-n\epsilon}}{\alpha + m - 1},
\end{aligned}$$

i.e., $\mathbb{1}_{a_1}$ can $\frac{\alpha+m-1-\alpha \mathrm{e}^{-n\epsilon}}{\alpha+m-1}$-dominates $f(P)$, which completes the proof. □

Theorem 5. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, η-Condorcet loser criterion, and γ-SD-efficiency with $\gamma > \frac{\mathrm{e}^{n\epsilon}-\eta}{\mathrm{e}^{n\epsilon}}$.*

Proof. Let $m > 2$. Consider the following profile P with $k(m-2)$ voters ($k \geqslant 1$).

- k voters: $y \succ a_1 \succ a_2 \succ \cdots \succ x \succ a_{m-2}$,
- k voters: $y \succ a_2 \succ a_3 \succ \cdots \succ x \succ a_1$,
- k voters: $y \succ a_3 \succ a_4 \succ \cdots \succ x \succ a_2$,
- k voters: $\cdots$,
- k voters: $y \succ a_{m-2} \succ a_1 \succ \cdots \succ x \succ a_{m-3}$.

Then it is quite evident that $\mathrm{CL}(P) = x$. Since f satisfies η-Condorcet loser criterion, we have $\mathbb{P}[f(P) = a] \geqslant \eta \cdot \mathbb{P}[f(P) = x]$, for all $a \in A\backslash\{x\}$. By Equation (1), we have $\mathbb{P}[f(P) = x] \geqslant \mathrm{e}^{-n\epsilon}$. Since f satisfies η-Condorcet loser criterion,

$$\mathbb{P}[f(P) = a] \geqslant \eta \cdot \mathbb{P}[f(P) = x] \geqslant \alpha \mathrm{e}^{-n\epsilon}, \quad \text{for all } a \in A\backslash\{x\}.$$

However, the unique SD-efficient lottery of P is $\mathbb{1}_y$, since $\mathbb{1}_y$ can SD-dominates any other lotteries on A. Further,

$$\begin{aligned}
\frac{\sum\limits_{b \succ a} \mathbb{P}[\mathbb{1}_y = b]}{\sum\limits_{b \succ a} \mathbb{P}[f(P) = b]} \geqslant & \frac{\inf\limits_{a \in A} \sum\limits_{b \succ a} \mathbb{P}[\mathbb{1}_y = b]}{\sup\limits_{a \in A} \sum\limits_{b \succ a} \mathbb{P}[f(P) = b]} \\
= & \frac{1}{1 - \inf \min\limits_{1 \leqslant i \leqslant m-2} \mathbb{P}[f(P) = a_i]} \\
\geqslant & \frac{1}{1 - \alpha \mathrm{e}^{-n\epsilon}}.
\end{aligned}$$

In other words, $\mathbb{1}_y$ can $\frac{\mathrm{e}^{n\epsilon}-\eta}{\mathrm{e}^{n\epsilon}}$-SD-dominates $f(P)$, which completes the proof. □

Theorem 6. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, γ-SD-efficiency, and β-Pareto efficiency with $\gamma > \frac{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}}{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}+\beta-1}$.*

Proof. Consider the profile P, where all voters' preferences are the same, i.e.,

$$a_1 \succ_j a_2 \succ_j \cdots \succ_j a_m, \quad \text{for all } j \in N.$$

By definition, for all $i < j$, any a_i Pareto dominates a_j in profile P. In other words, we have the following diagram

$$a_1 \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_m.$$

Since f satisfies β-Pareto efficiency, $\mathbb{P}[f(P) = a_{i+1}] \leqslant \beta \cdot \mathbb{P}[f(P) = a_i]$ holds for any $i < m$. By Equation (1), $\mathbb{P}[f(P) = a_m] \geqslant \mathrm{e}^{-n\epsilon} \cdot \mathbb{P}[f(P) = a_1]$. Further,

$$\begin{aligned}
&\mathbb{P}[f(P) = a_1] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = a_m], \\
&\mathbb{P}[f(P) = a_2] \leqslant \frac{1}{\beta} \cdot \mathbb{P}[f(P) = a_1] \leqslant \frac{\mathrm{e}^{n\epsilon}}{\beta} \cdot \mathbb{P}[f(P) = a_m], \\
&\cdots \\
&\mathbb{P}[f(P) = a_{m-1}] \leqslant \frac{1}{\beta^{m-2}} \cdot \mathbb{P}[f(P) = a_1] \leqslant \frac{\mathrm{e}^{n\epsilon}}{\beta^{m-2}} \cdot \mathbb{P}[f(P) = a_m].
\end{aligned}$$

By summing up the above inequalities, we have

$$1 = \sum_{a \in A} \mathbb{P}[f(P) = a] \leqslant \left(1 + \left(1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{m-2}}\right) \mathrm{e}^{n\epsilon}\right) \cdot \mathbb{P}[f(P) = a_m],$$

i.e., $\mathbb{P}[f(P) = a_m] \geqslant \frac{\beta - 1}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}$. However, the unique SD-efficient lottery of P is $\mathbb{1}_{a_1}$, since it can SD-dominate any other lottery. Further, we have

$$\begin{aligned}\frac{\sum_{b \succ a} \mathbb{P}[\mathbb{1}_y = b]}{\sum_{b \succ a} \mathbb{P}[f(P) = b]} \geqslant & \frac{\inf_{a \in A} \sum_{b \succ a} \mathbb{P}[\mathbb{1}_y = b]}{\sup_{a \in A} \sum_{b \succ a} \mathbb{P}[f(P) = b]} \\ = & \frac{1}{1 - \mathbb{P}[f(P) = a_m]} \\ \geqslant & \frac{1}{1 - \frac{\beta - 1}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}}.\end{aligned}$$

In other words, $\mathbb{1}_{a_1}$ can $\frac{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m}}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}$-SD-dominates $f(P)$, which completes the proof. $\square$

C More Experimental Figures

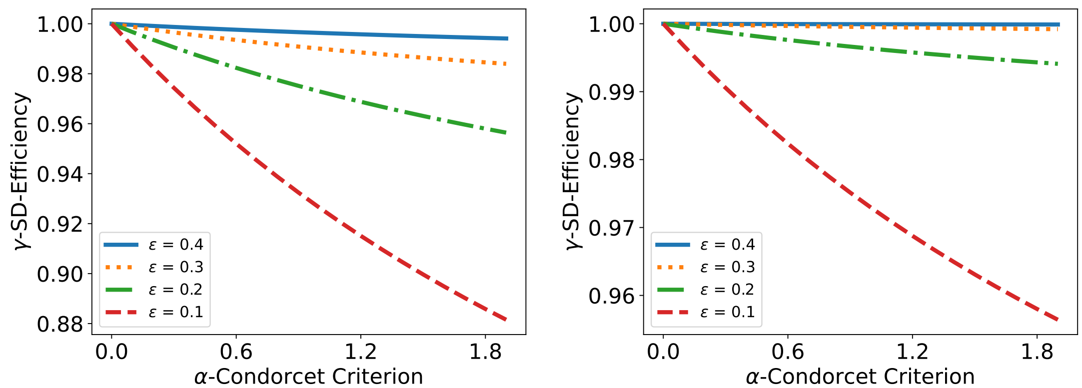

Fig. 1. Tradeoff curves between α-Condorcet criterion and γ-SD-efficiency under ϵ-DP (upper bounds). Left: $m = 5, n = 10$. Right: $m = 5, n = 20$.

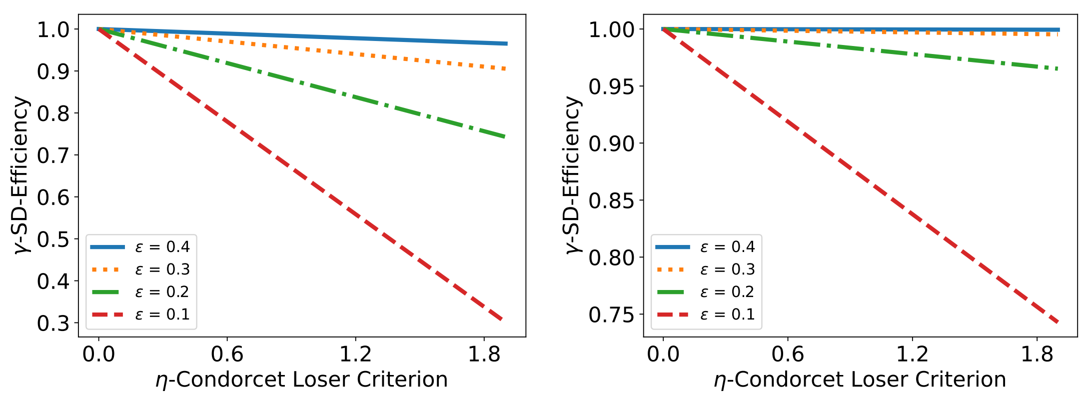

Fig. 2. Tradeoff curves between η-Condorcet loser criterion and γ-SD-efficiency under ϵ-DP (upper bounds). Left: $m = 5, n = 10$. Right: $m = 5, n = 20$.