A RULES DERIVATION

At any point during the out-of-order simulation, the state includes each agent's location and the step they have executed. For a given state, let $dist(A, B)$ represent the distance between agents A and B, $radius\_p$ denote the radius of an agent's perception, and max_vel denote the maximum speed of agent movement and information propagation per step. According to Section 3.2, a valid execution needs to make sure the following condition holds at any state:

$$\begin{aligned}&\forall \text{ agents } A, B, \text{ and their current steps } Step_A \text{ and } Step_B,\\ &\text{if } Step_A \neq Step_B, \text{then } dist(A, B) > radius\_p\\ &\qquad + (|Step_A - Step_B| - 1) \times max\_vel\end{aligned}$$

To satisfy the condition, we derive the following simulation conditions to ensure the state remains valid. Notice that our simulation conditions are over-estimations, which is sound in correctness but not necessarily complete.

Starting from any valid state, given any two agents A and B at steps $Step_A$ and $Step_B$, respectively, we derive the simulation conditions of A by case study. Let A' denote the new state of A after one more step so that $dist(A', B)$ denotes the new distance between A and B after the next step of A.

- Assume steps $Step_A = Step_B$. A is allowed to proceed to the next step if a valid state is reached after one further step of A. Formally, there should be

  $$\begin{aligned}dist(A', B) >&(Step_A - Step_B + 1 - 1) \times max\_vel\\ &+ radius\_p\end{aligned}$$

  Since $dist(A', B) \geq dist(A, B) - max\_vel$, we need:

  $$dist(A, B) - max\_vel > radius\_p$$

  Therefore, on the other side, A and B must stay at the same step if:

  $$dist(A, B) \leq max\_vel + radius\_p$$

  which means they are *coupled* and can either wait together or proceed together.

- Assume steps $Step_A > Step_B$. There should be

  $$\begin{aligned}dist(A', B) >&(Step_A - Step_B + 1 - 1) \times max\_vel\\ &+ radius\_p\end{aligned}$$

  Since $dist(A', B) \geq dist(A, B) - max\_vel$, we need:

  $$\begin{aligned}&dist(A, B) - max\_vel\\ &\quad > radius\_p + (Step_A - Step_B) \times max\_vel\end{aligned}$$

  Therefore, on the other side, A got *blocked* by B if

  $$\begin{aligned}dist(A, B) \leq&(Step_A - Step_B + 1) \times max\_vel\\ &+ radius\_p\end{aligned}$$

- Assume steps $Step_A < Step_B$. There should be

  $$\begin{aligned}dist(A', B) >&(Step_B - Step_A - 1 - 1) \times max\_vel\\ &+ radius\_p\end{aligned}$$

  Since $dist(A', B) \geq dist(A, B) - max\_vel$, we need:

  $$\begin{aligned}&dist(A, B) - max\_vel\\ &\quad > (Step_B - Step_A - 2) \times max\_vel + radius\_p\end{aligned}$$

  $$\begin{aligned}\Rightarrow dist(A, B) >&(Step_B - Step_A - 1) \times max\_vel\\ &+ radius\_p\end{aligned}$$

  which is the same valid condition of the current state. Therefore, A is not *blocked* by any future agents.