Appendix

In the appendix, $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n$ means: given a candidate plan $\pi = [a_1, \cdots, a_n]$, starting from an initial state s_0, by executing each step of action a_k that transfers state s_{k-1} to s_k, the final state s_n is a goal state. $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} // \cdots \xrightarrow{a_n} s_n$ means: the candidate plan $\pi = [a_1, \cdots, a_n]$ is invalid: either $pre(a_k)$ is not satisfied, or s_n is not a goal state.

Proof Sketch of Lemma 1

Lemma 1 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two classical planning problems. The following holds: $\Pi(\mathcal{P}_1 \otimes \mathcal{P}_2) = \Pi(\mathcal{P}_1) \cap \Pi(\mathcal{P}_2)$*

Proof sketch:

Part A We first need to prove $\Pi(\mathcal{P}_1) \cap \Pi(\mathcal{P}_2) \subseteq \Pi(\mathcal{P}_1 \otimes \mathcal{P}_2)$. We do that by picking any plan that is valid of $\mathcal{P}_1$ and $\mathcal{P}_2$ and show that it is valid for $\mathcal{P}_1 \otimes \mathcal{P}_2$.

Let $\pi = a_1, \cdots, a_n$. Assume π is valid in $\mathcal{P}_1$: $s_{1,0} \xrightarrow{a_1} s_{1,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_{1,n}$ and also valid in $\mathcal{P}_2$: $s_{2,0} \xrightarrow{a_1} s_{2,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_{2,n}$.

For all $j \in \{0, \ldots, k\}$, let $s_j = s_{1,j} \cup s_{2,j}$. We shall prove $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n$. We need to prove the four following points:

1. Prove s_0 is the initial state i of $\mathcal{P}_1 \otimes \mathcal{P}_2$. Based on Definition 3, we can get $s_0 = s_{1,0} \cup s_{2,0} = i$.
2. Prove $\forall k, a_k$ is applicable in s_{k-1}. The precondition of a_k is the conjunction of the preconditions from $\mathcal{P}_1$ and $\mathcal{P}_2$ which are independent (rely on different facts). Since they are satisfied in $s_{1,k-1}$ and $s_{2,k-1}$ are these states rely on different facts, the conjunction is satisfied in the union state.
3. Prove $\forall k,\ a_k[s_{k-1}] = s_k$. Again, the conditions of the conditional effects are independent for the two sub-problems $\mathcal{P}_1$ and $\mathcal{P}_2$, and their effects are independent. The effects of the action therefore stack up independently.
4. Prove s_n is a goal state. Similar to point 2 on preconditions, the goal condition is the conjunction of the goal conditions in either sub-problem, and the union of the two independents sets $s_{1,n}$ and $s_{2,n}$ satisfies the conjunction.

Part B We then need to prove $\Pi(\mathcal{P}_1 \otimes \mathcal{P}_2) \subseteq \Pi(\mathcal{P}_1) \cap \Pi(\mathcal{P}_2)$. Here we only provide the prove on behalf of $\mathcal{P}_1$, since $\mathcal{P}_2$ has the same prove procedures.

Let $\pi = [a_1, \ldots, a_n]$ be a valid plan for $\mathcal{P}_1 \otimes \mathcal{P}_2$, i.e., such that $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n$. For all $j \in \{0, \ldots, k\}$, let $s_{1,j} = s_j \cap F_1$. Again, we need to prove the four properties that guarantee $s_{1,0} \xrightarrow{a_1} s_{1,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_{1,n}$.

1. Prove $s_{1,0}$ is the initial state i_1 of $\mathcal{P}_1$. $s_0 \cap F_1 = (i_{1,0} \cup i_{2,0}) \cap F_1 = (i_{1,0} \cap F_1) \cup (i_{2,0} \cap F_1) = (i_{1,0} \cap F_1) = i_{1,0}$.
2. Prove $\forall k, a_k$ is applicable in $s_{1,k-1}$. Since the precondition of a_k in $\mathcal{P}_1 \otimes \mathcal{P}_2$ is the conjunction of the precondition of a_k in $\mathcal{P}_1$ with another formula, the precondition of a_k in $\mathcal{P}_1$ is satisfied by s_{k-1}. Furthermore, s_{k-1} is identical to $s_{1,k-1}$ except for some facts from F_2 which are irrelevant. Therefore, the precondition is satisfied.
3. Prove $\forall k,\ a_k[s_{1,k-1}] = s_{1,k}$. Again, we see that the conditional effects of a_k that modify the truth value of the facts in F_1 are the same for both problems $\mathcal{P}_1 \otimes \mathcal{P}_2$ and $\mathcal{P}_1$. Therefore, the state reached after applying the effects is the same: $s_{k-1}[a_k] \cap F_1 = ((s_{k-1} \cap F_1)[a_k]$.
4. Prove $s_{1,n}$ is a goal state of $\mathcal{P}_1$. This is similar to point 2 on precondition.

Proof Sketch of Lemma 2

Lemma 2 *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two classical planning problems. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are such that their action preconditions are trivial ($\forall j \in \{1, 2\}.\ \forall a \in A_j.\ pre(a) = true$), then the following holds: $\Pi(\mathcal{P}_1 \oplus \mathcal{P}_2) = \Pi(\mathcal{P}_1) \cup \Pi(\mathcal{P}_2)$*

Proof sketch: Let $\pi = a_1, \cdots, a_n$ and π is valid in $\mathcal{P}_1$: $s_{1,0} \xrightarrow{a_1} s_{1,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_{1,n}$. π is also valid in $\mathcal{P}_2$: $s_{2,0} \xrightarrow{a_1} s_{2,1} \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_{2,n}$.

The proof is very similar to that of Lemma 1. However, we need to make the following changes.

Part A only assume that the plan is valid for one of the problems $\mathcal{P}_p$ (but still assume that the states $s_{q,j}$ are the result of applying the actions in the other problem $q = 3 - p$ (even when they are not applicable). One can easily check that the four conditions are satisfied.

Part B. We assume the plan of $\mathcal{P}_1 \oplus \mathcal{P}_2$ leads to a goal state, and we need to check that the plan is valid for one of the problems $\mathcal{P}_1$ and $\mathcal{P}_2$. Because the goal condition is the disjunction $G_1 \vee G_2$ where G_1 and G_2 pertain to two different sets of facts F_1 and F_2, it is immediate that $s_n \cap F_p$ satisfies G_p for some p. Wlog, we assume $p = 1$.

Similar to Lemma 1, Conditions 1 and 3 are easy to verify. Condition 2 is trivial because the preconditions are trivial. Condition 4 is satisfied by the choice of p.

Proof Sketch of Lemma 3

Lemma 3 Let $\mathcal{P}$ be a planning problem. The following holds: $\Pi(\mathcal{P}) = \Pi(Triv(\mathcal{P}))$.

We write $\mathcal{P}' = Triv(\mathcal{P})$. For this proof, we concentrate on a single initial state i for $\mathcal{P}$ and the corresponding initial state $i \cup \{\zeta\}$; the result then generalises to the initial probabilistic distribution.

It should be clear that, for any sequence of actions $\pi = a_1, \ldots, a_k$, the state $i[\pi]$ equals $i[\pi] \setminus \{\zeta\}$ (but note that this does not imply $\zeta \in i[\pi]$) since the conditional effects of these actions are similar except for ζ.

Part A Consider a plan $\pi = a_1, \cdots, a_n$ that is not valid from i'. Since all actions in $\mathcal{P}'$ are trivial, and since both the states $i[\pi]$ and $i'[\pi]$ and goals G and G' differ only on ζ, we have $\zeta \notin i'[\pi]$. Furthermore, since $\zeta \in i'$, there must be an action, a_j, that makes has ζ as a negative effect. This means that the condition φ that triggers this negative effect is not satisfied; however, φ is the negation of the precondition of a_j in $\mathcal{P}$. Therefore, since the states $i[a_1, \ldots, a_{k-1}]$ and $i'[a_1, \ldots, a_{k-1}]$ are identical except for ζ, the action a_k is not applicable in $\mathcal{P}$ and π is not valid for i.

Part B Assume that π is valid for i'. Then, because π is valid, ζ is true in the final state. This implies that none of the conditions that have ζ has a negative effect trigger (there is action that would make ζ positive again), i.e., the negations of these conditions are always true. These negations are exactly the preconditions of the actions in $\mathcal{P}$, and we know that the states $i[a_1, \ldots, a_k]$ and $i'[a_1, \ldots, a_k]$ are identical except for ζ. Therefore, π is valid in $\mathcal{P}$.

Proof Sketch of Theorem 1

Theorem 1 *The set of states in which candidate plan π is invalid is $[\![CTags(\pi)]\!]$.*

Proof sketch:

Part A We prove if an initial state $i \in [\![CTags(\pi)]\!]$, then π is invalid for that initial state i. Based on $[\![CTags(\pi)]\!] = \bigcup_{ct \in CTags(\pi)} [\![ct]\!]$, we can have $i \in \bigcup_{ct \in CTags(\pi)} [\![ct]\!]$. Since $\forall ct \in CTags(\pi)$, $\pi \notin \Pi(Proj(\mathcal{P}, ct))$, we can have π is invalid for i.

Part B We prove if π is invalid for an initial state i, then $i \in [\![CTags(\pi)]\!]$. When i is an initial state that is invalid for π, there must be a subgoal φ that is not satisfied during the execution of the plan. Let c be the context of this subgoal. By definition of a tag, the validity of the subgoal at any time of the execution is defined entirely by the tag that was true in the initial state. Therefore, any other state that has the same tag will also fail the validity test. Hence, this tag is a counter-tag, i.e., the state i belongs to $[\![CTags(\pi)]\!]$.