Trading Off Voting Axioms for Privacy

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Abstract

In this paper, we investigate tradeoffs among differential privacy (DP) and several important voting axioms: Pareto efficiency, SD-efficiency, PC-efficiency, Condorcet criterion, and Condorcet loser criterion. We provide upper and lower bounds on the two-way tradeoffs between DP and each axiom. We also provide upper and lower bounds on three-way tradeoffs among DP and every pairwise combination of all the axioms, showing that, while the axioms are compatible without DP, their upper bounds cannot be achieved simultaneously under DP. Our results illustrate the effect of DP on the satisfaction and compatibility of voting axioms.

1 INTRODUCTION

Voting is a popular method for collective decision making. In a typical voting procedure, voters express their preferences over alternatives, and a winner is determined according to a voting rule. In modern social choice theory, voting rules are evaluated and compared by the axiomatic approach [Plott, 1976], w.r.t. their satisfaction of various normative properties, known as voting axioms. For example, Pareto efficiency mandates that any alternative which is Pareto-dominated—that is, an alternative deemed inferior to another by all voters—must not be selected as the winner.

Differential privacy (DP) has emerged as a *de facto* standard for privacy preservation, with widespread applications in machine learning [Abadi et al., 2016, Vasa and Thakkar, 2023, Sarwate and Chaudhuri, 2013], data mining [Friedman and Schuster, 2010, Zhang et al., 2011], and recommendation systems [Berlioz et al., 2015, Li et al., 2020], among others. Recently, privacy concerns have also gained significant attention in the context of voting schemes. For instance, Liu et al. [2020] demonstrate that traditional voting rules are susceptible to background knowledge attacks. In their example, Alice submits an anonymous ballot in an election. However, by analyzing other voters' social media activity, an adversary infers that the remaining votes result in a tie. Consequently, the adversary can deduce Alice's vote, as it must be the deciding vote that breaks the tie. This illustrates that even the disclosure of voting outcomes, such as the announcement of the winner, can compromise voter privacy.

To date, a significant body of research has explored differential privacy in the context of voting and rank aggregation [Shang et al., 2014, Hay et al., 2017, Yan et al., 2020]. These studies primarily focus on the privacy-utility tradeoff in voting rules (or rank aggregations), where utility is typically measured by accuracy or mean square error. Although some prior work [Lee, 2015, Li et al., 2023] has investigated the tradeoff between DP and certain voting axioms through specific mechanisms, the broader tradeoff between DP and voting axioms remains largely unexplored. Additionally, the relationship between voting axioms may be altered by the introduction of DP. For instance, in traditional social choice theory, the Condorcet criterion is compatible with Pareto efficiency, where the Condorcet winner (the alternative that defeats all others in pairwise comparisons) must win the election. However, under DP, the optimal approximations of the Condorcet criterion and Pareto efficiency become incompatible. This incompatibility arises because DP necessitates randomness in voting rules, meaning that all alternatives must have a nonzero probability to win. Consequently, under DP, a stronger Condorcet criterion requires the Condorcet winner to win with a higher probability, while a stronger Pareto efficiency focuses on the pairwise differences in winning probabilities (i.e., a Pareto-dominated alternative must have a lower winning probability than its dominating counterpart). Till now, the incompatibility between axioms induced by DP has not been systematically studied. Thus, the following question remains open.

What is the tradeoff among DP and voting axioms?

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Table 1: Two-way tradeoffs between ϵ-DP and approximate voting axioms.

Axiom	Upper Bound	Lower Bound	Reference
β-Pareto efficiency[†]	$\sup\beta \leqslant \mathrm{e}^{\frac{n\epsilon}{m-1}}$	$\sup\beta \geqslant \mathrm{e}^{\frac{n\epsilon}{2m-2}}$	Mechanism 1 and Propositions 1–2
γ-SD-efficiency	$\sup\gamma \leqslant \dfrac{(m-1)\mathrm{e}^{n\epsilon}}{(m-1)\mathrm{e}^{n\epsilon}+1}$	$\sup\gamma \geqslant \dfrac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}$	Mechanism 2 and Propositions 3–4
κ-PC-efficiency	$\sup\kappa = 0$	Not applicable[‡]	Proposition 5
α-Condorcet criterion	$\sup\alpha \leqslant \mathrm{e}^{\epsilon}$ [Li et al., 2023]	$\sup\alpha = \mathrm{e}^{\epsilon}$	Mechanism 3 and Proposition 6
η-Condorcet loser criterion	$\sup\eta \leqslant \mathrm{e}^{\epsilon}$	$\sup\eta = \mathrm{e}^{\epsilon}$	Mechanism 4 and Propositions 7–8

[†] This approximate axiom requires that the winning probability of each Pareto dominated alternative never exceeds $1/\beta$ of the winning probability of its dominant alternative, where $\beta \in (0, +\infty)$. It is evident that a larger β represents a higher level of Pareto efficiency. Other approximate axioms are defined in the similar way (by the ratio), see Definitions 2–6 for details.

[‡] Since the upper bound is 0, there is no lower bound.

Table 2: Three-way tradeoffs between approximate voting axioms under ϵ.

Combination of Axioms[†]		Compatibility[‡]	Upper Bound under ϵ-DP[‡]	Reference
α-Condorcet criterion	η-Condorcet loser criterion	✓	$\alpha \cdot \eta \leqslant \mathrm{e}^{\epsilon}$	Theorem 1
α-Condorcet criterion	β-Pareto efficiency	✓	$\alpha \cdot \beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$	Theorem 2
η-Condorcet loser criterion	β-Pareto efficiency	✓	$\eta \cdot \beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$	Theorem 3
α-Condorcet criterion	γ-SD-efficiency	✓	$\gamma \leqslant \dfrac{\alpha+m-1-\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}$	Theorem 4
η-Condorcet loser criterion	γ-SD-efficiency	✓	$\gamma \leqslant \dfrac{\mathrm{e}^{n\epsilon}-\eta}{\mathrm{e}^{n\epsilon}}$	Theorem 5
β-Pareto efficiency	γ-SD-efficiency	✓	$\gamma \leqslant \dfrac{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}}{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}+\beta-1}$	Theorem 6

[†] PC-efficiency is not considered in 3-way tradeoffs, since even its approximate version is not incompatible with DP.

[‡] A "✓" indicates that the standard versions of the axioms are compatible in non-private settings (where DP is not required). The compatibility between Condorcet criterion and SD-efficiency is shown in Proposition 9. The other compatibilities are quite evident.

[‡] The lower bounds of 3-way tradeoffs are also given by Mechanisms 1–4. Please refer to Table 3 and Appendix B.1 for details.

Since DP inherently requires randomness in voting rules, standard axioms are generally incompatible with DP [Li et al., 2023]. Therefore, we propose the approximate versions of voting axioms and investigate the tradeoff among DP and these approximate axioms.

Our contributions Our conceptual contribution involves proposing approximate versions of Pareto efficiency, SD-efficiency, PC-efficiency, and the Condorcet loser criterion (Definitions 2–6).

Our theoretical contributions are two-fold. First, we explore the 2-way tradeoff between DP and a single (approximate) axiom. We establish tradeoff theorems with both upper and lower bounds for the approximate Pareto, SD, and PC efficiencies (Propositions 1–5). For the Condorcet criterion and Condorcet loser criterion, we derive tight bounds for their tradeoffs with DP (Propositions 6–8). These results are summarized in Table 1, where the expressions on the lines illustrate the bounds of the approximate axioms under ϵ-DP. Second, we investigate the 3-way tradeoff between DP and pairs of axioms (referred to as tradeoff axioms under DP in this paper). Since even the approximate version of PC-efficiency cannot be achieved under DP, we only focus on the Condorcet criterion, Condorcet loser criterion, Pareto efficiency, and SD-efficiency in 3-way tradeoffs. We derive lower bounds for these tradeoffs through several mechanisms, as summarized in Table 3 in Section 4. Then we establish upper bounds for all pairwise tradeoffs under DP (Theorems 1–6). Our results demonstrate that while these axioms are compatible without DP, a tradeoff exists between each pair under DP. These findings are summarized in Table 2, where the expressions on the lines illustrate the upper bounds of the three-way tradeoffs between the axioms.

Algorithmically, we propose a family of randomized voting rules capable of achieving smooth two-way tradeoffs between DP and the aforementioned axioms (Mechanisms 1–4). For three-way tradeoffs, we introduce a mechanism that flexibly balances the Condorcet criterion and Condorcet loser criterion.

Related work and discussions To the best of our knowledge, the application of DP to rank aggregation (a generalized problem of voting) was first introduced by Shang et al. [2014], who derived upper bounds on the error rate under DP. In a similar vein, Lee [2015] proposed a tournament voting rule that achieves both DP and robustness to strategic manipulations. Hay et al. [2017] employed the Laplace and exponential mechanisms to enhance the privacy of Quicksort and Kemeny-Young methods. Kohli and Laskowski [2018] investigated DP, strategyproofness, and anonymity in voting on single-peaked preferences. Torra [2019] analyzed the privacy-preserving properties of random dictatorship, a well-known randomized voting rule, using DP. Their study identified conditions under which random dictatorship satisfies DP and proposed improvements for general cases. Wang et al. [2019] examined the privacy of positional voting and introduced a noise-adding mechanism that outperforms the naive Laplace mechanism in accuracy. Yan et al. [2020] addressed the tradeoff between accuracy and privacy in rank aggregation by achieving local DP through the Laplace mechanism and randomized response mechanism.

Most prior works have not thoroughly examined the tradeoff between privacy and voting axioms, and their privacy bounds are often not tight. Liu et al. [2020] introduced distributional DP [Bassily et al., 2013] to voting, analyzing the privacy levels of several common voting rules but without proposing methods to enhance privacy. Li et al. [2023] proposed a novel family of DP voting rules, which satisfies several "probabilistic" voting axioms, including (probabilistic) Condorcet criterion, Pareto efficiency, monotonicity, strategyproofness, and participation. However, among them, only the approximate version (with an approximation parameter) of Condorcet criterion was considered. Though the authors discussed the DP-axiom tradeoff shortly, their discussions were limited to some incompatibility results and an upper bound of approximate Condorcet criterion under DP, which are incomplete. A more comprehensive investigation remains necessary. Beyond social choice, DP has been applied to other economic domains, including mechanism design [Pai and Roth, 2013, Xiao, 2013], matching [Hsu et al., 2016], and resource allocation [Kannan et al., 2018, Chen et al., 2023].

In social choice theory, randomized voting has been extensively studied [Brandt, 2017], with much of the literature focusing on standard axiomatic properties such as manipulation complexity [Walsh and Xia, 2012], strategyproofness [Aziz et al., 2014, 2015], Pareto efficiency [Brandl et al., 2015, Gross et al., 2017], participation [Brandl et al., 2019], and monotonicity [Brandl et al., 2022]. Fairness properties in sortition have also been investigated [Benadè et al., 2019, Flanigan et al., 2020, 2021]. For approximate axiomatic properties, Procaccia [2010] examined how closely a strategyproof randomized rule could approximate a deterministic rule, while Birrell and Pass [2011] studied approximate strategyproofness for randomized rules. However, these approximate axioms are not naturally aligned with DP, as they rely on utility differences rather than ratios.

2 PRELIMINARIES

Let $A = \{a_1, a_2, \dots, a_m\}$ denote a set of $m \geqslant 2$ alternatives. For any $n \in \mathbb{N}$, let $N = \{1, 2, \dots, n\}$ be a set of voters. For each $j \in N$, the vote of voter j is a linear order $\succ_j \in \mathcal{L}(A)$, where $\mathcal{L}(A)$ denotes the set of all linear orders over A, i.e., all transitive, anti-reflexive, anti-symmetric, and complete binary relations. $P = \{\succ_1, \succ_2, \dots, \succ_n\}$ denotes the *(preference) profile*, which is the collection of n votes. For any $j \in N$, let $P_{-j} = \{\succ_1, \dots, \succ_{j-1}, \succ_{j+1}, \dots \succ_n\}$ denote the profile of removing the j-th vote from P.

Under the settings above, a *(randomized) voting rule* can be defined as a mapping $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the set of all random variables on A (usually called *lotteries* in social choice theory). Given a voting rule f and a profile P, the winning probability of alternative $a \in A$ is denoted by $\mathbb{P}[f(P) = a]$. A voting rule f is *neutral* if for any profile P and any permutation σ on A, $\sigma \cdot f(P) = f(\sigma \cdot P)$.

Differential privacy (DP, [Dwork, 2006]) At a high level, DP requires a function to have similar output (distribution) when inputting *neighboring databases*. Here, we say two databases are *neighboring* if one can be gotten by replacing no more than one entry from the other database.

Definition 1 (ϵ-Differential Privacy [Dwork, 2006]). *Given a privacy budget $\epsilon \in [0, +\infty)$, mechanism $f : \mathcal{D} \to \mathcal{O}$ satisfies ϵ-differential privacy (ϵ-DP for short) if for all $O \subseteq \mathcal{O}$ and each pair of neighboring databases $P, P' \in \mathcal{D}$,*

$$\mathbb{P}[f(P) \in O] \leqslant \mathrm{e}^{\epsilon} \cdot \mathbb{P}[f(P') \in O].$$

The probability of the above inequality is taken over the randomness of the mechanism. The smaller ϵ is, the better privacy guarantee can be offered. In the context of social choice, the mechanism f in Definition 1 is a voting rule and its domain is $\mathcal{D} = \mathcal{L}(A)^n$. Further, the pair of neighboring databases P, P' are two profiles differing on no more than one vote, i.e., there exists a voter $j \in N$ that $P_{-j} = P'_{-j}$.

Axioms of voting The axioms considered in the paper can be roughly divided into two parts: axioms that only depend on preferences over alternatives and axioms that depend on preferences over lotteries in $\mathcal{R}(A)$. We adopt the notions in [Brandt, 2017] here.

For clarity, we define the following notions before presenting the axioms. Given a profile P and two distinct alternatives $a, b \in A$, let $w_P[a, b]$ denote the *majority margin* of a

over b, which equals to the number of voter that consider $a \succ b$ minus the number of voter that consider $b \succ a$, i.e.,

$$w_P[a,b] = |\{j \in N : a \succ_j b\}| - |\{j \in N : b \succ_j a\}|.$$

Here, $a \succ_j b$ represents that the j-th voter prefers a to b. The *Condorcet winner* of P (denoted as $\mathrm{CW}(P)$) is an alternative $a \in A$ such that $w_P[a,b] > 0$ for all $b \neq a$. Similarly, the *Condorcet loser* of P (denoted as $\mathrm{CL}(P)$) is the alternative $a \in A$ which satisfies $w_P[a,b] < 0$ for all $b \neq a$. Based on these notions, the first part of the axioms are listed below.

- *Condorcet criterion (CC)*: A voting rule f satisfies Condorcet criterion if $\mathbb{P}[f(P) = \mathrm{CW}(P)] = 1$ holds for every profile P that $\mathrm{CW}(P)$ exists;
- *Condorcet loser criterion (CLC)*: A voting rule f satisfies Condorcet loser criterion if $\mathbb{P}[f(P) = \mathrm{CL}(P)] = 0$ holds for every profile P that $\mathrm{CL}(P)$ exists;
- *Pareto efficiency (PE)*: A voting rule f satisfies Pareto efficiency if $\mathbb{P}[f(P) = b] = 0$ for every profile $P \in \mathcal{L}(A)^n$, where $b \in A$ is Pareto dominated by some $a \in A$, i.e., $a \succ_j b$ for all $j \in N$.

The second part of axioms (efficiency notions except Pareto efficiency) considers the relationship between lotteries. For clarity, we define the following relationships before presenting the axioms.

Stochastic Dominance (SD): Given the j-th vote $\succ_j \in \mathcal{L}(A)$ and two lotteries $\xi, \zeta \in \mathcal{R}(A)$, ξ is more desirable under SD for the j-th voter (denoted by $\xi \succeq_j^{SD} \zeta$) if and only if for every $y \in A$, the probability of ξ selecting an alternative better than y is no less than the probability for ζ to select such an alternative, i.e.,

$$\sum_{\substack{x \in A, \\ x \succ y}} \mathbb{P}[\xi = x] \geqslant \sum_{\substack{x \in A, \\ x \succ y}} \mathbb{P}[\zeta = x], \quad \text{for all } y \in A. \tag{1}$$

We say ξ is strictly more desirable than ζ by means of SD for the j-th voter (denoted by $\xi \succ_j^{SD} \zeta$) if and only if $\xi \succeq_j^{SD} \zeta$ holds and $\zeta \succeq_j^{SD} \xi$ does not hold.

Pairwise Comparison (PC): Given the j-th vote $\succ_j \in \mathcal{L}(A)$ and two lotteries $\xi, \zeta \in \mathcal{R}(A)$, ξ is more desirable under PC for the j-th voter (denoted by $\xi \succeq_j^{PC} \zeta$) if and only if the probability that ξ yields a better alternative than ζ is no less than the other way round, i.e.,

$$\sum_{\substack{x, y \in A, \\ x \succ y}} \mathbb{P}[\xi = x] \cdot \mathbb{P}[\zeta = y] \geqslant \sum_{\substack{x, y \in A, \\ x \succ y}} \mathbb{P}[\zeta = x] \cdot \mathbb{P}[\xi = y].$$

Similarly, ξ is strictly more desirable than ζ by means of PC for the j-th voter (denoted by $\xi \succ_j^{PC} \zeta$) if and only if $\xi \succeq_j^{PC} \zeta$ holds and $\zeta \succeq_j^{PC} \xi$ does not hold.

Based on SD and PC, the second part of axioms (SD-efficiency and PC-efficiency) are defined as follows.

- *SD-Efficiency:* A voting rule f satisfies *SD-efficiency* if for every profile $P \in \mathcal{L}(A)^n$, there does not exist $\xi \in \mathcal{R}(A)$ satisfying both of the following two conditions.
  1. For all $j \in N$, $\xi \succeq_j^{SD} f(P)$.
  2. There exists some $j \in N$ that $\xi \succ_j^{SD} f(P)$.
- *PC-Efficiency:* A voting rule f satisfies *PC-efficiency* if for all profile $P \in \mathcal{L}(A)^n$, there does not exist $\xi \in \mathcal{R}(A)$ satisfying both of the following two conditions.
  1. For all $j \in N$, $\xi \succeq_j^{PC} f(P)$.
  2. There exists some $j \in N$ that $\xi \succ_j^{PC} f(P)$.

The relationship among the notions of efficiency mentioned in our paper can be visualized as the following diagram, where $a \to b$ indicates that a implies b.

$$\overset{\text{(Strongest)}}{\text{PC-efficiency}} \to \text{SD-efficiency} \to \overset{\text{(Weakest)}}{\text{Pareto efficiency}}$$

3 DP-AXIOMS TRADEOFF

This section investigates the tradeoff between privacy and voting axioms (2-way tradeoff). The axioms considered here can be divided into two parts, efficiency (Pareto efficiency, SD-efficiency, and PC-efficiency), and Condorcet consistency (Condorcet criterion and Condorcet loser criterion). As all five axioms are not compatible with DP, we propose their approximate versions (if has not yet been proposed in literature). With the approximate axioms, we establish both upper and lower bounds about their tradeoffs with DP, i.e., the upper and lower bounds of approximate axioms with a given privacy budget ϵ. All of the missing proofs in this section can be found in Appendix A.

3.1 DP-EFFICIENCY TRADEOFF

First of all, we discuss the tradeoff between DP and Pareto efficiency. Li et al. [2023] introduced the concept of probabilistic Pareto efficiency to address the inherent incompatibility between DP and Pareto efficiency. Probabilistic Pareto efficiency stipulates that each Pareto dominating alternative must have a higher probability of winning compared to the dominated alternative (standard Pareto efficiency requires "always winning" instead of "a higher probability of winning"). We further extend this notion by introducing a parameter β to quantify the level of Pareto efficiency.

Definition 2 (β-Pareto Efficiency, β-PE for short). *Given $\beta \in (0, +\infty)$, a voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies β-Pareto efficiency, if for each pair of alternatives $a, b \in A$ that $a \succ_j b$ for all $j \in N$, it holds that*

$$\mathbb{P}[f(P) = a] \geqslant \beta \cdot \mathbb{P}[f(P) = b].$$

Mechanism 1: BordaEXP

Input: Profile P, Noise level parameter ϵ

Output: Winning alternative a_{win}

Get Borda score $\mathsf{Borda}_P(a)$ of each alternative $a \in A$;

Compute the probability distribution $p \in \Delta(A)$, such that $p(a) \propto \mathrm{e}^{\mathsf{Borda}_P(a)\epsilon/(2m-2)}$ for all $a \in A$;

Sample $a_{win} \sim p$;

return a_{win}

In words, a voting rule satisfies β-PE if the winning probability of each Pareto dominated alternative never exceeds $1/\beta$ of the winning probability of its dominant alternative. Therefore, a larger β is more desirable and represents a higher level of Pareto efficiency. It's easy to check that 1-PE is equivalent to the probabilistic Pareto efficiency in [Li et al., 2023] and ∞-PE is equivalent to standard PE.

Next, we trade off Pareto efficiency with DP under the notion of β-PE. The following proposition provides an upper bound of β-PE under the constraint of ϵ-DP.

Proposition 1 (β-PE, Upper Bound). *For any ϵ, there is no neutral rule satisfying ϵ-DP and β-PE with $\beta > \mathrm{e}^{\frac{n\epsilon}{m-1}}$.*

Proof Sketch. Let f be a neutral rule satisfying ϵ-DP and β-Pareto efficiency. By ϵ-DP and neutrality, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = b], \quad \text{for all } P, a, b. \quad (2)$$

Then, by considering the profile P where all voters' preferences are the same, i.e., $a_1 \succ_j a_2 \succ_j \cdots \succ_j a_m$ for all $j \in N$, we have

$$\mathbb{P}[f(P) = a_1] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = a_m].$$

Theorefore, we have $\beta^{m-1} \leqslant \mathrm{e}^{n\epsilon}$, i.e., $\beta \leqslant \mathrm{e}^{\frac{n\epsilon}{m-1}}$. □

Proposition 1 provides an upper bound on the achievable level of Pareto efficiency when subject to the constraint of ϵ-DP. Next, we propose a mechanism (Mechanism 1, Borda score exponential mechanism or BordaEXP for short) to show a lower bound of the achievable approximate Pareto efficiency under DP. Technically, Mechanism 1 is an exponential mechanism that employs the Borda score as the utility metric, where the Borda score of an alternative $a \in A$ for a given profile P is defined as follows.

$$\mathsf{Borda}_P(a) = \sum_{\succ_j \in P} |\{b \in A : a \succ_j b\}|.$$

The following Proposition illustrates the lower bound of approximate Pareto efficiency achieved by Mechanism 1. Both upper and lower bound for β-PE are $\exp(\Theta(n\epsilon/m))$. We plot them in Figure 1, where we set $m = 5, n = 10$.

Proposition 2 (β-PE, Lower Bound). *Given $\epsilon \in \mathbb{R}_+$, Mechanism 1 satisfies ϵ-DP and $\mathrm{e}^{\frac{n\epsilon}{2m-2}}$-PE.*

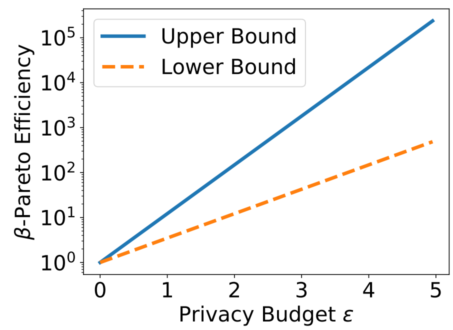

Figure 1: Tradeoff curves (upper and lower bounds) between β-PE and ϵ-DP, where $m = 5$, $n = 10$.

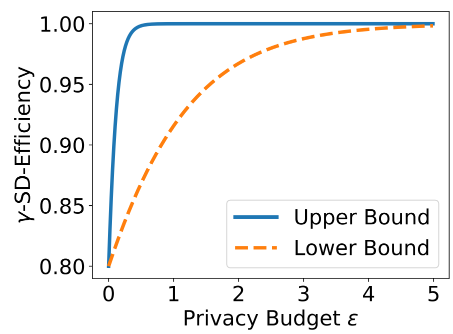

Figure 2: Tradeoff curves (upper and lower bounds) between γ-SDE and ϵ-DP, where $m = 5$, $n = 10$.

Secondly, we discuss the tradeoff between DP and SD-efficiency. As a notion stronger than Pareto efficiency, SD-efficiency is also incompatible with DP. In order to capture the incompatibility between DP and SD-efficiency, an approximate version of SD-efficiency is needed. Li et al. [2023] proposed approximate SD-strategyproofness, where they introduced a parameter in Inequality (1) (the definition of SD relationship). Their method inspires us to consider the approximate SD relationship, which is defined as follows.

Let $\xi, \zeta \in \mathcal{R}(A)$ be two lotteries and $\succ \in \mathcal{L}(A)$ be a linear order on A. Then ξ is said to γ-stochastically dominate ζ (denoted by $\xi \succeq^{\gamma-SD} \zeta$), if and only if for any $y \in A$,

$$\sum_{x \in A, x \succ y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x \in A, x \succ y} \mathbb{P}[\zeta = x].$$

Then, we define γ-SD-efficiency as follows based on the approximate SD relationship.

Definition 3 (γ-SD-Efficiency, γ-SDE for short). *Given $\gamma \in (0, 1]$, a voting rule $f : \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies γ-SD-efficiency if for each profile P, $f(P)$ is not γ-SD-dominated.*

In the definition, a larger value of γ is more desirable , since a larger γ imposes stricter conditions for a lottery to γ-SD-dominate another one. Consequently, achieving γ-SDE becomes comparatively easier as γ decreases. Especially, when $\gamma = 1$, γ-SDE reduces to the standard SD-efficiency.

Mechanism 2: RD-Anti

Input: Profile P, Noise level parameter ϵ

Output: Winning alternative a_{win}

Select a ballot $\succ_j \in P$ randomly;

Get the last-ranked alternative of $\succ_j$, denoted by a;

Compute the probability distribution $p \in \Delta(A)$, where $p(a) = \frac{1}{(m-1)\mathrm{e}^{\epsilon}+1}$ and $p(b) = \mathrm{e}^{\epsilon} \cdot p(a)$, for all $b \neq a$;

Sample $a_{win} \sim p$;

return a_{win}

Compared to the β-PE (Definition 2), the definition of γ-SDE is a bit more sophisticated, which brings obstacles to our exploration. Therefore, we develop a tool to help us judge whether a voting rule satisfies γ-SDE with a given γ, as shown in the following lemma.

Lemma 1. *Given $\gamma \in \mathbb{R}_+$, a voting rule $f \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies γ-SD-efficiency if and only if*

$$\frac{1}{\gamma} \geqslant \sup_{P \in \mathcal{L}(A)^n,\, \xi \in \mathcal{R}(A)} \inf_{j \in N,\, y \in A} \frac{\sum\limits_{x \in A, x \succ_j y} \mathbb{P}[\xi = x]}{\sum\limits_{x \in A, x \succ_j y} \mathbb{P}[f(P) = x]}.$$

Using Lemma 1, we are ready to capture the upper bound of γ-SDE under ϵ-DP, as shown in the following proposition.

Proposition 3 (γ-SDE, Upper Bound). *For any ϵ, there is no neutral voting rule satisfying ϵ-DP and γ-SDE with $\gamma > \frac{(m-1)\mathrm{e}^{n\epsilon}}{(m-1)\mathrm{e}^{n\epsilon}+1}$.*

Proposition 3 establishes an upper bound for SD-efficiency under DP. Further, building on the well-known voting rule of random dictatorship* (RD), we propose a mechanism (Mechanism 2) to explore the lower bound of achievable approximate SD-efficiency under DP. Technically, Mechanism 2 replaces the dictatorial process in RD with an exponential mechanism that adopts the anti-plurality score as its utility measure, where the anti-plurality score is defined as

$$\mathsf{Anti}_{\succ}(a) = \begin{cases} 0, & a \text{ is the last-ranked in } \succ, \\ 1, & \text{otherwise.} \end{cases}$$

Then the following proposition shows the lower bound of approximate SD-efficiency under ϵ-DP achieved by Mechanism 2. Both the upper bound and lower bound for γ-SDE are plotted in Figure 2, where we set $m = 5, n = 10$.

Proposition 4 (γ-SDE, Lower Bound). *Given $\epsilon \in \mathbb{R}_+$, Mechanism 2 satisfies ϵ-DP and $\frac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}$-SDE.*

Finally, we investigate the tradeoff between DP and PC-efficiency. Similar to SD, the approximate version of PC relationship is also needed.

*Select a voter uniformly at random and declare their top-ranked alternative as the winner.

Given $\kappa \in \mathbb{R}_+$, $\succ \in \mathcal{L}(A)$, and two lotteries $\xi, \zeta \in \mathcal{R}(A)$, ξ is more desirable than ζ by means of κ-PC (denoted by $\xi \succeq^{\kappa\text{-}PC} \zeta$) if and only

$$\sum_{\substack{x,y \in A,\\ x \succ y}} \mathbb{P}[\xi = x] \cdot \mathbb{P}[\zeta = y] \geqslant \frac{1}{\kappa} \sum_{\substack{x,y \in A,\\ x \succ y}} \mathbb{P}[\zeta = x] \cdot \mathbb{P}[\xi = y].$$

Further, the approximate PC-efficiency can be defined.

Definition 4 (κ-PC-Efficiency). *Given $\kappa \in (0, 1]$, a voting rule $f \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies κ-PC-efficiency if $f(P)$ is never κ-PC dominated for any profile $P \in \mathcal{L}(A)^n$.*

Similar to the case of SD, a larger κ in Definition 4 is also more desirable, and 1-PC-efficiency also reduces to standard PC-efficiency. In addition, our generalization of SD-efficiency and PC-efficiency keeps the relationship between them (PC-efficiency implies SDE), shown as follows.

Lemma 2. *For all $\gamma \in (0, 1]$, any voting rule satisfying γ-PC-efficiency satisfies γ-SDE.*

Therefore, the upper bound of κ-PC-efficiency under DP must be smaller than the upper bound of γ-SDE. However, the following proposition shows that even the approximate version of PC-efficiency cannot be achieved under DP.

Proposition 5 (κ-PC-Efficiency, Upper Bound). *Given any $\kappa, \epsilon \in \mathbb{R}_+$, there is no voting rule satisfying ϵ-DP and κ-PC-efficiency simultaneously.*

Proposition 5 also implies that PC-efficiency is the limit of tradeoff between DP and efficiency. Any notion stronger than PC-efficiency cannot be even approximately achieved when DP is required.

3.2 DP-CONDORCET CONSISTENCY TRADEOFF

The Condorcet winner and Condorcet loser are fundamental concepts in social choice theory. Based on these concepts, Condorcet consistency typically refers to two axioms: the Condorcet criterion and the Condorcet loser criterion. However, DP requires that the support set of $f(P)$ equals A for every profile P, i.e., $\Pr[f(P) = a] \neq 0$ holds for all $P \in \mathcal{L}(A)^n$ and $a \in A$. This implies that neither of these axioms can be fully satisfied under the constraint of DP. Li et al. [2023] introduced an approximate version of the Condorcet criterion (α-pCondorcet) to quantify the level of satisfaction of the Condorcet criterion. Furthermore, they proved that if a voting rule satisfies ϵ-DP and α-pCondorcet, then $\alpha \leq e^{\epsilon}$. This axiom is referred to as the α-Condorcet criterion in this work, and its formal definition is as follows.

Definition 5 (α-Condorcet Criterion, α-CC for short [Li et al., 2023]). *Given $\alpha \in \mathbb{R}_+$, a voting rule $f \colon \mathcal{L}(A)^n \to$*

Mechanism 3: CWRR

Input: Profile P, Noise level parameter ϵ

Output: Winning alternative a_{win}

if $\mathrm{CW}(P)$ *exists* **then**
  $p(\mathrm{CW}(P)) = \frac{\mathrm{e}^\epsilon}{\mathrm{e}^\epsilon+m-1}$;
  $p(a) = \frac{1}{\mathrm{e}^\epsilon+m-1}$, for all $a \in A\backslash\{\mathrm{CW}(P)\}$;
else
  $p(a) = \frac{1}{m}$, for all $a \in A$;
Sample $a_{win} \sim p$;
return a_{win}

Mechanism 4: CLRR

Input: Profile P, Noise level parameter ϵ

Output: Winning alternative a_{win}

if $\mathrm{CL}(P)$ *exists* **then**
  $p(\mathrm{CL}(P)) = \frac{1}{(m-1)\mathrm{e}^\epsilon+1}$;
  $p(a) = \frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}$, for all $a \in A\backslash\{\mathrm{CL}(P)\}$;
else
  $p(a) = \frac{1}{m}$, for all $a \in A$;
Sample $a_{win} \sim p$;
return a_{win}

$\mathcal{R}(A)$ satisfies α-Condorcet criterion if for all profiles P that $\mathrm{CW}(P)$ exists and each alternative $a \in A\backslash\{\mathrm{CW}(P)\}$,

$$\mathbb{P}[f(P) = \mathrm{CW}(P)] \geqslant \alpha \cdot \mathbb{P}[f(P) = a].$$

Next, we show the upper bound of approximate Condorcet criterion (e^ϵ-CC) is achievable under DP (Proposition 6). Technically, we study Mechanism 3 (Condorcet Winner Randomized Response, abbreviated as CWRR), which applies randomized response to $\mathrm{CW}(P)$.

Proposition 6 (α-CC, Lower Bound). *Given $\epsilon \in \mathbb{R}_+$, Mechanism 3 satisfies both ϵ-DP and e^ϵ-Condorcet.*

Similar to α-CC, we propose the approximate Condorcet loser criterion to measure the level of satisfaction of Condorcet loser criterion.

Definition 6 (η-Condorcet Loser Criterion, η-CLC for short). *Given $\eta \in \mathbb{R}_+$, a voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies η-Condorcet loser criterion if for all profiles P such that $\mathrm{CL}(P)$ exists and each alternative $a \in A\backslash\{\mathrm{CL}(P)\}$,*

$$\mathbb{P}[f(P) = a] \geqslant \eta \cdot \mathbb{P}[f(P) = \mathrm{CL}(P)].$$

Notably, a larger η is more desirable, and ∞-CLC is equivalent to the standard Condorcet loser criterion. Further, the following proposition shows an upper bound of η under DP.

Proposition 7 (η-CLC, Upper Bound). *For any ϵ, there is no voting rule satisfying ϵ-DP and η-CLC with $\eta > \mathrm{e}^\epsilon$.*

Proposition 8 shows the upper bound in Proposition 7 can be achieved by Mechanism 4 (Condorcet loser randomized response, CLRR), formally stated as follows.

Proposition 8 (η-CLC, Lower Bound). *Given $\epsilon \in \mathbb{R}_+$, Mechanism 4 satisfies ϵ-DP and e^ϵ-CLC.*

4 TRADEOFF BETWEEN AXIOMS UNDER DP

This section investigates the 3-way tradeoffs among DP and voting axioms. Concretely, we examine the distinction between the axiom tradeoffs in classical social choice theory and the axiom tradeoffs under DP. Since even the approximation of PC-efficiency cannot be achieved under DP, we only take α-CC, η-CLC, β-PE, and γ-SDE into consideration. On the one hand, we capture the lower bounds of 3-way tradeoffs by proving the levels of satisfaction to all the approximate axioms achieved by Mechanisms 1–4, which are summarized in Table 3. All of the detailed results and their proofs corresponding to Table 3 are shown in Appendix B.1.

On the other hand, we investigate the upper bounds of the 3-way tradeoff between DP and each pairwise combination of the axioms. All of the missing proofs for the upper bounds in this section can be found in Appendix B.2.

Lower bounds of 3-way tradeoffs In the previous section, we showed that BordaExp, RD-Anti, CWRR, and CLRR (Mechanisms 1–4) currently reach the best achievable levels of β-PE, γ-SDE, α-CC, and η-CLC, respectively. Therefore, these voting rules can provide the lower bounds of 3-way tradeoffs. For example, CWRR satisfies 1-PE and e^ϵ-CC, which indicates that the 3-way tradeoff among ϵ-DP, β-PE, and α-CC has a lower bound, i.e., $\alpha = \mathrm{e}^\epsilon$ and $\beta = 1$.

Especially, the smoothed tradeoff between α-CC and η-CLC can be achieved by the probability mixture of CWRR and CLRR, which performs CWRR with probability $\omega \in [0, 1]$ and performs CLRR with probability $1 - \omega$.

Let $f_\epsilon^\omega(P)$ denote $\omega \cdot \mathrm{CWRR}(P, \epsilon) + (1-\omega) \cdot \mathrm{CLRR}(P, \epsilon)$. Then for any profile P,

$$\mathbb{P}[f_\epsilon^\omega(P) = a] = \begin{cases} \frac{\omega \cdot \mathrm{e}^\epsilon}{\mathrm{e}^\epsilon+m-1} + \frac{(1-\omega)\cdot \mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}, & a = \mathrm{CW}(P) \\ \frac{\omega}{\mathrm{e}^\epsilon+m-1} + \frac{1-\omega}{(m-1)\mathrm{e}^\epsilon+1}, & a = \mathrm{CL}(P) \\ \frac{\omega}{\mathrm{e}^\epsilon+m-1} + \frac{(1-\omega)\cdot \mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}, & \text{otherwise.} \end{cases}$$

Therefore, f_ϵ^ω satisfies ϵ-DP, α-CC, and η-CLC, where we have $\alpha \cdot \eta = \mathrm{e}^\epsilon$ exactly. Please refer to Appendix B.1 for the full results of lower bounds. In the rest of this section, we will show the upper bounds of 3-way tradeoffs.

Tradeoff between Condorcet consistency First of all, we discuss the tradeoff between Condorcet criterion and Condorcet loser criterion. In fact, the standard forms of these two axioms are compatible in standard social choice

Table 3: Lower bounds of 3-way tradeoff achieved by Mechanisms 1–4.

Voting Rule	**β-PE**	**γ-SDE**	**α-CC**	**η-CLC**	**Reference**
BordaEXP	$\mathrm{e}^{\frac{n\epsilon}{2m-2}}$	$\dfrac{\mathrm{e}^{\frac{n}{2}}+(m-2)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}{\mathrm{e}^{\frac{n}{2}}+(m-1)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}$	$\mathrm{e}^{\left(\lfloor\frac{n}{2}\rfloor+1\right)\cdot\frac{m}{2m-2}-\frac{n}{2}}$	$\mathrm{e}^{\frac{n}{2m-2}-\left(\lceil\frac{n}{2}\rceil-1\right)\frac{m}{2m-2}}$	Mechanism 1
RD-Anti	1	$\dfrac{(m-1)\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}$	$\dfrac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^{\epsilon}+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^{\epsilon}}$	$\dfrac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^{\epsilon}+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^{\epsilon}}$	Mechanism 2
CWRR	1	$\dfrac{m-1}{m}$	e^{ϵ}	1	Mechanism 3
CLRR	1	$\dfrac{(m-2)\mathrm{e}^{\epsilon}+1}{(m-1)\mathrm{e}^{\epsilon}+1}$	1	e^{ϵ}	Mechanism 4

theory (without DP), since for any profile P, the Condorcet winner $\mathrm{CW}(P)$ will never coincide with the Condorcet loser $\mathrm{CL}(P)$. However, when DP is required, the best α-CC ($\alpha = \mathrm{e}^{\epsilon}$) is not compatible with the best η-CLC ($\eta = \mathrm{e}^{\epsilon}$), since Condorcet winner can sometimes be converted to Condorcet loser by reversing only one voter's vote (e.g., when P_{-j} are tied, reversing $\succ_j$ exchanges Condorcet winner and Condorcet loser). Formally, we have the following theorem.

Theorem 1. *There is no voting rule satisfying ϵ-DP, α-CC and η-CLC with $\alpha \cdot \eta > \mathrm{e}^{\epsilon}$.*

Proof Sketch. Consider the profile P ($n = 2k+1$), where $k+1$ voters consider $a_1 \succ a_2 \succ \cdots \succ a_m$ and k voters consider $a_m \succ a_{m-1} \succ \cdots \succ a_1$. Let P' be another profile ($n = 2k+1$), where k voters consider $a_1 \succ' a_2 \succ' \cdots \succ' a_m$, and $k+1$ voters consider $a_m \succ a_{m-1} \succ \cdots \succ a_1$. Then P and P' are neighboring, and $\mathrm{CW}(P) = \mathrm{CL}(P')$. By the definition of DP, we have $\alpha \cdot \eta \leqslant \mathrm{e}^{\epsilon}$. □

Surprisingly, the upper bound shown in Theorem 1 coincides with the lower bound achieved by the probability mixture of CWRR and CLRR. In other words, this bound is tight for Condorcet criterion and Condorcet loser criterion.

Condorcet consistency against Pareto efficiency Secondly, we discuss the three-way tradeoff between Condorcet consistency and Pareto efficiency. In standard social choice theory, Pareto efficiency and the Condorcet criterion are compatible, as selecting the original Condorcet method (i.e., choosing $\mathrm{CW}(P)$ as the winner with probability 1) satisfies Pareto efficiency. However, under ϵ-DP, α-CC focuses only on maximizing the winning probability of $\mathrm{CW}(P)$, while β-PE requires consideration of each pair of Pareto-dominating and Pareto-dominated alternatives. Consequently, when DP is required, the optimal β-PE and α-CC may not be achieved simultaneously. Formally, we present the following theorem.

Theorem 2. *There is no neutral voting rule satisfying ϵ-DP, β-PE, and α-CC with $\alpha \cdot \beta^{m-2} > \mathrm{e}^{n\epsilon}$.*

Remark 1. According to Proposition 1 and [Li et al., 2023], the upper bounds of β-PE and α-CC are $\beta \leqslant \mathrm{e}^{\frac{n}{m-1}}$ and $\alpha \leqslant \mathrm{e}^{\epsilon}$, respectively. Therefore, there is a natural upper bound of $\alpha\beta^{m-2}$,

$$\alpha\beta^{m-2} \leqslant \sup\alpha \cdot (\sup\beta)^{m-2} = \mathrm{e}^{1+n-\frac{n}{m-1}},$$

i.e., Theorem 2 is non-trivial only when $n \leqslant m-1$.

Similarly, the same phenomenon also occurs to the η-CLC. Condorcet loser criterion is also compatible with Pareto efficiency in social choice theory, since the Condorcet loser will never be a Pareto dominator. However, due to the same reason as Condorcet criterion, the best η-CLC may not be compatible with the best β-PE under the same condition. Formally, we have the following theorem.

Theorem 3. *There is no neutral voting rule ϵ-DP, β-PE, and η-CLC with $\eta \cdot \beta^{m-2} > \mathrm{e}^{n\epsilon}$.*

Condorcet consistency against SD-efficiency The relationship between SD-efficiency and Condorcet criterion is also affected by ϵ-DP. The next proposition shows SD-efficiency is compatible with Condorcet criterion in social choice theory. To simplify notations, we let Condorcet domain $\mathcal{D}_C$ denote the set of all profiles P where $\mathrm{CW}(P)$ exists. We say a voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ is a Condorcet method if it satisfies $\mathbb{P}[f(P) = \mathrm{CW}(P)] = 1$.

Proposition 9. *Condorcet method is SD-efficient on $\mathcal{D}_C$.*

However, when DP is required, we can not achieve the best α-CC and γ-SDE simultaneously. The upper bound of this tradeoff is shown in the following theorem.

Theorem 4. *There is no neutral voting rule satisfying ϵ-DP, α-CC, and γ-SDE with $\gamma > \frac{\alpha+m-1-\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}$.*

The tradeoff curves between γ-SDE and α-CC subject to ϵ-DP are shown in Figure 3, where we set $m = 5, n = 10$.

For Condorcet loser criterion, the situation is quite similar. On the one hand, the compatibility between SD-efficiency and Condorcet loser criterion without DP is proved via the maximal-lottery mechanism [Brandt, 2017]. On the

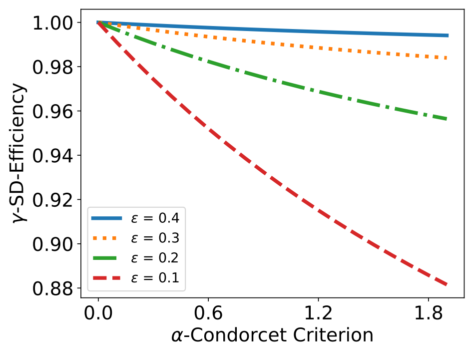

Figure 3: Tradeoff curves (upper bounds) between γ-SDE against α-CC under ϵ-DP, where $m = 5, n = 10$.

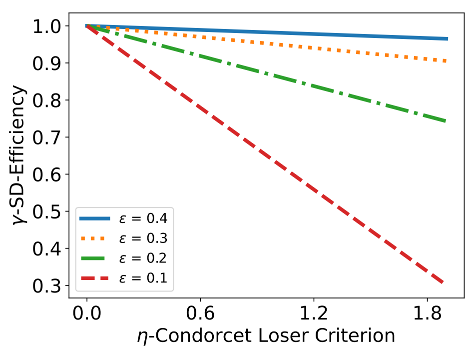

Figure 4: Tradeoff curves (upper bounds) between γ-SDE against η-CLC under ϵ-DP, where $m = 5, n = 10$.

other hand, there is a difference between optimizing the SD-efficiency and minimizing the winning probability of the Condorcet loser under DP, which leads to a 3-way tradeoff among them. Formally, we have the following theorem.

Theorem 5. *There is no neutral voting rule satisfying ϵ-DP, η-CLC, and γ-SDE with $\gamma > \frac{e^{n\epsilon} - \eta}{e^{n\epsilon}}$.*

The tradeoff curves between γ-SDE and η-CLC under ϵ-DP are shown in Figure 4, where we set $m = 5$ and $n = 10$.

Pareto efficiency against SD-efficiency Finally, we investigate the 3-way tradeoff between Pareto efficiency, SD-efficiency, and DP. Although the standard SD-efficiency implies the standard Pareto efficiency in social choice theory, their best approximate bounds are incompatible under DP. Formally, we have the following theorem.

Theorem 6. *There is no neutral voting rule satisfying ϵ-DP, γ-SDE, and β-PE with $\gamma > \frac{e^{n\epsilon} - e^{n\epsilon}\beta^{2-m}}{e^{n\epsilon} - e^{n\epsilon}\beta^{2-m} + \beta - 1}$.*

The upper bounds established in Theorem 6 are illustrated in Figures 5–6. From the figure, we make the following two observations:

1. Curves corresponding to larger values of ϵ (weaker privacy guarantees) are positioned toward the top-right compared to those with smaller ϵ, indicating that the incompatibility between these axioms diminishes as the privacy guarantee weakens.

2. Curves corresponding to larger values of n (more voters) are positioned toward the top-right compared to those with smaller n (fewer voters), indicating that the incompatibility decreases as the number of voters increases. This trend is consistent across other figures. Additional figures illustrating different values of n for other axioms are provided in Appendix C.

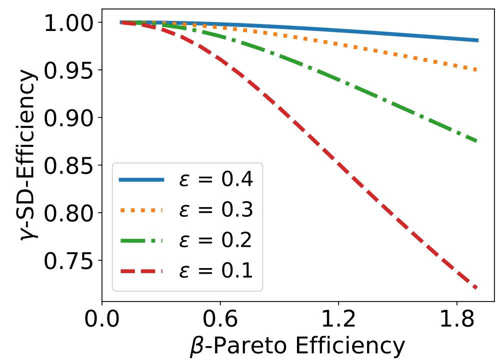

Figure 5: Tradeoff curves (upper bounds) between γ-PE against α-SDE under ϵ-DP, where $m = 5, n = 10$.

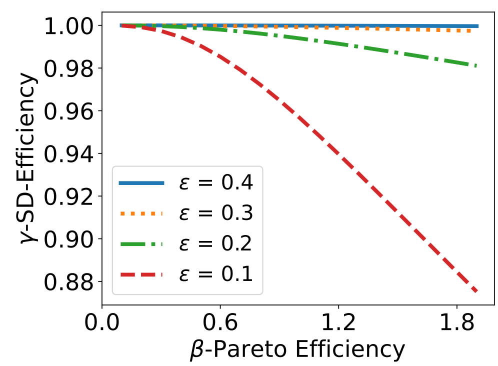

Figure 6: Tradeoff curves (upper bounds) between γ-PE against α-SDE under ϵ-DP, where $m = 5, n = 20$.

5 CONCLUSION AND FUTURE WORK

This paper investigated the tradeoff between DP and varieties of voting axioms, including Condorcet consistency and three efficiencies. We found that DP is significantly incompatible with all of these axioms and quantified their 2-way tradeoffs against DP. Further, we explored the 3-way tradeoffs among DP and these axioms. Our results show that the tradeoffs between axioms are different with or without DP. It would be an interesting future direction to study the tradeoffs between DP and other axioms. Besides, it is also interesting to develop tighter bounds for the tradeoffs.

6 ACKNOWLEDGMENTS

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A MISSING PROOFS IN SECTION 3

Proposition 1 (**β-Pareto Efficiency, Upper Bound**). *There is no neutral rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ that satisfies ϵ-DP and β-Pareto efficiency with $\beta > \mathrm{e}^{\frac{n\epsilon}{m-1}}$.*

Proof. Let $f\colon \mathcal{L}(A)^n \to A$ be a voting rule satisfying ϵ-DP and β-Pareto efficiency. Let P be a profile, where $a_1 \succ a_2 \succ \cdots \succ a_m$, for all $i \in N$. Since a_1 Pareto dominates a_2, a_2 Pareto dominates a_3, etc., we have

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \beta \cdot \mathbb{P}[f(P) = a_2]\\ &\geqslant \beta^2 \cdot \mathbb{P}[f(P) = a_3]\\ &\cdots\\ &\geqslant \beta^{m-1} \cdot \mathbb{P}[f(P) = a_m].\end{aligned}$$

Then we claim that for profile P,

$$\mathbb{P}[f(P) = a_1] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = a_m].$$

Theorefore, we have $\beta^{m-1} \leqslant \mathrm{e}^{n\epsilon}$, i.e., $\beta \leqslant \mathrm{e}^{\frac{n\epsilon}{m-1}}$, as desired.

Finally, we prove the claim above. In fact, for any voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP and neutrality, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = b], \quad \text{for all } a, b \in A. \tag{3}$$

Now, we prove Equation (3). For any profile $P, P' \in \mathcal{L}(A)^n$, let $\ell_0(\cdot,\cdot)$ represents the ℓ_0-distance between them, i.e., $\ell_0(P, P') = \{j \in N : \succ_j \neq \succ'_j\}$. Then, by considering the following operation **Op**, we can see that P can be transferred to P' through $k = \ell_0(P, P')$ times of operations.

- **Op**: Choose a voter $i \in N$ that $\succ_i \neq \succ'_i$, let $\succ_i = \succ'_i$.

Letting $P_0, P_1, \ldots, P_k$ denote all of the profiles, we have the following diagram.

$$P = P_0 \xrightarrow{\textbf{Op}} P_1 \xrightarrow{\textbf{Op}} P_2 \xrightarrow{\textbf{Op}} \cdots \xrightarrow{\textbf{Op}} P_k = P'.$$

Notice that in each step, only one voter's preference is changed. Consequently, for each i, P_i and P_{i+1} are neighboring profiles. Since f satisfies ϵ-DP, we have

$$\begin{aligned}\mathbb{P}[f(P) = a] &\leqslant \mathrm{e}^{\epsilon} \cdot \mathbb{P}[f(P_1) = a]\\ &\leqslant \mathrm{e}^{2\epsilon} \cdot \mathbb{P}[f(P_2) = a]\\ &\leqslant \cdots\\ &\leqslant \mathrm{e}^{k\cdot\epsilon} \cdot \mathbb{P}[f(P') = a].\end{aligned}$$

Besides, for any given $P, P' \in \mathcal{L}(A)^n$, there are at most n distinct voters $j \in N$ that $\succ_j \neq \succ'_j$. Therefore, for any profile $P, P' \in \mathcal{L}(A)^n$ and any $a \in A$, we have

$$\mathbb{P}[f(P) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P') = a].$$

Now, for any profile $P \in \mathcal{L}(A)^n$ and an arbitrarily chosen paif of alternatives $a, b \in A$, let P' be the profile transferred from P by swapping a and b in each voter's preference. By the neutrality of f, we have

$$\begin{aligned}\mathbb{P}[f(P) = a] &= \mathbb{P}[f(P') = b],\\ \mathbb{P}[f(P') = a] &= \mathbb{P}[f(P) = b].\end{aligned}$$

Then it follows that

$$\begin{aligned}\mathbb{P}[f(P) = a] &\leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P') = a]\\ &= \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = b],\end{aligned}$$

which completes the proof. $\square$

Proposition 2 (**β-Pareto Efficiency, Lower Bound**). *Given $\epsilon \in \mathbb{R}_+$, Mechanism 1 satisfies ϵ-DP and $\mathrm{e}^{\frac{n\epsilon}{2m-2}}$-Pareto efficiency.*

Proof. Let $\mathfrak{E}_{\mathsf{Borda}}\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ denote the mapping introduced by BordaEXP. Then for any profile $P \in \mathcal{L}(A)^n$ and alternative $a \in A$, we have

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P) = a] = \frac{\mathrm{e}^{\mathsf{Borda}_P(a)\epsilon/(2m-2)}}{\sum\limits_{c \in A} \mathrm{e}^{\mathsf{Borda}_P(c)\epsilon/(2m-2)}}.$$

First, we establish the bound for Pareto efficiency. Given profile $P \in \mathcal{L}(A)^n$, suppose $a, b \in A$ are a pair of alternatives that $a \succ_j b$ for all $j \in N$. It follows that, for each voter $j \in N$, the number of alternatives that are considered worse than b according to her preference order $\succ_j$ is strictly less than the number of alternatives considered worse than a. Formally, we have

$$|\{c \in A : a \succ_j c\}| - |\{c \in A : b \succ_j c\}| \geqslant 1, \quad \text{for all } j \in N.$$

By the definition of Borda score, we have

$$\mathsf{Borda}_P(a) - \mathsf{Borda}_P(b) \geqslant n.$$

Then it follows that

$$\begin{aligned}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P) = a] &= \frac{\mathrm{e}^{\mathsf{Borda}_P(a)\cdot\epsilon/(2m-2)}}{\sum\limits_{c \in A} \mathrm{e}^{\mathsf{Borda}_P(c)\cdot\epsilon/(2m-2)}}\\ &\geqslant \frac{\mathrm{e}^{\mathsf{Borda}_P(b)} \cdot \mathrm{e}^{n\epsilon/(2m-2)}}{\sum\limits_{c \in A} \mathrm{e}^{\mathsf{Borda}_P(c)\cdot\epsilon/(2m-2)}}\\ &= \mathrm{e}^{n\epsilon/(2m-2)} \cdot \mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P) = b],\end{aligned}$$

which indicates that $\mathfrak{E}_{\mathsf{Borda}}$ satisfies $\mathrm{e}^{\frac{n\epsilon}{2m-2}}$-Pareto efficiency.

Then we prove the DP-bound. For all neighboring profiles

$P, P' \in \mathcal{L}(A)^n$,

$$\begin{aligned} &\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P')=a]} \\ &= \frac{\mathrm{e}^{\mathsf{Borda}_P(a)\cdot\frac{\epsilon}{2m-2}}}{\mathrm{e}^{\mathsf{Borda}_{P'}(a)\cdot\frac{\epsilon}{2m-2}}} \cdot \frac{\sum_{c\in A} \mathrm{e}^{\mathsf{Borda}_{P'}(c)\cdot\frac{\epsilon}{2m-2}}}{\sum_{c\in A} \mathrm{e}^{\mathsf{Borda}_{P}(c)\cdot\frac{\epsilon}{2m-2}}} \\ &\leqslant \mathrm{e}^{\epsilon/2} \cdot \sup_{P\in\mathcal{L}(A)^n} \frac{\sum_{c\in A} \mathrm{e}^{\mathsf{Borda}_{P'}(c)\cdot\frac{\epsilon}{2m-2}}}{\sum_{c\in A} \mathrm{e}^{\mathsf{Borda}_{P}(c)\cdot\frac{\epsilon}{2m-2}}} \\ &\leqslant \mathrm{e}^{\epsilon}, \end{aligned}$$

which completes the proof. □

Lemma 1. *Given $\gamma > 0$, a voting rule f satisfies γ-SD-efficiency if and only if*

$$\frac{1}{\gamma} \geqslant \sup_{P\in\mathcal{L}(A)^n, \xi\in\mathcal{R}(A)} \inf_{j\in N, y\in A} \frac{\sum_{x:x\succ_j y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ_j y} \mathbb{P}[f(P) = x]}.$$

Proof. Suppose that f is not γ-SD-efficient. Then there must be some profile $P \in \mathcal{L}(A)^n$ that $f(P)$ is γ-SD-dominated by some $\xi \in \mathcal{R}(A)$, i.e., for all $y \in A$ and $\succ_j \in P$,

$$\sum_{x:x\succ_j y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x:x\succ_j y} \mathbb{P}[f(P) = x],$$

which is equivalent to

$$\frac{1}{\gamma} \leqslant \inf_{j\in N, y\in A} \frac{\sum_{x:x\succ_j y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ_j y} \mathbb{P}[f(P) = x]}.$$

Therefore, f is γ-SD-efficient if and only if for each $P \in \mathcal{L}(A)^n$, there does not exist such ξ, i.e., for all $y \in A$ and $P \in \mathcal{L}(A)^n$,

$$\frac{1}{\gamma} \geqslant \inf_{j\in N, y\in A} \frac{\sum_{x:x\succ_j y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ_j y} \mathbb{P}[f(P) = x]},$$

which is equivalent to

$$\frac{1}{\gamma} \geqslant \sup_{P\in\mathcal{L}(A)^n, \xi\in\mathcal{R}(A)} \inf_{j\in A, y\in A} \frac{\sum_{x:x\succ_j y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ_j y} \mathbb{P}[f(P) = x]}.$$

That completes the proof. □

Proposition 3 (γ-SD-Efficiency, Upper Bound). *Given $\gamma \in \mathbb{R}_+$, there is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP and γ-SD-efficiency with $\gamma > \frac{(m-1)\mathrm{e}^{n\epsilon}}{(m-1)\mathrm{e}^{n\epsilon}+1}$.*

Proof. Consider two profiles, P_1 and P_2, where all voters in P_1 share the same preference order $a_1 \succ a_2 \succ \cdots \succ a_m$. In contrast, in P_2, the voters' preferences are $a_m \succ' a_2 \succ' a_3 \succ' \cdots \succ' a_{m-1} \succ' a_1$. Then the unique SD-efficient lottery for P_1 and P_2 should be $\mathbb{I}_{a_1}$ and $\mathbb{I}_{a_m}$, respectively. Here, $\mathbb{I}_{a_1}$ and $\mathbb{I}_{a_m}$ represent the indicator functions of a_1 and a_m, formally defined as follows.

$$\mathbb{P}[\mathbb{I}_{a_1} = a] = \begin{cases} 1 & a = a_1 \\ 0 & \text{otherwise} \end{cases},$$

$$\mathbb{P}[\mathbb{I}_{a_m} = a] = \begin{cases} 1 & a = a_m \\ 0 & \text{otherwise} \end{cases}.$$

Let f be any neutral voting rule satisfying ϵ-DP. By Equation (3), we have

$$\begin{aligned} \mathbb{P}[f(P_1) = a_1] &\leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P_2) = a_1] && \text{(by } \epsilon\text{-DP)} \\ &= \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P_1) = a_m]. && \text{(by neutrality)} \end{aligned}$$

By symmetry, for any $a \neq a_m$, $\mathbb{P}[f(P_1) = a] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P_1) = a_m]$. Therefore,

$$\begin{aligned} 1 &= \sum_{a\in A} \mathbb{P}[f(P_1) = a] \\ &\leqslant ((m-1)\mathrm{e}^{n\epsilon} + 1) \cdot \mathbb{P}[f(P_1) = a_m], \end{aligned}$$

i.e., $\mathbb{P}[f(P_1) = a_m] \geqslant \frac{1}{(m-1)\mathrm{e}^{n\epsilon}+1}$. If there exists some γ that f satisfies γ-SD-efficiency, there does not exist any $\xi \in \mathcal{R}(A)$, such that

$$\sum_{x:x\succ y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x:x\succ y} \mathbb{P}[f(P_1) = x], \quad \text{for all } y \in A.$$

Therefore, we have

$$\begin{aligned} \frac{1}{\gamma} &\geqslant \sup_{P\in\mathcal{L}(A)^n} \sup_{\xi\in\mathcal{R}(A)} \inf_{y\in A} \frac{\sum_{x:x\succ y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ y} \mathbb{P}[f(P) = x]} \\ &\geqslant \sup_{\xi\in\mathcal{R}(A)} \inf_{y\in A} \frac{\sum_{x:x\succ y} \mathbb{P}[\xi = x]}{\sum_{x:x\succ y} \mathbb{P}[f(P_1) = x]} \\ &\geqslant \inf_{y\in A} \frac{\sum_{x:x\succ y} \mathbb{P}[\mathbb{I}_{a_1} = x]}{\sum_{x:x\succ y} \mathbb{P}[f(P_1) = x]} \\ &= \frac{1}{\max_{y\in A} \sum_{x:x\succ y} \mathbb{P}[f(P_1) = x]} \\ &= \frac{1}{1 - \mathbb{P}[f(P_1) = a_m]} \\ &\geqslant \frac{(m-1)\mathrm{e}^{n\epsilon} + 1}{(m-1)\mathrm{e}^{n\epsilon}}. \end{aligned}$$

In other words, we have $\gamma \leqslant \frac{(m-1)\mathrm{e}^{n\epsilon}}{(m-1)\mathrm{e}^{n\epsilon}+1}$, as desired. □

Proposition 4 (**γ-SD-Efficiency, Lower Bound**). *Mechanism 2 satisfies ϵ-DP and $\frac{(m-1)\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}$-SD-efficiency.*

Proof. Letting $\mathfrak{E}_{\mathsf{Anti}} \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ denote the mapping introduced by Mechanism 2.

For any neighboring profiles $P, P' \in \mathcal{L}(A)^n$ that $P_{-j} = P'_{-j}$ and $\succ_j \neq \succ'_j$, suppose that the chosen ballot in the mechanism is $\succ_i$. Then

$$\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a \mid i \neq j] = \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i \neq j] \qquad \text{(use } C \text{ to denote them)}$$

Further, for any $a \in A$,

$$\begin{aligned}
&\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a]} \\
&= \frac{\mathbb{P}[i = j \wedge \mathfrak{E}_{\mathsf{Anti}}(P) = a] + \mathbb{P}[i \neq j \wedge \mathfrak{E}_{\mathsf{Anti}}(P) = a]}{\mathbb{P}[i = j \wedge \mathfrak{E}_{\mathsf{Anti}}(P') = a] + \mathbb{P}[i \neq j \wedge \mathfrak{E}_{\mathsf{Anti}}(P') = a]} \\
&= \frac{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a \mid i = j] + \frac{n-1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a \mid i \neq j]}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i = j] + \frac{n-1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i \neq j]} \\
&= \frac{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a \mid i = j] + \frac{n-1}{n} \cdot C}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i = j] + \frac{n-1}{n} \cdot C} \quad (P_{-j} = P'_{-j}) \\
&\leqslant \frac{\mathrm{e}^\epsilon \cdot \frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i = j] + \frac{n-1}{n} \cdot C}{\frac{1}{n}\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P') = a \mid i = j] + \frac{n-1}{n} \cdot C} \\
&\leqslant \mathrm{e}^\epsilon,
\end{aligned}$$

which indicates that $\mathfrak{E}_{\mathsf{Anti}}$ satisfies ϵ-DP. On the other hand, given profile P, suppose the top-ranked and the last-ranked alternative of $\succ_i$ are $a_\top$ and $a_\bot$, respectively. Then, for any $\xi \in \mathcal{R}(A)$, we have

$$\frac{\sum_{x : x \succ_i y} \mathbb{P}[\xi = x]}{\sum_{x : x \succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} \leqslant \frac{1}{\sum_{x : x \succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} = \frac{\sum_{x : x \succ_i y} \mathbb{P}[\mathbb{I}_{a_\top} = x]}{\sum_{x : x \succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]}.$$

Theorefore,

$$\begin{aligned}
&\sup_{\xi \in \mathcal{R}(A)} \inf_{y \in A} \frac{\sum_{x : x \succ_i y} \mathbb{P}[\xi = x]}{\sum_{x : x \succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} \\
&= \inf_{y \in A} \frac{1}{\sum_{x : x \succ_i y} \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = x]} \\
&= \frac{1}{1 - \mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P) = a_\bot]} \\
&= \frac{(m-1)\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon + 1}.
\end{aligned}$$

By Lemma 1, $\mathfrak{E}_{\mathsf{Anti}}$ satisfies $\frac{(m-1)\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}$-SD-efficiency, which completes the proof. □

Lemma 2. *Given $\gamma \leqslant 1$, γ-PC-efficiency implies γ-SD-efficiency.*

Proof. In fact, we only need to prove that for any $\xi, \zeta \in \mathcal{R}(A)$, $\xi \succeq^{\gamma-SD} \zeta$ implies $\xi \succeq^{\gamma-PC} \zeta$. Let ξ and ζ be two lotteries satisfying $\xi \succeq^{\gamma-SD} \zeta$, i.e.,

$$\sum_{x \succ y} \mathbb{P}[\xi = x] \geqslant \frac{1}{\gamma} \cdot \sum_{x \succ y} \mathbb{P}[\zeta = y], \qquad \text{for any } y \in A.$$

Then, on the one hand, we have

$$\begin{aligned}
&\sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\xi = x] \cdot \mathbb{P}[\zeta = y] \\
&= \sum_{y \in A} \mathbb{P}[\zeta = y] \cdot \sum_{x \succ y} \mathbb{P}[\xi = x] \\
&\geqslant \sum_{y \in A} \mathbb{P}[\zeta = y] \cdot \frac{1}{\gamma} \sum_{x \succ y} \mathbb{P}[\zeta = y] \\
&= \frac{1}{\gamma} \cdot \sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\zeta = y] \cdot \mathbb{P}[\zeta = y].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\frac{1}{\gamma} \cdot \sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\zeta = y]\mathbb{P}[\xi = y] \\
&= \sum_{x \in A} \mathbb{P}[\zeta = y] \cdot \sum_{y \prec x} \mathbb{P}[\xi = y] \\
&= \frac{1}{\gamma} \cdot \sum_{x \in A} \mathbb{P}[\zeta = y] \cdot \sum_{y \prec x} \left(1 - \sum_{x \succeq y} \mathbb{P}[\xi = x]\right) \\
&\leqslant \frac{1}{\gamma} \cdot \sum_{x \in A} \mathbb{P}[\zeta = y] \cdot \sum_{y \prec x} \left(1 - \frac{1}{\gamma} \sum_{x \succeq y} \mathbb{P}[\zeta = y]\right) \\
&\leqslant \frac{1}{\gamma} \cdot \sum_{x \in A} \mathbb{P}[\zeta = y] \cdot \sum_{y \prec x} \left(1 - \sum_{x \succeq y} \mathbb{P}[\zeta = y]\right) \\
&= \frac{1}{\gamma} \cdot \sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\zeta = y] \cdot \mathbb{P}[\zeta = y].
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\xi = x] \cdot \mathbb{P}[\zeta = y] \\
&\geqslant \frac{1}{\gamma} \cdot \sum_{x,y \in A \wedge x \succ y} \mathbb{P}[\zeta = x] \cdot \mathbb{P}[\xi = y],
\end{aligned}$$

which completes the proof. □

Proposition 5 (**κ-PC-Efficiency, Upper Bound**). *Given any $\kappa, \epsilon \in \mathbb{R}_+$, there is no voting rule $f \colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP and κ-PC-efficiency.*

Proof. Consider the profile P, where all voters share the same prefereoce

$$a_1 \succ a_2 \succ \cdots \succ a_m.$$

Then the unique PC-efficient distribution on A is $\mathbb{I}_{a_1}$. Further, we have

$$\begin{aligned}&\sum_{x,y\colon x\succ_i y} \mathbb{P}[\mathbb{I}_{a_1}=x]\cdot\mathbb{P}[f(P)=y]\\ =&\sum_{y\colon a_1\succ_i y}\mathbb{P}[f(P)=y]=1-\mathbb{P}[f(P)=a_1].\end{aligned}$$

However,

$$\begin{aligned}&\sum_{x,y\colon x\succ_i y}\mathbb{P}[f(P)=x]\cdot\mathbb{P}[\mathbb{I}_{a_1}=y]\\ =&\sum_{x\in A}\mathbb{P}[f(P)=x]\cdot\sum_{y\colon x\succ_i y}\mathbb{P}[\mathbb{I}_{a_1}=y]=0.\end{aligned}$$

In other words, for all $\kappa\in\mathbb{R}_+$, the lottery $\mathbb{I}_{a_1}$ can κ-PC-dominate any $f(P)$, which completes the proof. □

Proposition 6 (α-Condorcet Criterion, Lower Bound). *Mechanism 3 satisfies* e^{ϵ}*-Condorcet criterion and* ϵ*-DP.*

Proof. Let $\mathfrak{R}_{\mathrm{CW}}\colon \mathcal{L}(A)^n\to\mathcal{R}(A)$ denote the mapping introduced by CWRR. Then for any profile $P\in\mathcal{L}(A)^n$ and alternative $a\in A$, we have

$$\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]=\begin{cases}\frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon}+m-1}, & a=\mathrm{CW}(P)\\ \frac{1}{\mathrm{e}^{\epsilon}+m-1}, & \text{otherwise}\end{cases},$$

for all $P\in\mathcal{L}(A)^n$ that $\mathrm{CW}(P)$ exists.

By definition, it is not hard to see that CWRR satisfies e^{ϵ}-Condorcet criterion. Thus, we only need to prove that CWRR satisfies ϵ-DP. In fact, for any neighboring profiles $P,P'\in\mathcal{L}(A)^n$ and $a\in A$,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]}{\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P')=a]}&\leqslant\frac{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P)=a]}{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P')=a]}\\ &\leqslant\frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon}+m-1}\Big/\frac{1}{\mathrm{e}^{\epsilon}+m-1}\\ &=\mathrm{e}^{\epsilon},\end{aligned}$$

which completes the proof. □

Proposition 7 (η-Condorcet Loser Criterion, Upper Bound). *There is no voting rule satisfying ϵ-DP and η-Condorcet loser criterion with $\eta>\mathrm{e}^{\epsilon}$.*

Proof. Suppose $f\colon\mathcal{L}(A)^n\to A$ be a voting rule satisfying ϵ-DP and η-Condorcet loser criterion. Consider the profile P ($n=2k+1$):

- $k+1$ voters: $a_1\succ a_2\succ\cdots\succ a_m$,
- k voters: $a_m\succ a_{m-1}\succ\cdots\succ a_1$.

By definition, we have $w_P[a_m,a_i]=-1$, for all $a_i\in A\backslash\{a_m\}$, i.e., a_m is the Condorcet loser. Now, letting one voter change her preferece from $a_1\succ a_2\succ\cdots\succ a_m$ to $a_m\succ a_{m-1}\succ\cdots\succ a_1$, we can obtain another profile P':

- k voters: $a_1\succ' a_2\succ'\cdots\succ' a_m$,
- $k+1$ voters: $a_m\succ' a_{m-1}\succ'\cdots\succ' a_1$.

Now we have $w_{P'}[a_1,a_i]=-1$, for all $a_i\in A\backslash\{a\}$, i.e., a_1 is the Condorcet loser for P'. Then

$$\begin{aligned}\mathbb{P}[f(P)=a_1]&\geqslant\eta\cdot\mathbb{P}[f(P)=a_m] && \text{(By η-Condorcet loser)}\\ &\geqslant\mathrm{e}^{-\epsilon}\cdot\eta\cdot\mathbb{P}[f(P')=a_m] && \text{(By ϵ-DP)}\\ &\geqslant\mathrm{e}^{-\epsilon}\cdot\eta^2\cdot\mathbb{P}[f(P')=a_1] && \text{(By η-Condorcet loser)}\\ &\geqslant\mathrm{e}^{-2\epsilon}\cdot\eta^2\cdot\mathbb{P}[f(P)=a_1], && \text{(ϵ-DP)}\end{aligned}$$

which indicates that $\mathrm{e}^{-2\epsilon}\cdot\eta^2\leqslant 1$, i.e., $\eta\leqslant\mathrm{e}^{\epsilon}$. That completes the proof. □

Proposition 8 (η-Condorcet Loser Criterion, Lower Bound). *Mechanism 4 satisfies* e^{ϵ}*-Condorcet loser criterion and* ϵ*-DP.*

Proof. Let $\mathfrak{R}_{\mathrm{CL}}\colon\mathcal{L}(A)^n\to\mathcal{R}(A)$ denote the mapping introduced by CLRR, we have

$$\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P)=a]=\begin{cases}\frac{1}{(m-1)\mathrm{e}^{\epsilon}+1}, & a=\mathrm{CL}(P)\\ \frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}, & \text{otherwise}\end{cases},$$

for all $P\in\mathcal{L}(A)^n$ that $\mathrm{CL}(P)$ exists.

By definition, it is not hard to see that CLRR satisfies e^{ϵ}-Condorcet criterion. Thus, we only need to prove that CLRR satisfies ϵ-DP. In fact, for any neighboring profiles $P,P'\in\mathcal{L}(A)^n$ and $a\in A$,

$$\begin{aligned}\frac{\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P)=a]}{\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P')=a]}&\leqslant\frac{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P)=a]}{\max_{a\in A}\mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P')=a]}\\ &\leqslant\frac{\mathrm{e}^{\epsilon}}{(m-1)\mathrm{e}^{\epsilon}+1}\Big/\frac{1}{(m-1)\mathrm{e}^{\epsilon}+1}\\ &=\mathrm{e}^{\epsilon},\end{aligned}$$

which completes the proof. □

B MISSING PROOFS IN SECTION 4

B.1 RESULTS IN TABLE 3 AND THEIR PROOFS

Proposition 9. *Given $\epsilon\in\mathbb{R}_+$, BordaEXP satisfies*

(1) $\frac{\mathrm{e}^{\frac{n}{2}}+(m-2)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}{\mathrm{e}^{\frac{n}{2}}+(m-1)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}$*-SD-efficiency,*

(2) $\mathrm{e}^{\left(\lfloor \frac{n}{2} \rfloor+1\right)\cdot \frac{m}{2m-2}-\frac{n}{2}}$-*Condorcet criterion,*

(3) $\mathrm{e}^{\frac{n}{2m-2}-\left(\lceil \frac{n}{2} \rceil-1\right)\frac{m}{2m-2}}$-*Condorcet loser criterion.*

Proof. Let $\mathfrak{E}_{\mathsf{Borda}}$ denote the voting rule introduced by BordaEXP. First, we prove (1). In fact,

$$
\begin{aligned}
&\sup_{P,\xi}\inf_{j,y}\frac{\sum_{x\succ_j y}\mathbb{P}[\xi=x]}{\sum_{x\succ_j y}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=x]}\\
\leqslant\ &\sup_{P}\inf_{j,y}\frac{1}{\sum_{x\succ_j y}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=x]}\\
=\ &\sup_{P}\inf_{j}\frac{1}{1-\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a_\perp^j]}\\
\leqslant\ &\frac{1}{1-\sup_{P}\inf_{j}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a_\perp^j]}.
\end{aligned}
$$

where $a_\perp^j$ denote the last-ranked alternative in $\succ_j$. By symmetry, we have

$$
\sup_{P}\inf_{j}\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a_\perp^j]=\frac{\mathrm{e}^{\frac{n(m-2)}{4m-4}}}{\mathrm{e}^{\frac{n}{2}}+(m-1)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}.
$$

Then BordaEXP satisfies $\frac{\mathrm{e}^{\frac{n}{2}}+(m-2)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}{\mathrm{e}^{\frac{n}{2}}+(m-1)\cdot\mathrm{e}^{\frac{n(m-2)}{4m-4}}}$. Second, we prove (2). By definition, for any profile P that $\mathrm{CW}(P)$ exists, $\mathrm{CW}(P)$ must defeat each alternative $a\neq\{\mathrm{CW}(P)\}$ in at least half of the votes, i.e., $\mathsf{Borda}_P(\mathrm{CW}(P))\geqslant(m-1)\left(\lfloor\frac{n}{2}\rfloor+1\right)$. And for each $a\neq\mathrm{CW}(P)$, $\mathsf{Borda}_P(a)\leqslant(m-1)n-(\lfloor\frac{n}{2}\rfloor+1)$. Therefore,

$$
\begin{aligned}
&\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=\mathrm{CW}(P)]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}\\
\geqslant\ &\mathrm{e}^{\frac{(m-1)\left(\lfloor\frac{n}{2}\rfloor+1\right)}{2m-2}-\frac{(m-1)n-(\lfloor\frac{n}{2}\rfloor+1)}{2m-2}}\\
=\ &\mathrm{e}^{\left(\lfloor \frac{n}{2} \rfloor+1\right)\cdot \frac{m}{2m-2}-\frac{n}{2}},
\end{aligned}
$$

which indicates that BordaEXP satisfies $\mathrm{e}^{\left(\lfloor \frac{n}{2} \rfloor+1\right)\cdot \frac{m}{2m-2}-\frac{n}{2}}$-Condorcet criterion. Finally, we prove (3). By definition, for any profile P that $\mathrm{CL}(P)$ exists, $a\neq\mathrm{CL}(P)$ must be ranked than $\mathrm{CL}(P)$ in at least a half of votes, i.e., $\mathsf{Borda}_P(\mathrm{CL}(P))\leqslant(m-1)\left(\lfloor\frac{n}{2}\rfloor-1\right)$. And for each $a\neq\mathrm{CL}(P)$, $\mathsf{Borda}_P(a)\geqslant n-\lceil\frac{n}{2}\rceil+1$. Therefore,

$$
\begin{aligned}
\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Borda}}(P)=\mathrm{CL}(P)]}&\geqslant\mathrm{e}^{\frac{n-\lceil\frac{n}{2}\rceil+1}{2m-2}-\frac{(m-1)\left(\lfloor\frac{n}{2}\rfloor-1\right)}{2m-2}}\\
&=\mathrm{e}^{\frac{n}{2m-2}-\left(\lceil \frac{n}{2} \rceil-1\right)\frac{m}{2m-2}},
\end{aligned}
$$

which indicates that the BordaEXP mechanism satisfies $\mathrm{e}^{\frac{n}{2m-2}-\left(\lceil \frac{n}{2} \rceil-1\right)\frac{m}{2m-2}}$-Condorcet loser criterion. □

Proposition 10. *Given $\epsilon\in\mathbb{R}_+$, RD-Anti satisfies*

(1) 1-*Pareto efficiency,*

(2) $\frac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^\epsilon+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^\epsilon}$-*Condorcet criterion,*

(3) $\frac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^\epsilon+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^\epsilon}$-*Condorcet loser criterion.*

Proof. First, given profile P, for any $a,b\in A$, a Pareto dominates b means that $a\succ_j b$ for all $j\in N$. Then a, the Pareto dominator, is never ranked last in any $\succ_j$. Therefore, $\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]\geqslant\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=b]$, which completes the proof of (1). Second, we prove (2). For any profile $P\in\mathcal{L}(A)^n$,

$$
|\{j\in N:a_\perp^j=\mathrm{CW}(P)\}|\leqslant\lceil\frac{n}{2}\rceil-1,
$$

otherwise $\mathrm{CW}(P)$ will be the Condorcet loser. Therefore,

$$
\begin{aligned}
&\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=\mathrm{CW}(P)]\\
\geqslant\ &\frac{\lceil\frac{n}{2}\rceil-1}{n}\cdot\frac{1}{(m-1)\mathrm{e}^\epsilon+1}+\frac{\lfloor\frac{n}{2}\rfloor+1}{n}\cdot\frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}.
\end{aligned}
$$

For any $a\neq\mathrm{CW}(P)$,

$$
\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]\leqslant\frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}.
$$

Hence, we have

$$
\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=\mathrm{CW}(P)]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]}\geqslant\frac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^\epsilon+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^\epsilon},
$$

which completes the proof. Finally, we prove (3). Given a profile P that $\mathrm{CL}(P)$ exists,

$$
|\{j\in N:a_\perp^j=a\}|\leqslant\lceil\frac{n}{2}\rceil-1,
$$

otherwise a will be the Condorcet loser. Therefore,

$$
\begin{aligned}
&\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]\\
\geqslant\ &\frac{\lceil\frac{n}{2}\rceil-1}{n}\cdot\frac{1}{(m-1)\mathrm{e}^\epsilon+1}+\frac{\lfloor\frac{n}{2}\rfloor+1}{n}\cdot\frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}.
\end{aligned}
$$

However,

$$
\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=\mathrm{CL}(P)]\leqslant\frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon+1}.
$$

Hence we have

$$
\frac{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=a]}{\mathbb{P}[\mathfrak{E}_{\mathsf{Anti}}(P)=\mathrm{CL}(P)]}\geqslant\frac{\left(\lfloor\frac{n}{2}\rfloor-1\right)\mathrm{e}^\epsilon+\lceil\frac{n}{2}\rceil+1}{n\mathrm{e}^\epsilon},
$$

which completes the proof of (3). □

Proposition 11. *Given $\epsilon\in\mathbb{R}_+$, CWRR satisfies 1-Pareto efficiency, $\frac{m-1}{m}$-SD-efficiency, and 1-Condorcet loser criterion.*

Proof. The bounds of Pareto efficiency and Condorcet loser criterion are evident, since for any profile P, neither a Pareto dominated alternative nor the Condorcet loser can be the Condorcet winner. Then we only need to prove the bound of SD-efficiency. Given profile P, we have

$$\sup_{P,\xi} \inf_{j,y} \frac{\sum_{x \succ_j y} \mathbb{P}[\xi = x]}{\sum_{x \succ_j y} \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = x]} \leqslant \frac{1}{1 - \sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a_\perp^j]}.$$

Then there are two possible cases for the profile, discussed as follows

1. If $\mathrm{CW}(P)$ exists, then there must exist some j that $a_\perp^j \neq \mathrm{CW}(P)$. Therefore
$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a_\perp^j] = \frac{1}{\mathrm{e}^\epsilon + m - 1}.$$
2. If $\mathrm{CW}(P)$ does not exist, then
$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a_\perp^j] = \frac{1}{m} \geqslant \frac{1}{\mathrm{e}^{\epsilon + m - 1}}.$$

In other words, we have

$$\sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CW}}(P) = a_\perp^j] = \frac{1}{m},$$

which indicates that CWRR satisfies $\frac{m-1}{m}$-SD-efficiency. That completes the proof. $\square$

Proposition 12. *Given $\epsilon \in \mathbb{R}_+$, CLRR satisfies 1-Pareto efficiency, $\frac{(m-2)\mathrm{e}^\epsilon+1}{(m-1)\mathrm{e}^\epsilon+1}$-SD-efficiency, and 1-Condorcet criterion.*

Proof. The bounds of Pareto efficiency and Condorcet criterion are evident, since for any profile P, neither a Pareto dominator nor the Condorcet winner can be the Condorcet loser. Then we only need to prove the bound of SD-efficiency. Given profile P, we have

$$\sup_{P,\xi} \inf_{j,y} \frac{\sum_{x \succ_j y} \mathbb{P}[\xi = x]}{\sum_{x \succ_j y} \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = x]} \leqslant \frac{1}{1 - \sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a_\perp^j]}.$$

Then there are two possible cases for the profile, discussed as follows

1. If $\mathrm{CL}(P)$ exists, considering the profile P, where each $a_\perp^j \neq \mathrm{CL}(P)$ for each $j \in N$, we have
$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a_\perp^j] = \frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon + 1}.$$
2. If $\mathrm{CL}(P)$ does not exist, then
$$\inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a_\perp^j] = \frac{1}{m} \leqslant \frac{\mathrm{e}^\epsilon}{(m-1)\mathrm{e}^\epsilon + 1}.$$

In other words, we have

$$\sup_P \inf_j \mathbb{P}[\mathfrak{R}_{\mathrm{CL}}(P) = a_\perp^j] = \frac{1}{m},$$

i.e., CWRR satisfies $\frac{(m-2)\mathrm{e}^\epsilon+1}{(m-1)\mathrm{e}^\epsilon+1}$-SD-efficiency. That completes the proof. $\square$

B.2 PROOFS OF THEOREMS 1-6

Theorem 1. *There is no voting rule satisfying ϵ-DP, α-Condorcet criterion and η-Condorcet loser criterion with $\alpha \cdot \eta > \mathrm{e}^\epsilon$.*

Proof. Consider the profile P ($n = 2k+1$):

- $k+1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$,
- k voters: $a_m \succ a_{m-1} \succ \cdots \succ a_1$.

By definition, we have $\mathrm{CW}(P) = a_1$, since $w_P[a_1, a_i] = 1$, for all $a_i \neq a_1$. Now consider another profile P' with the same number of voters:

- k voters: $a_1 \succ' a_2 \succ' \cdots \succ' a_m$,
- $k+1$ voters: $a_m \succ a_{m-1} \succ \cdots \succ a_1$.

Then $w_P[a_1, a_i] = -1$, for all $a_i \in A \backslash \{a\}$, i.e., a_1 is a Condorcet loser. Since there is only one voter changes her preference from P to P', we have

$$\begin{aligned}
\mathbb{P}[f(P) = a_1] &\geqslant \alpha \cdot \mathbb{P}[f(P) = a_2] && (\alpha\text{-Condorcet criterion}) \\
&\geqslant \alpha\eta \cdot \mathbb{P}[f(P) = a_m] && (\eta\text{-Condorcet loser criterion}) \\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha\eta \cdot \mathbb{P}[f(P') = a_m] && (\epsilon\text{-DP}) \\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha^2 \cdot \eta \cdot \mathbb{P}[f(P') = a_2] && (\alpha\text{-Condorcet criterion}) \\
&\geqslant \mathrm{e}^{-\epsilon} \cdot \alpha^2 \cdot \eta^2 \cdot \mathbb{P}[f(P') = a_1] && (\eta\text{-Condorcet loser criterion}) \\
&\geqslant \mathrm{e}^{-2\epsilon} \cdot \alpha^2 \cdot \eta^2 \cdot \mathbb{P}[f(P') = a_1], && (\epsilon\text{-DP})
\end{aligned}$$

which indicates that $\mathrm{e}^{-2\epsilon}\alpha^2\eta^2 \leqslant 1$, i.e., $\alpha\eta \leqslant \mathrm{e}^\epsilon$. That completes the proof. $\square$

Theorem 2. *If a neutral voting rule $f \colon \mathcal{L}(A)^n \to A$ satisfies ϵ-DP, β-Pareto efficiency, and α-Condorcet criterion, then $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$.*

Proof. Consider the following profile P ($n = 2k+1$):

- $k+1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$;
- k voters: $a_2 \succ \cdots \succ a_m \succ a_1$.

By definition, we have $w_P[a_1, a_i] = 1$, for all $a_i \in A \backslash \{a_1\}$, which indicates that $\mathrm{CW}(P) = a_1$. Also notice that $w_P[a_i, a_j] = n$ for all $i < j$. Thus, a_i Pareto dominates a_j for all $i < j$. The relations among all alternatives are shown in the following graph.

$$a_1 \xrightarrow{\textbf{CW}} a_2 \xrightarrow{\textbf{Pareto}} a_3 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_m. \quad (4)$$

Since f satisfies α-Condorcet criterion and β-Pareto efficiency, we have

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \alpha \cdot \mathbb{P}[f(P) = a_2] \\ &\geqslant \alpha\beta \cdot \mathbb{P}[f(P) = a_3] \\ &\geqslant \cdots \\ &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m].\end{aligned}$$

Now, consider another profile P', where all voters' preferences are exactly the same:

$$a_m \succ a_{m-1} \succ \cdots \succ a_1.$$

Then we have the following graph.

$$a_m \xrightarrow{\textbf{CW}} a_{m-1} \xrightarrow{\textbf{Pareto}} a_{m-2} \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_1.$$

Similarly, we have

$$\mathbb{P}[f(P') = a_m] \geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_1].$$

Notice that $|\{j \in N : \succ_j \neq \succ'_j\}| = n$. Therefore,

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m] \\ &\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_m] \quad (\epsilon\text{-DP}) \\ &\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P') = a_1] \\ &\geqslant \mathrm{e}^{-2n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P) = a_1]. \\ &\qquad (\epsilon\text{-DP})\end{aligned}$$

Then $\mathrm{e}^{-2n\epsilon}\alpha^2\beta^{2m-4} \leqslant 1$, i.e., $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof. □

Theorem 3. *If a neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfies ϵ-DP, β-Pareto efficiency, and α-Condorcet loser criterion, then $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$.*

Proof. Consider the following profile P $(n = 2k+1)$:

- $k+1$ voters: $a_1 \succ a_2 \succ \cdots \succ a_m$;
- k voters: $a_2 \succ \cdots \succ a_m \succ a_1$.

By definition, we have $w_P[a_1, a_i] = 1$, for all $a_i \in A \backslash \{a_1\}$, which indicates that $\mathrm{CL}(P) = a_1$. Also notice that $w_P[a_i, a_j] = n$ for all $i < j$. Thus, a_i Pareto dominates a_j for all $i < j$. The relations among all alternatives are shown in the following graph.

$$a_1 \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_{m-1} \xrightarrow{\textbf{CL}} a_m.$$

Since f satisfies α-Condorcet loser criterion and β-Pareto efficiency, we have

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \beta \cdot \mathbb{P}[f(P) = a_2] \\ &\geqslant \cdots \\ &\geqslant \beta^{m-2} \cdot \mathbb{P}[f(P) = a_{m-1}] \\ &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m].\end{aligned}$$

Now, consider another profile P', where all voters' preferences are exactly the same:

$$a_m \succ a_{m-1} \succ \cdots \succ a_1.$$

Then we have the following graph.

$$a_m \xrightarrow{\textbf{Pareto}} a_{m-1} \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{CL}} a_1.$$

Similarly, we have

$$\mathbb{P}[f(P') = a_m] \geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_1].$$

Notice that $|\{j \in N : \succ_j \neq \succ'_j\}| = n$. Therefore,

$$\begin{aligned}\mathbb{P}[f(P) = a_1] &\geqslant \alpha\beta^{m-2} \cdot \mathbb{P}[f(P) = a_m] \\ &\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha\beta^{m-2} \cdot \mathbb{P}[f(P') = a_m] \quad (\epsilon\text{-DP}) \\ &\geqslant \mathrm{e}^{-n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P') = a_1] \\ &\geqslant \mathrm{e}^{-2n\epsilon} \cdot \alpha^2\beta^{2m-4} \cdot \mathbb{P}[f(P) = a_1]. \\ &\qquad (\epsilon\text{-DP})\end{aligned}$$

Then $\mathrm{e}^{-2n\epsilon}\alpha^2\beta^{2m-4} \leqslant 1$, i.e., $\alpha\beta^{m-2} \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof. □

Proposition 13. *Condorcet method satisfies SD-efficiency on $\mathcal{D}_C$.*

Proof. Let P be an arbitrarily chosen profile in $\mathcal{D}_C$. Then we only need to proof that there does not exist $\xi \in \mathcal{R}(A)$ that SD-dominates $\mathsf{CM}(P)$.

In fact, if there exists such a ξ, we can obtain by definition that for all $j \in N$ and $a \in A$,

$$\sum_{b \succ_j a} \mathbb{P}[\xi = b] \geqslant \sum_{b \succ_j a} \mathbb{P}[\mathsf{CM}(P) = b],$$

Since for any $a \in A$ that $\mathrm{CW}(P) \succ_j a$, we have

$$\sum_{b \succ_j a} \mathbb{P}[\mathsf{CM}(P) = b] = \mathbb{P}[\mathsf{CM}(P) = \mathrm{CW}(P)] = 1,$$

which indicates that

$$\sum_{b \succ_j a} \mathbb{P}[\xi = b] \geqslant 1.$$

Therefore, for any $a \in A$ that $\mathrm{CW}(P) \succ_j a$, $\mathbb{P}[\xi = a] = 0$. However, according to the definition of $\mathrm{CW}(P)$, each $a \in A$ must be ranked behind $\mathrm{CW}(P)$ in some $\succ_j$. Hence we have $\xi = \mathsf{CM}(P)$, a contradiction. $\square$

Theorem 4. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, α-Condorcet criterion, and γ-SD efficiency with $\gamma > \frac{\alpha+m-1-\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}$.*

Proof. Consider the profile P, where all voters' vote are exactly the same, i.e.,

$$a_1 \succ_j a_2 \succ_j \cdots \succ_j a_m, \quad \text{for all } j \in N.$$

It is not hard to see that $\mathrm{CW}(P) = a_1$. Since f satisfies α-Condorcet criterion, we have $\mathbb{P}[f(P) = a] \leqslant \mathbb{P}[f(P) = a_1]/\alpha$, for all $a \in A\backslash\{a_1\}$. Therefore,

$$\begin{aligned} 1 &= \mathbb{P}[f(P) = a_1] + \sum_{a \in A\backslash\{a_1\}} p[f(P) = a] \\ &\leqslant \left(1 + \frac{m-1}{\alpha}\right)\mathbb{P}[f(P) = a_1], \end{aligned}$$

i.e., $\mathbb{P}[f(P) = a_1] \geqslant \frac{\alpha}{\alpha+m-1}$. Further, by Equation (3), we have

$$\mathbb{P}[f(P) = a_m] \geqslant \mathrm{e}^{-n\epsilon} \cdot \mathbb{P}[f(P) = a_1] \geqslant \frac{\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}.$$

However, for profile P, the unique SD-efficient lottery is $\mathbb{I}_{a_1}$. In other words, all lotteries $\xi \in \mathcal{R}(A)$ that $\xi \neq \mathbb{I}_{a_1}$ are γ-SD-dominated by $\mathbb{I}_{a_1}$ with $\gamma > 1$. Further,

$$\begin{aligned} \frac{\sum_{x \succ y} \mathbb{P}[\mathbb{I}_{a_1} = x]}{\sum_{x \succ y} \mathbb{P}[f(P) = x]} \geqslant& \frac{\inf_{y \in A} \sum_{x \succ y} \mathbb{P}[\mathbb{I}_{a_1} = x]}{\sup_{y \in A} \sum_{x \succ y} \mathbb{P}[f(P) = x]} \\ =& \frac{1}{1 - \mathbb{P}[f(P) = a_m]} \\ \geqslant& \frac{1}{1 - \frac{\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}} \\ =& \frac{\alpha+m-1-\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}, \end{aligned}$$

i.e., $\mathbb{I}_{a_1}$ can $\frac{\alpha+m-1-\alpha\mathrm{e}^{-n\epsilon}}{\alpha+m-1}$-dominates $f(P)$, which completes the proof. $\square$

Theorem 5. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, η-Condorcet loser criterion, and γ-SD-efficiency with $\gamma > \frac{\mathrm{e}^{n\epsilon}-\eta}{\mathrm{e}^{n\epsilon}}$.*

Proof. Let $m > 2$. Consider the following profile P with $k(m-2)$ voters ($k \geqslant 1$).

- k voters: $y \succ a_1 \succ a_2 \succ \cdots \succ x \succ a_{m-2}$,
- k voters: $y \succ a_2 \succ a_3 \succ \cdots \succ x \succ a_1$,
- k voters: $y \succ a_3 \succ a_4 \succ \cdots \succ x \succ a_2$,
- k voters: $\cdots$,
- k voters: $y \succ a_{m-2} \succ a_1 \succ \cdots \succ x \succ a_{m-3}$.

Then it is quite evident that $\mathrm{CL}(P) = x$. Since f satisfies η-Condorcet loser criterion, we have $\mathbb{P}[f(P) = a] \geqslant \eta \cdot \mathbb{P}[f(P) = x]$, for all $a \in A\backslash\{x\}$. By Equation (3), we have $\mathbb{P}[f(P) = x] \geqslant \mathrm{e}^{-n\epsilon}$. Since f satisfies η-Condorcet loser criterion, for all $a \in A\backslash\{x\}$, we have

$$\mathbb{P}[f(P) = a] \geqslant \eta \cdot \mathbb{P}[f(P) = x] \geqslant \alpha\mathrm{e}^{-n\epsilon}.$$

However, the unique SD-efficient lottery of P is $\mathbb{I}_y$, since $\mathbb{I}_y$ can SD-dominates any other lotteries on A. Further,

$$\begin{aligned} \frac{\sum_{b \succ a} \mathbb{P}[\mathbb{I}_y = b]}{\sum_{b \succ a} \mathbb{P}[f(P) = b]} &\geqslant \frac{\inf_{a \in A} \sum_{b \succ a} \mathbb{P}[\mathbb{I}_y = b]}{\sup_{a \in A} \sum_{b \succ a} \mathbb{P}[f(P) = b]} \\ &= \frac{1}{1 - \inf \min_{1 \leqslant i \leqslant m-2} \mathbb{P}[f(P) = a_i]} \\ &\geqslant \frac{1}{1 - \alpha\mathrm{e}^{-n\epsilon}}. \end{aligned}$$

In other words, $\mathbb{I}_y$ can $\frac{\mathrm{e}^{n\epsilon}-\eta}{\mathrm{e}^{n\epsilon}}$-SD-dominates $f(P)$, which completes the proof. $\square$

Theorem 6. *There is no neutral voting rule $f\colon \mathcal{L}(A)^n \to \mathcal{R}(A)$ satisfying ϵ-DP, γ-SD-efficiency, and β-Pareto efficiency with $\gamma > \frac{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}}{\mathrm{e}^{n\epsilon}-\mathrm{e}^{n\epsilon}\beta^{2-m}+\beta-1}$.*

Proof. Consider the profile P, where all voters' preferences are the same, i.e.,

$$a_1 \succ_j a_2 \succ_j \cdots \succ_j a_m, \quad \text{for all } j \in N.$$

By definition, for all $i < j$, any a_i Pareto dominates a_j in profile P. In other words, we have the following diagram

$$a_1 \xrightarrow{\textbf{Pareto}} a_2 \xrightarrow{\textbf{Pareto}} \cdots \xrightarrow{\textbf{Pareto}} a_m.$$

Since f satisfies β-Pareto efficiency, $\mathbb{P}[f(P) = a_{i+1}] \leqslant \beta \cdot \mathbb{P}[f(P) = a_i]$ holds for any $i < m$. By Equation (3), $\mathbb{P}[f(P) = a_m] \geqslant \mathrm{e}^{-n\epsilon} \cdot \mathbb{P}[f(P) = a_1]$. Further,

$$\begin{aligned} \mathbb{P}[f(P) = a_1] &\leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P) = a_m], \\ \mathbb{P}[f(P) = a_2] &\leqslant \frac{1}{\beta} \cdot \mathbb{P}[f(P) = a_1] \\ &\leqslant \frac{\mathrm{e}^{n\epsilon}}{\beta} \cdot \mathbb{P}[f(P) = a_m], \\ &\cdots \\ \mathbb{P}[f(P) = a_{m-1}] &\leqslant \frac{1}{\beta^{m-2}} \cdot \mathbb{P}[f(P) = a_1] \\ &\leqslant \frac{\mathrm{e}^{n\epsilon}}{\beta^{m-2}} \cdot \mathbb{P}[f(P) = a_m]. \end{aligned}$$

By summing up the above inequalities, we have

$$
\begin{aligned}
1 &= \sum_{a \in A} \mathbb{P}[f(P) = a] \\
&\leqslant \left(1 + \left(1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{m-2}}\right) \mathrm{e}^{n\epsilon}\right) \mathbb{P}[f(P) = a_m],
\end{aligned}
$$

i.e., $\mathbb{P}[f(P) = a_m] \geqslant \frac{\beta-1}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}$. However, the unique SD-efficient lottery of P is $\mathbb{I}_{a_1}$, since it can SD-dominate any other lottery. Further, we have

$$
\begin{aligned}
\frac{\sum_{b \succ a} \mathbb{P}[\mathbb{I}_y = b]}{\sum_{b \succ a} \mathbb{P}[f(P) = b]} &\geqslant \frac{\inf_{a \in A} \sum_{b \succ a} \mathbb{P}[\mathbb{I}_y = b]}{\sup_{a \in A} \sum_{b \succ a} \mathbb{P}[f(P) = b]} \\
&= \frac{1}{1 - \mathbb{P}[f(P) = a_m]} \\
&\geqslant \frac{1}{1 - \frac{\beta - 1}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}}.
\end{aligned}
$$

In other words, $\mathbb{I}_{a_1}$ can $\frac{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m}}{\mathrm{e}^{n\epsilon} - \mathrm{e}^{n\epsilon}\beta^{2-m} + \beta - 1}$-SD-dominates $f(P)$, which completes the proof. $\square$

C MORE FIGURES FOR THE TRADEOFF CURVES

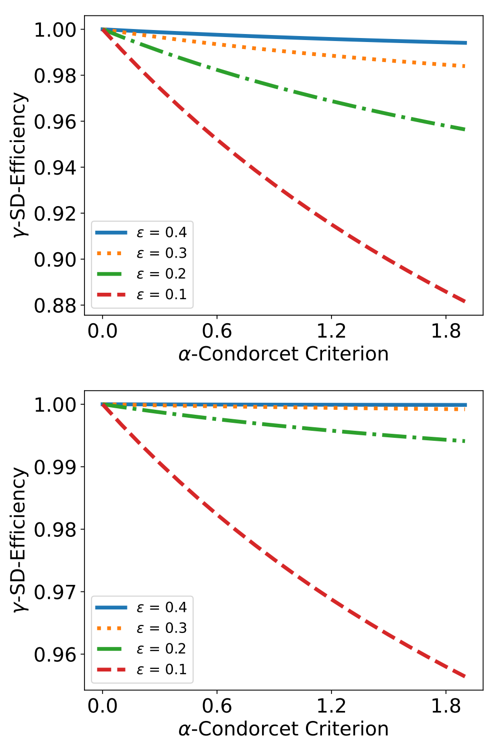

Figure 7: Tradeoff curves between α-Condorcet criterion and γ-SD-efficiency under ϵ-DP (upper bounds). Above: $m = 5, n = 10$. Below: $m = 5, n = 20$.

The Python codes for drawing these curves are included in the supplementary materials.

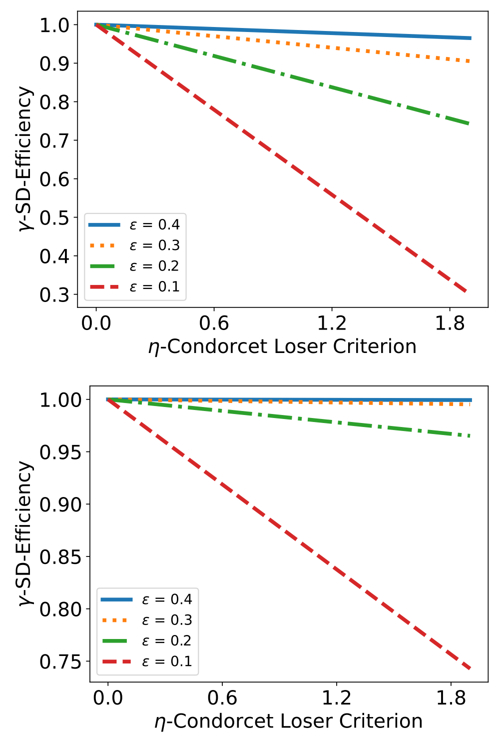

Figure 8: Tradeoff curves between η-Condorcet loser criterion and γ-SD-efficiency under ϵ-DP (upper bounds). Above: $m = 5, n = 10$. Below: $m = 5, n = 20$.