A Pseudocode of Algorithm 2

Algorithm 2: Meta-Expert Learning Algorithm

Input: Time horizon T; support size K; accuracy of error information $(\sigma_1, \cdots, \sigma_T)$; norm parameter $q \in [1, \infty]$.
Output: A bidding policy π.
Initialization: Construct T^K base experts $\{f_i\}$ $(i = 1, 2, \cdots, T^K)$ that cover the oracle with cumulative reward difference at most $O(1)$ (as in the proof of Theorem 5);

```
for i = 1, 2, ··· T^K do
    Initialize R_{0,f_i} ← 0;
end
Initialize R_{0,h} ← 0, R_{0,g} ← 0, R_{0,f} ← 0;
Initialize L_0 ← 0 ;
for t ∈ {1, 2, ··· , T} do
    The bidder receives private value v_t ∈ [0, 1];
    Set learning rate η_{t,1} ← min{ 1/4, sqrt(K log T / L_t) };
    The bidder observes hint h_t ∈ [0, 1], along with its accuracy σ_t;
    L_t ← L_{t-1} + σ_t^{q/(q+1)};
    Set b_{t,h} ← h_t + σ_t^{q/(q+1)};
    Sample b_{t,g} according to ChEW policy;
    for i = 1, 2, ··· , T^K do
        Let b_{t,f} ← f_i(v_t) with probability
```

$$p_{t,i} := \frac{\exp\left(\eta_{t,1} R_{t-1,f_i}\right)}{\exp\left(\eta_{t,1} R_{t-1,h}\right) + \sum_{i'=1}^{T^K} \exp\left(\eta_{t,1} R_{t-1,f_{i'}}\right)}.$$

```
    end
```

Let $b_{t,f} \leftarrow b_{t,h}$ with probability $p_{t,T^K+1} := \frac{\exp(\eta_{t,1} R_{t-1,h})}{\exp(\eta_{t,1} R_{t-1,h}) + \sum_{i'=1}^{T^K} \exp\left(\eta_{t,1} R_{t-1,f_{i'}}\right)}$;

```
    Sample b_{t,f} ~ p_t ;
    Set learning rate η_{t,2} ← min{ 1/4, sqrt(log 3 / L_t) };
    for i ∈ {f, g, h} do
```

$$P_{t,i} = \frac{\exp\left(\eta_{t,2} R_{t-1,i}\right)}{\sum_{i' \in \{f,g,h\}} \exp\left(\eta_{t,2} R_{t-1,i'}\right)};$$

```
    end
    The bidder samples policy i ~ P_t and bids b_{t,i};
    The bidder receives others' highest bid m_t;
    for i = 1, 2, ··· , T^K do
```

$$R_{t,f_i} \leftarrow R_{t-1,f_i} + r(f_i(v_t); v_t, m_t).$$

```
    end
    for i ∈ {f, g, h} do
        Update R_{t,i} ← R_{t-1,i} + r(b_{t,i}; v_t, m_t);
    end
end
```

The algorithm has a tree structure with the nodes in the upper layer representing algorithms instead of specific oracles. In Algorithm 2, the upper nodes are respectively: the algorithm that achieves the regret upper bound in Theorem 5 described in Appendix C.1, "ChEW" algorithm to achieve $\widetilde{O}(\sqrt{T})$ regret bound proposed in [HZF+20], and a single expert which bids $h_t + \sigma_t^{q/(q+1)}$ each time. The probability distribution $P_{t,i}$ runs the multiplicative weights update on the above strategies (see details in Appendix C.2).

B Proof of Main Result in Section 3

B.1 Proof of Regret Upper Bounds in Theorem 1 and Theorem 2

B.1.1 Proof of Upper Bound in Theorem 1.

We prove a slightly stronger result than Theorem 1:

Lemma 1. *If $v_t \equiv 1$ and the bidder observes σ_t at each time t, then the following regret upper bound holds for Algorithm 1:*

$$\sup_{\{m_t, h_t, \sigma_t\}} \text{Reg}(\pi_1) = O\left(\log T + \sqrt{\log T \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}}\right),$$

with $\text{Reg}(\pi)$ *defined in (2), and the supremum is taken over all m_t sequences and hints that satisfy (3), and the infimum is taken over all possible policies π.*

Proof. The following is similar to proof of Theorem 3 in [HZF+20]. As in the standard analysis of multiplicative weights [CBL06], define:

$$\phi_t = \frac{1}{K}\sum_{a=1}^{K} \exp\left(\eta_t \cdot \sum_{s<t} r_{s,a}\right), \quad t = 1, \dots, T+1.$$

Recall that $K = T+1$ and a^* is the extra expert. We translate every $r_{t,a}$ by $-r_{t,a^*}$ to ensure that $r_{t,a} \in [-1, 1]$ and $r_{t,a^*} = 0$. Then for $t \in [T]$, Jensen's inequality with $\eta_t/\eta_{t+1} \geq 1$ gives

$$\begin{aligned}
(\phi_{t+1})^{\frac{\eta_t}{\eta_{t+1}}} &= \left[\frac{1}{K}\sum_{a=1}^{K} \exp\left(\eta_{t+1} \cdot \sum_{s<t+1} r_{s,a}\right)\right]^{\frac{\eta_t}{\eta_{t+1}}} \\
&\leq \frac{1}{K}\sum_{a=1}^{K}\left[\exp\left(\eta_{t+1} \cdot \sum_{s<t+1} r_{s,a}\right)^{\frac{\eta_t}{\eta_{t+1}}}\right] \\
&= \phi_t \sum_{a=1}^{K} p_{t,a} \cdot \exp\left(\eta_t \cdot r_{t,a}\right) =: \phi_t \mathbb{E}[\exp\left(\eta_t X_t\right)].
\end{aligned}$$

Here X_t is a random variable that takes value $r_{t,a}$ with probability $p_{t,a}$. Now using Bernstein's inequality

$$\mathbb{E}[\exp(\lambda X)] \leq \exp\left(\lambda \mathbb{E}[X] + (e^\lambda - \lambda - 1)\mathsf{Var}(X)\right),$$

with $|X - \mathbb{E}[X]| \leq 1$ almost surely, we have

$$\frac{\log \phi_{t+1}}{\eta_{t+1}} - \frac{\log \phi_t}{\eta_t} \leq \mathbb{E}[X_t] + \frac{e^{\eta_t} - \eta_t - 1}{\eta_t}\mathsf{Var}(X_t) \leq \mathbb{E}[X_t] + \eta_t \mathsf{Var}(X_t),$$

where the last inequality is due to $e^x - x - 1 \leq x^2$ for $x \in [0, 1]$. Define $r_t^* := \max_{a \in [K]} r_{t,a}$, we have

$$\mathsf{Var}(X_t) \leq \mathbb{E}[(r_t^* - X_t)^2] \leq 1 \cdot \mathbb{E}[r_t^* - X_t] = r_t^* - \mathbb{E}[X_t].$$

By telescoping and defining $\eta_{T+1} := \eta_T$,

$$\frac{\log \phi_{T+1}}{\eta_T} = \sum_{t=1}^{T}\left[\frac{\log \phi_{t+1}}{\eta_{t+1}} - \frac{\log \phi_t}{\eta_t}\right] \leq \sum_{t=1}^{T} \mathbb{E}[X_t] + \sum_{t=1}^{T} \eta_t \left(r_t^* - \mathbb{E}[X_t]\right). \tag{5}$$

For the left-hand side of (5), we also have

$$\log \phi_{T+1} \geq \eta_T \cdot \max_{a \in [K]} \sum_{s=1}^{T} r_{t,a} - \log K. \tag{6}$$

Combining (5) and (6),

$$\max_{a\in[K]}\sum_{t=1}^{T} r_{t,a} \le \frac{\log K}{\eta_T} + \sum_{t=1}^{T}(1-\eta_t)\cdot\mathbb{E}[X_t] + \sum_{t=1}^{T}\eta_t\cdot r_t^*. \tag{7}$$

Rearranging (7) leads to the following upper bound on the cumulative regret:

$$\max_{a\in[K]}\sum_{t=1}^{T} r_{t,a} - \sum_{t=1}^{T}\mathbb{E}[X_t] \le \frac{\log K}{\eta_T} + \sum_{t=1}^{T}\eta_t r_t^* - \sum_{t=1}^{T}\eta_t\cdot\mathbb{E}[X_t]. \tag{8}$$

Let $V_T := (\log K)/\eta_T + \sum_{t=1}^{T}\eta_t r_t^*$, it remains to upper bound the last term of (8). To do so, note that (7) holds for any intermediate value of $t\in[T]$ as well. Since $\max_{a\in[K]}\sum_{t=1}^{T} r_{t,a} \ge \sum_{t=1}^{T} r_{t,a^*} = 0$, for every $t\in[T]$ we have

$$S_t := \sum_{s=1}^{t}(1-\eta_s)\cdot\mathbb{E}[X_s] \ge -\frac{\log K}{\eta_{t+1}} - \sum_{s=1}^{t}\eta_s\cdot r_s^* = -V_t \ge -V_T,$$

where the last inequality is due to $\eta_{t+1}\ge\eta_T$ and $r_t^*\ge r_{t,a^*}=0$ for every $t\in[T]$. Consequently,

$$\begin{aligned}
-\sum_{t=1}^{T}\eta_t\cdot\mathbb{E}[X_t] &= -\sum_{t=1}^{T}(S_t - S_{t-1})\cdot\frac{\eta_t}{1-\eta_t}\\
&= -\sum_{t=1}^{T-1} S_t\cdot\left(\frac{\eta_t}{1-\eta_t} - \frac{\eta_{t+1}}{1-\eta_{t+1}}\right) - S_T\cdot\frac{\eta_T}{1-\eta_T}\\
&\le V_T\sum_{t=1}^{T-1}\left(\frac{\eta_t}{1-\eta_t} - \frac{\eta_{t+1}}{1-\eta_{t+1}}\right) + V_T\cdot\frac{\eta_T}{1-\eta_T}\\
&= \frac{V_T\eta_1}{1-\eta_1} \le V_T,
\end{aligned}$$

where we have used that $1/4\ge\eta_1\ge\eta_2\ge\ldots\ge\eta_T>0$. Plugging this inequality back into (7) gives

$$\max_{a\in[K]}\sum_{t=1}^{T} r_{t,a} - \sum_{t=1}^{T}\mathbb{E}[X_t] \le 2V_T. \tag{9}$$

Finally it remains to upper bound $\mathbb{E}[V_T]$, where the expectation is taken with respect to the randomness in the hint sequence $\{h_t\}_{t=1}^{T}$. Since the definition of the expert a^* gives that

$$\begin{aligned}
r_t^* &\le (1-m_t) - (1-h_t-\sigma_t^{q/(q+1)})\mathbb{1}(h_t+\sigma_t^{q/(q+1)}\ge m_t)\\
&\le \begin{cases} h_t+\sigma_t^{q/(q+1)} - m_t & \text{if } h_t+\sigma_t^{q/(q+1)}\ge m_t\\ 1 & \text{if } h_t+\sigma_t^{q/(q+1)} < m_t\end{cases},
\end{aligned}$$

we conclude that

$$\begin{aligned}
\mathbb{E}[r_t^*] &\le \mathbb{P}(h_t+\sigma_t^{q/(q+1)}<m_t) + \mathbb{E}[|h_t+\sigma_t^{q/(q+1)}-m_t|]\\
&\le \frac{\mathbb{E}[|h_t-m_t|^q]}{(\sigma_t^{q/(q+1)})^q} + (\mathbb{E}[|h_t-m_t|^q])^{1/q} + \sigma_t^{q/(q+1)}\\
&\le 2\sigma_t^{q/(q+1)} + \sigma_t \le 3\sigma_t^{q/(q+1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[V_T] &\le \frac{\log K}{\eta_T} + \sum_{t=1}^{T}\eta_t\mathbb{E}[r_t^*]\\
&\le 4\log K + \sqrt{\sum_{t=1}^{T}\sigma_t^{q/(q+1)}\log K} + 3\sum_{t=1}^{T}\sqrt{\frac{\log K}{\sum_{s\le t}\sigma_s^{q/(q+1)}}}\cdot\sigma_t^{q/(q+1)}\\
&\le 4\log K + 7\sqrt{\sum_{t=1}^{T}\sigma_t^{q/(q+1)}\log K},
\end{aligned}$$

where the last inequality follows from

$$\sum_{i=1}^{n} \frac{a_i}{\sqrt{\sum_{j\le i} a_j}} \le \sum_{i=1}^{n} \int_{\sum_{j\le i-1} a_j}^{\sum_{j\le i} a_j} \frac{\mathrm{d}x}{\sqrt{x}} = \int_{0}^{\sum_{i=1}^{n}} \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{\sum_{i=1}^{n} a_i}$$

for any non-negative reals $a_1, \cdots, a_n$. Plugging the above upper bound of $\mathbb{E}[V_T]$ into (9) completes the proof of the lemma. □

Theorem 1 follows from Lemma 1 and the following Jensen's inequality:

$$\sqrt{\log T \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}} \le \sqrt{\log T \cdot T \cdot \left(\frac{\sum_{t=1}^{T} \sigma_t}{T}\right)^{\frac{q}{q+1}}} \le \sqrt{\log T \cdot L^{q/(q+1)} \cdot T^{1/(q+1)}}.$$

B.1.2 Proof of Upper Bound in Theorem 2.

To achieve the upper bound of Theorem 2, we construct the same T base experts as Algorithm 1, as well as T additional experts who bid $h_t + i/T, i \in [T]$ at each time t. Then at an additional $O(1)$ cost in the final regret, the additional experts include an expert who bids $h_t + \sqrt{L/T}$ at each time t. Using the same analysis in the proof of Lemma 1, this algorithm achieves a regret upper bound

$$\text{Reg}(\pi_2) \le 2\left(\frac{\log(2T)}{\eta} + \eta \cdot \sum_{t=1}^{T} \mathbb{E}[r_t^*]\right),$$

where $\eta > 0$ is a fixed learning rate, and

$$r_t^* \le \begin{cases} h_t + \sqrt{L/T} - m_t & \text{if } h_t + \sqrt{L/T} \ge m_t \\ 1 & \text{if } h_t + \sqrt{L/T} < m_t \end{cases}.$$

Consequently,

$$\begin{aligned} \sum_{t=1}^{T} \mathbb{E}[r_t^*] &\le \sum_{t=1}^{T} \mathbb{E}[|h_t + \sqrt{L/T} - m_t|] + \sum_{t=1}^{T} \mathbb{P}(h_t + \sqrt{L/T} < m_t) \\ &\le \sqrt{LT} + \mathbb{E}\left[\sum_{t=1}^{T} |h_t - m_t|\right] + \frac{1}{\sqrt{L/T}} \mathbb{E}\left[\sum_{t=1}^{T} |h_t - m_t|\right] \\ &\le 2\sqrt{LT} + L \le 3\sqrt{LT}, \end{aligned}$$

as $1 \le L \le T$. Now choosing $\eta = \min\{1/4, \sqrt{(\log T)/\sqrt{LT}}\}$ leads to the regret upper bound $O\left((\log T)^{\frac{1}{2}} (T \cdot L)^{\frac{1}{4}}\right)$.

B.2 Proof of Regret Lower Bounds in Theorem 1 and Theorem 2

B.2.1 Proof of Lower Bound in Theorem 1.

Proof. We use Le Cam's Two-Point method. Construct hint and minimum bid to win as follows: Let $h_t = \frac{1}{2}, t = 1, \ldots, T$ and σ_t be the same for all t such that $\sigma^{\frac{q}{q+1}} \le \frac{1}{4}$. Consider the following two CDFs for $m_t \in [0, 1]$:

$$G_1(x) = \begin{cases} 0, & \text{if } 0 < x < \frac{1}{2} \\ 2 \cdot (1 - \bar{x} + \delta), & \text{if } \frac{1}{2} < x < \bar{x} \\ 1, & \text{if } \bar{x} < x < 1 \end{cases}, \quad G_2(x) = \begin{cases} 0, & \text{if } 0 < x < \frac{1}{2} \\ 2 \cdot (1 - \bar{x} - \delta), & \text{if } \frac{1}{2} < x < \bar{x} \\ 1, & \text{if } \bar{x} < x < 1 \end{cases},$$

where $\bar{x} := \frac{1}{2} + \frac{1}{2} \cdot \sigma^{\frac{q}{q+1}}$ and let $\delta < \frac{1}{2} \cdot \sigma^{\frac{q}{q+1}}$. Easy to observe the above construction satisfies:

$$\mathbb{E}[|m_t - h_t|^q] \leq 2 \cdot (\frac{1}{2} \cdot \sigma^{\frac{q}{q+1}} + \delta) \cdot \left(\frac{1}{2} \cdot \sigma^{\frac{q}{q+1}}\right)^q \leq \sigma^{\frac{q}{q+1}} \cdot \left(\sigma^{\frac{q}{q+1}}\right)^q = \sigma^q.$$

Let $r_1(v_t, b_t)$ and $r_2(v_t, b_t)$ be the expected instantaneous reward under CDFs G_1 and G_2. Then under the above construction:

$$\begin{aligned}
&\max_{b\in[0,1]} r_1(1,b) = r_1(1,\frac{1}{2}) = \frac{1}{2} \cdot \frac{1-\bar{x}+\delta}{1-\frac{1}{2}} = 1-\bar{x}+\delta,\\
&\max_{b\in[0,1]} r_2(1,b) = r_2(1,\bar{x}) = 1-\bar{x},\\
&\max_{b\in[0,1]} (r_1(1,b)+r_2(1,b)) = r_1(1,\bar{x}) + r_2(1,\bar{x}) = 2\cdot(1-\bar{x}).
\end{aligned}$$

Therefore, for any $b_t \in [0,1]$,

$$\begin{aligned}
&\left(\max_{b\in[0,1]} r_1(1,b) - r_1(1,b_t)\right) + \left(\max_{b\in[0,1]} r_2(1,b) - r_2(1,b_t)\right)\\
&\geq \left(\max_{b\in[0,1]} r_1(1,b)\right) + \left(\max_{b\in[0,1]} r_2(1,b)\right) - \max_{b\in[0,1]}(r_1(1,b)+r_2(1,b))\\
&= (1-\bar{x}+\delta) + (1-\bar{x}) - 2\cdot(1-\bar{x}) = \delta.
\end{aligned}$$

Thus we have for any policy π,

$$\begin{aligned}
\sup_G \text{Reg}(\pi) &\geq \frac{1}{2}\mathbb{E}_{G_1}[\text{Reg}(\pi)] + \frac{1}{2}\mathbb{E}_{G_2}[\text{Reg}(\pi)]\\
&= \frac{1}{2}\sum_{t=1}^{T}\left(\mathbb{E}_{P_1^t}\left[\max_{b\in[0,1]} r_1(1,b) - r_1(1,b_t)\right] + \mathbb{E}_{P_2^t}\left[\max_{b\in[0,1]} r_2(1,b) - r_2(1,b_t)\right]\right) \qquad (10)\\
&\geq \frac{1}{2}\sum_{t=1}^{T}\delta \cdot \int \min\{dP_1^t, dP_2^t\}\\
&\geq \frac{1}{2}\sum_{t=1}^{T}\delta \cdot \left(1 - \|P_1^t - P_2^t\|_{\text{TV}}\right)\\
&\geq \frac{1}{2}T\delta \cdot \left(1 - \|P_1^T - P_2^T\|_{\text{TV}}\right),
\end{aligned}$$

where b_t in (10) denotes the bid of the oracle chosen by policy π at time t and P_i^t ($i \in \{1,2\}$) denotes the distribution of all observables $(m_1, \ldots, m_{t-1})$ at the beginning of time t. The KL divergence:

$$\begin{aligned}
D_{\text{KL}}(P_1^T\|P_2^T) &= (T-1)\cdot D_{\text{KL}}(G_1\|G_2)\\
&= (T-1)\cdot\left(2\cdot(1-\bar{x}+\delta)\cdot\log\frac{1-\bar{x}+\delta}{1-\bar{x}-\delta} + 2\cdot\left(\bar{x}-\frac{1}{2}-\delta\right)\cdot\log\frac{\bar{x}-\frac{1}{2}-\delta}{\bar{x}-\frac{1}{2}+\delta}\right)\\
&\leq (T-1)\cdot\left(2\cdot(1-\bar{x}+\delta)\cdot\left(\frac{1-\bar{x}+\delta}{1-\bar{x}-\delta}-1\right) + 2\cdot\left(\bar{x}-\frac{1}{2}-\delta\right)\cdot\left(\frac{\bar{x}-\frac{1}{2}-\delta}{\bar{x}-\frac{1}{2}+\delta}-1\right)\right)\\
&= 4\cdot\delta\cdot(T-1)\cdot\left(\frac{1-\bar{x}+\delta}{1-\bar{x}-\delta} - \frac{\bar{x}-\frac{1}{2}-\delta}{\bar{x}-\frac{1}{2}+\delta}\right)\\
&\leq \frac{4T\cdot\delta^2}{(\bar{x}-\frac{1}{2}+\delta)(1-\bar{x}-\delta)}\\
&\leq \frac{16T\cdot\delta^2}{\frac{1}{2}\cdot\sigma^{\frac{q}{q+1}}+\delta}.\\
&\leq \frac{32T\cdot\delta^2}{\sigma^{\frac{q}{q+1}}}.
\end{aligned}$$

Taking the separation parameter $\delta = \min\left\{\frac{1}{2} \cdot \sigma^{\frac{q}{q+1}}, \frac{1}{8} \cdot \sigma^{\frac{q}{2(q+1)}} \cdot T^{-\frac{1}{2}}\right\}$ and substituting into (6) leads to the regret lower bound in Theorem 1:

$$\Omega\left(\sqrt{T\sigma^{\frac{q}{q+1}}}\right) = \Omega\left(\sqrt{L^{\frac{q}{q+1}} \cdot T^{\frac{1}{q+1}}}\right).$$

□

B.2.2 Proof of Lower Bound in Theorem 2.

Proof. At each time t, let $v_t = 1$ and point estimation equals to $\frac{1}{2}$. Define $\varepsilon \in [0, \frac{1}{8}]$ to be some parameter relevant to L. Consider the following two scenarios: (each with probability $\frac{1}{2}$)

- σ_t equals to 0 with probability $p_1 := 1 - 2(\varepsilon - \delta)$, and equals to ε with probability $1 - p_1$, in which case m_t always takes value $h_t + \varepsilon$.
- σ_t equals to 0 with probability $p_2 := 1 - 2(\varepsilon + \delta)$, and equals to ε with probability $1 - p_2$, in which case m_t always takes value $h_t + \varepsilon$.

Easy to observe under this construction the expected value of L:

$$\bar{L} = \sum_{t=1}^{T} \frac{\varepsilon}{2} \cdot (2(\varepsilon + \delta) + 2(\varepsilon - \delta)) = 2\varepsilon^2 \cdot T.$$

The above construction also satisfies:

$$\begin{aligned}
&\max_{b\in[0,1]} R_1(1,b) = R_1\left(1, \frac{1}{2}\right) = \frac{1}{2} - \varepsilon + \delta, \\
&\max_{b\in[0,1]} R_2(1,b) = R_2\left(1, \frac{1}{2} + \varepsilon\right) = \frac{1}{2} - \varepsilon, \\
&\max_{b\in[0,1]} (R_1(1,b) + R_2(1,b)) = R_1\left(1, \frac{1}{2} + \varepsilon\right) + R_2\left(1, \frac{1}{2} + \varepsilon\right) = 2 \cdot \left(\frac{1}{2} - \varepsilon\right),
\end{aligned}$$

where R_1 and R_2 are expected rewards under the two scenarios. The following steps are similar to previous subsection, for any policy π,

$$\begin{aligned}
\sup_{\{m_t,h_t,\sigma_t\}} \text{Reg}(\pi) &\geq \frac{1}{2}\mathbb{E}_1[\text{Reg}(\pi)] + \frac{1}{2}\mathbb{E}_2[\text{Reg}(\pi)] \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(\mathbb{E}_{P_1^t}\left[\max_{b\in[0,1]} R_1(1,b) - R_1(1,b_t)\right] + \frac{1}{2}\mathbb{E}_{P_2^t}\left[\max_{b\in[0,1]} R_2(1,b) - R_2(1,b_t)\right]\right) \\
&\geq \frac{1}{2}\sum_{t=1}^{T} \delta \cdot \int \min\{dP_1^t, dP_2^t\} \\
&\geq \frac{1}{2}\sum_{t=1}^{T} \delta \cdot \left(1 - \|P_1^t - P_2^t\|_{\text{TV}}\right) \\
&\geq \frac{1}{2}T\delta \cdot \left(1 - \|P_1^T - P_2^T\|_{\text{TV}}\right), \qquad (11)
\end{aligned}$$

with P_1^t and P_2^t defined the same as (10). And the KL divergence

$$\begin{aligned}
D_{\mathrm{KL}}(P_1^T \| P_2^T) &= \sum_{t=1}^{T} \left(2(\varepsilon - \delta) \cdot \log \frac{\varepsilon - \delta}{\varepsilon + \delta} + (1 - 2(\varepsilon - \delta)) \cdot \log \frac{1 - 2(\varepsilon - \delta)}{1 - 2(\varepsilon + \delta)} \right) \\
&\le \sum_{t=1}^{T} \left(2(\varepsilon - \delta) \cdot \frac{-2\delta}{\varepsilon + \delta} + (1 - 2(\varepsilon - \delta)) \cdot \frac{4\delta}{1 - 2(\varepsilon + \delta)} \right) \\
&\le 4\delta T \cdot \left(-\frac{\varepsilon - \delta}{\varepsilon + \delta} + \frac{1 - 2\varepsilon + 2\delta}{1 - 2\varepsilon - 2\delta} \right) \\
&= 8\delta^2 T \cdot \frac{1}{(\varepsilon + \delta)(1 - 2\varepsilon - 2\delta)} \\
&\le \frac{16T \cdot \delta^2}{\varepsilon}.
\end{aligned}$$

Taking $\delta = \min\left\{\varepsilon, \frac{1}{4}\sqrt{\frac{\varepsilon}{2T}}\right\}$ and substitute in (11), we have:

$$\sup_{\{m_t, h_t, \sigma_t\}} \mathrm{Reg}(\pi) \ge \frac{1}{4} \min\left\{\varepsilon T, \frac{1}{4\sqrt{2}}\sqrt{T \cdot \varepsilon}\right\},$$

which leads to a lower bound of $\Omega((T \cdot L)^{\frac{1}{4}})$. Note that the construction above requires σ_t to be unknown, otherwise one can achieve 0 regret by bidding hint for $\sigma_t = 0$ and bidding hint + ε for $\sigma = \varepsilon$, which is a technical explanation for the separation in Section 3. □

B.3 Proof of Theorem 3.

Proof. If $L > \left(\sqrt{T}\right)^{\frac{q-1}{q}}$, then $T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} > \sqrt{T}$ and the regret can be lower bounded by $\Omega\left(\sqrt{T}\right)$. So in the following construction, we assume $L \le \left(\sqrt{T}\right)^{\frac{q-1}{q}}$. First we divide time horizon to $\left\lfloor T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rfloor$ equal parts and let σ_t be the same for all t. Construct private values and hints as follows: For $t = i \cdot \left\lfloor \left(\frac{T}{L}\right)^{\frac{q}{q+1}} \right\rfloor + 1, i \cdot \left\lfloor \left(\frac{T}{L}\right)^{\frac{q}{q+1}} \right\rfloor + 2, \ldots, (i+1) \cdot \left\lfloor \left(\frac{T}{L}\right)^{\frac{q}{q+1}} \right\rfloor$,

$$\begin{aligned}
v_t &= \frac{1}{2} + \frac{1}{2} \cdot \frac{i}{T^{\frac{1}{q+1}} L^{\frac{q}{q+1}}}, \\
h_t &= \frac{1}{4} + \frac{i}{4} \cdot \sigma^{\frac{q}{q+1}}, \\
m_t &= \begin{cases} \frac{1}{4} + \frac{i}{4} \cdot \sigma^{\frac{q}{q+1}}, & w.p. \quad 1 - \frac{1}{4} \cdot \left(\sigma^{\frac{q}{q+1}} \pm \delta\right) \\ \frac{1}{4} + \frac{i+1}{4} \cdot \sigma^{\frac{q}{q+1}}, & w.p. \quad \frac{1}{4} \cdot \left(\sigma^{\frac{q}{q+1}} \pm \delta\right) \end{cases}
\end{aligned}$$

where $i = 0, 1, 2, \ldots, \left\lfloor T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rfloor - 1$ and δ is the separation parameter similarly defined in the proof of Theorem 1. Since $L \le \left(\sqrt{T}\right)^{\frac{q-1}{q}}$, we have

$$\frac{1}{T^{\frac{1}{q+1}} L^{\frac{q}{q+1}}} \ge \left(\frac{L}{T}\right)^{\frac{q}{q+1}} = \sigma^{\frac{q}{q+1}},$$

which ensures any strategy π that bids in $\left[\frac{1}{4} + \frac{i}{4} \cdot \sigma^{\frac{q}{q+1}}, \frac{1}{4} + \frac{i+1}{4} \cdot \sigma^{\frac{q}{q+1}}\right]$ for the i-th part belongs to 1-Lipschitz and monotone oracle. Therefore, we can now consider the whole time horizon as $\left\lfloor T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rfloor$ independent problems, each of which consists of $\left\lfloor \left(\frac{T}{L}\right)^{\frac{q}{q+1}} \right\rfloor$ time steps and has fixed v_t. Substituting $L_i := \left\lfloor \left(\frac{T}{L}\right)^{\frac{q}{q+1}} \right\rfloor \cdot \sigma$, which is L for the i-th subproblem, and applying similar method to the proof of Theorem 1, we can get:

$$\sup_G \mathrm{Reg}_i(\pi) = \Omega\left(\sqrt{\left(\frac{T}{\left\lceil T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rceil}\right)^{\frac{1}{q+1}} \cdot \left(\frac{L}{\left\lceil T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rceil}\right)^{\frac{q}{q+1}}}\right)$$

$$= \Omega\left(\frac{T^{\frac{1}{q+1}} L^{\frac{q}{q+1}}}{\left\lceil T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rceil}\right) = \Omega(1),$$

for each independent problem. Summing over all subproblems leads to the lower bound $\Omega\left(T^{\frac{1}{q+1}} L^{\frac{q}{q+1}}\right)$. □

B.4 Proof of Theorem 4

Proof. We prove that even when L takes expected value $\Theta(1)$, the minimax regret is still lower bounded by $\Omega(\sqrt{T})$. The proof is similar to that of Theorem 3, but by dividing time horizon into $\sqrt{T}$ subproblems. At each time t inside the i-th subproblem, the bidder observes $h_t = \frac{1}{4} + \frac{i \cdot \varepsilon}{4}$ (where $\varepsilon = \frac{1}{\sqrt{T}}$). In the construction of the lower bound in Theorem 2, σ_t equals to 0 with probability $1 - \Theta(\varepsilon)$ and equals to ε with probability $\Theta(\varepsilon)$. Thus,

$$\bar{L} = \mathbb{E}\left[\sum_{t=1}^{T} \sigma_t\right] = T \cdot \varepsilon^2 = \Theta(1).$$

Meanwhile, applying similar method to the proof of Theorem 2, we can get a lower bound of $\Omega\left(\sqrt{\sqrt{T} \cdot \frac{1}{\sqrt{T}}}\right) = \Omega(1)$ for each independent problem, leading to the final lower bound $\Omega\left(\sqrt{T}\right)$. □

C Proof of Main Result in Section 4

C.1 Proof of Theorem 5

C.1.1 Proof of Upper Bounds in Theorem 5

Proof. In the following subsection, we provide a way to achieve $O\left(\sqrt{\log T \cdot T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}} \cdot K}\right)$ regret upper bound. [*]

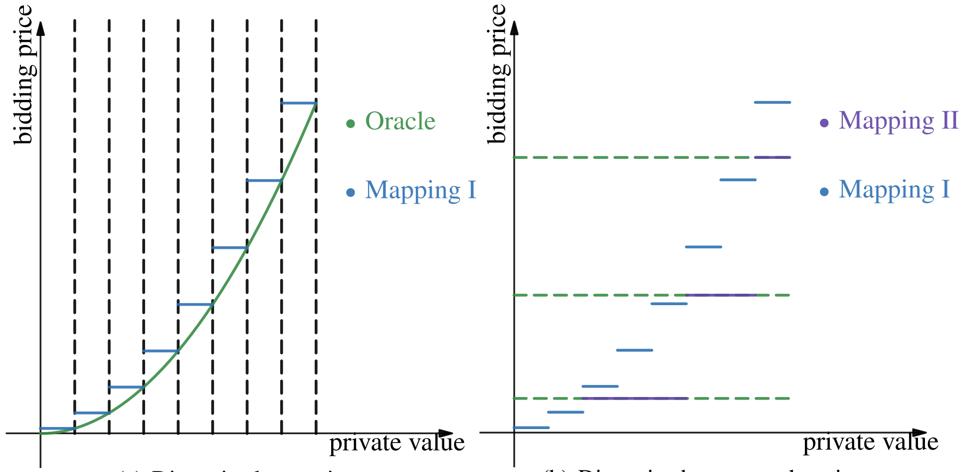
Figure 3 shows any function in oracle can be mapped to a piecewise constant function whose value only takes those in the support set, define this mapped function set to be A. We prove in the appendix that the number of functions in the converted set A is smaller than T^K, then applying the algorithm in Theorem 1's proof directly leads to an upper bound of [*]

$$O\left(\sqrt{\log\left(|\text{expert set}|\right) \cdot T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}}}\right) = O\left(\sqrt{\log T \cdot T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}} \cdot K}\right).$$

To show set A is small enough, let's first imagine walking from $(1, 0)$ to (T, K) with each step either to the positive direction of x-axis or y-axis exactly by 1. There are $T + K - 1$ steps in total and one may choose K of them to go up. Now given any function in A, suppose at $x = 1$ the value equals to the i-th support and at $x = T$ the value equals to the j-th support, which can be considered as points $(1, i)$ and (T, j), $i, j \in \mathbb{Z}$, $0 \le i \le j \le K$. Without loss of monotonicity, we add points $(0, 0)$ and $(T + 1, K)$ to the interval-support pairs of this function, i.e. the function takes value of the

[*] The other two are described in Appendix A.

[*] Although in the proof of Theorem 1 we show an upper bound of $O\left(\sqrt{\log T \cdot T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}}}\right)$, the proof schetch can indeed be applied to any finite set of experts.



(a) Discretize by x axis (b) Discretize by support location

Figure 3: Given any 1-Lipschitz and monotone oracle, we first discretize the x-axis into T small intervals, changing the oracle to a piecewise constant function that bids the maximum point for each interval in the oracle; Secondly, we map this piecewise constant function to a piecewise function that only takes support value as bidding price. Easy to verify step 1 leads to $T \cdot O(\frac{1}{T}) = O(1)$ loss, while step 2 leads to a non-negative change to the cumulative reward.

i-th support for the t-th interval, $i \in [K], t \in [T]$, iff we pass point (t, i) in the route from $(0, 0)$ to $(T + 1, K)$. The set of routes and set A forms a bijection, both have cardinality:

$$\binom{T+K-1}{K} = \frac{T+K-1}{K} \cdot \frac{T+K-2}{K-1} \cdots \frac{T}{1} \leq T^K.$$

□

C.1.2 Proof of Lower Bounds in Theorem 5

Proof. Consider the three cases separately:

- If $L < \frac{K^{\frac{q+1}{q}}}{T^{\frac{1}{q}}}$, then as in the proof of Theorem 3 we can construct $N = \left\lfloor T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rfloor$ independent problems since $N < K$ in this case. For each independent problem the lower bound is $\Omega(1)$, leading to a total lower bound of $\Omega\left(T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}}\right)$.

- If $\frac{K^{\frac{q+1}{q}}}{T^{\frac{1}{q}}} \leq L \leq \frac{T}{K^{\frac{q+1}{q}}}$, we cannot divide into $\left\lfloor T^{\frac{1}{q+1}} L^{\frac{q}{q+1}} \right\rfloor$ subproblems since there are only K values m_t can take. So instead, we divide time horizon into K subproblems: For $t = i \cdot \lfloor \frac{T}{K} \rfloor + 1, i \cdot \lfloor \frac{T}{K} \rfloor + 2, \ldots, (i+1) \cdot \lfloor \frac{T}{K} \rfloor$,

$$v_t = \frac{1}{2} + \frac{1}{2} \cdot \frac{i}{K},$$
$$h_t = \frac{1}{4} + \frac{i}{4} \cdot \sigma^{\frac{q}{q+1}},$$
$$m_t = \begin{cases} \frac{1}{4} + \frac{i}{4} \cdot \sigma^{\frac{q}{q+1}}, & w.p. \quad 1 - \frac{1}{4} \cdot \left(\sigma^{\frac{q}{q+1}} \pm \delta\right) \\ \frac{1}{4} + \frac{i+1}{4} \cdot \sigma^{\frac{q}{q+1}}, & w.p. \quad \frac{1}{4} \cdot \left(\sigma^{\frac{q}{q+1}} \pm \delta\right) \end{cases}$$

where $i = 0, 1, 2, \ldots, K-1$. Observe that the difference between v_t for adjacent subproblem is $\frac{1}{2} \cdot \frac{1}{K}$ and the difference between bid value for adjacent subproblem is at most

$$2 \cdot \frac{\sigma^{\frac{q}{q+1}}}{4} = \frac{\sigma^{\frac{q}{q+1}}}{2} = \frac{L^{\frac{q}{q+1}}}{2 \cdot T^{\frac{q}{q+1}}} \leq \frac{1}{2} \cdot \frac{1}{K},$$

ensuring the $N = K$ subproblems are indeed independent from each other. Additionally, the separation parameter δ for each subproblem equals to $\sqrt{\frac{\sigma^{\frac{q}{q+1}}}{\frac{T}{K}}} = \sqrt{\frac{K \cdot L^{\frac{q}{q+1}}}{T^{\frac{1}{q+1}}}}$, which is smaller than the separation of m_t: $\sigma^{\frac{q}{q+1}} = \left(\frac{L}{T}\right)^{\frac{q}{q+1}}$. Thus substituting Theorem 1, finally the lower bound is,

$$K \cdot \Omega\left(\left(\frac{T}{K}\right)^{\frac{1}{q+1}} \cdot \left(\frac{L}{K}\right)^{\frac{q}{q+1}}\right) = \Omega\left(\sqrt{K \cdot T^{\frac{1}{q+1}} \cdot L^{\frac{q}{q+1}}}\right).$$

- If $L > \frac{T}{K^{\frac{q+1}{q}}}$, a traditional lower bound gives $\Omega\left(\sqrt{T}\right)$.

□

C.2 Proof of Theorem 6

Proof. Let the learning rate for the upper level $\eta_{t,2} = \min\left\{\frac{1}{4}, \sqrt{\frac{\log 3}{\left\lceil \sum_{s=1}^{t-1} \sigma_s^{\frac{q}{q+1}} \right\rceil + 1}}\right\}$ and apply similar analysis as in Appendix B.1.1:

$$\begin{aligned}
\sum_{t=1}^{T} \mathbb{E}[X_t] &\geq \max_{i \in \{f,g,h\}} \sum_{t=1}^{T} r_{t,i} - 2 \cdot \left(\frac{\log 3}{\eta_{T,2}} + 2\sum_{t=1}^{T} \eta_{t,2} \cdot 2 \cdot \sigma_t^{\frac{q}{q+1}}\right) \\
&= \max_{i \in \{f,g,h\}} \sum_{t=1}^{T} r_{t,i} - 2 \cdot \left(\sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}} + 4\sqrt{\log 3} \cdot \sum_{t=1}^{T} \frac{\sigma_t^{\frac{q}{q+1}}}{\sqrt{\left\lceil \sum_{s=1}^{t} \sigma_s^{\frac{q}{q+1}} \right\rceil + 1}}\right) \\
&\overset{(a)}{\geq} \max_{i \in \{f,g,h\}} \sum_{t=1}^{T} r_{t,i} - 2 \cdot \left(\sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}} + 8\sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}}\right) \\
&= \max_{i \in \{f,g,h\}} \sum_{t=1}^{T} r_{t,i} - 18 \cdot \sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}},
\end{aligned} \tag{12}$$

where $\sum_{t=1}^{T} \mathbb{E}[X_t]$ is the expected total reward by running Algorithm 1, with expectation taken over both policy randomness and possible m_t sequences. (a) can be considered as taking integral of function $f(x) = \frac{1}{\sqrt{x}}$, but with another piecewise function smaller than it instead. And applying similar method to the lower level of the first node we have:

$$\sum_{t=1}^{T} r_{t,f} \geq \max_{a \in [T^K]} \sum_{t=1}^{T} r_{t,a} - 18 \cdot \sqrt{K \log T \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}}. \tag{13}$$

Combining (12) and (13) and the regret upper bound of ChEW algorithm and choosing hint expert:

$$\begin{aligned}
\sum_{t=1}^{T} \mathbb{E}[X_t] &\geq \max_{i \in \{f,g,h\}} \left(\max_{a \in [T^K]} \sum_{t=1}^{T} r_{t,a} - \text{Reg}(i)\right) - 18 \cdot \sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}} \\
&= \max_{a \in [T^K]} \sum_{t=1}^{T} r_{t,a} - \min\left\{18 \cdot \sqrt{K \log T \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}}, 2 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}, C \cdot \sqrt{T}\right\} - 18 \cdot \sqrt{\log 3 \cdot \sum_{t=1}^{T} \sigma_t^{\frac{q}{q+1}}},
\end{aligned}$$

where C is a constant number. Therefore, we have:

$$\begin{aligned}\mathrm{Reg}(\pi) &= O\left(\min\left\{\sqrt{\log T\cdot\sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}}\cdot K}, \sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}}, \sqrt{T}\right\}\right) + O\left(\sqrt{\log 3\cdot\sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}}}\right)\\ &\overset{\text{(b)}}{=} O\left(\min\left\{\sqrt{\log T\cdot\sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}}\cdot K}, \sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}}, \sqrt{T}\right\}\right),\end{aligned}$$

while (b) holds since $\sum_{t=1}^{T}\sigma_t^{\frac{q}{q+1}} > L > 1$. □

C.3 Proof of Theorem 7

C.3.1 Proof of Upper Bound in Theorem 7

Proof. Instead of one single hint expert in Algorithm 2, construct T hint experts, with each one bidding a constant gap over h_t, i.e. with the first hint expert bidding $h_t + \frac{1}{T}$ for $t = 1, \dots, T$; the second hint expert bidding $h_t + \frac{2}{T}$ for $t = 1, \dots, T$; etc. The upper layer then consists of T hint experts and two super nodes, representing ChEW algorithm (g) and modified Algorithm 1 (f). The lower layer of f consists of T^K base experts (constructed as in Appendix C.1) and T hint experts. Let the learning rate for the upper level $\eta_2 = \min\left\{\frac{1}{4}, \sqrt{\frac{\log(T+2)}{\sqrt{TL}}}\right\}$,

$$\begin{aligned}\sum_{t=1}^{T}\mathbb{E}[X_t] &\geq \max_{i\in\{f,g,h\}}\sum_{t=1}^{T} r_{t,i} - 2\cdot\left(\frac{\log(T+2)}{\eta_2} + 4\eta_2\sqrt{LT}\right)\\ &= \max_{i\in\{f,g,h\}}\sum_{t=1}^{T} r_{t,i} - 10\cdot\sqrt{\log(T+2)\cdot\sqrt{LT}},\end{aligned}\tag{14}$$

And applying similar method to super node f:

$$\sum_{t=1}^{T} r_{t,f} \geq \max_{a\in[T^K]}\sum_{t=1}^{T} r_{t,a} - 10\cdot\sqrt{K\log T\cdot\sqrt{TL}}.\tag{15}$$

Combining (14) and (15),

$$\begin{aligned}\sum_{t=1}^{T}\mathbb{E}[X_t] &\geq \max_{i\in\{f,g,h\}}\left(\max_{a\in[T^K]}\sum_{t=1}^{T} r_{t,a} - \mathrm{Reg}(i)\right) - 20\cdot\sqrt{\log T\cdot\sqrt{TL}}\\ &= \max_{a\in[T^K]}\sum_{t=1}^{T} r_{t,a} - \min\left\{10\cdot\sqrt{K\log T\cdot\sqrt{TL}}, 2\cdot\sqrt{TL}, C\cdot\sqrt{T}\right\} - 20\cdot\sqrt{\log T\cdot\sqrt{TL}},\end{aligned}$$

where C is a constant number. Therefore, we have:

$$\begin{aligned}\mathrm{Reg}(\pi) &= O\left(\min\left\{\sqrt{K\log T\cdot\sqrt{TL}}, \sqrt{T}\right\}\right) + O\left(\sqrt{\log T\cdot\sqrt{TL}}\right)\\ &= O\left(\min\left\{\sqrt{K\log T\cdot\sqrt{TL}}, \sqrt{T\log T}\right\}\right).\end{aligned}$$

□

C.3.2 Proof of Lower Bound in Theorem 7

The following is similar to proof of lower bound in Theorem 5.

Proof.

- If $L > \frac{T}{K^2}$, as in the proof of Theorem 4 construct $N_0 = \left\lfloor\sqrt{\frac{T}{L}}\right\rfloor < K$ independent sub-problems, while for each sub-problem

$$L' = \frac{L}{\sqrt{T/L}} = \sqrt{\frac{L^3}{T}}, \quad T' = \frac{T}{\sqrt{T/L}} = \sqrt{TL},$$

and for each sub-problem regret is lower bounded by $\Omega\left(\left(\sqrt{LT}\cdot\sqrt{\frac{L^3}{T}}\right)^{1/4}\right)$, leading to a total lower bound of $\Omega\left(\left(\sqrt{LT}\cdot\sqrt{\frac{L^3}{T}}\right)^{1/4}\cdot\sqrt{\frac{T}{L}}\right)=\Omega(\sqrt{T})$.

- If $L\le\frac{T}{K^2}$, it is not feasible to construct N_0 independent sub-problems as the optimal bidding value can not take $N_0>K$ values. Instead construct K independent problems, with the separation parameter (see Appendix B.2.2): $\delta=\sqrt{\frac{L}{T}}\cdot\frac{K}{T}<\frac{1}{T}$, leading to a total regret lower bound of $\Omega\left(\sqrt{\sqrt{\frac{T}{K}\cdot\frac{L}{K}}}\cdot K=\Omega\left(\sqrt{K\cdot\sqrt{TL}}\right)\right)$.

□

D Experimental Details

D.1 Description of Experiment 1 in Section 5

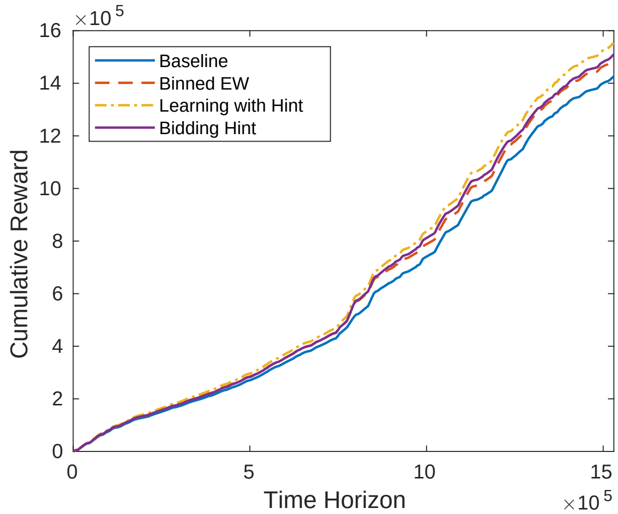
Divide the whole range of private value to D bins, each of which contains v_t's that are close to each other. As long as the bidder observes v_t at time t, we reduce the problem to the bin focusing on the data points with private values close to v_t. Then each bin itself forms a sub-problem described in Section 3. Experiment 1 only serves as an illustration of the effect by hints. The role of hints is threefolds:

- We use hint to help allocating data to different bins. Instead of binning only by private values, we use hint as a side information and conduct binning also based on it. The total number of bins is $M_1\cdot M_2$, while M_1 is the number of discretization for v_t and M_2 is the number of discretization for hints. As for the result on empirical data, we observe $M_2=4$ already leads to rather good performance.
- We use hint to calculate the estimation of instantaneous reward for any given bid b_t' under the assumption that $m_t=b_t$: $r_{t,a}':=r(b_t;h_t,v_t)$, where b_t is the bid at time t according to oracle a. Then we add this estimated reward to each experts' reward history while sampling among these experts:
$$p_{t,a}=\frac{\exp\left(\eta_t\cdot\left(\sum_{s=1}^{t-1}r_{s,a}+r_{t,a}'\right)\right)}{\sum_{a'\in\mathcal{F}}\exp\left(\eta_t\cdot\left(\sum_{s=1}^{t-1}r_{s,a'}+r_{t,a'}'\right)\right)},t=2,3,\cdots,T.$$
And if σ_t is also observed, we define $r_{t,a}':=r(b_t;h_t+c_1\cdot\sigma_t,v_t)$ instead, where c_1 is a hyper-parameter to be tuned.
- We include a set of hint experts
$$b_t(a_i):=h_t+\sigma_t^{\Delta_i},\quad i=1,2,\cdots,k,$$
which is close to a combination of algorithms for whether knowing the error, since for real datasets q is often not observed.

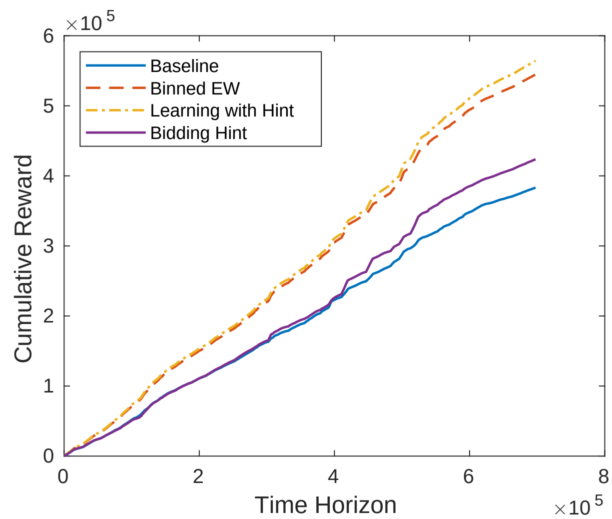
The results in Figure 4 shows the improvement by incorporating hint on other two datasets. The results implies that on datasets whose hint has rather small error, e.g. on dataset 1 bidding hint itself already beats simple online learning algorithm, the improvement by hint is more significant. Namely, 4.38% on dataset 1 with more accurate hint and 3.54% on dataset 2 whose hint is not so good.

D.2 Polynomial Algorithm in Section 5

Consider any 1-Lipschitz & monotone oracle f, since support size is finite, f can be mapped to a discontinuous function f' with $O(1)$ loss, which can be further represented by a series of interval-support pair:
$$\left(0,\frac{1}{D}\right]\leftrightarrow s_{i_1},\quad\left(\frac{1}{D},\frac{2}{D}\right]\leftrightarrow s_{i_2},\quad\dots,\quad\left(\frac{D-1}{D},1\right]\leftrightarrow s_{i_D},$$



(a) Results on Dataset 1



(b) Results on Dataset 2

Figure 4: Cumulative rewards as a function of time. The dashdot lines stands for incorporating hint into exponential weighting, and the purple solid lines are directly bidding hint. The dotted lines represent binned exponential algorithm.

Algorithm 3: DP algorithm without knowing support locations

Inputs: Time horizon T; support size K;
Initialization: $\text{Reward}_{T,K,T} \leftarrow 0$; $\text{P} \leftarrow 0$;
for $t = 1, 2, \ldots, T$ **do**
  % Calculate Sum_Forward&Sum_Backward Matrix
  $\text{Sum\_Forward}_{T,K,T} \leftarrow 1$; $\text{Sum\_Backward}_{T,K,T} \leftarrow 1$;
  for $i = 1, 2, \ldots T$ **do**
    for $j = 1, 2, \ldots, T$ **do**
      $\text{Sum\_Forward}_{i,1,j} \leftarrow \text{Sum\_Forward}_{i-1,1,j} \cdot \exp(\eta_t \cdot \text{Reward}_{i,1,j})$;
      $\text{Sum\_Backward}_{i,K,j} \leftarrow \text{Sum\_Backward}_{i+1,K,j} \cdot \exp(\eta_t \cdot \text{Reward}_{i,K,j})$;
      for $k = 2, 3, \ldots K - 1$ **do**

$$\begin{aligned}\text{Sum\_Forward}_{i,k,j} \leftarrow & \sum_{v=1}^{j-1} \Big(\text{Sum\_Forward}_{i-1,k-1,v} \cdot \exp(\eta_t \cdot \text{Reward}_{i,k,j})\Big) \\ & + \text{Sum\_Forward}_{i-1,k,j} \cdot \exp(\eta_t \cdot \text{Reward}_{i,k,j});\end{aligned}$$

$$\begin{aligned}\text{Sum\_Backward}_{i,k,j} \leftarrow & \sum_{v=j+1}^{T} \Big(\text{Sum\_Backward}_{i+1,k+1,v} \cdot \exp(\eta_t \cdot \text{Reward}_{i,k,j})\Big) \\ & + \text{Sum\_Backward}_{i+1,k,j} \cdot \exp(\eta_t \cdot \text{Reward}_{i,k,j});\end{aligned}$$

      end
    end
  end
  % Calculate Probability
  $i \leftarrow \lfloor v_t \cdot T \rfloor$;
  for $j = 1, 2, \ldots T$ **do**

$$\begin{aligned}\text{P}_j \leftarrow \sum_{k=1,2,\ldots K} & \Bigg(\Big(\text{Sum\_Forward}_{i-1,k,j} + \sum_{v=1}^{j-1} \text{Sum\_Forward}_{i-1,k-1,v}\Big) \cdot \exp(\eta_t \cdot \text{Reward}_{i,k,j}) \\ & \cdot \Big(\text{Sum\_Backward}_{i+1,k,j} + \sum_{v=j+1}^{T} \text{Sum\_Backward}_{i+1,k+1,v}\Big)\Bigg);\end{aligned}$$

  end
  for $k = 1, 2, \ldots K$ **do**
    $\text{P}_{\lfloor h_t \cdot T \rfloor} \leftarrow \text{P}_{\lfloor h_t \cdot T \rfloor} + \exp(\eta_t \cdot \text{RH})$;
  end
  Sample $b_t \sim (P / \sum(P))$;
  % Update Reward Matrix
  for $k = 1, 2, \ldots, K$ **do**
    for $j = 1, 2, \ldots T$ **do**
      if $m_t \leq j/T$ **then**
        $\text{Reward}_{i,k,j} \leftarrow \text{Reward}_{i,k,j} + (v_t - j/T)$;
    end
  end
  $\text{RH} \leftarrow \text{RH} + r(h_t; v_t, m_t)$;
end

where $0 \leq s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_K \leq 1$ are the locations of supports in increasing order and $0 \leq i_1 \leq i_2 \leq \cdots \leq i_D \leq K, i_1, i_2, \cdots, i_D \in \mathbb{Z}$. The main idea is to record the cumulative reward for all possible interval-support tuples and use dynamic programming to calculate total reward for some expert sets instead of keeping track of all T^K experts.

Reward[D][K][D]: The first two dimensions represent interval: $[d][k] : (d/D, (d+1)/D] \leftrightarrow s_k$. The third dimension represent the bidding, with steply update

$$\text{Reward}_{i,k,j} \leftarrow \text{Reward}_{i,k,j} + (v_t - j/D)$$

Then we use dynamic programming to calculate the sum of the rewards for several continuous intervals, instead of keeping track of all T^K experts.

Sum_Forward[D][K][D]: Forward DP recording array, representing combined intervals: $[d][K] : (0, d/D] \leftrightarrow \{1, \ldots, s_k\}$ and the third dimension represents bidding for the last interval: $(d/D, (d+1)/D]$. The update calculation is carried out per step before choosing an action.

Sum_Backward[D][K][D]: Backward DP recording array, representing combined intervals: $[d][K] : ((d+1)/D, 1] \leftrightarrow \{s_k + 1, \ldots, K\}$ and the third dimension represents bidding for the first interval: $((d+1)/D, (d+2)/D]$. The update calculation is carried out per step before choosing an action.

Combining the results of Sum_Forward and Sum_Backward, we can calculate reward history for a subset of the T^K experts, which is the only needed quantity for calculating probability in exponential weighting instead of keeping record with an exponential size.