Appendix

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A Proof of Proposition 1

Proposition 1. *No deterministic and strategyproof mechanism satisfies Strong Proportionality.*

Proof. For $n = 2$, consider the location profile with 1 agent at 0 and 1 agent at 0.5. Strong Proportionality requires that the facility be placed at 0.25. Now also consider the location profile with 1 agent at 0 and 1 agent at 1. Strong Proportionality requires that the facility be placed at 0.5. However, this means the agent at 0.5 in the first location profile can misreport their location as 1 to have the facility placed at their own location, violating strategyproofness. Thus strategyproofness and Strong Proportionality are incompatible in deterministic mechanisms. □

B Proof of Claim 2

Claim 2: $\Pr[Y_{(n+1)} = 1] = 1$ and $\Pr[Y_{(1)} = 0] = 1$.

Proof of Claim 2. We first show that $\Pr[Y_{(n+1)} = 1] = 1$. Suppose the contrary, that there exists $\beta < 1$ such that $\Pr[Y_{(n+1)} \leq \beta] > 0$. Under the location profile $x = (1, \cdots, 1)$, if f satisfies Proportionality in expectation we must have $\mathbb{E}[d(x_n, f(x))] = 0$. However, this leads to a contradiction since

$$\begin{aligned}\mathbb{E}[d(x_n, f(x))] &\geq (1 - \beta) \Pr[Y_{(n+1)} \leq \beta] \\ &> 0,\end{aligned}$$

where the first inequality follows from the fact that if $Y_{(n+1)} \leq \beta$ then $f(x) \leq \beta$, and thus $d(1, f(x)) \geq (1 - \beta)$.

A similar, symmetric argument can be applied to show that $\Pr[Y_{(1)} = 0] = 1$ holds. □

C Extension of Theorem 3 to the real line $\mathbb{R}$

In this section we extend the result of Theorem 3 to the real line. We use the following theorem which characterizes strategyproof and anonymous mechanisms on the real line as Phantom mechanisms.

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Theorem 7 (Moulin (1980))**.** *A mechanism f on the domain $X = \mathbb{R}$ is strategyproof and anonymous if and only if there exists $(n+1)$ real numbers $y_1, \cdots, y_{n+1} \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that*

$$f(x) = \text{med}(x_1, \cdots, x_n, y_1, \cdots, y_{n+1})$$

We also modify our definition of the Random Rank mechanism. Given a profile of locations $x \in \mathbb{R}^n$, we define

$$\text{rank}^k(x) := \text{med}(\underbrace{-\infty, \ldots, -\infty}_{n-k}, x_1, \ldots, x_n, \underbrace{+\infty, \ldots, +\infty}_{k-1}).$$

The Random Rank mechanism on the real line then chooses $k \in \{1, \cdots, n\}$ uniformly at random and outputs $\text{rank}^k(x)$.

Theorem 4. *A mechanism on the domain $X = \mathbb{R}$ is universally anonymous, universally truthful and Strong Proportional in expectation if and only if it is the Random Rank mechanism.*

Proof. ($\Longrightarrow$) By Theorem 7 we know that f is a probability distribution over Phantom mechanisms. For each $i \in \{1, \cdots, n+1\}$, denote Y_i as the random variable corresponding to the location of the i'th Phantom. Also denote $Y_{(i)}$ as the random variable corresponding to the i'th order statistic.

Claim 3: $\Pr[Y_{(n+1)} = +\infty] = 1$ and $\Pr[Y_{(1)} = -\infty] = 1$.

Proof of Claim 3. Suppose on the contrary that there exists $\lambda \in \mathbb{R}$ such that $\Pr[Y_{(n+1)} \leq \lambda] > 0$. Consider a location profile $x = (2\lambda, \cdots, 2\lambda)$. If f satisfies Strong Proportionality in expectation then we have $\mathbb{E}[d(x_1, f(x))] = 0$. However, this contradicts the following

$$\begin{aligned}\mathbb{E}[d(x_1, f(x))] &\geq |2\lambda - \lambda| \Pr[Y_{(n+1)} \leq \lambda] \\ &> 0,\end{aligned}$$

where the inequality follows since if $Y_{(n+1)} \leq \lambda$ then $f(x) \leq \lambda$, and thus $d(x_1, f(x)) \geq |\lambda|$.

A similar, symmetric argument can be used to obtain $\Pr[Y_{(1)} = -\infty] = 1$. □

By Claim 3 we see that only $n-1$ Phantoms are necessary since

$$\begin{aligned}f(x) &= \text{med}(-\infty, Y_{(2)}, \cdots, Y_{(n)}, x_1, \cdots, x_n, +\infty) \\ &= \text{med}(Y_{(2)}, \cdots, Y_{(n)}, x_1, \cdots, x_n)\end{aligned}$$

For notational convenience, we relabel the remaining $n-1$ Phantoms such that

$$f(x) = \text{med}(Y_{(1)}, \cdots, Y_{(n-1)}, x_1, \cdots, x_n).$$

Claim 4: $\Pr[Y_{(i)} = +\infty] = \frac{i}{n}$ and $\Pr[Y_{(i)} = -\infty] = \frac{n-i}{n}$ for each $i \in \{1, \cdots, n-1\}$.

Proof of Claim 4. Using the arguments presented in Claim 1, we see that Strong Proportionality implies

$$\begin{cases}\Pr[Y_{(i)} \leq \alpha] \leq \frac{n-i}{n}, \\ \Pr[Y_{(i)} \geq \beta] \leq \frac{i}{n},\end{cases} \qquad \text{for any} \quad \alpha < \beta, \quad \alpha, \beta \in \mathbb{R}. \tag{1}$$

From above we see that indeed $\Pr[Y_{(i)} = +\infty] = \frac{i}{n}$ and $\Pr[Y_{(i)} = -\infty] = \frac{n-i}{n}$. □

By Claim 4, we see that $Y_{(i)} \in \{-\infty, +\infty\}$ for each $i \in \{1, \cdots, n-1\}$ and furthermore,

$$\begin{aligned}\Pr[f(x) = &\text{med}(\underbrace{-\infty, \cdots, -\infty}_{n-k}, \underbrace{+\infty, \cdots, +\infty}_{k-1}, x_1, \cdots, x_n)] \\ &= \Pr[Y_{(n-k)} = -\infty,\ Y_{(n-k+1)} = +\infty] \\ &= \Pr[Y_{(n-k)} = -\infty] - \Pr[Y_{(n-k+1)} = -\infty] \\ &= \frac{n-(n-k)}{n} - \frac{n-(n-k+1)}{n} \\ &= \frac{1}{n}.\end{aligned}$$

The third equality follows from the fact that for any $i \in \{1, \cdots, n\}$, we have $\Pr[Y_{(i)} = -\infty, Y_{(i+1)} = +\infty] + \Pr[Y_{(i)} = -\infty, Y_{(i+1)} = -\infty] = \Pr[Y_{(i)} = -\infty]$ and $\Pr[Y_{(i)} = -\infty, Y_{(i+1)} = -\infty] = \Pr[Y_{(i+1)} = -\infty]$.

Hence we see that f is equivalent to running rank^k mechanism for each $k \in \{1 \cdots, n\}$ with probability $\frac{1}{n}$. Thus indeed f is the Random Rank mechanism.

($\Longleftarrow$) Similar to the case when $X = [0, 1]$, the Random Rank mechanism is universally anonymous and universally truthful when the domain is $X = \mathbb{R}$ as each realization of the mechanism, rank^k, is strategyproof and anonymous by Theorem 7. The proof that Random Rank satisfies Strong Proportionality in expectation is identical that in the proof of Theorem 3. ☐

Remark 1. *Note that the Phantoms are random variables on the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$, and thus a random variable Y may satisfy $\Pr[Y = +\infty] > 0$. This is in contrast to random variables defined on $\mathbb{R}$ in which every random variable Y must satisfy $\lim_{N \to \infty} \Pr[Y \geq N] = 0$.*

D I.I.D. Phantom Mechanisms

Definition 1 (I.I.D Phantom Mechanism)**.** *A mechanism is an* I.I.D Phantom *mechanism if it is a Phantom mechanism with $y_1 = 0$, $y_{n+1} = 1$ and the remaining phantoms $y_1, \ldots, y_{n-1}$ are drawn I.I.D according to some distribution D on $[0, 1]$*

The I.I.D Phantom mechanisms are universally truthful, ex-post efficient and universally anonymous, as they only give positive support to instances of deterministic Phantom mechanisms with $y_1 = 0$ and $y_{n+1} = 1$, which by Theorem 2 are strategyproof, efficient and anonymous. If the expected values of the Phantom distribution's order statistics are uniformly spaced on $[0, 1]$, then the mechanism also satisfies Proportionality in expectation.

Theorem 8. *An I.I.D Phantom mechanism with distribution D satisfies Proportionality in expectation if and only if the order statistics $D_{(i)}$ have expected value $\mathbb{E}[D_{(i)}] = \frac{i}{n}$ for each $i \in \{1, \cdots, n-1\}$.*

Proof. ($\Longrightarrow$) Fix any $i \in \{1, \cdots, n-1\}$. Consider a location profile $x = (\underbrace{0, \cdots, 0}_{n-i}, \underbrace{1, \cdots, 1}_{i})$ and let S^0 be the set of agents located at 0, thus $|S^0| = n - i$. Denote $D_{(i)}$ as the random variable corresponding to the location of the i'th order statistic of the Phantoms. Since our mechanism is a Phantom mechanism the output location of the mechanism is distributed as $D_{(i)}$. Thus for any $i \in S^0$ we have

$$\begin{aligned}
\mathbb{E}[D_{(i)}] &= \mathbb{E}[d(0, f(x))] \\
&= \mathbb{E}[d(x_i, f(x))] \\
&\leq \frac{n - |S^0|}{n} \\
&= \frac{i}{n}
\end{aligned}$$

where the second last equality holds since f satisfies Proportionality in expectation. Similarly let S^1 be the set of agents located at 1, and thus $|S^1| = i$. For $j \in S^1$, by proportionality in expectation we see that

$$\begin{aligned}
\mathbb{E}[d(x_j, f(x))] &= \mathbb{E}[d(1, f(x))] \\
&\leq \frac{n - |S^1|}{n} \\
&= \frac{n - i}{n}
\end{aligned}$$

Since $\mathbb{E}[d(1, f(x))] = 1 - \mathbb{E}[D_{(i)}]$, by rearranging above we see that $\mathbb{E}[D_{(i)}] \geq \frac{i}{n}$. Hence indeed $\mathbb{E}[D_{(i)}] = \frac{i}{n}$ for each $i \in \{1, \cdots, n-1\}$ as needed to show.

($\Longleftarrow$) For any $x \in \{0, 1\}^n$, let S^0 be the set of agents located at 0 and S^1 be the set of agents located at 1. Let $|S^0| = k$ and $|S^1| = n - k$, the location of the facility is distributed according to $D_{(n-k)}$. Hence for any $i \in S^0$, we have $\mathbb{E}[d(x_i, f(x))] = \mathbb{E}[D_{(n-k)}] = \frac{n-|S^0|}{n}$. Similarly for $j \in S^1$, we have $\mathbb{E}[d(x_j, f(x))] = 1 - \mathbb{E}[D_{(n-k)}] = 1 - \frac{n-k}{n} = \frac{n-|S^1|}{n}$ as desired. □

By Theorem 8, we know that the Random Phantom mechanism is Proportional in expectation.

E Proof of Theorem 5

Theorem 5. *The AverageOrRandomRank$-p$ mechanism satisfies Strong Proportionality in expectation and is strategyproof in expectation if and only if $p \in [0, \frac{1}{2}]$.*

Proof. We first show that the mechanism is Strong Proportional in expectation. Consider any location profile $x \in \{\alpha, \beta\}^n$, and let S_α denote the set of agents at α and $S_\beta = N \backslash S$ denote the set of agents at β. The AverageOrRandomRank$-p$ mechanism places the facility at:

- α with probability $(1-p)\frac{|S_\alpha|}{n}$,
- at β with probability $(1-p)\frac{|S_\beta|}{n}$,
- and at $\frac{|S_\alpha|\alpha + |S_\beta|\beta}{n}$ with probability p.

For all $i \in S_\alpha$, we have

$$\begin{aligned}
\mathbb{E}[d(x_i, f_{RR}(x))] &= (1-p)\frac{|S_\beta|}{n}(\beta - \alpha) + p\left(\frac{|S_\alpha|\alpha + |S_\beta|\beta}{n} - \alpha\right) \\
&= \frac{|S_\beta|}{n}\beta - \alpha(1-p)\frac{|S_\beta|}{n} + p\alpha\frac{|S_\alpha| - n}{n} \\
&= \frac{|S_\beta|}{n}(\beta - \alpha) = \frac{n - |S_\alpha|}{n}(\beta - \alpha),
\end{aligned}$$

and for all $j \in S_\beta$, we have

$$\begin{aligned}
\mathbb{E}[d(x_j, f_{RR}(x))] &= (1-p)\frac{|S_\alpha|}{n}(\beta - \alpha) + p\left(\beta - \frac{|S_\alpha|\alpha + |S_\beta|\beta}{n}\right) \\
&= -\frac{|S_\alpha|}{n}\alpha + \beta(1-p)\frac{|S_\alpha|}{n} + p\beta\frac{n - |S_\beta|}{n} \\
&= \frac{|S_\alpha|}{n}(\beta - \alpha) = \frac{n - |S_\beta|}{n}(\beta - \alpha).
\end{aligned}$$

Hence, AverageOrRandomRank$-p$ satisfies Strong Proportionality in expectation.

We now show that the mechanism is strategyproof in expectation. Suppose an agent at x_i deviates by distance d to attain a better expected distance. Its expected cost is reduced by $\frac{dp}{n}$ from the average location moving closer, but is also increased by $\frac{d(1-p)}{n}$ from its reported location moving away. For strategyproofness we require that $\frac{d(1-p)}{n} \geq \frac{dp}{n}$, which is satisfied for $p \in [0, \frac{1}{2}]$. Furthermore, it is easy to see that if $p > \frac{1}{2}$, an agent can improve its expected distance from the facility by misreporting its location. □

F Proof of Theorem 6

Theorem 6. *A mechanism is universally anonymous, universally truthful and SPF in expectation if and only if it is the Random Rank mechanism.*

Proof. Since SPF implies Strong Proportionality, by Theorem 3 it suffices to prove Random Rank satisfies SPF. Consider any location profile x within range R and subset of agents $S \subseteq N$ within range r. Denote X_S as the event that Random Rank places the facility at an agent in S. Then for any $i \in S$, we have

$$\begin{aligned}\mathbb{E}[d(x_i, f(x))] &\leq R(1 - \Pr[X_S]) + r \Pr[X_S] \\ &\leq R\left(\frac{n - |S|}{n}\right) + r\frac{|S|}{n} \\ &\leq R\left(\frac{n - |S|}{n}\right) + r.\end{aligned}$$

□