RISK ASSESSMENT AND STATISTICAL SIGNIFICANCE IN THE AGE OF FOUNDATION MODELS

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ABSTRACT

We propose a distributional framework for assessing socio-technical risks of foundation models with quantified statistical significance. Our approach hinges on a new statistical relative testing based on first and second order stochastic dominance of real random variables. We show that the second order statistics in this test are linked to mean-risk models commonly used in econometrics and mathematical finance to balance risk and utility when choosing between alternatives. Using this framework, we formally develop a risk-aware approach for foundation model selection given guardrails quantified by specified metrics. Inspired by portfolio optimization and selection theory in mathematical finance, we define a *metrics portfolio* for each model as a means to aggregate a collection of metrics, and perform model selection based on the stochastic dominance of these portfolios. The statistical significance of our tests is backed theoretically by an asymptotic analysis via central limit theorems instantiated in practice via a bootstrap variance estimate. We use our framework to compare various large language models regarding risks related to drifting from instructions and outputting toxic content.

1 INTRODUCTION

Foundation models such as large language models have shown remarkable capabilities and opportunities redefining the field of artificial intelligence. At the same time, they present pressing and challenging socio-technical risks regarding the trustworthiness of their outputs and their alignment with human values and ethics (Bommasani et al., 2021). Evaluating LLMs is therefore a multi-dimensional problem, where those risks are assessed across diverse tasks and domains (Chang et al., 2023).

In order to quantify these risks, Liang et al. (2022); Wang et al. (2023); Huang et al. (2023) proposed benchmarks of automatic metrics for probing the trustworthiness of LLMs including accuracy, robustness, fairness, toxicity of the outputs, etc. Human evaluation benchmarks can be even more nuanced, and are often employed when tasks surpass the scope of standard metrics. Notable benchmarks based on human and automatic evaluations include, among others, Chatbot Arena (Zheng et al., 2023), HELM (Bommasani et al., 2023), MosaicML's Eval, Open LLM Leaderboard (Wolf, 2023), and BIG-bench (Srivastava et al., 2022), each catering to specific evaluation areas such as chatbot performance, knowledge assessment, and domain-specific challenges. Traditional metrics, however, sometimes do not correlate well with human judgments. Aiming for a better alignment with human judgments, some approaches utilize ChatGPT/GPT-4 for natural language generation evaluations (Liu et al., 2023; Zhang et al., 2023; Hada et al., 2023).

Evaluating Machine Learning models is intimately connected to statistical significance testing (SST), although it is still underutilized: for example, Dror et al. (2018) reports almost 50% of ACL papers miss SST indicators. With the ever increasing parametric complexity of Large Language Models, obtaining a reliable SST in evaluating foundation models becomes ever more urgent. Focusing on robust comparison of DNNs specifically, Dror et al. (2019) defined criteria that valid and powerful tests should satisfy, and developed an assumption-less significance test utilizing Almost Stochastic Ordering (ASO) (Del Barrio et al., 2018). Besides theoretical guarantees, this method was shown to outperform traditional SST on a variety of NLP tasks. Building on the ASO work, Ulmer et al. (2022) re-implemented the method of Del Barrio et al. (2018); Dror et al. (2019) as an easy-to-use open-source package accompanied by guidelines and case studies.

In this paper, we propose a distributional framework for evaluating and comparing multiple foundation models across different metrics with quantified statistical significance. While Dror et al. (2018) relied on first order almost stochastic dominance and on a statistical testing against a fixed threshold, we propose a relative testing based on first and second stochastic orders. In econometrics and mathematical finance, the second order dominance is used to perform portfolio selection and optimization in risk averse settings (Ogryczak & Ruszczynski, 2002). We draw inspiration from this literature and propose a *metrics portfolio* for each model as a means to aggregate the metrics, and perform SST of model comparisons using these portfolios. We provide an efficient implementation of these new proposed statistical tests and risk assessments.

The paper is organized as follows: In Section 2 we review stochastic dominance and introduce our new notion of *relative* dominance. Section 3 presents our statistical tests, backed with central limit theorems, and in Section 4 we present our distributional framework for evaluating foundation models based on our relative tests. Finally in Section 5, we present experimental results of our framework, evaluating LLMs regarding risks of drift from instructions and toxicity of generations.

2 Stochastic Dominance

We first review notions of stochastic dominance and their relation to downside risk measures and risk averse preference modeling. Stochastic dominance between two random variables defines a partial ordering via considering a point-wise comparison of performance functions constructed from the variables' distributions. We use the notation of the seminal paper Ogryczak & Ruszczynski (2002), and assume that the random variables are standardized so that larger outcomes are preferable.

2.1 First and Second order Dominance and Mean-Risk Models

First Order Stochastic Dominance The First-order Stochastic Dominance (FSD) between real-valued random variables uses the right-continuous cumulative distribution (CDF) as a performance function. Specifically, for a real random variable X, define the first performance function $F_X^{(1)} : \mathbb{R} \to [0,1]$ as the CDF: $F_X^{(1)}(\eta) = \mathbb{P}(X \leq \eta), \forall \eta \in \mathbb{R}$. The FSD of X on Y is defined as follows:

$$X \underset{\text{FSD}}{\succcurlyeq} Y \iff F_X^{(1)}(\eta) \leq F_Y^{(1)}(\eta), \forall \eta \in \mathbb{R}, \tag{1}$$

this intuitively means that for all outcomes η, the probability of observing smaller outcomes than η is lower for X than Y. An equivalent definition can be expressed using the quantile $F_X^{(-1)}$:

$$X \underset{\text{FSD}}{\succcurlyeq} Y \iff F_X^{(-1)}(p) \geq F_Y^{(-1)}(p), \forall p \in (0,1], \tag{2}$$

where $F_X^{(-1)} : [0,1] \to \overline{\mathbb{R}}$ is the left-continuous inverse of $F_X^{(1)}$: $F_X^{(-1)}(p) = \inf\{\eta : F_X^{(1)}(\eta) \geq p\}$ for $p \in (0,1]$. We focus on this definition as it is more computationally and notationally friendly since the quantile function is always supported on $[0,1]$.

Second Order Stochastic Dominance The Second-order Stochastic Dominance (SSD) is defined via the second performance function $F_X^{(2)} : \mathbb{R} \to [0,1]$ that measures the area under the CDF: $F_X^{(2)}(\eta) = \int_{-\infty}^{\eta} F_X^{(1)}(x)dx$, for $x \in \mathbb{R}$, yielding:

$$X \underset{\text{SSD}}{\succcurlyeq} Y \iff F_X^{(2)}(\eta) \leq F_Y^{(2)}(\eta), \forall \eta \in \mathbb{R}. \tag{3}$$

Note that FSD implies SSD, hence SSD is a finer notion of dominance. While FSD implies that X is preferred to Y by any utility-maximizing agent preferring larger outcomes[1], Ogryczak & Ruszczynski (2002) showed that SSD implies that X is preferred to Y by any *risk-averse* agent preferring larger outcomes.[2] Similarly to FSD, SSD can be measured with quantile functions via introducing

[1] I.e. having an increasing utility function.

[2] I.e. having an increasing and *concave* utility function.

the second quantile function also known as *integrated quantiles* $F_X^{(-2)} : (0,1] \to \overline{\mathbb{R}}$

$$F_X^{(-2)}(p) = \int_0^p F_X^{(-1)}(t)dt, \text{ for } t \in (0,1]. \tag{4}$$

Similarly to the FSD case, a more computationally friendly definition can be expressed in terms of the second quantile function:

$$X \underset{\text{SSD}}{\succcurlyeq} Y \iff F_X^{(-2)}(p) \geq F_Y^{(-2)}(p), \forall p \in (0,1]. \tag{5}$$

This equivalence is not straightforward and is due to Fenchel duality between $F^{(2)}$ and $F^{(-2)}$. Using $p = 1$ we see that SSD implies $\mu_X \geq \mu_Y$, where μ_X and μ_Y are means of X and Y.

Mean – Risk Models (MRM) As noted earlier SSD is linked to risk assessment via the second performance function $F^{(2)}(.)$ measuring expected shortfall, and the negative second quantile function $-F^{(-2)}(p)$ that is an assessment of expected losses given outcomes lower than the p-quantile.

Definition 1 (Mean – Risk Models)**.** *A mean – risk model of a random variable X consists of the pair (μ_X, r_X), where μ_X is the mean of X, and r_X is a functional that measures the risk of the random outcome X.*

The consistency of a mean – risk model with SSD is defined as follows:

Definition 2 (SSD consistency of Mean – Risk Models)**.** *A mean – risk model (μ_X, r_X) is $\alpha-$consistent with SSD, if for $\alpha > 0$ the following is true:*

$$X \underset{SSD}{\succcurlyeq} Y \implies \mu_X - \alpha r_x \geq \mu_Y - \alpha r_Y. \tag{6}$$

The ubiquitous mean – risk model in machine learning is (μ_X, σ_X), where σ_X is the standard deviation. Unfortunately this model is not consistent with the SSD and has several limitations as it implies Gaussianity of the outcomes or a quadratic utility function. A large body of work was devoted to defining mean – risk models that are consistent with SSD in econometrics and mathematical finance. The summary statistics r_X quantifying risks consistent with SSD are of interest in order to assess risks of foundation models. We give in Table 1 a summary of these risks measurement and their $\alpha-$consistency and refer the reader to (Ogryczak & Ruszczynski, 2002) for proofs of these claims:

Name	Risk Measure	$\alpha-$ consistency with SSD
Standard deviation	$\sigma_X = \sqrt{\mathbb{E}(X - \mu_X)^2}$	not consistent
Absolute semi deviation	$\delta_X = \mathbb{E}(\mu_X - X)_+$	$1-$ consistent
Negative Tail Value at Risk	$-\text{TVAR}_X(p) = -\frac{F^{(-2)}(p)}{p}$	$1-$ consistent for all $p \in (0,1]$
Mean absolute deviation from a quantile	$h_X(p) = \mu_x - \frac{F_X^{(-2)}(p)}{p}$	$1-$ consistent for all $p \in (0,1]$
Gini Tail	$\Gamma_X = 2\int_0^1 (\mu_X p - F_X^{(-2)}(p))dp$	$1-$ consistent

Table 1: Risk models and their $\alpha-$consistency with SSD.

Note that several risks in Table 1 use the second quantile function as part of an assessment of the left tails of the outcomes.

2.2 Relaxations of Stochastic Dominance

Recalling the definitions of FSD and SSD in Equations (7) and (8), in the finite-sample regime it is hard to test for these relations as one needs to show the infinite-sample quantile or second quantile properties hold uniformly over all $p \in (0,1]$. This difficulty motivated the relaxation of stochastic dominance to an almost stochastic dominance pioneered by Leshno & Levy (2002). These relaxations were revisited for the first order by Alvarez-Esteban et al. (2014) who later proposed an optimal transportation approach to assess almost first stochastic order Del Barrio et al. (2018).

Almost FSD (ε-FSD) Following Leshno & Levy (2002), Del Barrio et al. (2018) relaxed FSD via the violation ratio of FSD:

Definition 3 (FSD Violation Ratio (Del Barrio et al., 2018)). *For $F_X \neq F_Y$ define the violation ratio:*

$$\varepsilon_{\mathsf{W}_2}(F_X, F_Y) = \frac{\int_{A_0^{(1)}} (F_X^{(-1)}(t) - F_Y^{(-1)}(t))^2 dt}{\int_0^1 (F_X^{(-1)}(t) - F_Y^{(-1)}(t))^2 dt} = \frac{\int_0^1 (F_Y^{(-1)}(t) - F_X^{(-1)}(t))_+^2 dt}{\mathsf{W}_2^2(F_X, F_Y)},$$

where $A_0^{(1)} = \left\{ t \in (0,1) : F_Y^{(-1)}(t) > F_X^{(-1)}(t) \right\}$ is the violation set the relation $X \underset{FSD}{\succcurlyeq} Y$, and W_2 is the Wasserstein-2 distance.

Note that $0 \leq \varepsilon_{\mathsf{W}_2}(F_X, F_Y) \leq 1$, with value 0 if $X \underset{\text{FSD}}{\succ} Y$ and 1 if $Y \underset{\text{FSD}}{\succ} X$. For $\varepsilon \in (0, \frac{1}{2}]$, the relaxed FSD can be therefore defined as follows

$$X \underset{\varepsilon-\text{FSD}}{\succcurlyeq} Y \iff \varepsilon_{\mathsf{W}_2}(F_X, F_Y) \leq \varepsilon. \tag{7}$$

Figure 1a in Appendix C illustrates ε-FSD, dashed areas represent the violation set.

Almost SSD (ε-SSD) Note that the original definition of ε-FSD of X on Y in Leshno & Levy (2002) is an L_1 definition and uses the CDF rather than quantiles: $\int_{-\infty}^{\infty} (F_X(x) - F_Y(x))_+ dx \leq \varepsilon \int_{\infty}^{\infty} |F_X(x) - F_Y(x)| dx$. Tzeng et al. (2013) gave a similar L_1 definition for ε-SSD using the second performance function $F^{(2)}(.)$. According to Tzeng et al. (2013), X dominates Y in the ε-SSD if $\int_{-\infty}^{\infty} (F_X^{(2)}(x) - F_Y^{(2)}(x))_+ dt \leq \varepsilon \int_{-\infty}^{+\infty} |F_X^{(2)}(x) - F_Y^{(2)}(x)| dx$. Following Del Barrio et al. (2018), we redefine ε-SSD using second quantiles and with a L_2 definition, this eases the analysis and practically the integration is on $(0, 1]$ rather than $(-\infty, \infty)$.

We define the SSD violation ratio as follows:

Definition 4 (SSD Violation Ratio). *For $F_X \neq F_Y$ define the violation ratio:*

$$\varepsilon_{IQ}(F_X, F_Y) = \frac{\int_{A_0^{(2)}} (F_X^{(-2)}(t) - F_Y^{(-2)}(t))^2 dt}{\int_0^1 (F_X^{(-2)}(t) - F_Y^{(-2)}(t))^2 dt} = \frac{\int_0^1 (F_Y^{(-2)}(t) - F_X^{(-2)}(t))_+^2 dt}{d_{IQ}^2(F_X, F_Y)},$$

where $A_0^{(2)} = \left\{ t \in (0,1) : F_Y^{(-2)}(t) > F_X^{(-2)}(t) \right\}$ is the violation set the relation $X \underset{SSD}{\succcurlyeq} Y$, and d_{IQ} is the L_2 distance between the Integrated Quantiles ($F^{(-2)}$).

We are now ready to define ε-SSD, for $\varepsilon \in (0, \frac{1}{2})$:

$$X \underset{\varepsilon-\text{SSD}}{\succcurlyeq} Y \iff \varepsilon_{IQ}(F_X, F_Y) \leq \varepsilon \tag{8}$$

Figure 1b in Appendix C illustrates the second order, dashed areas represent the violation set of SSD of X on Y. Integrated quantiles fully characterize one dimensional distributions as can be seen from the following theorem, proved in Appendix E:

Theorem 1 (d_{IQ} is a metric). *d_{IQ} is a metric on the space of univariate distributions with continuous CDF, moreover, it metrizes the weak topology.*

2.3 Relative Stochastic Dominance

In the remainder of the paper, we refer to the FSD violation ratio as $\varepsilon_{\mathsf{W}_2}(F_X, F_Y) \equiv \varepsilon^{(1)}(F_X, F_Y)$ and to the SSD violation ratio as $\varepsilon_{IQ}(F_X, F_Y) \equiv \varepsilon^{(2)}(F_X, F_Y)$. One of the shortcomings of almost

stochastic dominance is the need to fix a threshold ε on the violation ratio. When comparing two random variables, setting a threshold is a viable option. Nevertheless, when one needs to rank multiple variables $X_1, \dots, X_k$ (considering all pairwise comparisons), setting a single threshold that would lead to a consistent relative stochastic dominance among the k variables becomes challenging. To alleviate this issue, we draw inspiration from relative similarity and dependence tests (Bounliphone et al., 2016a;b) that circumvent the need for a threshold via relative pairwise testings.

For $\ell \in \{1, 2\}$ (i.e for FSD or SSD) we consider all pairs of violations ratios:

$$\varepsilon_{ij}^{(\ell)} = \varepsilon^{(\ell)}(F_{X_i}, F_{X_j}) \text{ for } i, j \in \{1 \dots k\}, i \neq j,$$

noting that $\varepsilon_{ij}^{(\ell)} + \varepsilon_{ji}^{(\ell)} = 1$. Let $F = (F_{X_1}, \dots F_{X_k})$. We define the one-versus-all violation ratio of the dominance of X_i on all other variables $X_j, j \neq i$:

$$\varepsilon_i^{(\ell)}(F) = \frac{1}{k-1} \sum_{j \neq i} \varepsilon_{ij}^{(\ell)}.$$

We then define relative stochastic dominance for both orders, r-FSD an r-SSD respectively:

$$X_{i_1} \underset{r-\text{FSD}}{\succcurlyeq} X_{i_2} \dots \underset{r-\text{FSD}}{\succcurlyeq} X_{i_k} \iff \varepsilon_{i_1}^{(1)}(F) \leq \dots \leq \varepsilon_{i_k}^{(1)}(F) \tag{9}$$

$$X_{i_1} \underset{r-\text{SSD}}{\succcurlyeq} X_{i_2} \dots \underset{r-\text{SSD}}{\succcurlyeq} X_{i_k} \iff \varepsilon_{i_1}^{(2)}(F) \leq \dots \leq \varepsilon_{i_k}^{(2)}(F) \tag{10}$$

In this definition of relative stochastic dominance, the most dominating model is the one with the lowest one-versus-all violation ratio and to test for relative dominance of X_i on X_j we can look at the following statistics:

$$\Delta\varepsilon_{ij}^{(\ell)}(F) = \varepsilon_i^{(\ell)}(F) - \varepsilon_j^{(\ell)}(F), \tag{11}$$

and we have the following threshold-free test for relative order:[3]

$$X_i \underset{r-\text{FSD}}{\succcurlyeq} X_j \iff \Delta\varepsilon_{ij}^{(1)}(F) \leq 0 \tag{12}$$

$$X_i \underset{r-\text{SSD}}{\succcurlyeq} X_j \iff \Delta\varepsilon_{ij}^{(2)}(F) \leq 0 \tag{13}$$

3 Testing For Almost and Relative Stochastic Dominance

Given empirical samples from F_X and F_Y we perform statistical testing of the almost and relative stochastic dominance of X on Y given empirical estimates of the statistics given in Sections 2.2 and 2.3. A key ingredient for quantifying the statistical significance of such tests is a central limit theorem that guarantees that the centered empirical statistics is asymptotically Gaussian at the limit of infinite sample size. Given n samples from F_X (m from F_Y respectively), we denote F_X^n and F_Y^m the corresponding empirical distributions. For ε_0- FSD, Del Barrio et al. (2018) studied the following hypothesis testing $H_0 : X \underset{\varepsilon_0-\text{SSD}}{\not\succcurlyeq} Y$ versus the alternative $H_a : X \underset{\varepsilon_0-\text{SSD}}{\succcurlyeq} Y$. Using (7), this amounts to the following null hypothesis : $H_0 : \varepsilon_{\mathsf{W}_2}(F_X^n, F_Y^m) > \varepsilon_0$. Del Barrio et al. (2018) showed the asymptotic normality of the empirical statistics:

$$\sqrt{\frac{mn}{m+n}} \left(\varepsilon_{\mathsf{W}_2}(F_X^n, F_Y^m) - \varepsilon_{\mathsf{W}_2}(F_X, F_Y)\right) \to \mathcal{N}(0, \sigma^2(F_X, F_Y)).$$

Given an estimate of the standard deviation $\hat{\sigma}_{m,n}$ of $\varepsilon_{\mathsf{W}_2}(F_X^n, F_Y^m)$ obtained via bootstrapping[4], Del Barrio et al. (2018); Ulmer et al. (2022) propose to reject H_0 with a confidence level $1 - \alpha$ if:

[3]For comparing $k = 2$ random variables, these r-FSD and r-SSD tests reduce to 0.5-FSD and 0.5-SSD absolute tests, respectively.

[4]While the CLT provides an asymptotic value for the variance, bootstrapping is used in practice since it is nonasymptotic and hence usually more accurate.

$$\varepsilon_{\mathsf{W}_2}(F_X^n, F_Y^m) \leq \varepsilon_0 + \sqrt{\frac{m+n}{mn}}\hat{\sigma}_{m,n}\Phi^{-1}(\alpha), \tag{14}$$

where Φ^{-1} is the quantile function of a standard normal.

ε-SSD Testing Similar to ε-FSD, using the definition in (8) we propose to test using the following null hypothesis for testing for ε_0-SSD:

$$H_0 : \varepsilon_{IQ}(F_X^n, F_Y^m) > \varepsilon_0$$

We state a Central Limit Theorem for the second order statistics in Appendix D (Theorem 2, proved in Appendix F.1). Similarly to (14), Theorem 2 suggests to reject H_0 with a confidence $1-\alpha$ if :

$$\varepsilon_{IQ}(F_X^n, F_Y^m) \leq \varepsilon_0 + \sqrt{\frac{m+n}{mn}}\hat{\sigma}_{m,n}\Phi^{-1}(\alpha), \tag{15}$$

where (for the same reasons as the FSD case) $\hat{\sigma}_{m,n}$ is an estimate of the standard deviation of $\varepsilon_{IQ}(F_X^n, F_Y^m)$ that we can obtain via bootstrapping (Efron & Tibshirani, 1993).

Relative Stochastic Dominance Testing We turn now to relative stochastic dominance that we introduced in (12) and (13) for first and second orders. Given n samples from k random variables $(X_1 \dots X_k)$. Let $F = (F_1, \dots, F_k)$ be the marginals of X_i and $F_n = (F_{1n}, \dots, F_{kn})$ denote the empirical marginals. To test for R-SSD of X_{i_1} on X_{i_2} we propose to test the following null hypothesis:

$$H_0 : \Delta\varepsilon_{ij}^{(\ell)}(F_n) > 0, \ell = 1 \text{ or } 2$$

We state in Appendix D a central limit theorem for the relative second order statistics (Theorem 3 proved in in Appendix F.2). A similar result holds for the relative first order statistics that we omit for brevity. Theorem 3 suggests to reject H_0 with a confidence $1-\alpha$ if:

$$\Delta\varepsilon_{i_1,i_2}^{(2)}(F_n) \leq \sqrt{\frac{1}{n}}\hat{\sigma}_n\Phi^{-1}(\alpha) \tag{16}$$

where σ_n is an estimate of the standard deviation of $\Delta\varepsilon_{i_1,i_2}^{(2)}(F_n)$ that we (again) can obtain via bootstrapping. A similar test can be performed for the relative FSD.

Multi-Testing Algorithm Algorithm 1 given in Appendix A summarizes the multi-testing setup for both relative and almost (absolute) FSD and SSD. The main idea behind Algorithm 1 is to turn multi-testing to pairwise testings and to correct for the confidence level by dividing by the number of all pairs (Bonferroni, 1936). Then in order to combine the pairwise rankings to a single rank, we use a simple Borda count (de Borda, 1781) rank aggregation algorithm.

4 Distributional Risk Assessment of Foundation Models

Setup We consider the assessment of a foundation model $A : \mathcal{X} \to \mathcal{O}$, using N metrics $m_i : \mathcal{O} \to \mathbb{R}, i = 1 \dots N$, where m_i are real valued functions evaluated on different test sets and tasks. We denote the respective data distribution of these sets and tasks by D_i. Without loss of generality, assume that each of the metrics are standardized such that higher values of m_i correspond to more desirable model performance. We model observed values for each metric m_i as a continuous random variable M_i with unknown CDF F_{M_i}. For a model $A : \mathcal{X} \to \mathcal{O}$ and a data sample $X \sim D_i$, we describe the evaluation of model A with m_i with the following random variable M_i: $M_i|A, X := m_i(A(X)),\ X \sim D_i, i = 1 \dots N$, where the randomness arises from the data sampling procedure $X \sim D_i$, and (if applicable) the stochasticity of the model A, for example if the model uses sampling to compute its output.

Stochastic Dominance In Dror et al. (2018; 2019); Ulmer et al. (2022); Simpson (2021) a distributional assessment of the models based on stochastic dominance was proposed to overcome the

limitations of the ubiquitous Mean-Variance Risk model used in machine learning. (Ulmer et al., 2022) used first order almost stochastic dominance and advocated for selecting a model A over B based on a metric m_i if: $M_i|A, X \underset{\varepsilon-\text{FSD}}{\succcurlyeq} M_i|B, X$. We expand this to the Relative-FSD. In natural language (and other) applications, it is often crucial to mitigate the risk of outputs with low metrics, especially when those metrics quantify important socio-technical guardrails such as model's toxicity, safety, or robustness. Unfortunately, the first stochastic ordering does not capture an assessment of the left tail behavior of $M_i|A, X$ and $M_i|B, X$ and hence does not provide a risk-aware assessment Ogryczak & Ruszczynski (2002). To alleviate this issue, we instead consider the *second* order stochastic ordering and use our second order *almost* or *relative* stochastic dominance tests introduced in Section 3 for selecting a model A if:$M_i|A, X \underset{\varepsilon \text{ or } R-\text{SSD}}{\succcurlyeq} M_i|B, X$.

Metrics Portfolio Selection using Stochastic Dominance Evaluation of foundation models is multi-dimensional in nature and multiple metrics assess the models on different socio-technical dimensions that probe the trustworthiness of their outputs and their adherence to shared values and ethics. In order to enable such an evaluation with stochastic orders, we propose the following method inspired from portfolio optimization in finance. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a probability vector that represents the importance of the m_i metrics to the model's end user. Inspired by the portfolio optimization literature, we model the user return from a model as a *portfolio of metrics* m_i *evaluated on a test set* D_i. Following (Ulan et al., 2021; Belgodere et al., 2023), we define this portfolio as an Archimedean copula, which forms a weighted geometric mean of the CDFs:

$$R_A(X) = \exp\left(\sum_{i=1}^{N} \lambda_i \log F_{M_i}\left(m_i(A(X))\right)\right) = \prod_{i=1}^{N} F_{M_i}^{\lambda_i}(m_i(A(X))). \tag{17}$$

Note that (17) normalizes the metrics using the CDF of the metric M_i, eliminating the issue of differing dynamic ranges. This CDF should be formed by pooling together the evaluations on all samples and from all models being compared, to ensure that the various R_A are comparable. The CDF normalization is monotonic and hence it preserves the order of each metrics and allow us to aggregate in the probability space the metrics using a simple weighted geometric mean. Computing $R_A(X)$ for all test samples X, we can therefore characterize the distribution of the metric portfolio of the model A. To compare two models it is enough to compare their corresponding portfolios, specifically, Model A is preferred to Model B using ε- or R-SSD:

$$R_A(X) \underset{\varepsilon- \text{ or } R-\text{SSD}}{\succcurlyeq} R_B(X). \tag{18}$$

Similar tests can be performed for FSD.

Multiple Models Comparison Given k models $A_\ell, \ell = 1 \ldots k$ and their evaluations $m_i(A_\ell(X))$, $X \sim D_i$, $i = 1 \ldots N$, we pool all model evaluations for a metric to estimate the CDF of each metric F_{M_i} and construct a portfolio for each model $R_{A_\ell}(X)$. We use our Relative Stochastic Dominance testing introduced in Section 3 and in Algorithm 1 to rank the models by their metrics portfolio in relative SSD or FSD with a confidence level $1 - \alpha$.

Per Metric Stochastic Dominance and Rank Aggregation We also explore another approach for multi-testing, by considering the stochastic dominance of the models on per-metric basis. This amounts to computing N relative stochastic orders for each $\mathcal{M}_i = (m_i(A_1(X)), \ldots, m_i(A_\ell(X)))$, $i = 1 \ldots N$. This amounts to producing via Algorithm 1 a relative ranking π_i of the models based on $\mathcal{M}_i$. A single rank π is then obtained via rank aggregation with uniform weighting on the per-metric rankings $\pi_i, i = 1 \ldots N$. We use for rank aggregation the R package of (Pihur et al., 2009). For more details on rank aggregation, the reader is referred to Appendix B.2.

5 Experiments

5.1 Validation of Statistical Significance

We examine the statistical properties of our tests as a function of sample size. We purposely design synthetic score distributions to represent challenging problems comprising large overlap between

the distributions and considerable violation ratio, but where one would still like to have an ordering among the variables. For this we consider the two Gaussian variables $X \sim \mathcal{N}(0, 1)$ and and $Y \sim \mathcal{N}(0.5, 2)$. Figure 2 in Appendix G.1 shows that our tests have desirable statistical properties.

5.2 LLM EVALUATION WITH STOCHASTIC DOMINANCE

We showcase LLM evaluation with stochastic dominance to assess two risks: drifting from instructions and outputting toxic content. The following datasets correspond to each risk we assess.

Mix-Instruct Evaluation Data We use the data from (Jiang et al., 2023), that consists of an instruction, an input sentence and an expected output from the user, as well as the output of a set of different LLMs. The dataset consists of a training set of 100K samples and a test set of 5K samples. (Jiang et al., 2023) used automatic metrics such as BARTscore and BLEU score comparing the LLM generation to the expected output in order to evaluate if each LLM followed the instruction. (Jiang et al., 2023) used also chatGPT to evaluate the generations. The total number of automatic metrics N is 8, the total number of evaluated models k is 12. All metrics are unified so that larger values are preferred.

Toxicity Evaluation We use the real toxicity prompts dataset of Gehman et al. (2020), and generate prompts completions from the Llama 2 7b , Llama 2 13b, Llama 2 70b , MosaicML MPT 30b and Tiiuae Falcon 40b models available in Opensource ($k = 5$ models). We select two sets of prompts: toxic prompts (toxicity > 0.8, that gives $\sim$10K prompts) and non-toxic prompts (toxicity < 0.2, from which we randomly sample 10K). We sample from each model, 10 completions per prompt using nucleus sampling (top-p sampling with $p = 0.9$ and a temperature of 1). This procedure yields a dataset of $\sim$200K sentence completions per model. We evaluate the toxicity of these generations using the Perspective API, on the following toxicity metrics ($N = 6$ metrics): Toxicity, Severe toxicity, Identity Attack, Insult, Profanity and Threat. Following Liang et al. (2022), we evaluate the toxicity of generated completions only and refer to this as ***Gen Only*** evaluation. In order to also give the context of the completion, we prepend the model generation with the prompt and evaluate the full sentence using Perspective API. We refer to this as ***Prompt+Gen***. The polarity of all toxicity metrics is unified so that high values refer to non toxic content (we use $-\log$ probabilities of Perspective API outputs).

Evaluation Protocol and Baselines We evaluate each of the use cases (instruction following and toxicity) using the following absolute stochastic dominance tests: (1) ε-FSD (corresponds to the ASO evaluation of Ulmer et al. (2022)) for $\varepsilon = 0.08, 0, 25, 0.4$. (2) our proposed ε-SSD using the same values for ε, (3) our relative stochastic dominance R-FSD and R-SSD tests, (4) the Mean – Risk models described in Table 1, and (5) the ranking produced by the Mean Win Rate (**MWR**) used by LLM leaderboards such as HELM (Liang et al., 2022). As noted in Section 4, we either perform these tests on a *metrics portfolio* (given in Equation (17)) – we refer to this as **test @ P**; or on a per metric basis leading to N rankings of the models that we reduce to a single ranking via Rank Aggregation (RA) (Pihur et al., 2009) – we refer to this as **RA(test @ M)**. In this naming convention, **test** takes values in {MWR, ε-FSD, ε-SSD, R-FSD, R-SSD, Mean – Risk Model ($\mu_X - r_X$)} where r_X is a chosen risk from Table 1. We perform all our statistical tests with a significance level $\alpha = 0.05$, and use 1000 bootstrap iterations.

Efficient Implementation We compare the computational complexity of our implementation for computing all stochastic orders to that of the `Deep-Significance` package (deepsig, 2022) which implements ε-FSD in the ASO framework (Ulmer et al., 2022), on the task of comparing models on the Mix-Instruct dataset (sample size 5K, $k = 12$ models). Using the `Deep-Significance` implementation of MULTI-ASO in (Ulmer et al., 2022) for $\varepsilon = 0.25$ with just 3 bootstrap iterations[5], the test completes in 15min50s (averaged over 7 runs). Our code for relative and absolute testing performs all tests at once and relies on caching vectorization and multi-threading of the operations. Our code completes all tests in an average of just 17.7 s with 1000 bootstrap iterations. Experiments were run on a CPU machine with 128 AMD cores, of which 2 were used.

Mix-Instruct Results and Analysis Table 2 summarizes the rankings we obtain for different models using the different tests described above. We see that the Mean Win Rate currently used in LLM

[5]Limited to 3 for computational reasons.

leaderboards such as HELM (Liang et al., 2022) leads to different orderings than FSD and SSD. For example the flan-t5 model is ranked 5 or 6 by MWR with rank aggregation and portfolio respectively. In contrast, for R-FSD and R-SSD it is given a low ranking $(8, 11)$ or 12. This is due to the fact that MWR only counts wins and does not take into account how fat is the left tail of the distribution of the metric being assessed, possibly leading to overevaluation of risky models. When comparing R-FSD and R-SSD to each other, we see some changes in the ranking in near or adjacent positions. Remarkably, the R-SSD ordering agrees with the rank aggregation of all (consistent) mean – risk models, confirming the theoretical link between second order dominance and risk averse decision making. Table 3 in Appendix G shows for this data that R-FSD and R-SSD are consistent with ε-FSD and SSD respectively for various values of ε. While it is common to give radar plots of MWR for a metric or an average of the metric, we give in G a radar plot (Figure 3) for each of the Mean – Risk models, to aid practitioners in visualizing and selecting models in a risk aware manner.

	Open assistant	koala	alpaca	llama 7b	flan-t5	stablelm	Vicuna	Dolly (v2)	Moss 6b	ChatGLM	mpt-7b instruct	mpt-7b
Mean Win Rates												
RA(MWR @ M)	**1**	6	**2**	8	5	7	**3**	10	9	4	11	12
MWR @ P	**1**	5	**2**	7	6	8	**3**	9	10	4	11	12
Relative FSD												
RA(R-FSD @ M)	**1**	6	**2**	5	8	11	4	10	7	**3**	9	12
R-FSD @ P	**1**	6	**2**	5	11	10	4	8	7	**3**	9	12
Relative SSD												
RA(R-SSD @ M)	**1**	7	**2**	5	12	10	4	9	6	**3**	8	11
R-SSD @ P	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10
Mean-Risk Models												
RA($\mu_X - \Gamma_X$) @ M	**1**	7	**2**	5	12	11	4	9	6	**3**	8	10
RA($\mu_X - r_X$) @ P	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10

Table 2: Rankings of models on following instructions according to all tests, with the top 3 ranks highlighted. We see that SSD and Mean – Risk models are consistent. Note that RA($\mu_X - r_X$) @ P denotes the aggregation of rankings produced by ($\mu_X - r_X$) @ P for each r_X in Table 1.

Toxicity Results and Analysis Table 4 in Appendix G summarizes the results of our tests. We make a few observations: First, overall the portfolio approach agrees well with the rank aggregation of per-metric rankings. The portfolio is more computationally efficient as it needs to run the stochastic dominance test only on the portfolio, rather than running N tests and aggregating them via rank aggregation. Secondly, on this dataset the R-FSD and R-SSD agree, with a few exceptions. Interestingly, when comparing models on model generation only, on toxic prompts MosaicML MPT stands out, while on non toxic prompts Llama2 7B stands out and on the combined set Mosaic ML MPT stands out. When evaluating the toxicity of the context (Prompt + Gen), Llama70B stands out on toxic prompts, Llama7b stands out on non toxic prompts and MosaicML MPT still stands out on the combined set. This study shows that the evaluation problem is not only challenging in terms of the statistical significance of the test, but also with regards to the conditioning on which data the evaluation is performed. The stability of the ranking across all methods, on the combined set suggests that rank stability can be a criterion to assess the representativity of the evaluation set.

6 Conclusion

In this paper we introduced a distributional framework for risk assessment and comparison of foundation models based on multi-metric evaluations. Our framework is of interest beyond the current applications presented here by providing statistical significance while ranking assets for decision making. We believe our tools for training models to be risk averse can be of significant use to practitioners and serve as a stepping stone towards solving the AI alignment problem.

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A Multi-Testing Algorithm for Relative and Almost Stochastic Dominance

Our multi-testing algorithm for relative and almost stochastic dominance is detailed in Algorithm 1.

B Supplement Discussions

B.1 $\delta-$ Consistency of Gini-Risk Models with ε-SSD

$\delta-$ Consistency of Gini-Risk Models with ε-SSD We relax the definition of $\alpha-$ consistency of mean-risk models with SSD to (α, δ) consistency with ε-SSD as follows:

Definition 5 ((α, δ) consistency of MRM with ε-SSD)**.** *A mean-risk model (μ_X, r_X) is (α, δ) consistent with ε-SSD, if there exists $\alpha, \delta > 0$ such that $X \underset{\varepsilon\text{-SSD}}{\succcurlyeq} Y \implies \mu_X - \alpha r_x + \delta \geq \mu_Y - \alpha r_Y$*

It is easy to see that the Mean-Gini tail MRM of X and Y is consistent with their ε-SSD:

Proposition 1. *The Mean-Gini Tail MRM is $(1, 2\varepsilon^{\frac{1}{2}} d_{IQ}(F_X, F_Y))$ consistent with ε-SSD.*

Proof of Proposition 1.

$$
\begin{aligned}
\mu_X - \Gamma_X &= \mu_X - 2\int_0^1 (\mu_X p - F_X^{(-2)}(p))dp = 2\int_0^1 (F_X^{(-2)}(p) - F_Y^{(-2)}(p) + F_Y^{(-2)}(p))dp \\
&= 2\int_0^1 F_Y^{(-2)}(p) + 2\int_{A_0^{(2)}} (F_X^{(-2)}(p) - F_Y^{(-2)}(p))dp + 2\underbrace{\int_{[0,1]/A_0^{(2)}} (F_X^{(-2)}(p) - F_Y^{(-2)}(p))dp}_{\geq 0} \\
&\geq 2\int_0^1 F_Y^{(-2)}(p) - 2\int_{A_0^{(2)}} |F_X^{(-2)}(p) - F_Y^{(-2)}(p))|dp \\
&= \mu_Y - \Gamma_Y - 2\int_0^1 (F_Y^{(-2)}(p) - F_X^{(-2)}(p))_+ dp \\
&\geq \mu_Y - \Gamma_Y - 2\left(\int_0^1 dp\right)^{\frac{1}{2}} \left(\int_0^1 (F_Y^{(-2)}(p) - F_X^{(-2)}(p))_+^2 dp\right)^{\frac{1}{2}} \text{ (Cauchy-Schwartz)} \\
&\geq \mu_Y - \Gamma_Y - 2\varepsilon^{\frac{1}{2}} d_{IQ}(F_X, F_Y) \text{(By assumption } X \underset{\varepsilon-\text{SSD}}{\succcurlyeq} Y\text{)}
\end{aligned}
$$

□

B.2 Rank Aggregation

Given N ranks $\pi_i, i = 1 \dots N$ represented as permutations in S_k, the rank aggregation in (Pihur et al., 2009) solves the following problem :

$$\min_{\pi \in S_k} \sum_{i=1}^{N} \alpha_i d(\pi, \pi_i),$$

where $\alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1$ represent importance of each ranking and d is a distance between permutations. (Pihur et al., 2009) have multiple choices of distance such as Pearson or Kendall's-Tau. We fixed through out our experiments the distance to Pearson.

B.3 Mean Win Rate and CDF normalizers in portfolio

To unpack the notations in (17), consider a distribution $\mathcal{A}$ on models space. For a sample $X \sim D_i$ and a model $A \sim \mathcal{A}$, the metric $m_i()$ normalization through its CDF can be written as follows:

$$F_{M_i}(m_i(A(X)) = \mathbb{E}_{B\sim\mathcal{A}}\mathbb{E}_{Y\sim D_i} \mathbb{1}_{m_i(B(Y)\leq m_i(A(X))}. \tag{19}$$

Algorithm 1 Stochastic Order Multi-testing (relative and **absolute**)

Input: $F_1, ..., F_k$, k models we want to rank corresponding to empirical measure $p_1 = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^1}, \dots p_k = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^k}$, **Threshold:** τ.
Input: Desired stochastic order $\in \{1, 2\}$, B number of bootstraps, $m = K^2$ number of comparisons, significance level α.
Cache the bootstraps samples and their statistics
for $j = 1$ **to** k **do**
  $p_j^0 \leftarrow p_j$
  Get Quantiles and Integrated Quantiles
  $\mathrm{Q}_{0,j} \leftarrow \textsc{GetQuantiles}(p_j)$
  $\mathrm{IQ}_{0,j} \leftarrow \textsc{GetIntegratedQuantiles}(p_j)$
  for $b = 1$ **to** B **do**
    Get Quantiles and Integrated Quantiles
    $p_j^b \leftarrow \textsc{ResampleWithReplacement}(p_j, n)$ {using quantiles and uniform}
    $\mathrm{Q}_{b,j} \leftarrow \textsc{GetQuantiles}(p_j^b)$
    $\mathrm{IQ}_{b,j} \leftarrow \textsc{GetIntegratedQuantiles}(p_j^b)$
  end for
end for
Compute all violation ratios
$\varepsilon_{b,i,j} \leftarrow \textsc{ComputeViolationRatios}(F_i^b, F_j^b, \text{order})$ for $b = 0 \dots B, \quad i, j = 1 \dots k, i \neq j$ {ratio of Q or IQ of $j > i$ by total area}
$\varepsilon_{b,i,i} = 0, \forall\, b, i$
Compute the sum statistics
for $b = 0$ **to** B **do**
  for $i = 1$ **to** k **do**
    $\varepsilon_b^i \leftarrow \frac{1}{k-1}\sum_j \varepsilon_{b,i,j}$
  end for
end for
Compute the relative statistics
$\Delta\varepsilon_b^{i,j} = \varepsilon_b^i - \varepsilon_b^j, \forall b, i, j$
Compute the Bootstrap Variance
for $i = 1$ **to** k **do**
  for $j = 1$ **to** k **do**
    $\sigma_{ij} = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\Delta\varepsilon_b^{i,j} - \textsc{Mean}(\Delta\varepsilon_b^{i,j}, b))^2}$
    $\sigma_{ij}^{\text{abs}} = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\varepsilon_{b,i,j} - \textsc{Mean}(\varepsilon_{b,i,j}, b))^2}$
  end for
end for
Compute the test
$\mathrm{Win}_{ij} = \mathrm{Win}_{ij}^{\text{abs}} = 0$
for $i = 1$ **to** k **do**
  for $j = 1$ **to** k **do**
    if $i \neq j$ and $\Delta\varepsilon_0^{i,j} - \frac{1}{\sqrt{n}}\sigma_{ij}\Phi^{-1}(\alpha/k^2) \leq 0$ **then**
      $\mathrm{Win}_{ij} = 1$ {with confidence level $1 - \alpha/k^2$}
    end if
    if $i \neq j$ and $\varepsilon_{0.i,j} - \frac{1}{\sqrt{n}}\sigma_{ij}^{\text{abs}}\Phi^{-1}(\alpha/k^2) \leq \tau$ **then**
      $\mathrm{Win}_{ij}^{\text{abs}} = 1$ {with confidence level $1 - \alpha/k^2$}
    end if
  end for
end for
$\text{rank} = \textsc{Borda}(\text{Win})$ {with confidence level $1 - \alpha$}
$\text{rank}_{\text{abs}} = \textsc{Borda}(\text{Win}^{\text{abs}})$ {with confidence level $1 - \alpha$}
return rank, rank_{abs}

Algorithm 2 COMPUTEVIOLATIONRATIOS(F_a,F_b,order)

if order =1 **then**
  return $\varepsilon_{\mathsf{W}_2}(F_a, F_b)$ in Definition 3
else if order=2 **then**
  return $\varepsilon_{IQ}(F_a, F_b)$ in Definition 4
end if

Hence for a model A on each evaluated sample the CDF normalizer computes a soft ranking of the evaluation of the model A with a metric m_i on the sample X with respect to all models and all samples.

Remark 1 (Mean Win Rate). *Note that in LLM leaderborads such as HELM and Hugging face, the performance of a model A evaluated with a metric m_i, is summarized via a Mean Win Rate (MWR) aggregated on models level looking on expected metrics*

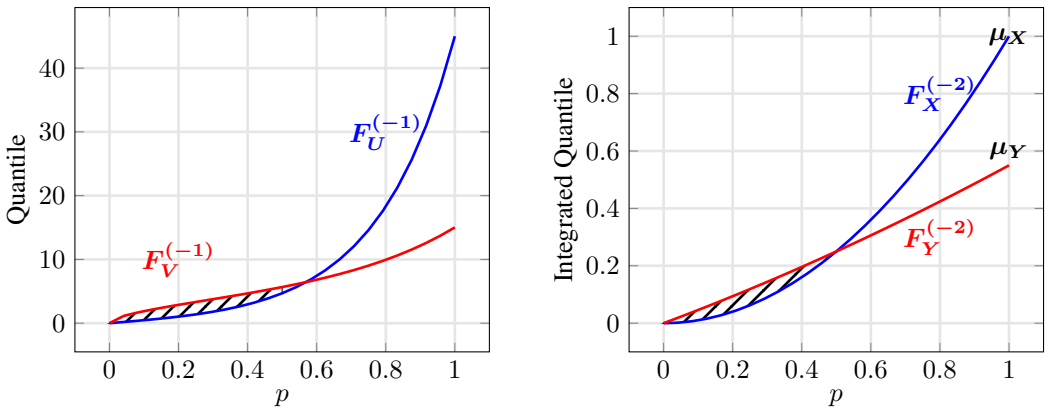
$$\mathrm{MWR}_{A,M_i} = \mathbb{E}_{B\sim\mathcal{A}}\mathbb{1}_{\mathbb{E}_{X\sim D_i}[m_i(B(X))]\leq\mathbb{E}_{X\sim D_i}[m_i(A(X))]}, \tag{20}$$

or aggregated on sample level marginalizing on models with a max*:*

$$\overline{\mathrm{MWR}}_{A,M_i} = \mathbb{E}_{X\sim D_i}\mathbb{1}_{\max_{B\neq A} m_i(B(X))\leq m_i(A(X))}, \tag{21}$$

Contrasting (19) *,* (20) *and* (21) *we see that instead of looking at the MWR summary statistics that does not allow to consider all order statistics and relative ordering as well the risks of tails events, we consider a full distributional assessment in the metrics portfolio approach.*

C FIGURES



(a) ε- FSD (first order): $U \underset{\varepsilon-\mathrm{FSD}}{\succcurlyeq} V$

(b) ε-SSD (second order): $X \underset{\varepsilon-\mathrm{SSD}}{\succcurlyeq} Y$

Figure 1: **(a) An Example of Almost First Order Stochastic Dominance:** Plots of quantile functions of U and V. Dashed areas is the violation set of first order stochastic dominance of U on V. **(b) An Example of Almost Second Order Stochastic Dominance:** Plots of integrated quantile functions; dashed area is the violation set for the second order stochastic dominance of X on Y.

D CENTRAL LIMIT THEOREMS

D.1 CLT FOR ε-SSD

Theorem 2 (Central Limit Theorem for ε-SSD)**.** *Assume that F_X, F_Y are supported on intervals[a] in $[-M, M]$, and have pdfs f_x, f_y such that $\frac{f_x'(p)}{f_x^3(p)}$, $\frac{f_y'(p)}{f_y^3(p)}$ are bounded almost everywhere on the support of f_x and f_y respectively. Assume we have n samples from F_X and m samples from F_Y, with $n, m \to \infty$ such that $\frac{n}{n+m} \to \lambda$ for some λ. Then*

$$\sqrt{\frac{mn}{m+n}}\left(\varepsilon_{IQ}(F_X^n, F_Y^m) - \varepsilon_{IQ}(F_X, F_Y)\right) \to \mathcal{N}(0, \sigma_\lambda^2(F_X, F_Y))$$

where

$$\sigma_\lambda^2(F_X, F_Y) = \frac{1}{d_{IQ}^8(F_X, F_Y)}\left[(1-\lambda)\mathrm{Var}(v_X(U)) + \lambda \mathrm{Var}(v_Y(U))\right],$$

for $U \sim \mathrm{Unif}[0,1]$, $v_Y(t) = 2\left(\frac{1}{f_y(F_Y^{-1}(t))}\right)\left(\int_t^1 (F_X^{(-2)}(p) - F_Y^{(-2)}(p))_+ dp\right)$, and $v_X(t) = 2\left(\frac{1}{f_x(F_X^{-1}(t))}\right)\left(\int_t^1 (F_X^{(-2)}(p) - F_Y^{(-2)}(p))_- dp\right)$.

[a] The interval for F and for G need not coincide.

Remark 2 (Non-independent samples)**.** *Theorem 2 assumes that the n-sample from F_X is independent of the m-sample for F_Y. Consider instead the setting where there are n samples from F_X and F_Y that are dependent (e.g. X, Y are evaluations of different models applied to the same data). We can describe general dependence structure as the following. Suppose (X, Y) has marginals $X \sim F_X$, $Y \sim F_Y$, with some unknown dependence structure (optionally described by the copula $C_{XY}(u_x, u_y) = \Pr(F_X(X) \le u_x, F_Y(Y) \le u_y)$). Let*

$$(U_x, U_y) = (F_X(X), F_Y(Y)) \sim C_{XY}.$$

Note that U_x and U_y have marginals equal to $\mathrm{Unif}([0,1])$, but U_x and U_y may be dependent. Hence the variances in each term of the decomposition (23) in the appendix cannot be added. Instead, one should modify the result of Theorem 2 to use

$$\bar{\sigma}_\lambda^2(F_X, F_Y) = \frac{1}{d_{IQ}^8(F_X, F_Y)}\mathrm{Var}\left[v_X(U_x) + v_Y(U_y)\right].$$

D.2 CLT FOR RELATIVE STATISTICS

We focus here on presenting the Central Limit Theorem for SSD. The relative FSD has a similar form and we omit its statement here.

Theorem 3 (Central limit Theorem for Relative SSD). *Assume $F_{1n}, \ldots, F_{kn}$ are available and independent, and each F_i satisfies the conditions of Theorem 2. Then*

$$\sqrt{n}\left(\Delta\varepsilon^{(2)}_{i_1,i_2}(F_n) - \Delta\varepsilon^{(2)}_{i_1,i_2}(F)\right) \to_w \mathcal{N}\left(0, \frac{1}{(k-1)^2}\sum_{i=1}^{k}\sigma_i^2(i_1,i_2)\right).$$

where

$$\sigma_i^2(i_1,i_2) = \begin{cases} \mathrm{Var}\left(\frac{2v^{(1)-}_{i_1i_2}(U_i)}{d^4_{IQ}(F_{i_1},F_{i_2})} + \sum_{j\neq i_1,i_2}\frac{v^{(1)-}_{i_1j}(U_i)}{d^4_{IQ}(F_{i_1},F_j)}\right) & i = i_1 \\ \mathrm{Var}\left(\frac{2v^{(2)+}_{i_1i_2}(U_i)}{d^4_{IQ}(F_{i_1},F_{i_2})} - \sum_{j\neq i_1,i_2}\frac{v^{(1)-}_{i_2j}(U_i)}{d^4_{IQ}(F_{i_2},F_j)}\right) & i = i_2 \\ \mathrm{Var}\left(\frac{v^{(2)+}_{i_1j}(U_i)}{d^4_{IQ}(F_{i_1},F_j)} - \frac{v^{(2)+}_{i_2j}(U_i)}{d^4_{IQ}(F_{i_2},F_j)}\right) & i \neq i_1, i_2 \end{cases}$$

for $U_i \sim \mathrm{Unif}([0,1])$ *all independent, and* $v^{(1),-}_{ij}(t) = 2\left(\frac{dF_i^{-1}(t)}{dt}\right)\left(\int_t^1 (F_i^{(-2)}(p) - F_j^{(-2)}(p))_- dp\right)$, $v^{(2),+}_{ij}(t) = 2\left(\frac{dF_j^{-1}(t)}{dt}\right)\left(\int_t^1 (F_i^{(-2)}(p) - F_j^{(-2)}(p))_+ dp\right)$.

Remark 3 (Dependent samples). *If the F_{in} are dependent, a similar expression to that shown in Remark 2 for the absolute testing case also holds here. The statement is omitted.*

E Proof of Theorem 1

First, we show that $d_{IQ}(F,G) = 0$ if and only if $F = G$. The forward direction is obvious. For the reverse direction, if $d_{IQ}(F,G) = 0$, then $F^{(-2)}(t) = G^{(-2)}(t)$ a.e. By the continuity of integrated quantiles, this implies $F^{(-2)} = G^{(-2)}$ everywhere. Then, since $F^{(-1)}(t)$ is simply the derivative of $F^{(-2)}(t)$ with respect to t[6], $F^{(-1)} = G^{(-1)}$ everywhere by differentiating both sides of $F^{(-2)}(t) = G^{(-2)}(t)$. Hence $F = G$ since distributions are uniquely determined by their quantile functions.

The triangle inequality follows from the triangle inequality of the L_2 norm, since $\sqrt{\int_0^1 (F^{(-2)}(t) - G^{(-2)}(t))^2 dt} = \|F^{(-2)}(t) - G^{(-2)}(t)\|_{L_2([0,1])}$. Hence d_{IQ} is a metric. By Theorem 10 in Gushchin & Borzykh (2017), we know that random variable $X_{(i)} \to_w X$ (with cdf $F_{(i)}$) if and only if $F^{(-2)}_{(i)}$ converges uniformly to $F^{(-2)}$. Hence d_{IQ} must metrize weak convergence.

F Proofs of Central Limit Theorems

F.1 Absolute Testing: Proof of Theorem 2

Note that for U_i and V_i an n-sample and an m-sample respectively from $\mathrm{Unif}([0,1])$, we can get X_i, Y_i as $X_i = F^{-1}(U_i)$, $Y_i = G^{-1}(V_i)$. Let $H_{n,1}$ and $H_{m,2}$ be the empirical d.f.s of the U_i and V_i respectively. We have

$$F_n^{-1}(t) = F^{-1}(H_{n,1}^{-1}(t)),$$

hence

$$F_n^{(-2)}(t) = \int_0^t F_n^{-1}(p)dp = \int_0^t F^{-1}(H_{n,1}^{-1}(p))dp.$$

We are interested in

$$\varepsilon_{IQ}(F_n, G_m) = \frac{\int_{A_0}(F_n^{(-2)}(t) - G_m^{(-2)}(t))^2 dt}{d^2_{IQ}(F_n, G_m)},$$

[6]This follows because F^{-2} is the integral of the finite-valued quantile function $F^{-1}(t)$.

where

$$A_0 = \left\{ t \in (0,1) : G_m^{(-2)}(t) > F_n^{(-2)}(t) \right\},$$

is the violation set.

It is shown in Gushchin & Borzykh (2017) (Theorem 10 therein) that integrated quantiles converge uniformly, i.e. $F_n^{(-2)}(t) \to F^{(-2)}(t)$ pointwise. As an immediate consequence, we have

$$\varepsilon_{IQ}(F_n, G_m) \to_{a.s.} \varepsilon_{IQ}(F, G).$$

We apply the following decomposition and bound the two terms separately:

$$\varepsilon_{IQ}(F_n, G_m) - \varepsilon_{IQ}(F, G) = (\varepsilon_{IQ}(F_n, G_m) - \varepsilon_{IQ}(F, G_m)) + (\varepsilon_{IQ}(F, G_m) - \varepsilon_{IQ}(F, G)). \quad (22)$$

We derive asymptotic normality of these terms for G_m, the proof for F_n is identical by symmetry.

We introduce the statistics

$$S_m = \int_0^1 (F^{(-2)}(t) - G_m^{(-2)}(t))^2 dt$$
$$S_m^+ = \int_0^1 (F^{(-2)}(t) - G_m^{(-2)}(t))_+^2 dt$$
$$S_m^- = \int_0^1 (F^{(-2)}(t) - G_m^{(-2)}(t))_-^2 dt$$

The nonrandom S, S^+, S^- are defined similarly with G instead of G_m.

Next, set

$$T_m = \sqrt{m}(S_m - S)$$
$$T_m^+ = \sqrt{m}(S_m^+ - S^+)$$
$$T_m^- = \sqrt{m}(S_m^- - S^-).$$

Theorem 4. *Assume that G is supported on an interval that is a subset of $[-M, M]$, and has pdf g such that $\frac{g'(p)}{g^3(p)}$ is bounded almost everywhere on the support of g. Then*

$$T_m = \alpha_{m,2}(v) + o_P(1)$$
$$T_m^+ = \alpha_{m,2}(v^+) + o_P(1)$$
$$T_m^- = \alpha_{m,2}(v^-) + o_P(1)$$

where we define $\alpha_{m,2}(t) = \sqrt{m}(t - H_{m,1}^{-1}(t))$ and $\alpha_{m,2}(v) = \int_0^1 v(t)\alpha_{m,2}(t)dt$, and

$$v(t) = 2\left(\frac{1}{g(G^{-1}(t))}\right)\left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right).$$
$$v^+(t) = 2\left(\frac{1}{g(G^{-1}(t))}\right)\left(\int_t^1 (F^{(-2)}(p) - G^{(-2)}(p))_+ dp\right),$$
$$v^-(t) = 2\left(\frac{1}{g(G^{-1}(t))}\right)\left(\int_t^1 (F^{(-2)}(p) - G^{(-2)}(p))_- dp\right).$$

Proof. We begin with T_m. Note that[7]

$$\begin{aligned}
T_m &= \sqrt{m}(S_m - S) \\
&= \sqrt{m}\int_0^1 (F^{(-2)}(t) - G_m^{(-2)}(t))^2 - (F^{(-2)}(t) - G^{(-2)}(t))^2 dt \\
&= \sqrt{m}\int_0^1 \left[2F^{(-2)}(t) - G_m^{(-2)}(t) - G^{(-2)}(t)\right](G^{(-2)}(t) - G_m^{(-2)}(t))dt \\
&\to 2\sqrt{m}\int_0^1 \left[F^{(-2)}(t) - G^{(-2)}(t)\right](G^{(-2)}(t) - G_m^{(-2)}(t))dt \\
&= 2\sqrt{m}\int_0^1 \left[F^{(-2)}(t) - G^{(-2)}(t)\right]\left[\int_0^t G^{(-1)}(p) - G^{(-1)}(H_{m,1}^{-1}(p)))dp\right]dt
\end{aligned}$$

[7] Convergence here is uniform convergence of the integrated quantiles.

Let us do integration by parts:

$$
\begin{aligned}
&2\sqrt{m}\int_0^1 \left[F^{(-2)}(t) - G^{(-2)}(t)\right]\left[\int_0^t G^{(-1)}(p) - G^{(-1)}(H_{m,1}^{-1}(p)))dp\right]dt = \\
&= 2\sqrt{m}\left[\left(\int_0^1 F^{(-2)}(t) - G^{(-2)}(t)dt\right)\left[\int_0^1 G^{(-1)}(t) - G^{(-1)}(H_{m,1}^{-1}(t)))dt\right]\right. \\
&\left. - \int_0^1 \left(\int_0^t F^{(-2)}(p) - G^{(-2)}(p)dp\right)\left[G^{(-1)}(t) - G^{(-1)}(H_{m,1}^{-1}(t)))\right]dt\right] \\
&= 2\sqrt{m}\int_0^1 \left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right)\left[G^{(-1)}(t) - G^{(-1)}(H_{m,1}^{-1}(t)))\right]dt \\
&= 2\sqrt{m}\int_0^1 \left(\frac{dG^{-1}(t)}{dt}\right)\left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right)(t - H_{m,1}^{-1}(t))dt \\
&+ O\left(\sqrt{m}\int_0^1\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp)(t - H_{m,1}^{-1}(t))^2dt\right) \\
&= 2\sqrt{m}\int_0^1 \left(\frac{dG^{-1}(t)}{dt}\right)\left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right)(t - H_{m,1}^{-1}(t))dt + o_P(1).
\end{aligned}
$$

In the penultimate step we have used a first-order Taylor series on $G^{-1}(t)$ via the assumption that $\frac{d^2G^{-1}(t)}{dt^2} = -\frac{g'(G^{-1}(t))}{g^3(G^{-1}(t))}$ is bounded almost everywhere, and in the final step we have noted that

$$
\begin{aligned}
\sqrt{m}\int_0^1 \left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right)(t - H_{m,1}^{-1}(t))^2dt &\leq 2\sqrt{m}\int_0^1 (t - H_{m,1}^{-1}(t))^2dt \\
&= o_P(1),
\end{aligned}
$$

since the support of F and G lie in $[-M, M]$ and $\int_0^1 (t - H_{m,1}^{-1}(t))^2dt = O_p(1/m)$.

We then have

$$T_m = \alpha_{m,2}(v) + o_P(1),$$

where $\alpha_{m,2}(t) = \sqrt{m}(t - H_{m,1}^{-1}(t))$, and $\alpha_{m,2}(v) = \int_0^1 v(t)\alpha_{m,2}(t)dt$ where

$$v(t) = 2\left(\frac{dG^{-1}(t)}{dt}\right)\left(\int_t^1 F^{(-2)}(p) - G^{(-2)}(p)dp\right).$$

Similarly,

$$T_m^+ = \alpha_{m,2}(v^+) + o_P(1), \quad T_m^- = \alpha_{m,2}(v^-) + o_P(1)$$

where

$$
\begin{aligned}
v^+(t) &= 2\left(\frac{dG^{-1}(t)}{dt}\right)\left(\int_t^1 (F^{(-2)}(p) - G^{(-2)}(p))_+dp\right), \\
v^-(t) &= 2\left(\frac{dG^{-1}(t)}{dt}\right)\left(\int_t^1 (F^{(-2)}(p) - G^{(-2)}(p))_-dp\right).
\end{aligned}
$$

□

Corollary 1. *Assume that G is supported on an interval in $[-M, M]$, and has pdf g such that $\frac{g'(p)}{g^3(p)}$ is bounded almost everywhere on the support of g. Then as $m \to \infty$*

$$\sqrt{m}(\epsilon_{IQ}(F, G_m) - \epsilon_{IQ}(F, G)) \to_w \mathcal{N}(0, \sigma^2)$$

and if additionally $n \to \infty$

$$\sqrt{m}(\epsilon_{IQ}(F_n, G_m) - \epsilon_{IQ}(F_n, G)) \to_w \mathcal{N}(0, \sigma^2),$$

if for $U \sim \mathrm{Unif}([0, 1])$

$$\sigma^2 = \frac{\mathrm{Var}(v^+(U))}{d_{IQ}^8(F, G)}$$

is finite.

Proof. Note that by Theorem 4

$$\sqrt{m}(\epsilon_{IQ}(F,G_m)-\epsilon_{IQ}(F,G))=\sqrt{m}\left(\frac{S_m^-}{S_m}-\frac{S^-}{S}\right)=\frac{\sqrt{m}}{SS_m}(T_m^- - T_m)\to -\frac{\alpha_{m,2}(v^+)}{S^2}$$

since $S_m \to S$ a.s. Recalling the definition of $\alpha_{m,2}$ yields asymptotic normality with zero mean as in Del Barrio et al. (2018), and variance as calculated in the corollary statement.

The case of $\sqrt{m}(\epsilon_{IQ}(F_n,G_m)-\epsilon_{IQ}(F_n,G))$ follows similarly since integrated quantiles weakly converge as $F_n \to F$. □

Continuing with the main proof, recalling (22) and using Corollary 1 along with the asymptotic independence of the two terms and the fact that $\frac{n}{n+m}\to\lambda$, we have

$$\begin{aligned}
&\sqrt{\frac{mn}{m+n}}\left(\varepsilon_{IQ}(F_n,G_m)-\varepsilon_{IQ}(F,G)\right)\\
&\qquad=\sqrt{(1-\lambda)n}(\varepsilon_{IQ}(F_n,G_m)-\varepsilon_{IQ}(F,G_m))+\sqrt{\lambda n}(\varepsilon_{IQ}(F,G_m)-\varepsilon_{IQ}(F,G)) \quad (23)\\
&\qquad\to\mathcal{N}(0,\sigma_\lambda^2(F,G))
\end{aligned}$$

where

$$\sigma_\lambda^2(F,G)=\frac{1}{d_{IQ}^8(F,G)}\left[(1-\lambda)\mathrm{Var}(v_F(U))+\lambda\mathrm{Var}(v_G(U))\right].$$

Here, we have defined

$$v_G(t)=2\left(\frac{1}{g(G^{-1}(t))}\right)\left(\int_t^1 (F^{(-2)}(p)-G^{(-2)}(p))_+dp\right),$$

and

$$v_F(t)=2\left(\frac{1}{f(F^{-1}(t))}\right)\left(\int_t^1 (F^{(-2)}(p)-G^{(-2)}(p))_-dp\right).$$

F.2 Relative Testing: Proof of Theorem 3

Note that

$$\begin{aligned}
\Delta\varepsilon_{IQ}^{i_1,i_2}(F)&=\epsilon_{IQ}^{i_1}(F)-\epsilon_{IQ}^{i_2}(F)\\
&=\frac{1}{k-1}\left[\sum_{j\neq i_1}\epsilon_{IQ}^{i_1 j}-\sum_{j\neq i_2}\epsilon_{IQ}^{i_2 j}\right]\\
&=\frac{1}{k-1}\left[2\epsilon_{IQ}^{i_1 i_2}-1+\sum_{j\neq i_1,i_2}(\epsilon_{IQ}^{i_1 j}-\epsilon_{IQ}^{i_2 j})\right].
\end{aligned}$$

For compactness, let us introduce the differencing notation $\phi(\cdot)|_{F}^{F_n} = \phi(F_n) - \phi(F)$. We seek a CLT on

$$\begin{aligned}
\sqrt{n}(\Delta\widehat{\varepsilon_{IQ}^{i_1,i_2}}(F_n) - \Delta\varepsilon_{IQ}^{i_1,i_2}(F)) &= \frac{\sqrt{n}}{k-1}\left(2\epsilon_{IQ}(\cdot, F_{i_2,n}) + \sum_{j\neq i_1,i_2}\epsilon_{IQ}(\cdot, F_{j,n})\right)\Bigg|_{F_{i_1}}^{F_{i_1,n}} \\
&\quad + \frac{\sqrt{n}}{k-1}\left(2\epsilon_{IQ}(F_{i_1}, \cdot) - \sum_{j\neq i_1,i_2}\epsilon_{IQ}(\cdot, F_{j,n})\right)\Bigg|_{F_{i_2}}^{F_{i_2,n}} \\
&\quad + \frac{\sqrt{n}}{k-1}\sum_{j\neq i_1,i_2}(\epsilon_{IQ}(F_{i_1}, \cdot) - \epsilon_{IQ}(F_{i_2}, \cdot))|_{F_j}^{F_{j,n}} \\
\to_w \underbrace{\frac{\sqrt{n}}{k-1}\left(2\epsilon_{IQ}(\cdot, F_{i_2}) + \sum_{j\neq i_1,i_2}\epsilon_{IQ}(\cdot, F_j)\right)\Bigg|_{F_{i_1}}^{F_{i_1,n}}}_{I} &+ \underbrace{\frac{\sqrt{n}}{k-1}\left(2\epsilon_{IQ}(F_{i_1}, \cdot) - \sum_{j\neq i_1,i_2}\epsilon_{IQ}(\cdot, F_j)\right)\Bigg|_{F_{i_2}}^{F_{i_2,n}}}_{II} \\
&\quad + \underbrace{\frac{\sqrt{n}}{k-1}\sum_{j\neq i_1,i_2}(\epsilon_{IQ}(F_{i_1}, \cdot) - \epsilon_{IQ}(F_{i_2}, \cdot))|_{F_j}^{F_{j,n}}}_{III}
\end{aligned}$$

where we have used the uniform convergence of integrated quantiles. Note that I, II, and each term in the sum in III are all independent.

Define

$$v_{ij}^{(1)}(t) = 2\left(\frac{dF_i^{-1}(t)}{dt}\right)\left(\int_t^1 F_i^{(-2)}(p) - F_j^{(-2)}(p)dp\right),$$

$$v_{ij}^{(2)}(t) = 2\left(\frac{dF_j^{-1}(t)}{dt}\right)\left(\int_t^1 F_i^{(-2)}(p) - F_j^{(-2)}(p)dp\right),$$

and $v_{ij}^{(1)+}$, $v_{ij}^{(2)+}$ similarly. Then by the proof of Corollary 1, each term in III converges to

$$\begin{aligned}
\frac{\sqrt{n}}{k-1}(\epsilon_{IQ}(F_{i_1}, \cdot) - \epsilon_{IQ}(F_{i_2}, \cdot))|_{F_j}^{F_{j,n}} &\to -\frac{\alpha_{m,j}(v_{i_1j}^{(2)+})}{(k-1)d_{IQ}^4(F_{i_1}, F_j)} + \frac{\alpha_{m,j}(v_{i_2j}^{(2)+})}{(k-1)d_{IQ}^4(F_{i_2}, F_j)} \\
&= \frac{1}{k-1}\alpha_{m,j}\left(-\frac{v_{i_1j}^{(2)+}}{d_{IQ}^4(F_{i_1}, F_j)} + \frac{v_{i_2j}^{(2)+}}{d_{IQ}^4(F_{i_2}, F_j)}\right) \\
&\to_w \mathcal{N}\left(0, \frac{1}{(k-1)^2}\sigma_j^2(i_1, i_2)\right), \quad \forall j \neq i_1, i_2.
\end{aligned}$$

where

$$\sigma_j^2(i_1, i_2) = \frac{1}{(k-1)^2}\mathrm{Var}\left(\frac{v_{i_1j}^{(2)+}(U)}{d_{IQ}^4(F_{i_1}, F_j)} - \frac{v_{i_2j}^{(2)+}(U)}{d_{IQ}^4(F_{i_2}, F_j)}\right), \quad \forall j \neq i_1, i_2,$$

and $U \sim \mathrm{Unif}([0,1])$. Similarly for I and II,

$$\begin{aligned}
I &\to_w \mathcal{N}\left(0, \frac{1}{(k-1)^2}\sigma_{i_1}^2(i_1, i_2)\right) \\
II &\to_w \mathcal{N}\left(0, \frac{1}{(k-1)^2}\sigma_{i_2}^2(i_1, i_2)\right)
\end{aligned}$$

where[8]

$$\sigma^2_{i_1}(i_1,i_2) = \text{Var}\left(\frac{2v^{(1)-}_{i_1i_2}(U)}{d^4_{IQ}(F_{i_1},F_{i_2})} + \sum_{j\neq i_1,i_2}\frac{v^{(1)-}_{i_1j}(U)}{d^4_{IQ}(F_{i_1},F_j)}\right),$$

$$\sigma^2_{i_2}(i_1,i_2) = \text{Var}\left(\frac{2v^{(2)+}_{i_1i_2}(U)}{d^4_{IQ}(F_{i_1},F_{i_2})} - \sum_{j\neq i_1,i_2}\frac{v^{(1)-}_{i_2j}(U)}{d^4_{IQ}(F_{i_2},F_j)}\right).$$

Putting everything together via independence,

$$\sqrt{n}\left(\widehat{\Delta\varepsilon^{i_1,i_2}_{IQ}}(F_n) - \Delta\varepsilon^{i_1,i_2}_{IQ}(F)\right) \to_w \mathcal{N}\left(0, \frac{1}{(k-1)^2}\sum_{i=1}^{k}\sigma^2_i(i_1,i_2)\right).$$

G Additional Experimental Results

G.1 Statistical Significance on Synthetic Data

We examine the statistical properties of our tests as a function of sample size. We purposely design synthetic score distributions to represent challenging problems with large overlap between the distributions and a considerable violation ratio, but where one would still like to have an ordering among the variables. For this we consider the two Gaussian distributions with mean $\mu = 0$ and standard deviation $\sigma = 1$, and with mean $\mu = 0.5$ and standard deviation $\sigma = 2$, respectively. In the top panels of Figure 2 we show the PDF, CDF and integrated quantile function of these two Gaussians, illustrating the large violation ratio. The orange distribution can be calculated to be 0.2-FSD and 0.45-SSD over the blue distribution. Note that these ε values are not comparable, due to the differences in definitions. In Figure 2, we conduct experiments illustrating the power of our tests for the absolute tests of the hypotheses $H_{0,FSD} = 0.45$-FSD and $H_{0,SSD} = 0.45$-SSD. We also use our relative tests, which in this 2-variable case (as noted in the main text) are equivalent to testing $H_{0,FSD} = 0.5$-FSD and $H_{0,SSD} = 0.5$-SSD. The bottom left panel in Figure 2 show the True Positive Rate for the different types of tests that we developed: relative test with quantile function, relative test with Integrated Quantile Function, absolute test with quantile function, and absolute test with Integrated Quantile Function. As expected, all tests quickly converge towards True Positive Rate of 1.0 for growing sample sizes.

G.2 Mix-Instruct

Results for the Mix-Instruct data are shown in Figures 3 and 4, as well as Table 3.

G.3 Toxicity

Toxicity results are in Table 4.

G.4 Fat Left Tails of Metrics and Inconsistency of Mean-Variance with SSD

When metrics evaluated have fat tails, the Mean-Variance ranking can be inconsistent with the SSD. See Table 5.

[8] This $U \sim \text{Unif}([0,1])$ is drawn simply for this variance calculation and is not dependent on anything outside of this equation.

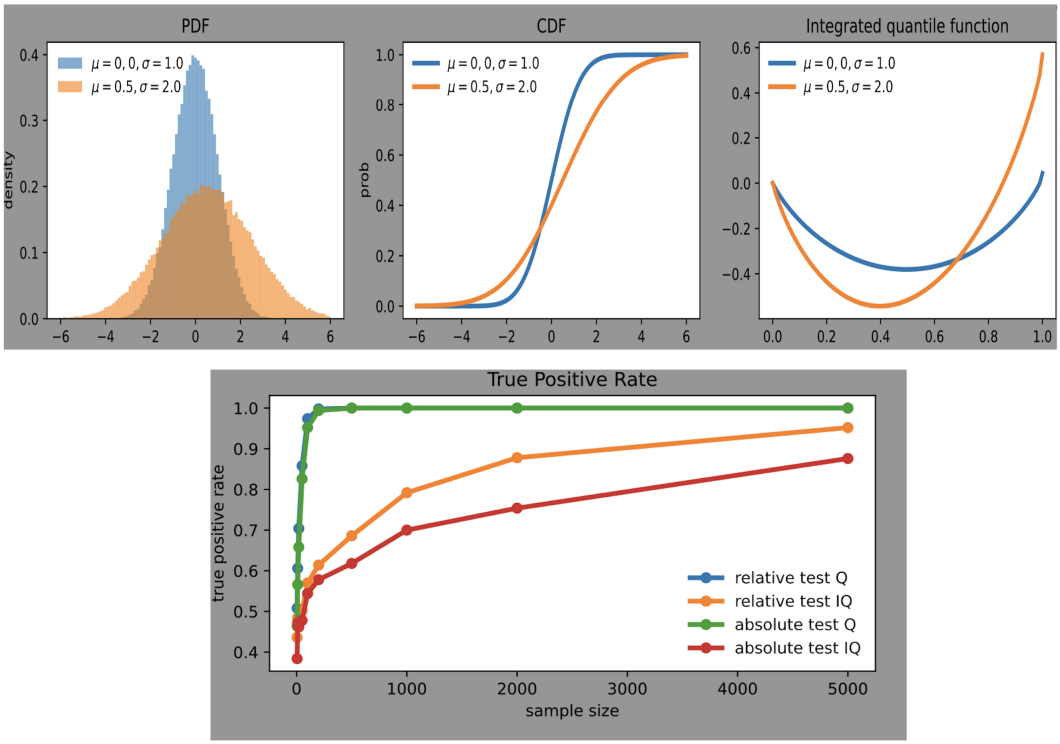


Figure 2: Comparing the True Positive Rate of our stochastic dominance methods on the test distributions in the top panels for different sample sizes. Decisions are made using a confidence threshold of $\alpha = 0.05$ and $\tau = 0.45$ (for the absolute tests). Note that the FSD and SSD curves should not be compared due to differences in the underlying hypotheses.

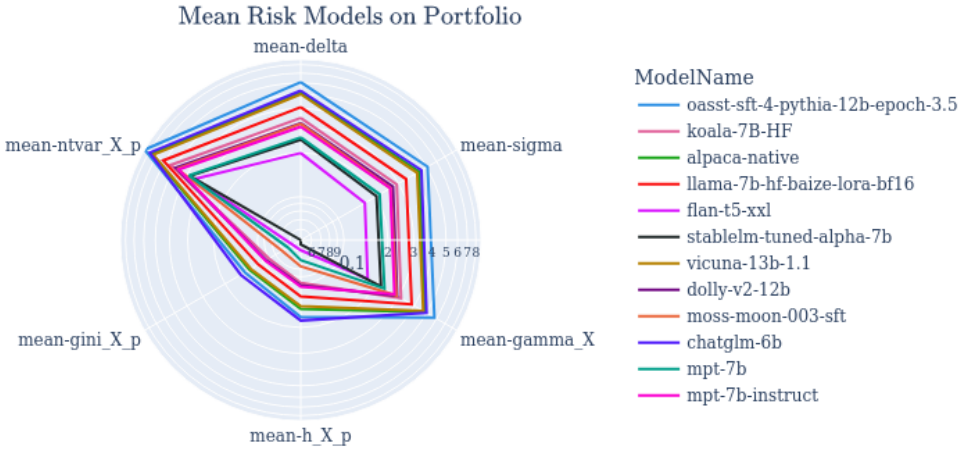


Figure 3: Radar plot of mean – risk models of the portfolio on Mix-Instruct data. Note that the outer models are indeed the ones preferred by SSD in Table 3.

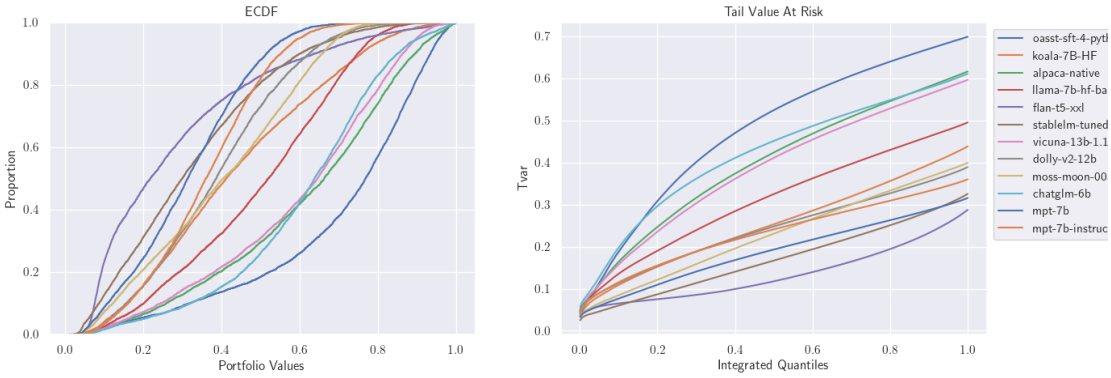


Figure 4: Empirical CDF and TvaR for portfolio on Mix-Instruct data

	Open assistant	koala	alpaca	llama 7b	flan-t5	stablelm	Vicuna	Dolly (v2)	Moss 6b	ChatGLM	mpt-7b instruct	mpt-7b
Mean Win Rates												
RA(MWR @ M)	**1**	6	**2**	8	5	7	**3**	10	9	4	11	12
MWR @ P	**1**	5	**2**	7	6	8	**3**	9	10	4	11	12
Relative FSD												
RA(R-FSD @ M)	**1**	6	**2**	5	8	11	4	10	7	**3**	9	12
R-FSD @ P	**1**	6	**2**	5	11	10	4	8	7	**3**	9	12
Relative SSD												
RA(R-SSD @ M)	**1**	7	**2**	5	12	10	4	9	6	**3**	8	11
R-SSD @ P	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10
Absolute FSD												
ε-FSD @ P ε=0.08	**1**	6	**2**	5	10	11	4	7	8	**3**	9	12
ε-FSD @ P ε=0.25	**1**	6	**2**	5	12	10	4	7	8	**3**	9	11
ε-FSD @ P ε=0.4	**1**	6	**2**	5	12	10	4	8	7	**3**	9	11
Absolute SSD												
ε-SSD @ P $\varepsilon = 0.08$	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10
ε-SSD @ P $\varepsilon = 0.25$	**1**	6	**3**	5	12	11	4	8	7	**2**	9	10
ε-SSD @ P ε=0.4	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10
Mean-Risk Models												
RA($\mu_X - r_X$) @ P	**1**	6	**3**	5	12	11	4	7	8	**2**	9	10

Table 3: Mix instruct Extended Results.

Scenario	Llama 2 7b	Llama 2 13b	Llama 2 70b	MosaicML MPT 30b	Tiiuae Falcon 40b
Toxic Prompts					
RA(R-FSD @M) (Gen Only)	**3**	**2**	4	**1**	5
R-FSD @ P(Gen Only)	**2**	**3**	4	**1**	5
RA(R-SSD @M) (Gen Only)	**3**	**2**	4	**1**	5
R-SSD@P (Gen Only)	**3**	**2**	4	**1**	5
RA(R-FSD @M) (Prompt + Gen)	**2**	**3**	**1**	4	5
R-FSD @P(Prompt + Gen)	**2**	**3**	**1**	4	5
RA(R-SSD @M) (Prompt + Gen)	**2**	**3**	**1**	4	5
R-SSD @P (Prompt + Gen)	**2**	**3**	**1**	4	5
Non-Toxic Prompts					
RA(R-FSD @M) (Gen Only)	**1**	**2**	4	**3**	5
R-FSD @P (Gen Only)	**1**	**2**	**3**	4	5
RA(R-SSD @M) (Gen Only)	**1**	**2**	**3**	4	5
R-SSD @P (Gen Only)	**1**	**2**	**3**	4	5
RA(R-FSD @M) (Prompt + Gen)	**3**	**2**	4	**1**	5
R-FSD @ P (Prompt + Gen)	**1**	**2**	4	**3**	5
RA(R-SSD @M) (Prompt + Gen)	**1**	**2**	**3**	4	5
R-SSD @P (Prompt + Gen)	**1**	**2**	4	**3**	5
All Combined (Toxic + Non-Toxic Prompts)					
RA(R-FSD @M) (Gen Only)	**2**	**3**	5	**1**	4
R-FSD @P (Gen Only)	**2**	**3**	5	**1**	4
RA(R-SSD @M) (Gen Only)	**2**	**3**	5	**1**	4
R-SSD @P (Gen Only)	**2**	**3**	5	**1**	4
RA(R-FSD @M) (Prompt + Gen)	**3**	4	5	**1**	**2**
RA(R-SSD @M) (Prompt + Gen)	**3**	4	5	**1**	**2**
R-FSD @P (Prompt + Gen)	**3**	4	5	**1**	**2**
R-SSD @P (Prompt + Gen)	**3**	4	5	**1**	**2**

Table 4: Toxicity Ranking

Scenario	Llama 2 7b	Llama 2 13b	Llama 2 70b	MosaicML MPT 30b	Tiiuae Falcon 40b
Non Toxic Prompts					
Identity Attack Metric Gen evaluation					
Mean - Sigma	**1**	**3**	4	**2**	5
Mean - Gamma	**2**	**3**	4	**1**	5
Mean - nTvAR	**2**	**3**	4	**1**	5
SSD	**2**	**3**	4	**1**	5
Threat Metric Prompt + Gen evaluation					
Mean - Sigma	**1**	**3**	**2**	4	5
Mean - Gamma	**1**	**2**	**3**	5	4
Mean - nTvAR	**1**	**2**	**3**	5	4
SSD	**1**	**2**	**3**	5	4

Table 5: Inconsistency of Mean - Sigma on Toxicity Metrics with SSD and other mean-risk models. This is a due to the fact the metric evaluated may a have a fat left tail see Figures 5 and 7.

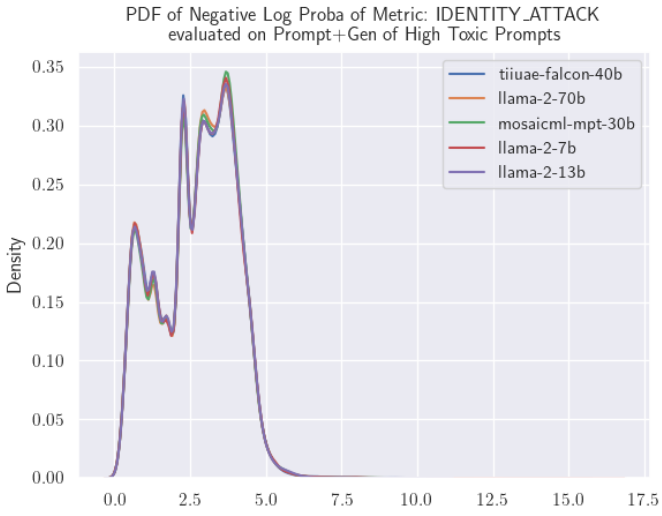


Figure 5: Identity Attack Metric distribution computed on Prompt+Generation output of Highly Toxic Prompts

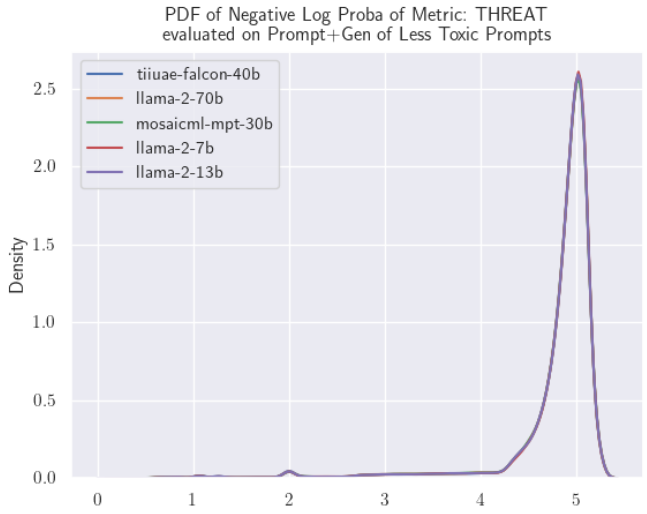


Figure 6: Threat Metric distribution computed on Prompt+Generation output of Less Toxic Prompts

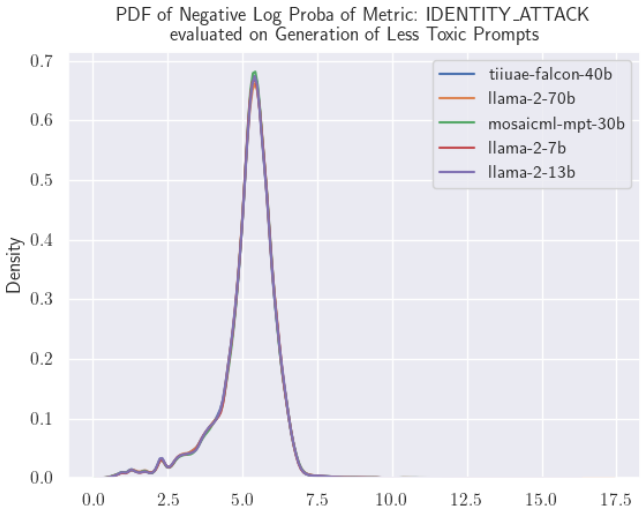


Figure 7: Identity Attack Metric distribution computed on Generation output of Less Toxic Prompts