A Optimization Details

A.1 Code Update Step

Algorithm 2 and 3 give details about the resolution of equation (21) by using proposition 3.1.

Algorithm 2 LRD-CODE. Solves the LRD problem by means of an alternated approach for every n-mode. $\mathcal{L}(\mathbf{X}_m^{(n)})$ is defined in eq. (21). The optimization problem is solved by means of algorithm 3.

```
1: Input: L(X_m^(n)), {X_{0,m}^(n)}_{n=1,m=1}^{N,M}
2: Output: {X_m^(n)}_{n=1,m=1}^{N,M}
3: while not converged do
4:     for n = 1 to N do
5:         for m = 1 to M do
6:             X_m^(n) = arg min_{X_m^(n)} L(X_m^(n))
7:         end for
8:     end for
9: end while
```

Algorithm 3 LRD-GD. Solves the optimization problem in algorithm (2) by means of a gradient descent approach in the DFT domain, gradient given by eq. (28).

```
1: Input: ∇L(x̂_m^(n)), {x̂_{0,m}^(n)}_{m=1}^M, T
2: Output: {x̂_{T,m}^(n)}_{m=1}^M
3: for t = 0 to T − 1 do
4:     x̂_{t+1,m}^(n) = x̂_{t,m}^(n) − η∇L(x̂_{t,m}^(n))
5: end for
```

A.2 Computational Complexity

The costly operations are the iterations of algorithm 2 which require to update $\hat{\mathbf{T}}_m^{(n)}$ that is an operation dominated by the computation of $\hat{\mathbf{F}}^{(n)}$ and $\hat{\mathbf{Q}}_m^{(n)}$. For a matrix completion case (two-dimensional) is

$$\mathcal{O}(2n^3 + Rn^2), \tag{32}$$

and for a three-dimensional tensor completion is

$$\mathcal{O}(n^5 + n^4 + (2 + R)n^3). \tag{33}$$

A.3 Existence of Solution

The existence of solution regarding the LRD decomposition is tied to the existence of the Kruskal tensor. It's existence is limited to $R < I_n$ for $n = 1, \cdots, N$.

B More about the Experiments

B.1 Synthetic data generation

As explained in section 4 synthetic data is generated using the LRD model, that is

$$\mathcal{Z} = \sum_m \mathcal{D}_m * \mathcal{K}_m = \sum_m \mathcal{D}_m * [\![\mathbf{X}_m^{(1)}, \dots, \mathbf{X}_m^{(N)}]\!], \tag{34}$$

where the dictionary $\{\mathcal{D}\}_{m=1}^M$ is the collection of filters learned and $\{\mathbf{X}_m^{(n)}\}_{m=1,n=1}^{M,N}$ the matrices that conform the kruskal activation map. For the two-dimensional case the dictionary is learned from the city and fruit testing dataset from Zeiler et al. (2010). For three-dimensional tensors we use the filters learned on the basketball sequence used by Reixach (2023). In both cases we generate random activation-map matrices drawn from the uniform distribution on the open interval $(0, 1)$ and learn the filters using the algorithm from Wohlberg (2017). Figures 3 and 4 show the used filters.

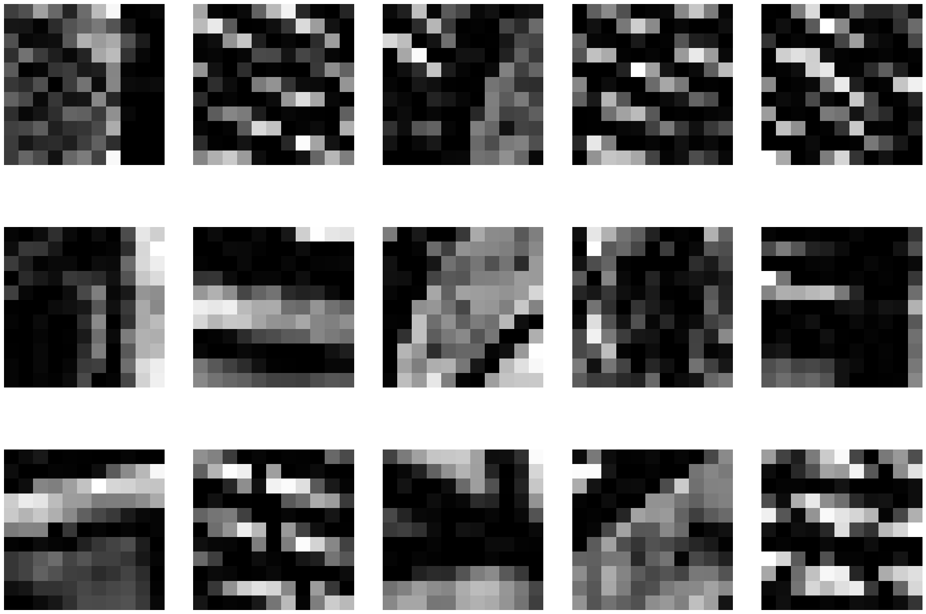

Figure 3: **Dictionary for the two-dimensional case.** From left to right and top to bottom the fifteen filters that conform the dictionary.

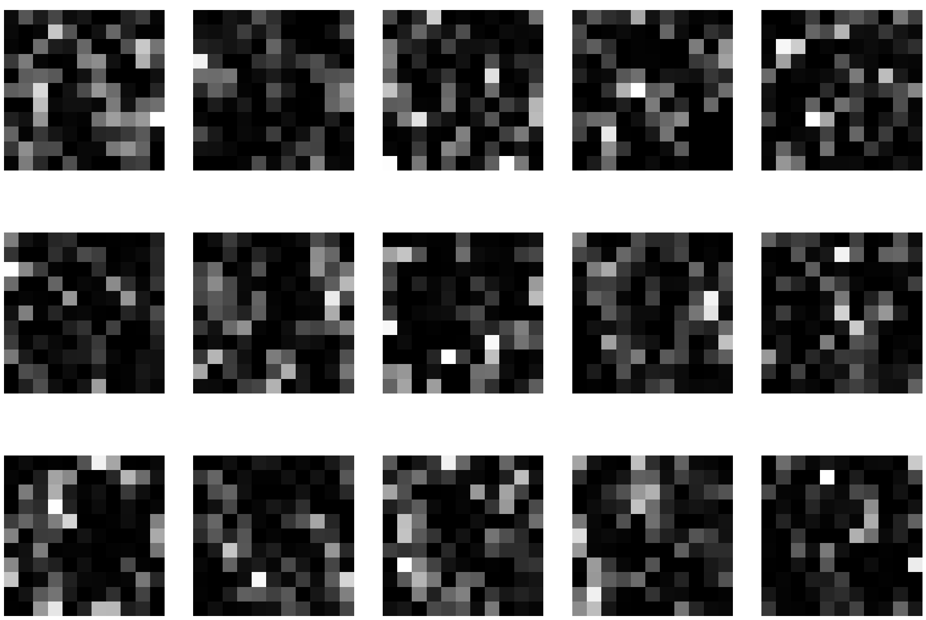

Figure 4: **Dictionary for the three-dimensional case.** From left to right and top to bottom the fifteen slides from each filter that conform the dictionary. As the filters are three-dimensional tensors here we picture the fifth slide in the third dimension of the filter.

B.2 MORE RESULTS

Figures 5, 6 and 7 report synthetic results for AWGN with $\sigma = 0$, $\sigma = 0.1$ and $\sigma = 0.5$ respectively. We observe the behaviour described in section 4. Tables 4, 5, 6, 7, 8 and 9 report results on real data for matrix and tensor completion. Regarding tensor completion, to the datasets considered in section 4 we also include SW-NIR kinetic data (Bijlsma & Smilde, 2000)($301 \times 241 \times 8$) that we cut to ($100 \times 100 \times 8$) due to computational limitations of our method. We observe the behaviour described in section 4.

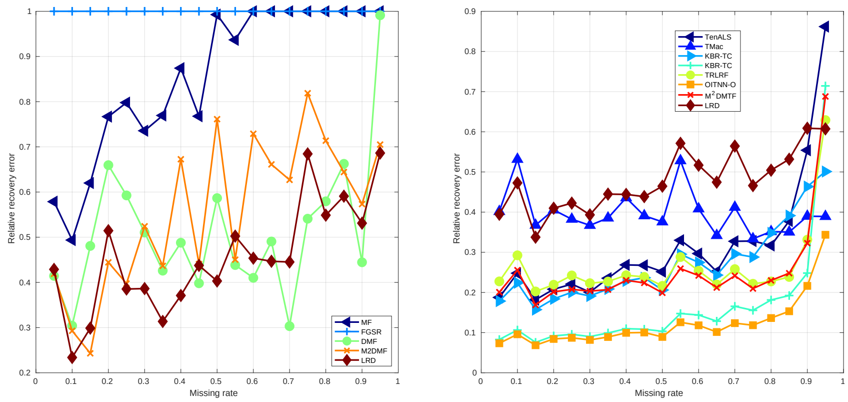

Figure 5: Performance evaluation of matrix and tensor completion on synthetic data ($\sigma = 0.0$).

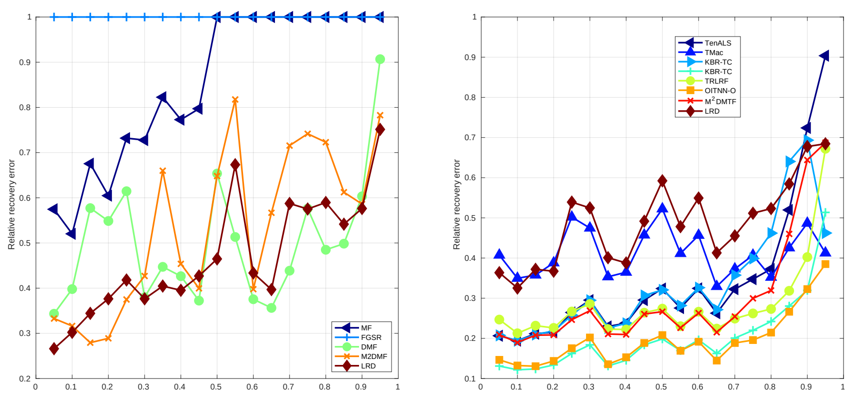

Figure 6: Performance evaluation of matrix and tensor completion on synthetic data ($\sigma = 0.1$).

Table 4: Relative recovery error of matrix completion on MovieLens-100k ($\sigma = 0.0$).

Missing ratio	MF	FGSR	DMF	M^{2}DMTF	LRD
10%	0.2513	**0.2413**	0.2548	**0.2454**	-
30%	0.2570	**0.2480**	0.2605	**0.2529**	0.2596
50%	0.2615	**0.2512**	0.2621	**0.2531**	0.2608
70%	0.2793	**0.2648**	0.2712	**0.2546**	0.2689
90%	0.3913	0.3374	**0.3157**	**0.2744**	0.8056

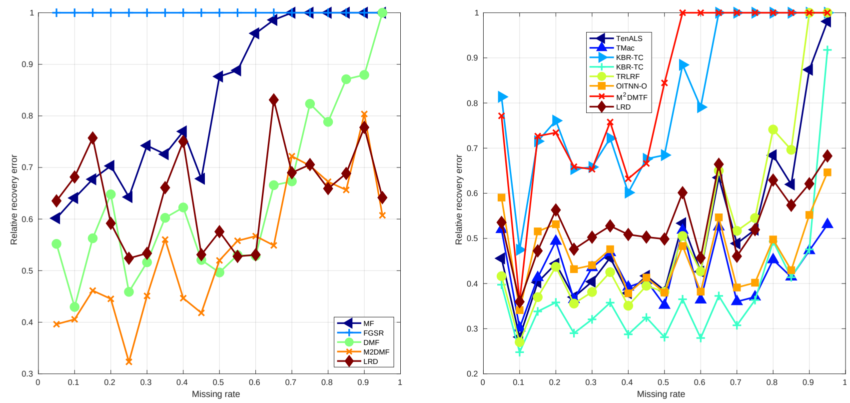

Figure 7: Performance evaluation of matrix and tensor completion on synthetic data ($\sigma = 0.5$).

Table 5: Relative recovery error of matrix completion on MovieLens-100k ($\sigma = 0.1$).

Missing ratio	MF	FGSR	DMF	M^2DMTF	LRD
10%	0.2472	**0.2376**	0.2479	**0.2450**	**0.2450**
30%	0.2531	**0.2436**	0.2567	**0.2472**	0.2547
50%	0.2606	**0.2495**	0.2616	**0.2543**	-
70%	0.2773	0.2621	0.2695	**0.2586**	**0.2615**
90%	0.3892	0.3356	**0.3170**	**0.2716**	0.5842

Table 6: Relative recovery error of matrix completion on MovieLens-100k ($\sigma = 0.5$).

Missing ratio	MF	FGSR	DMF	M^2DMTF	LRD
10%	**0.2562**	**0.2484**	0.2744	0.2600	0.2569
30%	**0.2585**	**0.2521**	0.2804	0.2699	-
50%	**0.2605**	**0.2511**	0.2781	0.2682	0.2770
70%	0.2806	**0.2654**	0.2854	0.2786	**0.2747**
90%	0.3999	0.3449	**0.3221**	**0.2803**	0.6660

B.3 Parameter Setting

B.3.1 Parameter Setting on Synthetic Data

- in MF, the factorization dimension d is 5. The α is set to 1 and the maximum iteration is 2000.
- in FGSR, we use the variational form of Schatten-$1/2$ norm, $d = 30$, $\lambda = 0.015$. The maximum iteration is 2000.
- in DMF, the network structure is $[10, 50, 100, I_n]$. The weight decay parameters are set to 0.1 and the maximum iteration is 1000.
- in M^2DMTF (three-dimensional case), $L = 2$, $d_i = 10\forall i$, $h_i = 50\forall i$, $m_i = I_i\forall i$ and $\lambda_i = 1\forall i$. The optimizer is iRprop+ and the maximum iteration is 2000.
- in LRD (two-dimensional case), $\alpha = 1 \cdot 10^{-16}$, γ is choosen from $\{1 \cdot 10^{-4}, 1 \cdot 10^{-2}\}$, $\lambda = 0$, $r = 3$, $\eta = 1/50$ and $T = 500$. Maximum iteration is 100.
- In FaLRTC we set $\alpha_i = 1\forall i$ and the maximum iteration is 200.

Table 7: Relative recovery error of tensor completion on three real tensors ($\sigma = 0.0$).

data	MR	FalRTC	TenALS	TMac	KBR-TC	TRLRF	OITNN-O	M^2DMTF	LRD
Amino	10%	0.0480	0.0197	**0.0112**	0.0186	0.0262	**0.0115**	0.0129	0.0441
	30%	0.0472	0.0199	**0.0116**	0.0203	0.0454	**0.0129**	0.0137	0.0229
	50%	0.0672	0.0203	**0.0134**	0.0197	0.0201	0.1360	**0.0149**	0.0380
	70%	0.1124	0.0206	**0.0169**	0.0251	0.0216	0.0188	**0.0157**	0.0381
	90%	0.01124	0.0221	**0.0163**	0.0251	0.0287	0.0188	**0.0161**	0.0504
Flow	10%	0.0788	0.3939	0.0047	**0.0014**	0.0450	**0.0021**	0.0135	0.0127
	30%	0.0998	0.3953	0.0057	**0.0014**	0.0460	**0.0017**	0.0150	0.0153
	50%	0.1353	0.3958	0.0076	**0.0016**	0.0372	**0.0021**	0.0118	0.0136
	70%	0.2121	0.3966	0.0103	**0.0019**	0.0234	**0.0048**	0.0142	0.0165
	90%	0.5275	0.4005	**0.0132**	**0.0039**	0.0375	0.0480	0.0260	0.0721
SW-NIR	10%	0.1285	0.1222	**0.0000**	**0.0012**	0.0055	0.0101	0.0057	0.0106
	30%	0.1617	0.1131	**0.0002**	> 0.95	0.0028	**0.0010**	0.0070	0.2030
	50%	0.2913	0.1507	0.1404	0.1403	0.0436	**0.0036**	**0.0059**	0.5486
	70%	0.6736	> 0.95	0.2362	0.2363	0.0877	**0.0062**	**0.0111**	> 0.95
	90%	0.9413	> 0.95	0.7050	> 0.95	0.5046	**0.4905**	**0.0459**	> 0.95

Table 8: Relative recovery error of tensor completion on three real tensors ($\sigma = 0.1$).

data	MR	FalRTC	TenALS	TMac	KBR-TC	TRLRF	OITNN-O	M^2DMTF	LRD
Amino	10%	0.1928	**0.0854**	0.2165	0.2634	**0.1091**	0.1554	0.2267	0.2782
	30%	0.1898	**0.0961**	0.2513	0.2517	**0.1555**	0.1701	0.2574	0.2504
	50%	0.1922	**0.1121**	0.3231	0.3080	0.1942	0.1858	0.3890	**0.1816**
	70%	0.2285	**0.1699**	0.4667	0.2268	**0.2040**	0.2284	0.5782	0.3354
	90%	0.2285	**0.1587**	0.4589	**0.2268**	0.2853	0.2284	0.5906	0.4117
Flow	10%	0.1329	0.3977	0.2143	0.1115	**0.1014**	0.1928	**0.0904**	0.1455
	30%	0.1488	0.3992	0.2290	**0.1124**	**0.1128**	0.1951	0.1116	0.1575
	50%	0.1697	0.3966	0.2571	0.1523	**0.1288**	0.1903	**0.1319**	0.1761
	70%	0.2237	0.3982	0.3367	0.2445	0.2099	**0.2001**	**0.1804**	0.2184
	90%	0.5243	0.4068	0.6801	**0.2524**	0.3942	**0.2876**	0.4299	0.3288
SW-NIR	10%	0.3469	**0.1266**	0.5085	**0.1688**	0.4394	0.4923	0.5717	0.4710
	30%	0.4176	**0.1380**	0.5612	**0.3251**	0.6218	0.4850	0.4725	0.5481
	50%	0.5336	**0.1682**	0.6547	**0.2140**	0.7099	0.4844	0.8324	0.4139
	70%	0.7896	> 0.95	0.9342	**0.2903**	> 0.95	**0.5755**	> 0.95	0.7776
	90%	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	**0.8742**	> 0.95	> 0.95

Table 9: Relative recovery error of tensor completion on three real tensors ($\sigma = 0.5$).

data	MR	FalRTC	TenALS	TMac	KBR-TC	TRLRF	OITNN-O	M^2DMTF	LRD
Amino	10%	> 0.95	**0.4625**	> 0.95	> 0.95	> 0.95	**0.5992**	> 0.95	0.6037
	30%	> 0.95	**0.5579**	> 0.95	> 0.95	> 0.95	0.6472	> 0.95	**0.4788**
	50%	> 0.95	**0.6784**	> 0.95	> 0.95	> 0.95	0.6982	> 0.95	**0.5671**
	70%	> 0.95	0.9343	> 0.95	> 0.95	> 0.95	**0.7254**	> 0.95	**0.6357**
	90%	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	**0.7254**	> 0.95	**0.7310**
Flow	10%	0.5109	**0.4069**	> 0.95	0.5551	0.7446	0.8524	0.7679	**0.3731**
	30%	0.5365	**0.4162**	> 0.95	0.5350	0.8835	0.8364	> 0.95	**0.3930**
	50%	0.5766	**0.4232**	> 0.95	0.5712	> 0.95	0.7671	> 0.95	**0.4138**
	70%	0.6765	**0.4456**	> 0.95	> 0.95	> 0.95	0.7086	> 0.95	**0.4535**
	90%	0.9100	**0.5281**	> 0.95	> 0.95	> 0.95	0.7164	> 0.95	**0.5263**
SW-NIR	10%	> 0.95	**0.6854**	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95
	30%	> 0.95	**0.6938**	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95
	50%	> 0.95	**0.8747**	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95
	70%	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95
	90%	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95	> 0.95

- in TenALS the initial rank is set to 1 and the maximum iteration is 20.
- in TMac, the rank is initialized to $[10, 10, 10]$ and adjusted adaptively. The maximum iteration is 1000.
- in KBR-TC, $\rho = 1.05$ and $\lambda = 0.01$. The maximum iteration is 300.

- in TRLRF the rank is set to $[3, 3, 3]$ and $\lambda = 10$. The maximum iteration is 500.
- in OITNN-O $\alpha = 1 \cdot 10^{-4}$. The maximum iteration is 300.
- in M^{2}DMTF (three-dimensional case), $L = 2$, $d_i = 10 \forall i$, $h_i = 20 \forall i$, $m_i = I_i \forall i$ and $\lambda_i = 1 \forall i$. The optimizer is iRprop+ and the maximum iteration is 3000.
- in LRD (three-dimensional case), $\alpha = 1 \cdot 10^{-16}$, $\gamma = 2 \cdot 10^{-5}$, $\lambda = 0$, $r = 3$, $\eta = 1/50$ and $T = 500$. Maximum iteration is 100.

B.3.2 Parameter Setting on Real Data

- in MF, the factorization dimension d is 5. The α is set to 1 and the maximum iteration is 2000.
- in FGSR, we use the variational form of Schatten-$1/2$ norm, $d = 50$, $\lambda = 0.015$. The maximum iteration is 2000.
- in DMF, the network structure is $[10, 50, 100, I_n]$. The weight decay parameters are set to 0.1 and the maximum iteration is 1000.
- in M^{2}DMTF (three-dimensional case), $L = 2$, $d_i = 10 \forall i$, $h_i = 50 \forall i$, $m_i = I_i \forall i$ and $\lambda_i = 1 \forall i$. The optimizer is iRprop+ and the maximum iteration is 2000.
- in LRD (two-dimensional case), $\alpha = 1 \cdot 10^{-10}$, $\gamma = 1 \cdot 10^{-8}$, $\lambda = 2$, $r = 5$, $\eta = 1/50$ and $T = 500$. Maximum iteration is 100.
- In FaLRTC we set $\alpha_i = 1 \forall i$ and the maximum iteration is 200.
- in TenALS the initial rank is set to 1 and the maximum iteration is 20.
- in TMac, the rank is initialized to $[10, 10, 10]$ and adjusted adaptively. The maximum iteration is 1000.
- in KBR-TC, $\rho = 1.05$ and $\lambda = 0.01$. The maximum iteration is 300.
- in TRLRF the rank is set to $[5, 5, 5]$ and $\lambda = 10$. The maximum iteration is 500.
- in OITNN-O $\alpha = 1 \cdot 10^{-4}$. The maximum iteration is 300.
- in M^{2}DMTF (three-dimensional case), $L = 2$, $d_i = 10 \forall i$, $h_i = 20 \forall i$, $m_i = I_i \forall i$ and $\lambda_i = 1 \forall i$. The optimizer is iRprop+ and the maximum iteration is 3000.
- in LRD (three-dimensional case), $\alpha = 1 \cdot 10^{-10}$, γ is chosen from $\{1 \cdot 10^{-8}, 1 \cdot 10^{-6}, 1 \cdot 10^{-5}, 5 \cdot 10^{-5}\}$, $\lambda = 0$, $r = 4$, $\eta = 1/50$ and $T = 500$. Maximum iteration is 100.

C Proofs

C.1 Proposition 3.1

The squared total variation regularization is given by

$$\frac{\gamma}{2} \|\mathcal{U}\|_{TV}^2 = \frac{\gamma}{2} \|\nabla \mathcal{U}\|_2^2 , \tag{35}$$

where,

$$\frac{\gamma}{2} \|\nabla \mathcal{U}\|_2^2 = \frac{\gamma}{2} \left\| \left[\left(\frac{\partial \mathbf{u}^{(n)}}{\partial \mathbf{t}_0} \right)^T, \left(\frac{\partial \mathbf{u}^{(n)}}{\partial \mathbf{t}_1} \right)^T, \dots, \left(\frac{\partial \mathbf{u}^{(n)}}{\partial \mathbf{t}_N} \right)^T \right]^T \right\|_2^2 . \tag{36}$$

By using the derivative property of the DFT transform,

$$\mathcal{F}\left\{ \frac{\partial \mathbf{u}^{(n)}}{\partial \mathbf{t}_i} \right\} = 2\pi j \xi_i \oplus \mathcal{F}\{\mathbf{u}^{(n)}\} = 2\pi j \xi_i \oplus \hat{\mathbf{W}}^{(n)} \hat{\mathbf{x}}^{(n)}, \tag{37}$$

with ξ_i and $\oplus$ defined in section 3.2. Together with the definition of $\hat{\Theta}^{(n)}$ leads us to the following expression,

$$\mathcal{F}\left\{\frac{\gamma}{2}\left\|\sum_{m=1}^{M}\mathfrak{D}_m * [\![\mathbf{X}_m^{(1)},\ldots,\mathbf{X}_m^{(N)}]\!]\right\|_{TV}^2\right\}= \frac{\gamma}{2}\left\|(\hat{\Theta}^{(n)})^T\hat{\mathbf{x}}^{(n)}\right\|_2^2, \tag{38}$$

which can be derived in the following manner (complex derivative),

$$\frac{\partial\frac{\gamma}{2}\left\|(\hat{\Theta}^{(n)})^T\hat{\mathbf{x}}^{(n)}\right\|_2^2}{\partial(\hat{\mathbf{x}}^{(n)})^H} = \frac{\gamma}{2}(\hat{\Theta}^{(n)})^H\hat{\Theta}^{(n)}. \tag{39}$$

The result above combined with the solution of eq. (20) and the derivative of the nuclear norm operator brings us to eq. (28).

C.2 Theorem 2.2

This proof and the ones that follow are obtained following the method from Fan (2022), we also introduce the following lemma from the same work (Lemma 1):

Lemma C.1. *Let S be a set defined over tensors of size $I_1 \times I_2 \times \cdots \times I_n$. Let $|S|$ be the e-covering number of S w.r.t. the Frobenius norm. Let $I_\pi = \prod_{i=1}^n I_i$. Suppose $\mathfrak{Z} \in S$ and $\max(\|\mathfrak{Y}\|_\infty, \|\mathfrak{Z}\|_\infty) \leq \xi$. Then with probability at least $1 - 2I_\pi^{-1}$:*

$$\sup_{\mathfrak{Z}\in S}\left|\frac{1}{\sqrt{I_\pi}}\left\|\mathfrak{Y}-\mathfrak{Z}\right\|_F - \frac{1}{\sqrt{|\Omega|}}\left\|P_\Omega(\mathfrak{Y}-\mathfrak{Z})\right\|_F\right| \leq \frac{2\epsilon}{\sqrt{|\Omega|}} + \left(\frac{8\xi^4\log(|S|I_\pi)}{|\Omega|}\right)^{1/4} \tag{40}$$

Proof. See Fan (2022) appendix D.1. □

After, we introduce a new lemma that shows an upper bound for the covering number of the Low-rank Deconvolution matrix set:

Lemma C.2. *Let $S = \{\mathbf{Z} \in \mathbb{R}^{I_2\times I_1} \colon \mathbf{Z} = \sum_{m=1}^M \mathbf{F}_2\,{}^{(2)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1, \left\|\hat{\mathbf{X}}_m^{(n)}\right\|_F \leq \beta_m^n, \left\|{}^{(2)}\hat{\mathbf{D}}_m\right\|_F \leq \beta_m^0, m=1,\ldots,M, n=1,2\}$ where $\hat{\mathbf{X}}_m^{(n)} \in \mathbb{C}^{I_n\times r}$, $\mathbf{F}_1 \in \mathbb{C}^{I_1\times I_1}$ and $\mathbf{F}_2 \in \mathbb{C}^{I_2\times I_2}$ are the inverse DFT matrices and ${}^{(2)}\hat{\mathbf{D}}_m \in \mathbb{C}^{I_2\times I_1}$. Then the covering number of S w.r.t. the Frobenius norm satisfy:*

$$\mathcal{N}(S, \|\cdot\|_F, \epsilon) \leq \left(\frac{3^4M^3\sum_{m=1}^M(\beta_m^0\beta_m^1\beta_m^2)^3}{\epsilon}\right)^{I_1r+2I_\pi} \tag{41}$$

Proof. See appendix C.6 □

With Lemma C.1 and Lemma C.2 we obtain

$$\begin{aligned}
\frac{1}{\sqrt{I_\pi}}\|\mathbf{Z}-\hat{\mathbf{Z}}\|_F &= \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\mathbf{E}-\hat{\mathbf{Z}}\|_F \\
&\leq \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\hat{\mathbf{Z}}\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F \\
&\leq \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon}{\sqrt{|\Omega|}} + \left(\frac{8\xi^4\log(|S|I_\pi)}{|\Omega|}\right)^{1/4} \\
&\leq \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon}{\sqrt{|\Omega|}} + \\
&\quad \left(\frac{8\xi^4(\log(I_\pi) + (I_1r+2I_\pi)\log(\frac{3^4M^3\sum_{m=1}^M(\beta_m^0\beta_m^1\beta_m^2)^3}{\epsilon}))}{|\Omega|}\right)^{1/4}
\end{aligned} \tag{42}$$

C.3 THEOREM 2.4

This is a special case of proof C.2 where ${}^{(2)}\hat{\mathbf{D}} = \mathbf{1}$ and DFT matrices are not longer needed as we consider spatial versions $\mathbf{X}^{(n)}$. With that we introduce a new lemma that shows an upper bound for the covering number of the classical Low-rank matrix set:

Lemma C.3. *Let* $S = \{\mathbf{Z} \in \mathbb{R}^{I_2 \times I_1} \colon \mathbf{Z} = \mathbf{X}^{(2)}(\mathbf{X}^{(1)})^T, \left\|\mathbf{X}^{(n)}\right\|_F \leq \beta^n, n = 1, 2\}$ *where* $\mathbf{X}^{(n)} \in \mathbb{R}^{I_n \times r}$. *Then the covering number of* S *w.r.t. the Frobenius norm satisfy:*

$$\mathcal{N}(S, \|\cdot\|_F, \epsilon') \leq \left(\frac{12(\beta^1\beta^2)^2}{\epsilon'}\right)^{I_1 r + I_\pi} \tag{43}$$

Proof. See appendix C.7 □

With Lemma C.1 and Lemma C.3 we obtain

$$\begin{aligned}
\frac{1}{\sqrt{I_\pi}}\|\mathbf{Z} - \hat{\mathbf{Z}}\|_F &= \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y} - \mathbf{E} - \hat{\mathbf{Z}}\|_F \\
&\leq \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y} - \hat{\mathbf{Z}}\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F \\
&\leq \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y} - \hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon'}{\sqrt{|\Omega|}} + \left(\frac{8\xi^4 \log(|S| I_\pi)}{|\Omega|}\right)^{1/4} \\
&\leq \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y} - \hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon'}{\sqrt{|\Omega|}} + \\
&\left(\frac{8\xi^4(\log(I_\pi) + (I_1 r + I_\pi)\log(\frac{12(\beta^1\beta^2)^2}{\epsilon'})}{|\Omega|}\right)^{1/4}
\end{aligned} \tag{44}$$

C.4 THEOREM 2.6

This is the three-dimensional variation of proof C.2. To that end we introduce a new lemma that shows an upper bound for the covering number of the three-dimensional Low-rank Deconvolution matrix set:

Lemma C.4. *Let* $S = \{\mathbf{Z} \in \mathbb{R}^{I_3 \times I_1 I_2} \colon \mathbf{Z} = \sum_{m=1}^{M} \mathbf{F}_3 \, {}^{(3)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)} \odot \hat{\mathbf{X}}_m^{(2)})^T \mathbf{F}_{12}, \left\|\hat{\mathbf{X}}_m^{(n)}\right\|_F \leq \beta_m^n, \left\|{}^{(3)}\hat{\mathbf{D}}_m\right\|_F \leq \beta_m^0, m = 1, \ldots, M, n = 1, 2, 3\}$ *where* $\hat{\mathbf{X}}_m^{(n)} \in \mathbb{C}^{I_n \times r}$, $\mathbf{F}_{12} \in \mathbb{C}^{I_1 I_2 \times I_1 I_2}$ *and* $\mathbf{F}_3 \in \mathbb{C}^{I_3 \times I_3}$ *are the inverse DFT matrices and* ${}^{(3)}\hat{\mathbf{D}}_m \in \mathbb{C}^{I_3 \times I_1 I_2}$. *Then the covering number of* S *w.r.t. the Frobenius norm satisfy:*

$$\mathcal{N}(S, \|\cdot\|_F, \epsilon) \leq \left(\frac{3 \cdot 4^4 M^4 \sum_{m=1}^{M}(\beta_m^0\beta_m^1\beta_m^2\beta_m^3 k_r)^4}{\epsilon}\right)^{(1+I_2)I_1 r + 2I_\pi} \tag{45}$$

Proof. See appendix C.8 □

With Lemma C.1 and Lemma C.4 we obtain

$$\begin{aligned}
\frac{1}{\sqrt{I_\pi}}\|\mathbf{Z}-\hat{\mathbf{Z}}\|_F &= \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\mathbf{E}-\hat{\mathbf{Z}}\|_F \\
&\le \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\hat{\mathbf{Z}}\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F \\
&\le \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon}{\sqrt{|\Omega|}} + \left(\frac{8\xi^4\log(|S|I_\pi)}{|\Omega|}\right)^{1/4} \\
&\le \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon}{\sqrt{|\Omega|}} + \\
&\left(\frac{8\xi^4(\log(I_\pi)+((1+I_2)I_1r+2I_\pi)\log(\frac{3\cdot 4^4 M^4\sum_{m=1}^{M}(\beta_m^0\beta_m^1\beta_m^2\beta_m^3k_r)^4}{\epsilon}))}{|\Omega|}\right)^{1/4}
\end{aligned} \tag{46}$$

C.5 THEOREM 2.8

This is a special case of proof C.4 where ${}^{(2)}\hat{\mathbf{D}} = \mathbf{1}$ and DFT matrices are not longer needed as we consider spatial versions $\mathbf{X}^{(n)}$. With that we introduce a new lemma that shows an upper bound for the covering number of the classical three-dimensional Low-rank matrix set:

Lemma C.5. *Let* $S = \{\mathbf{Z} \in \mathbb{R}^{I_3\times I_1I_2} \colon \mathbf{Z} = \mathbf{X}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T, \left\|\mathbf{X}^{(n)}\right\|_F \le \beta^n, n = 1,2,3\}$ *where* $\mathbf{X}^{(n)} \in \mathbb{R}^{I_n\times r}$. *Then the covering number of* S *w.r.t. the Frobenius norm satisfy:*

$$\mathcal{N}(S, \|\cdot\|_F, \epsilon') \le \left(\frac{3^4(\beta^1\beta^2\beta^3k_r)^3}{\epsilon'}\right)^{(1+I_2)I_1r+I_\pi} \tag{47}$$

Proof. See appendix C.9 ☐

With Lemma C.1 and Lemma C.5 we obtain

$$\begin{aligned}
\frac{1}{\sqrt{I_\pi}}\|\mathbf{Z}-\hat{\mathbf{Z}}\|_F &= \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\mathbf{E}-\hat{\mathbf{Z}}\|_F \\
&\le \frac{1}{\sqrt{I_\pi}}\|\mathbf{Y}-\hat{\mathbf{Z}}\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F \\
&\le \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon'}{\sqrt{|\Omega|}} + \left(\frac{8\xi^4\log(|S|I_\pi)}{|\Omega|}\right)^{1/4} \\
&\le \frac{1}{\sqrt{|\Omega|}}\|P_\Omega(\mathbf{Y}-\hat{\mathbf{Z}})\|_F + \frac{1}{\sqrt{I_\pi}}\|\mathbf{E}\|_F + \frac{2\epsilon'}{\sqrt{|\Omega|}} + \\
&\left(\frac{8\xi^4(\log(I_\pi)+((1+I_2)I_1r+I_\pi)\log(\frac{3^4(\beta^1\beta^2\beta^3k_r)^3}{\epsilon'})}{|\Omega|}\right)^{1/4}
\end{aligned} \tag{48}$$

C.6 LEMMA C.2

Let $\mathbf{Z} = \sum_{m=1}^{M}\mathbf{F}_2\,{}^{(2)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1$ where $\hat{\mathbf{X}}_m^{(n)} \in \mathbb{C}^{I_n\times r}$, $\mathbf{F}_1 \in \mathbb{C}^{I_1\times I_1}$ and $\mathbf{F}_2 \in \mathbb{C}^{I_2\times I_2}$ are the inverse DFT matrices and $\hat{\mathbf{D}}_m^{(2)} \in \mathbb{C}^{I_2\times I_1}$. We give the following two lemmas:

Lemma C.6. *Let* $S_{ab} \coloneqq \{\hat{\mathbf{A}} \in \mathbb{C}^{a\times b} \colon \hat{\mathbf{A}} = \mathbf{F}_n\mathbf{A},\ \ \mathbf{A} \in \mathbb{R}^{a\times b}, \|\mathbf{A}\|_F \le \beta, \mathbf{F}_2 \in \mathbb{C}^{a\times a}$ *the inverse DFT matrix*, $\|\mathbf{F}_n\|_F = 1\}$. *Then there exist an* ϵ-*net* $\tilde{S}_{ab}$ *obeying*

$$\mathcal{N}(S_{ab}, \|\cdot\|_F, \epsilon) \le \left(\frac{3\beta}{\epsilon}\right)^{ab} \tag{49}$$

such that $\left\|\hat{\mathbf{A}} - \tilde{\mathbf{A}}\right\|_F \le \epsilon$.

Proof. See appendix C.10 □

Lemma C.7. *Let* $\hat{\mathbf{A}} \in \mathbb{C}^{a\times b}$ *and* $\hat{\mathbf{B}} \in \mathbb{C}^{a\times b}$ *then*

$$\left\|\hat{\mathbf{A}} \oplus \hat{\mathbf{B}}\right\|_F \leq tr(\hat{\mathbf{A}}\hat{\mathbf{B}}^T) \leq \left\|\hat{\mathbf{A}}\right\|_F \left\|\hat{\mathbf{B}}\right\|_F \tag{50}$$

Where $\oplus$ *is defined above and depicts hadamard product.*

Now replace ϵ with ϵ/ζ_m^n and let $\left\|\hat{\mathbf{X}}_m^{(n)} - \tilde{\mathbf{X}}_m^{(n)}\right\|_F \leq \frac{\epsilon}{\zeta_m^n}, m = 1, \ldots, M, \quad n = 1, 2, \quad \|^{(2)}\hat{\mathbf{D}}_m - {}^{(2)}\tilde{\mathbf{D}}_m\|_F \leq \frac{\epsilon}{\zeta_m^0}$. Let $\zeta_m^0 = 3M\beta_m^1\beta_m^2$, $\zeta_m^1 = 3M\beta_m^0\beta_m^2$ and $\zeta_m^2 = 3M\beta_m^0\beta_m^1$.

$$\begin{aligned}
\left\|\mathbf{Z} - \bar{\mathbf{Z}}\right\|_F &= \left\|\sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 - \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\tilde{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right\|_F \\
&= \left\|\sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 \pm \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right. \\
&\left.\pm \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 - \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\tilde{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right\|_F \\
&\leq \left\|\sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 - \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right\|_F \\
&+ \left\|\sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 - \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right\|_F \\
&+ \left\|\sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\hat{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1 - \sum_{m=1}^{M} \mathbf{F}_2\, {}^{(2)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(2)}(\tilde{\mathbf{X}}_m^{(1)})^T\mathbf{F}_1\right\|_F \\
&\leq \sum_{m=1}^{M} \left\|\mathbf{F}_2\right\|_F \left\|{}^{(2)}\hat{\mathbf{D}}_m - {}^{(2)}\tilde{\mathbf{D}}_m\right\|_F \left\|\hat{\mathbf{X}}_m^{(2)}\right\|_F \left\|\hat{\mathbf{X}}_m^{(1)}\right\|_F \left\|\mathbf{F}_1\right\|_F \\
&+ \sum_{m=1}^{M} \left\|\mathbf{F}_2\right\|_F \left\|{}^{(2)}\tilde{\mathbf{D}}_m\right\|_F \left\|\hat{\mathbf{X}}_m^{(2)} - \tilde{\mathbf{X}}_m^{(2)}\right\|_F \left\|\hat{\mathbf{X}}_m^{(1)}\right\|_F \left\|\mathbf{F}_1\right\|_F \\
&+ \sum_{m=1}^{M} \left\|\mathbf{F}_2\right\|_F \left\|{}^{(2)}\tilde{\mathbf{D}}_m\right\|_F \left\|\tilde{\mathbf{X}}_m^{(2)}\right\|_F \left\|\hat{\mathbf{X}}_m^{(1)} - \tilde{\mathbf{X}}_m^{(1)}\right\|_F \left\|\mathbf{F}_1\right\|_F \\
&\leq \sum_{m=1}^{M} \frac{\epsilon}{\zeta_m^0}\beta_m^1\beta_m^2 + \sum_{m=1}^{M} \frac{\epsilon}{\zeta_m^2}\beta_m^0\beta_m^1 + \sum_{m=1}^{M} \frac{\epsilon}{\zeta_m^1}\beta_m^0\beta_m^2 \\
&= \epsilon.
\end{aligned} \tag{51}$$

The second inequality utilized the submultiplicativity of the Frobenius norm and Lemma C.7. Therefore, $\bar{S}$ is an ϵ-cover of S. Then, we have

$$\begin{aligned}
\mathcal{N}(S, \|\cdot\|_F, \epsilon) &\leq \sum_{m=1}^{M} \prod_{n=0}^{2} \left(\frac{3\beta_m^n\zeta_m^n}{\epsilon}\right)^{I_1 r + 2I_\pi} \\
&= \sum_{m=1}^{M} \left(\frac{3^4M^3(\beta_m^0\beta_m^1\beta_m^2)^3}{\epsilon}\right)^{I_1 r + 2I_\pi} \\
&= \left(\frac{3^4M^3\sum_{m=1}^{M}(\beta_m^0\beta_m^1\beta_m^2)^3}{\epsilon}\right)^{I_1 r + 2I_\pi}
\end{aligned} \tag{52}$$

C.7 LEMMA C.3

Let $\mathbf{Z} = \mathbf{X}^{(2)}(\mathbf{X}^{(1)})^T$ where $\mathbf{X}^{(n)} \in \mathbb{C}^{I_n \times r}$. Using Lemma C.6, replace ϵ with $\epsilon'/\bar{\zeta}^n$ and let $\left\|\mathbf{X}^{(n)} - \tilde{\mathbf{X}}^{(n)}\right\|_F \leq \frac{\epsilon}{\bar{\zeta}^n}, n = 1, 2$. Let $\bar{\zeta}^1 = 2\beta^2$ and $\bar{\zeta}^2 = 2\beta^1$.

$$\begin{aligned}
\left\|\mathbf{Z} - \bar{\mathbf{Z}}\right\|_F &= \left\|\mathbf{X}^{(2)}(\mathbf{X}^{(1)})^T - \bar{\mathbf{X}}^{(2)}(\bar{\mathbf{X}}^{(1)})^T\right\|_F \\
&= \left\|\mathbf{X}^{(2)}(\mathbf{X}^{(1)})^T \pm \bar{\mathbf{X}}^{(2)}(\mathbf{X}^{(1)})^T - \bar{\mathbf{X}}^{(2)}(\bar{\mathbf{X}}^{(1)})^T\right\|_F \\
&\leq \left\|\mathbf{X}^{(2)}(\mathbf{X}^{(1)})^T - \bar{\mathbf{X}}^{(2)}(\mathbf{X}^{(1)})^T\right\|_F + \left\|\bar{\mathbf{X}}^{(2)}(\mathbf{X}^{(1)})^T - \bar{\mathbf{X}}^{(2)}(\bar{\mathbf{X}}^{(1)})^T\right\|_F \\
&\leq \left\|\mathbf{X}^{(2)}_m - \bar{\mathbf{X}}^{(2)}_m\right\|_F \left\|\mathbf{X}^{(1)}_m\right\|_F + \left\|\bar{\mathbf{X}}^{(2)}_m\right\|_F \left\|\mathbf{X}^{(1)}_m - \bar{\mathbf{X}}^{(1)}_m\right\|_F \\
&\leq \frac{\epsilon'}{\bar{\zeta}^2}\beta^1 + \frac{\epsilon'}{\bar{\zeta}^1}\beta^2 = \epsilon'.
\end{aligned} \tag{53}$$

The second inequality utilized the submultiplicativity of the Frobenius norm. Therefore, $\bar{S}$ is an ϵ'-cover of S. Then, we have

$$\begin{aligned}
\mathcal{N}(S, \|\cdot\|_F, \epsilon') &\leq \prod_{n=0}^{1} \left(\frac{3\beta^n\bar{\zeta}^n}{\epsilon'}\right)^{I_1 r + I_\pi} \\
&= \left(\frac{3 \cdot 2^2(\beta^1\beta^2)^2}{\epsilon'}\right)^{I_1 r + I_\pi} = \left(\frac{12(\beta^1\beta^2)^2}{\epsilon'}\right)^{I_1 r + I_\pi}.
\end{aligned} \tag{54}$$

C.8 LEMMA C.4

Let $\mathbf{Z} = \sum_{m=1}^{M} \mathbf{F}_3{}^{(3)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}^{(3)}_m(\hat{\mathbf{X}}^{(1)}_m \odot \hat{\mathbf{X}}^{(2)}_m)^T\mathbf{F}_{12}$ where $\hat{\mathbf{X}}^{(n)}_m \in \mathbb{C}^{I_n \times r}$, $\mathbf{F}_{12} \in \mathbb{C}^{I_1 I_2 \times I_1 I_2}$ and $\mathbf{F}_3 \in \mathbb{C}^{I_3 \times I_3}$ are the inverse DFT matrices and $\hat{\mathbf{D}}^{(3)}_m \in \mathbb{C}^{I_3 \times I_1 I_2}$.We give the following lemma:

Lemma C.8. *Let $\hat{\mathbf{A}} \in \mathbb{C}^{a \times b}$ and $\hat{\mathbf{B}} \in \mathbb{C}^{a \times b}$ then*

$$\left\|\hat{\mathbf{A}} \odot \hat{\mathbf{B}}\right\|_F = \left\|\hat{\mathbf{A}}\right\|_F \left\|\hat{\mathbf{B}}\right\|_F \left\|\hat{\mathbf{I}}_r\right\|_F \tag{55}$$

Where $\odot$ is defined above and depicts khatri-rao product.

Now replace ϵ with ϵ/ζ^n_m and let $\left\|\hat{\mathbf{X}}^{(n)}_m - \tilde{\mathbf{X}}^{(n)}_m\right\|_F \leq \frac{\epsilon}{\zeta^n_m}, m = 1, \ldots, M, \quad n = 1, 2, \quad \|{}^{(2)}\hat{\mathbf{D}}_m - {}^{(2)}\tilde{\mathbf{D}}_m\|_F \leq \frac{\epsilon}{\zeta^0_m}$. Let $\zeta^0_m = 4M\beta^1_m\beta^2_m\beta^3_m k_r$, $\zeta^1_m = 4M\beta^0_m\beta^2_m\beta^3_m k_r$, $\zeta^2_m = 4M\beta^0_m\beta^1_m\beta^3_m k_r$ and

$\zeta_m^3 = 4M\beta_m^0\beta_m^1\beta_m^2 k_r$.

$$
\begin{aligned}
\left\|\mathbf{Z}-\bar{\mathbf{Z}}\right\|_F &= \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} - \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\tilde{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&= \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} \pm \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right. \\
&\pm \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} \pm \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} \\
&\left.- \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\tilde{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&\le \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\hat{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} - \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&+ \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \hat{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} - \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&+ \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\hat{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} - \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&+ \left\|\sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\hat{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12} - \sum_{m=1}^{M}\mathbf{F}_3\,{}^{(3)}\tilde{\mathbf{D}}_m \oplus \tilde{\mathbf{X}}_m^{(3)}(\tilde{\mathbf{X}}_m^{(1)}\odot\tilde{\mathbf{X}}_m^{(2)})^T\mathbf{F}_{12}\right\|_F \\
&\le \sum_{m=1}^{M}\left\|\mathbf{F}_3\right\|_F\left\|{}^{(3)}\hat{\mathbf{D}}_m - {}^{(3)}\tilde{\mathbf{D}}_m\right\|_F\left\|\hat{\mathbf{X}}_m^{(3)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(1)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(2)}\right\|_F\left\|\mathbf{I}_r\right\|_F\left\|\mathbf{F}_{12}\right\|_F \\
&+ \sum_{m=1}^{M}\left\|\mathbf{F}_3\right\|_F\left\|{}^{(3)}\tilde{\mathbf{D}}_m\right\|_F\left\|\hat{\mathbf{X}}_m^{(3)} - \tilde{\mathbf{X}}_m^{(3)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(1)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(2)}\right\|_F\left\|\mathbf{I}_r\right\|_F\left\|\mathbf{F}_{12}\right\|_F \\
&+ \sum_{m=1}^{M}\left\|\mathbf{F}_3\right\|_F\left\|{}^{(3)}\tilde{\mathbf{D}}_m\right\|_F\left\|\tilde{\mathbf{X}}_m^{(3)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(1)} - \tilde{\mathbf{X}}_m^{(1)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(2)}\right\|_F\left\|\mathbf{I}_r\right\|_F\left\|\mathbf{F}_{12}\right\|_F \\
&+ \sum_{m=1}^{M}\left\|\mathbf{F}_3\right\|_F\left\|{}^{(3)}\tilde{\mathbf{D}}_m\right\|_F\left\|\tilde{\mathbf{X}}_m^{(3)}\right\|_F\left\|\tilde{\mathbf{X}}_m^{(1)}\right\|_F\left\|\hat{\mathbf{X}}_m^{(2)} - \tilde{\mathbf{X}}_m^{(2)}\right\|_F\left\|\mathbf{I}_r\right\|_F\left\|\mathbf{F}_{12}\right\|_F \\
&\le \sum_{m=1}^{M}\frac{\epsilon}{\zeta_m^0}\beta_m^1\beta_m^2\beta_m^3 k_r + \sum_{m=1}^{M}\frac{\epsilon}{\zeta_m^3}\beta_m^0\beta_m^1\beta_m^2 k_r + \sum_{m=1}^{M}\frac{\epsilon}{\zeta_m^1}\beta_m^0\beta_m^2\beta_m^3 k_r + \sum_{m=1}^{M}\frac{\epsilon}{\zeta_m^3}\beta_m^0\beta_m^1\beta_m^3 k_r \\
&= \epsilon.
\end{aligned}
\tag{56}
$$

The second inequality utilized the submultiplicativity of the Frobenius norm and Lemma C.8. Therefore, $\bar{S}$ is an ϵ-cover of S. Then, we have

$$
\begin{aligned}
\mathcal{N}(S, \|\cdot\|_F, \epsilon) &\le \sum_{m=1}^{M}\prod_{n=0}^{3}\left(\frac{3\beta_m^n\zeta_m^n}{\epsilon}\right)^{(1+I_2)I_1 r + 2I_\pi} \\
&= \sum_{m=1}^{M}\left(\frac{3\cdot 4^4 M^4(\beta_m^0\beta_m^1\beta_m^2\beta_m^3 k_r)^4}{\epsilon}\right)^{(1+I_2)I_1 r + 2I_\pi} \\
&= \left(\frac{3\cdot 4^4 M^4\sum_{m=1}^{M}(\beta_m^0\beta_m^1\beta_m^2\beta_m^3 k_r)^4}{\epsilon}\right)^{(1+I_2)I_1 r + 2I_\pi}
\end{aligned}
\tag{57}
$$

C.9 LEMMA C.5

Let $\mathbf{Z} = \mathbf{X}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T$ where $\mathbf{X}^{(n)} \in \mathbb{C}^{I_n \times r}$. Using Lemma C.6, replace ϵ with $\epsilon'/\bar{\zeta}^n$ and let $\left\|\mathbf{X}^{(n)} - \tilde{\mathbf{X}}^{(n)}\right\|_F \leq \frac{\epsilon}{\bar{\zeta}^n}, n = 1, 2, 3$. Let $\bar{\zeta}^1 = 3\beta^2\beta^3$, $\bar{\zeta}^2 = 3\beta^1\beta^3$ and $\bar{\zeta}^3 = 3\beta^1\beta^2$.

$$
\begin{aligned}
\left\|\mathbf{Z} - \bar{\mathbf{Z}}\right\|_F &= \left\|\mathbf{X}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T - \bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \bar{\mathbf{X}}^{(2)})^T\right\|_F \\
&= \left\|\mathbf{X}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T \pm \bar{\mathbf{X}}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T \pm \bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \mathbf{X}^{(2)})^T - \bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \bar{\mathbf{X}}^{(2)})^T\right\|_F \\
&\leq \left\|\mathbf{X}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T - \bar{\mathbf{X}}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T\right\|_F + \left\|\bar{\mathbf{X}}^{(3)}(\mathbf{X}^{(1)} \odot \mathbf{X}^{(2)})^T - \bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \mathbf{X}^{(2)})^T\right\|_F \\
&+ \left\|\bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \mathbf{X}^{(2)})^T - \bar{\mathbf{X}}^{(3)}(\bar{\mathbf{X}}^{(1)} \odot \bar{\mathbf{X}}^{(2)})^T\right\|_F \\
&\leq \left\|\mathbf{X}_m^{(3)} - \bar{\mathbf{X}}_m^{(3)}\right\|_F \left\|\mathbf{X}_m^{(1)}\right\|_F \left\|\mathbf{X}_m^{(2)}\right\|_F + \left\|\bar{\mathbf{X}}_m^{(3)}\right\|_F \left\|\mathbf{X}_m^{(1)} - \bar{\mathbf{X}}_m^{(1)}\right\|_F \left\|\mathbf{X}_m^{(2)}\right\|_F \\
&+ \left\|\bar{\mathbf{X}}_m^{(3)}\right\|_F \left\|\bar{\mathbf{X}}_m^{(1)}\right\|_F \left\|\mathbf{X}_m^{(2)} - \bar{\mathbf{X}}_m^{(2)}\right\|_F \\
&\leq \frac{\epsilon'}{\bar{\zeta}^3}\beta^1\beta^2 + \frac{\epsilon'}{\bar{\zeta}^1}\beta^3\beta^2 + \frac{\epsilon'}{\bar{\zeta}^2}\beta^3\beta^1 = \epsilon'.
\end{aligned} \tag{58}
$$

The second inequality utilized the submultiplicativity of the Frobenius norm and Lemma C.8. Therefore, $\bar{S}$ is an ϵ'-cover of S. Then, we have

$$
\begin{aligned}
\mathcal{N}(S, \|\cdot\|_F, \epsilon') &\leq \prod_{n=0}^{2}\left(\frac{3\beta^n\bar{\zeta}^n}{\epsilon'}\right)^{(1+I_2)I_1r+I_\pi} \\
&= \left(\frac{3 \cdot 3^3(\beta^1\beta^2\beta^3)^3}{\epsilon'}\right)^{(1+I_2)I_1r+I_\pi} = \left(\frac{3^4(\beta^1\beta^2\beta^3)^3}{\epsilon'}\right)^{(1+I_2)I_1r+I_\pi}.
\end{aligned} \tag{59}
$$

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$$
\left\|\hat{\mathbf{A}} - \tilde{\mathbf{A}}\right\|_F = \left\|\mathbf{F}_n\mathbf{A} - \mathbf{F}_n\bar{\mathbf{A}}\right\|_F \leq \left\|\mathbf{A} - \bar{\mathbf{A}}\right\|_F. \tag{60}
$$

The inequality utilized the submultiplicativity of the Frobenius norm.