The First Book of Euclid's Elements—Commentary and Lean Formalization

Logan Murphy Kaiyu Yang

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Chapter 1

Introduction

Chapter 2

Definitions

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with the straight-lines on itself.
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right-angle.
12. And an acute angle (is) one less than a right-angle.
13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.

18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.
21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.
22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.
23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

Chapter 3

Postulates

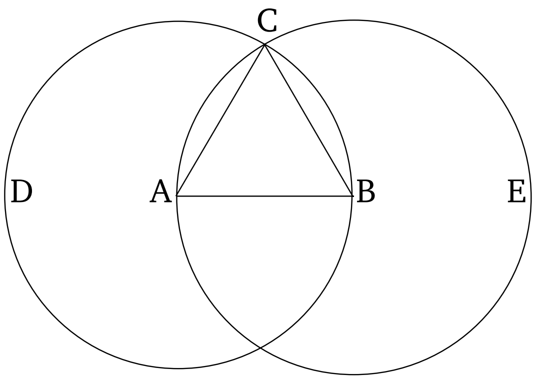
1. Let it have been postulated to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

Chapter 4

Common Notions

1. Things equal to the same thing are also equal to one another.
2. And if equal things are added to equal things then the wholes are equal.
3. And if equal things are subtracted from equal things then the remainders are equal.
4. And things coinciding with one another are equal to one another.
5. And the whole [is] greater than the part.

Proposition 1



To construct an equilateral triangle on a given finite straight-line.

Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another, to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 5]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 5]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Commentary

Proposition 4.1. *Given two disctinct points A and B on a line AB, there must be a point C, s.t. $\triangle$ ABC is an equilateral triangle.*

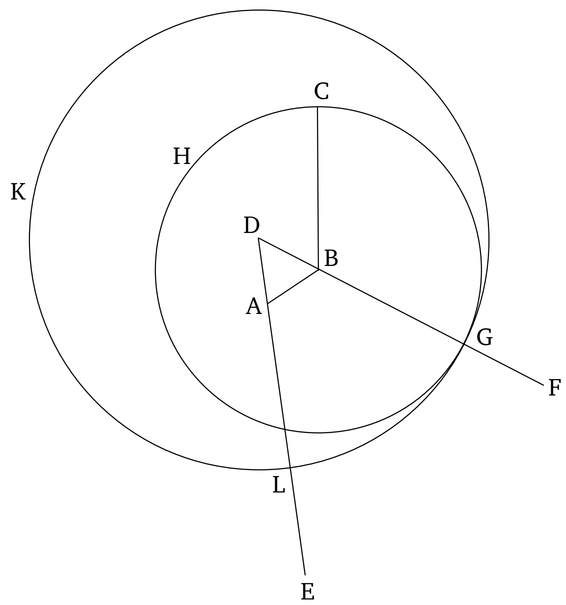
Proof. See the original proof by Euclid. □

Euclid omitted the fact that point C can be constructed on either side of AB, which is required for proving latter propositions. We state and prove this fact as Prop. 4.2.

Proposition 4.2. *A and B are two disctinct points on a line AB. X is a point not on AB. Then there must be a point C on the opposite side of AB from X, s.t. $\triangle\ ABC$ is an equilateral triangle.*

Proof. Similar to Euclid's proof but note that the point C can be constructed on either side. $\square$

Proposition 2



To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let A be the given point, and BC the given straight-line. So it is required to place a straight-line at point A equal to the given straight-line BC.

For let the straight-line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been been constructed upon it [Prop. 1.1]. And let the straight-lines AE and BF have been produced in a straight-line with DA and DB (respectively) [Post. 2]. And let the circle CGH with center B and radius BC have been drawn [Post. 3], and again let the circle GKL with center D and radius DG have been drawn [Post. 3].

Therefore, since the point B is the center of (the circle) CGH, BC is equal to BG [Def. 5]. Again, since the point D is the center of the circle GKL, DL is equal to DG [Def. 5]. And within these, DA is equal to DB. Thus, the remainder AL is equal to the remainder BG [C.N. 3]. But BC was also shown (to be) equal to BG. Thus, AL and BC are each equal to BG. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, AL is also equal to BC.

Thus, the straight-line AL, equal to the given straight-line BC, has been placed at the given point A. (Which is) the very thing it was required to do.

Commentary

Proposition 4.3. *B and C are two distinct points on a line BC. A is a point different from B. There must be a point L, s.t. $|AL| = |BC|$.*

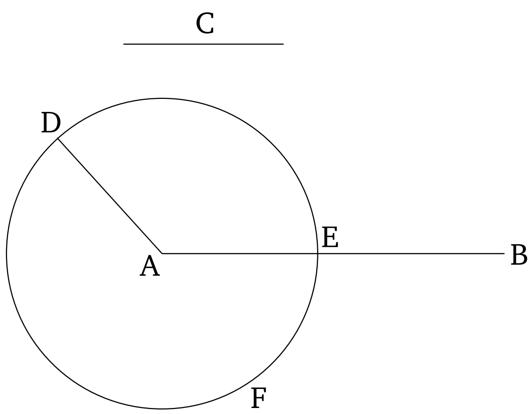
Proof. See the original proof by Euclid. $\square$

Euclid omitted the degenerated case where A is the same as B.

Proposition 4.4. *B and C are two distinct points on a line BC. For any point A, there must be a point L, s.t. $|AL| = |BC|$.*

Proof. When $A = B$, we can just take $L = C$. When $A \neq B$, we apply Prop. 4.3. $\square$

Proposition 3



For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let AB and C be the two given unequal straight-lines, of which let the greater be AB. So it is required to cut off a straight-line equal to the lesser C from the greater AB.

Let the line AD, equal to the straight-line C, have been placed at point A [Prop. 1.2]. And let the circle DEF have been drawn with center A and radius AD [Post. 3].

And since point A is the center of circle DEF, AE is equal to AD [Def. 5]. But, C is also equal to AD. Thus, AE and C are each equal to AD. So AE is also equal to C [C.N. 1].

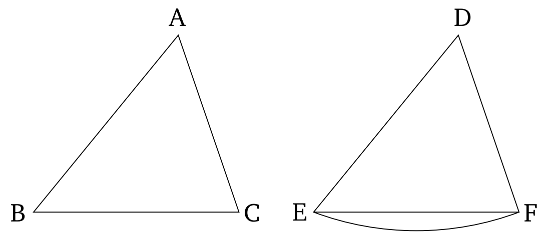
Thus, for two given unequal straight-lines, AB and C, the (straight-line) AE, equal to the lesser C, has been cut off from the greater AB. (Which is) the very thing it was required to do.

Commentary

Proposition 4.5. *A and B are two distinct points on a line AB. C_0 and C_1 are two distinct points on a line C. $A \neq C_0$, and $|AB| > |C_0C_1|$. There must be a point E between A and B, s.t., $|AE| = |C_0C_1|$*

Proof. See the original proof by Euclid. $\square$

Proposition 4



If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. And (let) the angle BAC (be) equal to the angle EDF. I say that the base BC is also equal to the base EF, and triangle ABC will be equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ABC to DEF, and ACB to DFE.

For if triangle ABC is applied to triangle DEF, the point A being placed on the point D, and the straight-line AB on DE, then the point B will also coincide with E, on account of AB being equal to DE. So (because of) AB coinciding with DE, the straight-line AC will also coincide with DF, on account of the angle BAC being equal to EDF. So the point C will also coincide with the point F, again on account of AC being equal to DF. But, point B certainly also coincided with point E, so that the base BC will coincide with the base EF. For if B coincides with E, and C with F, and the base BC does not coincide with EF, then two straight-lines will encompass an area. The very thing is impossible [Post. 1]. Thus, the base BC will coincide with EF, and will be equal to it [C.N. 4]. So the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) ABC to DEF, and ACB to DFE [C.N. 4].

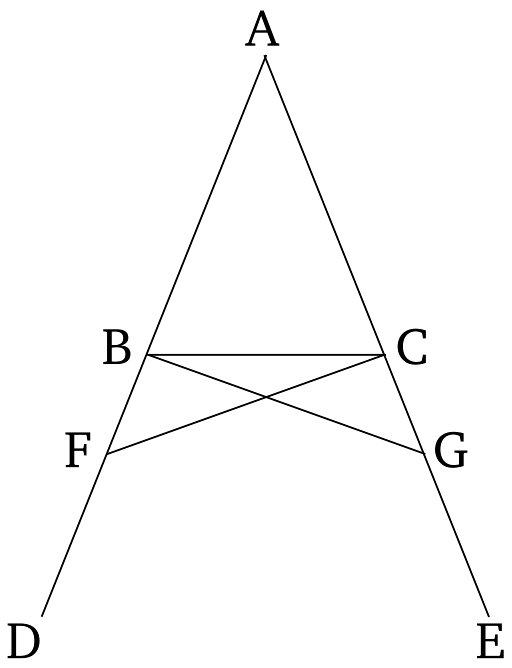
Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.6. *$\triangle$ ABC and $\triangle$ DEF are two triangles with $|AB| = |DE|$, $|AC| = |DF|$, and $\angle$ $BAC = \angle$ EDF. Then, $\triangle$ ABC and $\triangle$ DEF must be congruent. That is, $|BC| = |EF|$, $\angle$ $ABC = \angle$ DEF, and $\angle$ $ACB = \angle$ DFE.*

Proof. See the original proof by Euclid. $\square$

Proposition 5



For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.

Let ABC be an isosceles triangle having the side AB equal to the side AC, and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively) [Post. 2]. I say that the angle ABC is equal to ACB, and (angle) CBD to BCE.

For let the point F have been taken at random on BD, and let AG have been cut off from the greater AE, equal to the lesser AF [Prop. 1.3]. Also, let the straight-lines FC and GB have been joined [Post. 1].

In fact, since AF is equal to AG, and AB to AC, the two (straight-lines) FA, AC are equal to the two (straight-lines) GA, AB, respectively. They also encompass a common angle, FAG. Thus, the base FC is equal to the base GB, and the triangle AFC will be equal to the triangle AGB, and the remaining angles subtendend by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG, and AFC to AGB. And since the whole of AF is equal to the whole of AG, within which AB is equal to AC, the remainder BF is thus equal to the remainder CG [C.N. 3]. But FC was also shown (to be) equal to GB. So the two (straight-lines) BF, FC are equal to the two (straight-lines) CG, GB, respectively, and the angle BFC (is) equal to the angle CGB, and the base BC is common to them. Thus, the triangle BFC will be equal to the triangle CGB, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, FBC is equal to GCB, and BCF to CBG. Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF, within which CBG is equal to BCF, the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC. And FBC was also shown (to be) equal to GCB. And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.7. *$|AB| = |AC|$ in $\triangle\ ABC$. AB is extended to D, and AC is extended to E. Then, $\angle\ ABC = \angle\ ACB$ and $\angle\ CBD = \angle\ BCE$.*

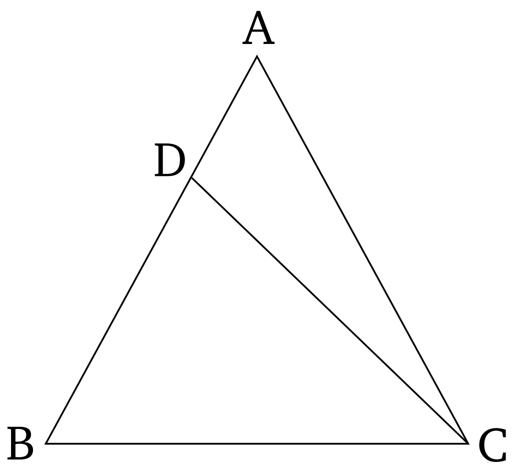
Proof. See the original proof by Euclid. □

Euclid often use the following restricted version of Prop. 4.7 in later proofs.

Proposition 4.8. *For $\triangle\ ABC$, if $|AB| = |AC|$, then $\angle\ ABC = \angle\ ACB$.*

Proof. Extend AB to D and AC to E. Then, $\angle\ ABC = \angle\ ACB$ by Prop. 4.7. □

Proposition 6



If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB. I say that side AB is also equal to side AC.

For if AB is unequal to AC then one of them is greater. Let AB be greater. And let DB, equal to the lesser AC, have been cut off from the greater AB [Prop. 1.3]. And let DC have been joined [Post. 1].

Therefore, since DB is equal to AC, and BC (is) common, the two sides DB, BC are equal to the two sides AC, CB, respectively, and the angle DBC is equal to the angle ACB. Thus, the base DC is equal to the base AB, and the triangle DBC will be equal to the triangle ACB [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, AB is not unequal to AC. Thus, (it is) equal.

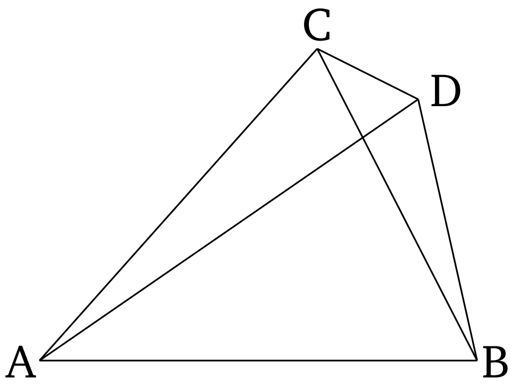
Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.9. *In $\triangle ABC$, if $\angle\, ABC = \angle\, ACB$, then $|AB| = |AC|$.*

Proof. See the original proof by Euclid. $\square$

Proposition 7



On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.

For, if possible, let the two straight-lines AC, CB, equal to two other straight-lines AD, DB, respectively, have been constructed on the same straight-line AB, meeting at different points, C and D, on the same side (of AB), and having the same ends (on AB). So CA is equal to DA, having the same end A as it, and CB is equal to DB, having the same end B as it. And let CD have been joined [Post. 1].

Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC [Prop. 1.5]. Thus, ADC (is) greater than DCB [C.N. 5]. Thus, CDB is much greater than DCB [C.N. 5]. Again, since CB is equal to DB, the angle CDB is also equal to angle DCB [Prop. 1.5]. But it was shown that the former (angle) is also much greater (than the latter). The very thing is impossible.

Thus, on the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

Commentary

Proposition 4.10. *A and B are two distinct points on a line AB. It is impossible to construct two distinct points C and D on the same side of AB, s.t., $|AC| = |AD|$ and $|CB| = |DB|$.*

Proof. Euclid's proof only works with the last two conditions, though he probably intended to prove a stronger version of Prop. 4.10 without these conditions, i.e., Prop. 4.10 below. To get there, we need to enumerate four possible cases (or two modulo permutations), whereas Euclid only considered the first one.

A, B are on the same of CD; B, D are on the same side of AC: See Euclid's original proof.

A, B are on different sides of CD; B, D are on the same side of AC: As Fig. 4 shows, extend AC to E and AD to F. Apply Prop. 4.7 to $\triangle ACD$ to derive $\angle\, DCE = \angle\, CDF$. Apply Prop. 4.8 to $\triangle\, BCD$ to derive $\angle\, BCD = \angle\, BDC$. Note that $\angle\, DCE\ >\ \angle\, BCD = \angle\, BDC\ >\ \angle\, CDF$. Contradiction.

A, B are on the same side of CD; B, D are on different sides of AC: Same as Euclid's proof, with C and D swapped.

A, B are on different sides of CD; B, D are on different sides of AC: Same as Case 2, with C and D swapped.

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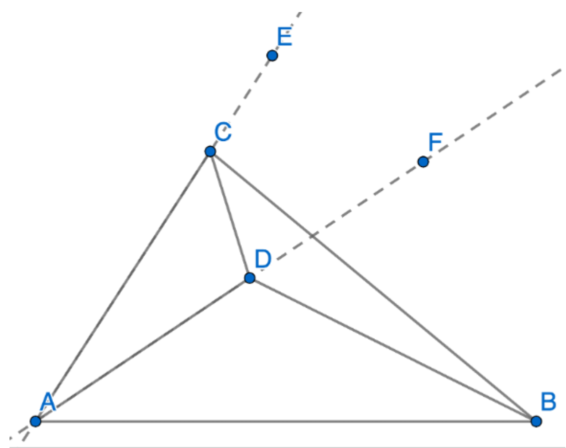


Figure 4.1: A and B are on different sides of CD. B and D are on the same side of AC.

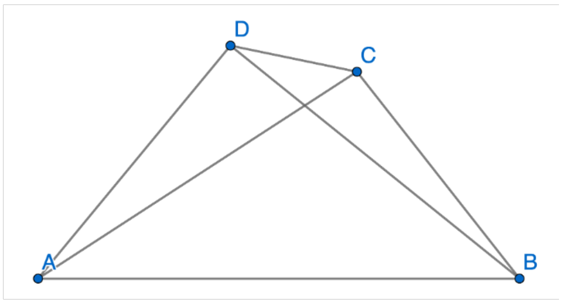


Figure 4.2: A and B are on the same side of CD. B and D are on different sides of AC.

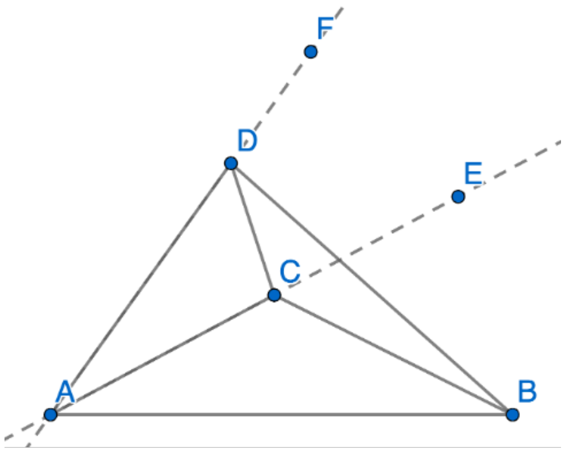
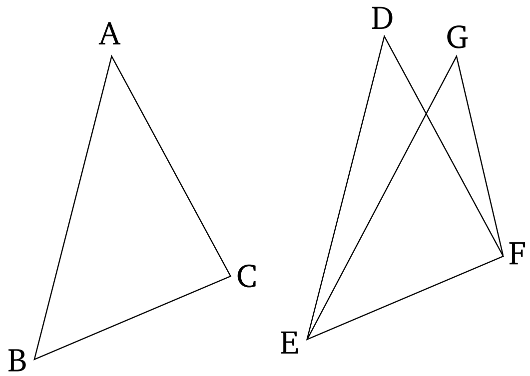


Figure 4.3: A and B are on different sides of CD. B and D are on different sides of AC.

Proposition 8



If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines.

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is) AB to DE, and AC to DF. Let them also have the base BC equal to the base EF. I say that the angle BAC is also equal to the angle EDF.

For if triangle ABC is applied to triangle DEF, the point B being placed on point E, and the straight-line BC on EF, then point C will also coincide with F, on account of BC being equal to EF. So (because of) BC coinciding with EF, (the sides) BA and CA will also coincide with ED and DF (respectively). For if base BC coincides with base EF, but the sides AB and AC do not coincide with ED and DF (respectively), but miss like EG and GF (in the above figure), then we will have constructed upon the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base BC being applied to the base EF, the sides BA and AC cannot not coincide with ED and DF (respectively). Thus, they will coincide. So the angle BAC will also coincide with angle EDF, and will be equal to it [C.N. 4].

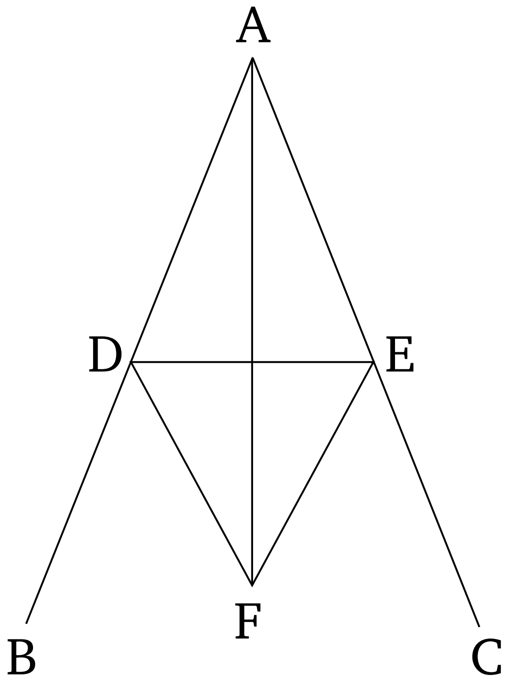
Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

Commentary

Proposition 4.11. *$\triangle ABC$ and $\triangle DEF$ are two triangles with $AB = DE$, $AC = DF$, and $BC = EF$. Then, $\angle BAC = \angle EDF$.*

Proof. See the original proof by Euclid. $\square$

Proposition 9



To cut a given rectilinear angle in half.

Let BAC be the given rectilinear angle. So it is required to cut it in half.

Let the point D have been taken at random on AB, and let AE, equal to AD, have been cut off from AC [Prop. 1.3], and let DE have been joined. And let the equilateral triangle DEF have been constructed upon DE [Prop. 1.1], and let AF have been joined. I say that the angle BAC has been cut in half by the straight-line AF.

For since AD is equal to AE, and AF is common, the two (straight-lines) DA, AF are equal to the two (straight-lines) EA, AF, respectively. And the base DF is equal to the base EF. Thus, angle DAF is equal to angle EAF [Prop. 1.8].

Thus, the given rectilinear angle BAC has been cut in half by the straight-line AF. (Which is) the very thing it was required to do.

Commentary

Proposition 4.12. *Given an* $\angle BAC$*, there must exist a point* F*, s.t.,* $F \neq A$ *and* $\angle BAF = \angle CAF$*.*

Proof. Euclid's proof has two problems. First, when constructing F, it fails to state the requirement that F and A must be on different sides of DE. Second, Euclid did not rule out the possibility that F lies on AB or AC. If that could happen, $\triangle DAF$ or $\triangle EAF$ wouldn't have existed, and the original proof would have failed.

To prove F is not on AB, let's first assume F is on AB (Fig. 4). Apply Prop. 4.7 to $\triangle ADE$ to derive $\angle FDE = \angle CED$. Apply Prop. 4.8 to $\triangle FDE$ to derive $\angle FDE = \angle FED$. Note that $\angle CED > \angle FED$. Contradiction.

Similarly, we can prove F is not on AC. □

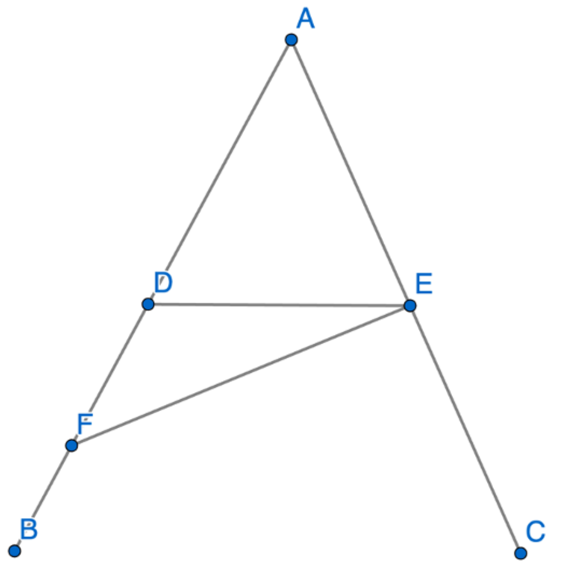


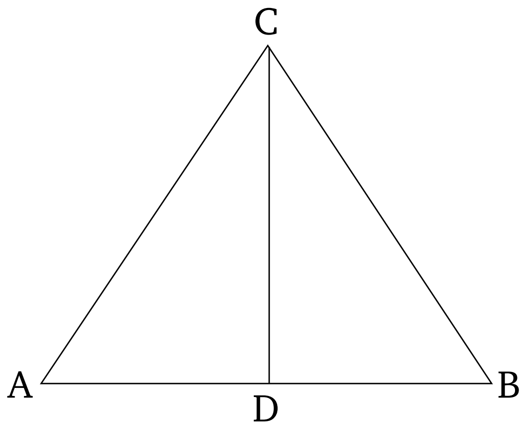
Figure 4.4: F on AB cannot be true.

We can derive two additional properties of F: First, F and C must be on the same side of AB. Second, F and B must be on the same side of AC. Euclid did not include them in Prop. 4.12, but they are necessary for later proofs.

Proposition 4.13. *Given an* $\angle BAC$*, there must exist a point* F*, s.t.,* $F \neq A$*,* $\angle BAF = \angle CAF$*;* F*,* C *are on the same side of* AB*; and* F*,* B *are on the same side of* AC*.*

Proof. Same as the proof of Prop. 4.12. □

Proposition 10



To cut a given finite straight-line in half.

Let AB be the given finite straight-line. So it is required to cut the finite straight-line AB in half.

Let the equilateral triangle ABC have been constructed upon (AB) [Prop. 1.1], and let the angle ACB have been cut in half by the straight-line CD [Prop. 1.9]. I say that the straight-line AB has been cut in half at point D.

For since AC is equal to CB, and CD (is) common, the two (straight-lines) AC, CD are equal to the two (straight-lines) BC, CD, respectively. And the angle ACD is equal to the angle BCD. Thus, the base AD is equal to the base BD [Prop. 1.4].

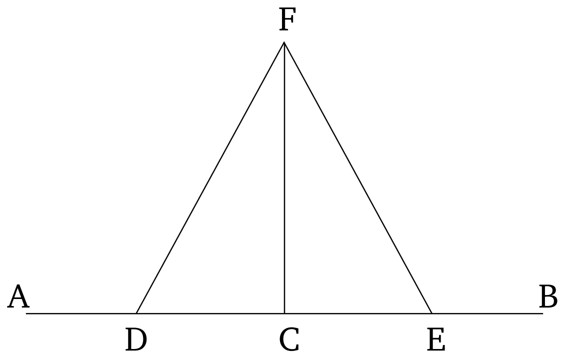
Thus, the given finite straight-line AB has been cut in half at (point) D. (Which is) the very thing it was required to do.

Commentary

Proposition 4.14. *A and B are two distinct points on a line AB. There must exist a point D between them, s.t.,* $|AD| = |DB|$.

Proof. See the original proof by Euclid. $\square$

Proposition 11



To draw a straight-line at right-angles to a given straight-line from a given point on it.

Let AB be the given straight-line, and C the given point on it. So it is required to draw a straight-line from the point C at right-angles to the straight-line AB.

Let the point D be have been taken at random on AC, and let CE be made equal to CD [Prop. 1.3], and let the equilateral triangle FDE have been constructed on DE [Prop. 1.1], and let FC have been joined. I say that the straight-line FC has been drawn at right-angles to the given straight-line AB from the given point C on it.

For since DC is equal to CE, and CF is common, the two (straight-lines) DC, CF are equal to the two (straight-lines), EC, CF, respectively. And the base DF is equal to the base FE. Thus, the angle DCF is equal to the angle ECF [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 10]. Thus, each of the (angles) DCF and FCE is a right-angle.

Thus, the straight-line CF has been drawn at right-angles to the given straight-line AB from the given point C on it. (Which is) the very thing it was required to do.

Commentary

Proposition 4.15. *A, B are two distinct points on a line AB. C is a point between them. Then, there must exist a point F not on AB, s.t., $\angle\ ACF$ is a right angle.*

Proof. See the original proof by Euclid. □

Euclid did not explicitly state that F can be constructed on any side of AB, which is useful for later proofs.

Proposition 4.16. *A, B are two distinct points on a line AB. C is a point between them, and X is a point not on AB. Then, there must exist a point F on the same side of AB with X, s.t., $\angle ACF$ is a right angle.*

Proof. Similar to the original proof by Euclid. □

Euclid's proof seems to assume A and B to be infinite points (so any point C on the line AB is between A and B). However, infinite points are not supported in E. We need the following proposition to get around this issue.

Proposition 4.17. *For two distinct points A and B on a line AB, there must exist a point F not on AB, s.t., $\angle FAB$ is a right angle.*

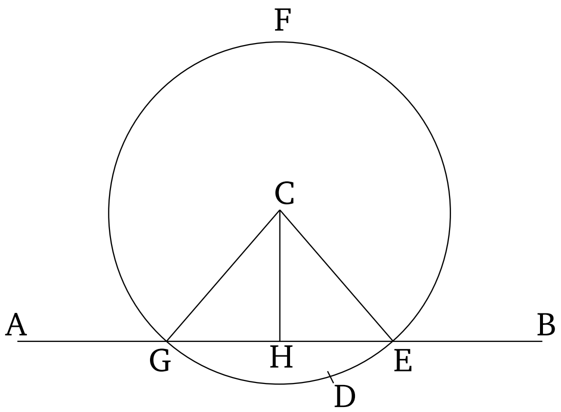
Proof. Let C be a point on AB, s.t., A is between B and C. Apply Prop. 4.15. □

Similarly, F can be constructed on either side.

Proposition 4.18. *A, B are two distinct points on a line AB. X is a point not on AB. Then, there must exist a point F on a different side of AB from X, s.t., $\angle FAB$ is a right angle.*

Proof. Let C be a point on AB, s.t., A is between B and C. Let Y be any point on the different side of AB from X. Apply Prop. 4.16. □

Proposition 12



To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.

Let AB be the given infinite straight-line and C the given point, which is not on (AB). So it is required to draw a straight-line perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

For let point D have been taken at random on the other side (to C) of the straight-line AB, and let the circle EFG have been drawn with center C and radius CD [Post. 3], and let the straight-line EG have been cut in half at (point) H [Prop. 1.10], and let the straight-lines CG, CH, and CE have been joined. I say that the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB).

For since GH is equal to HE, and HC (is) common, the two (straight-lines) GH, HC are equal to the two (straight-lines) EH, HC, respectively, and the base CG is equal to the base CE. Thus, the angle CHG is equal to the angle EHC [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands [Def. 10].

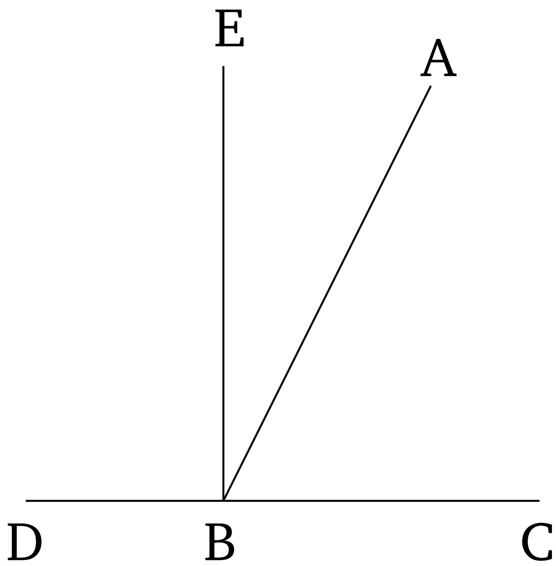
Thus, the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C, which is not on (AB). (Which is) the very thing it was required to do.

Commentary

Proposition 4.19. *A and B are two distinct points on a line AB. C is a point not on AB. Then, there must exist a point H on AB s.t., $\angle AHC$ or $\angle BHC$ is a right angle.*

Proof. See the original proof by Euclid. $\square$

Proposition 13



If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.

For let some straight-line AB stood on the straight-line CD make the angles CBA and ABD. I say that the angles CBA and ABD are certainly either two right-angles, or (have a sum) equal to two right-angles.

In fact, if CBA is equal to ABD then they are two right-angles [Def. 10]. But, if not, let BE have been drawn from the point B at right-angles to [the straight-line] CD [Prop. 1.11]. Thus, CBE and EBD are two right-angles. And since CBE is equal to the two (angles) CBA and ABE, let EBD have been added to both. Thus, the (sum of the angles) CBE and EBD is equal to the (sum of the) three (angles) CBA, ABE, and EBD [C.N. 2]. Again, since DBA is equal to the two (angles) DBE and EBA, let ABC have been added to both. Thus, the (sum of the angles) DBA and ABC is equal to the (sum of the) three (angles) DBE, EBA, and ABC [C.N. 2]. But (the sum of) CBE and EBD was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) CBE and EBD is also equal to (the sum of) DBA and ABC. But, (the sum of) CBE and EBD is two right-angles. Thus, (the sum of) ABD and ABC is also equal to two right-angles.

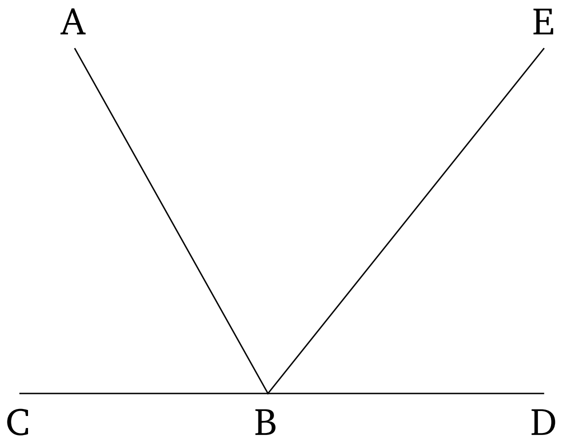
Thus, if a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.20. *A, B are two distinct points on a line AB. C, D are two distinct points on a different line CD. B is between C and D. Then, we have $\angle CBA + \angle ABD$ is 180 degrees.*

Proof. See the original proof by Euclid. $\square$

Proposition 14



If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another.

For let two straight-lines BC and BD, not lying on the same side, make adjacent angles ABC and ABD (whose sum is) equal to two right-angles with some straight-line AB, at the point B on it. I say that BD is straight-on with respect to CB.

For if BD is not straight-on to BC then let BE be straight-on to CB.

Therefore, since the straight-line AB stands on the straight-line CBE, the (sum of the) angles ABC and ABE is thus equal to two right-angles [Prop. 1.13]. But (the sum of) ABC and ABD is also equal to two right-angles. Thus, (the sum of angles) CBA and ABE is equal to (the sum of angles) CBA and ABD [C.N. 1]. Let (angle) CBA have been subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, BE is not straight-on with respect to CB. Similarly, we can show that neither (is) any other (straight-line) than BD. Thus, CB is straight-on with respect to BD.

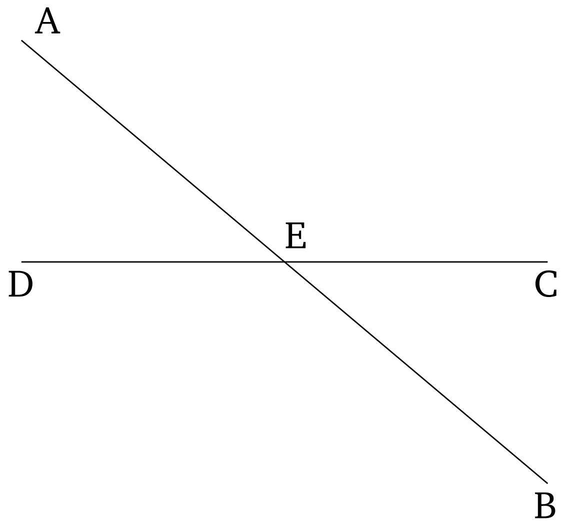
Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.21. *A, B are two distinct points on a line AB. C and D are two points on different sides of AB. B, C are on a line BC. B, D are on a line BD. $\angle ABC + \angle ABD$ is 180 degrees. Then, BC and BD must be the same line.*

Proof. Euclid only discussed the case where E and A are on the same side of BD, though the proof for other case is almost identical. $\square$

Proposition 15



If two straight-lines cut one another then they make the vertically opposite angles equal to one another.

For let the two straight-lines AB and CD cut one another at the point E. I say that angle AEC is equal to (angle) DEB, and (angle) CEB to (angle) AED.

For since the straight-line AE stands on the straight-line CD, making the angles CEA and AED, the (sum of the) angles CEA and AED is thus equal to two right-angles [Prop. 1.13]. Again, since the straight-line DE stands on the straight-line AB, making the angles AED and DEB, the (sum of the) angles AED and DEB is thus equal to two right-angles [Prop. 1.13]. But (the sum of) CEA and AED was also shown (to be) equal to two right-angles. Thus, (the sum of) CEA and AED is equal to (the sum of) AED and DEB [C.N. 1]. Let AED have been subtracted from both. Thus, the remainder CEA is equal to the remainder BED [C.N. 3]. Similarly, it can be shown that CEB and DEA are also equal.

Thus, if two straight-lines cut one another then they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

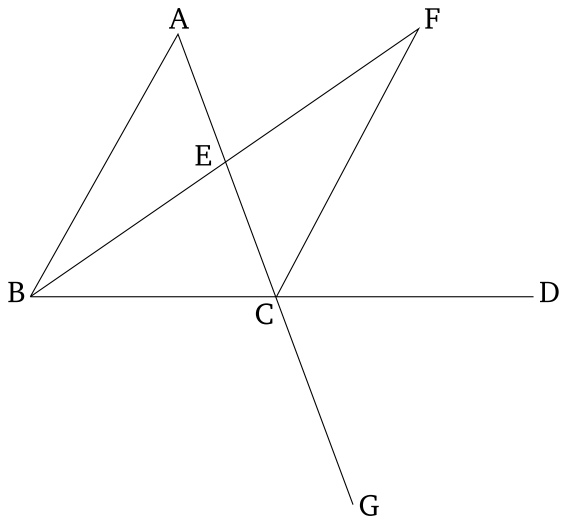
Commentary

Proposition 4.22. *AB and CD are two different lines intersecting at E. A and B are two distinct points on AB. C and D are two distinct points on CD. E is between A and B, and E*

is also between C and D. Then, we have $\angle\, AEC = \angle\, DEB$ and $\angle\, CEB = \angle\, AED$.

Proof. See the original proof by Euclid. $\square$

Proposition 16



For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC.

Let the (straight-line) AC have been cut in half at (point) E [Prop. 1.10]. And BE being joined, let it have been produced in a straight-line to (point) F. And let EF be made equal to BE [Prop. 1.3], and let FC have been joined, and let AC have been drawn through to (point) G.

Therefore, since AE is equal to EC, and BE to EF, the two (straight-lines) AE, EB are equal to the two (straight-lines) CE, EF, respectively. Also, angle AEB is equal to angle FEC, for (they are) vertically opposite [Prop. 1.15]. Thus, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle FEC, and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, BAE is equal to ECF. But ECD is greater than ECF. Thus, ACD is greater than BAE. Similarly, by having cut BC in half, it can be shown (that) BCG—that is to say, ACD—(is) also greater than ABC.

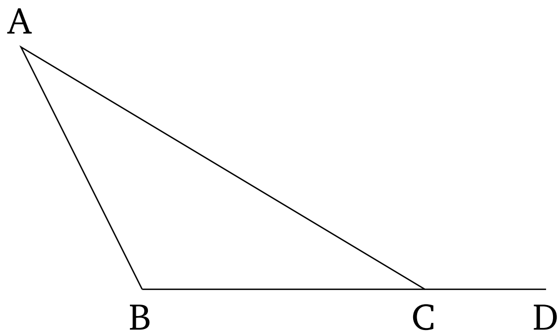
Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.23. *BC is an edge of $\triangle ABC$ and is extended to D. Then, we have $\angle ACD > \angle CBA$ and $\angle ACD > \angle BAC$.*

Proof. Euclid only proved $\angle ACD > \angle CBA$, though the proof of $\angle ACD > \angle BAC$ is almost identical. □

Proposition 17



For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.

Let ABC be a triangle. I say that (the sum of) two angles of triangle ABC taken together in any (possible way) is less than two right-angles.

For let BC have been produced to D.

And since the angle ACD is external to triangle ABC, it is greater than the internal and opposite angle ABC [Prop. 1.16]. Let ACB have been added to both. Thus, the (sum of the angles) ACD and ACB is greater than the (sum of the angles) ABC and BCA. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ABC and BCA is less than two right-angles. Similarly, we can show that (the sum of) BAC and ACB is also less than two right-angles, and further (that the sum of) CAB and ABC (is less than two right-angles).

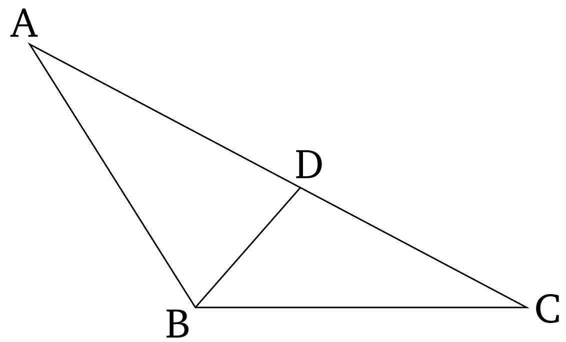
Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.24. *In $\triangle\ ABC$, we have $\angle\ ABC + \angle\ BCA$ is less than 180 degrees.*

Proof. See the original proof by Euclid. □

Proposition 18



In any triangle, the greater side subtends the greater angle.

For let ABC be a triangle having side AC greater than AB. I say that angle ABC is also greater than BCA.

For since AC is greater than AB, let AD be made equal to AB [Prop. 1.3], and let BD have been joined.

And since angle ADB is external to triangle BCD, it is greater than the internal and opposite (angle) DCB [Prop. 1.16]. But ADB (is) equal to ABD, since side AB is also equal to side AD [Prop. 1.5]. Thus, ABD is also greater than ACB. Thus, ABC is much greater than ACB.

Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

Commentary

Proposition 4.25. *In* $\triangle\ ABC$, *if* $|AC| > |AB|$, *then* $\angle\ ABC > \angle\ BCA$

Proof. See the original proof by Euclid. $\square$

Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than BCA. I say that side AC is also greater than side AB.

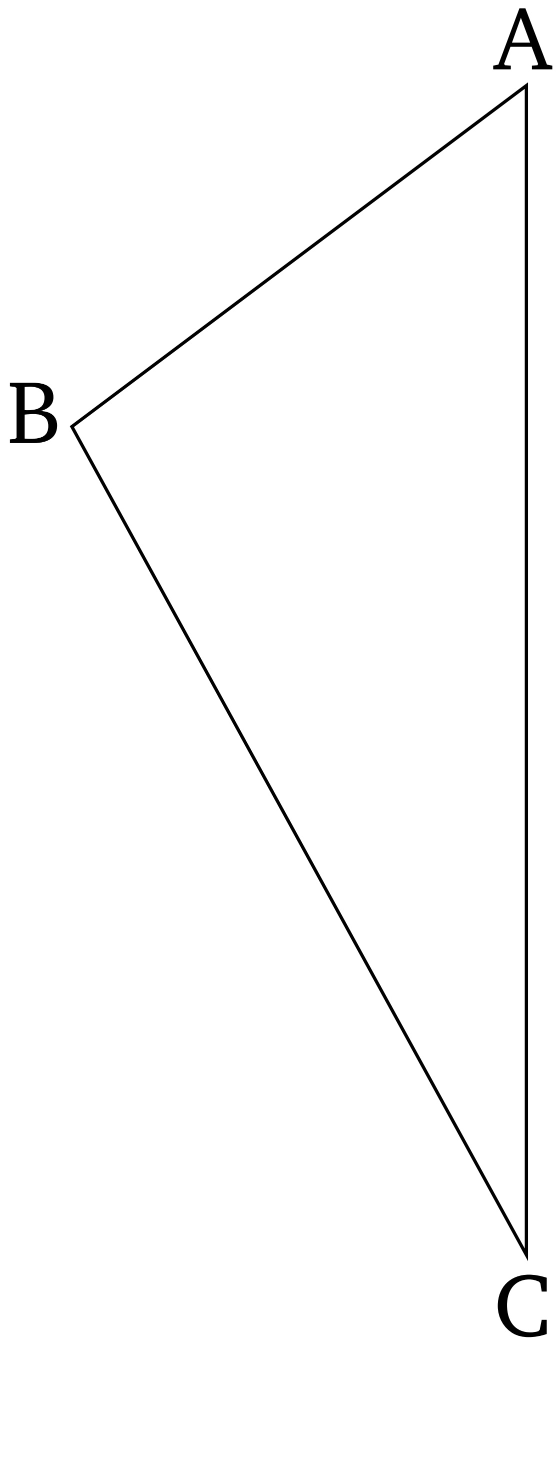
For if not, AC is certainly either equal to, or less than, AB. In fact, AC is not equal to AB. For then angle ABC would also have been equal to ACB [Prop. 1.5]. But it is not. Thus, AC is not equal to AB. Neither, indeed, is AC less than AB. For then angle ABC would also have been less than ACB [Prop. 1.18]. But it is not. Thus, AC is not less than AB. But it was shown that (AC) is not equal (to AB) either. Thus, AC is greater than AB.

Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

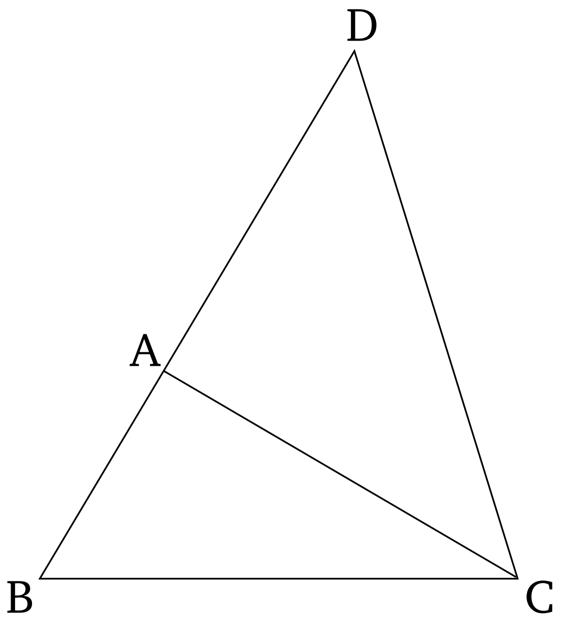
Commentary

Proposition 4.26. *In* $\triangle\ ABC$*, if* $\angle\ ABC\ >\ \angle\ BCA$*, then* $|AC| > |AB|$*.*

Proof. See the original proof by Euclid. $\square$


A
B
C

Proposition 20



In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).

For let ABC be a triangle. I say that in triangle ABC (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) BA and AC (is greater) than BC, (the sum of) AB and BC than AC, and (the sum of) BC and CA than AB.

For let BA have been drawn through to point D, and let AD be made equal to CA [Prop. 1.3], and let DC have been joined.

Therefore, since DA is equal to AC, the angle ADC is also equal to ACD [Prop. 1.5]. Thus, BCD is greater than ADC. And since DCB is a triangle having the angle BCD greater than BDC, and the greater angle subtends the greater side [Prop. 1.19], DB is thus greater than BC. But DA is equal to AC. Thus, (the sum of) BA and AC is greater than BC. Similarly, we can show that (the sum of) AB and BC is also greater than CA, and (the sum of) BC and CA than AB.

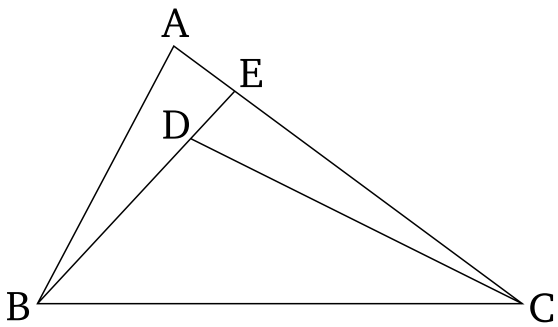
Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

Commentary

Proposition 4.27. *In* $\triangle\ ABC$*, we have* $|BA| + |AC| \ > \ |BC|$.

Proof. See the original proof by Euclid. $\square$

Proposition 21



If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.

For let the two internal straight-lines BD and DC have been constructed on one of the sides BC of the triangle ABC, from its ends B and C (respectively). I say that BD and DC are less than the (sum of the) two remaining sides of the triangle BA and AC, but encompass an angle BDC greater than BAC.

For let BD have been drawn through to E. And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle ABE the (sum of the) two sides AB and AE is thus greater than BE. Let EC have been added to both. Thus, (the sum of) BA and AC is greater than (the sum of) BE and EC. Again, since in triangle CED the (sum of the) two sides CE and ED is greater than CD, let DB have been added to both. Thus, (the sum of) CE and EB is greater than (the sum of) CD and DB. But, (the sum of) BA and AC was shown (to be) greater than (the sum of) BE and EC. Thus, (the sum of) BA and AC is much greater than (the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

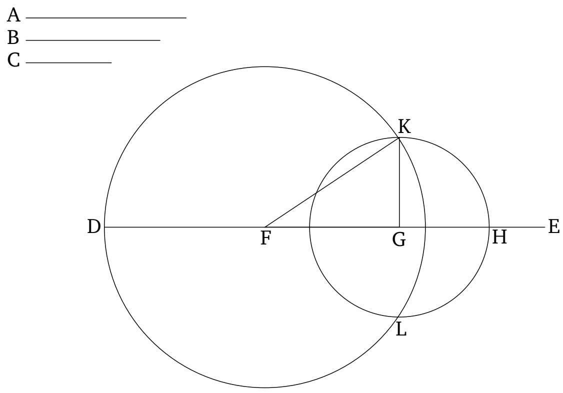
Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

Commentary

Proposition 4.28. *D is a point inside $\triangle\ ABC$. We have $|BD| + |DC| \ <\ |BA| + |AC|$, and $\angle\ BDC\ >\ \angle\ BAC$*

Proof. See the original proof by Euclid. $\square$

Proposition 22



To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has been constructed from three straight-lines equal to A, B, and C.

For since point F is the center of the circle DKL, FD is equal to FK. But, FD is equal to A. Thus, KF is also equal to A. Again, since point G is the center of the circle LKH, GH is equal to GK. But, GH is equal to C. Thus, KG is also equal to C. And FG is also equal to B. Thus, the three straight-lines KF, FG, and GK are equal to A, B, and C (respectively).

Thus, the triangle KFG has been constructed from the three straight-lines KF, FG, and GK, which are equal to the three given straight-lines A, B, and C (respectively). (Which is) the very thing it was required to do.

Commentary

Proposition 4.29. *A and A' are two disctinct points on a line AA'. B and B' are two distinct points on a line BB'. C and C' are two distinct points on a line CC'. $|AA'|+|BB'| > |CC'|$, $|AA'|+|CC'| > |BB'|$, and $|BB'|+|CC'| > |AA'|$. Then, there must exist $\triangle KFG$, s.t., $|FK| = |AA'|$, $|FG| = |BB'|$, and $|KG| = |CC'|$.*

Proof. See the original proof by Euclid. □

Euclid only proved that $\triangle KFG$ can be constructed. However, in later proofs, we need $\triangle KFG$ to be constructed on a given line and a given side of the line.

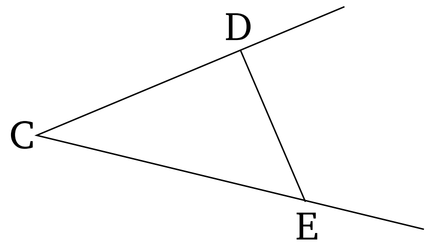
Proposition 4.30. *A and A' are two disctinct points on a line AA'. B and B' are two distinct points on a line BB'. C and C' are two distinct points on a line CC'. F and E are two distinct points on a line FE. $|AA'|+|BB'| > |CC'|$, $|AA'|+|CC'| > |BB'|$, and $|BB'|+|CC'| > |AA'|$. Then, there must exist $\triangle KFG$, s.t., G and F are on FE, F not between G and E, $|FK| = |AA'|$, $|FG| = |BB'|$, and $|KG| = |CC'|$.*

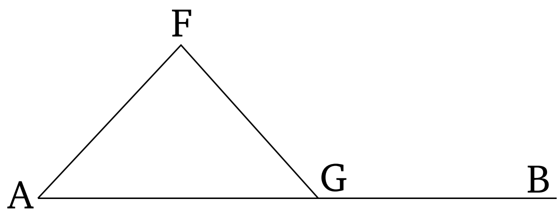
Proof. The same proof as Euclid's. □

Proposition 4.31. *A and A' are two disctinct points on a line AA'. B and B' are two distinct points on a line BB'. C and C' are two distinct points on a line CC'. F and E are two distinct points on a line FE. X is a point not on FE. $|AA'|+|BB'| > |CC'|$, $|AA'|+|CC'| > |BB'|$, and $|BB'|+|CC'| > |AA'|$. Then, there must exist $\triangle KFG$, s.t., G and F are on FE, F not between G and E, K on the same side of FE with X, $|FK| = |AA'|$, $|FG| = |BB'|$, and $|KG| = |CC'|$.*

Proof. The same proof as Euclid's. □

Proposition 23





To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.

Let AB be the given straight-line, A the (given) point on it, and DCE the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the (given) point A on the given straight-line AB.

Let the points D and E have been taken at random on each of the (straight-lines) CD and CE (respectively), and let DE have been joined. And let the triangle AFG have been constructed from three straight-lines which are equal to CD, DE, and CE, such that CD is equal to AF, CE to AG, and further DE to FG [Prop. 1.22].

Therefore, since the two (straight-lines) DC, CE are equal to the two (straight-lines) FA, AG, respectively, and the base DE is equal to the base FG, the angle DCE is thus equal to the angle FAG [Prop. 1.8].

Thus, the rectilinear angle FAG, equal to the given rectilinear angle DCE, has been constructed at the (given) point A on the given straight-line AB. (Which is) the very thing it was required to do.

Commentary

Proposition 4.32. *$\angle DCE$ is an angle. A and B are two distinct points on a line AB. Then, there must exists a point F different from A, s.t., $\angle FAB = \angle DCE$.*

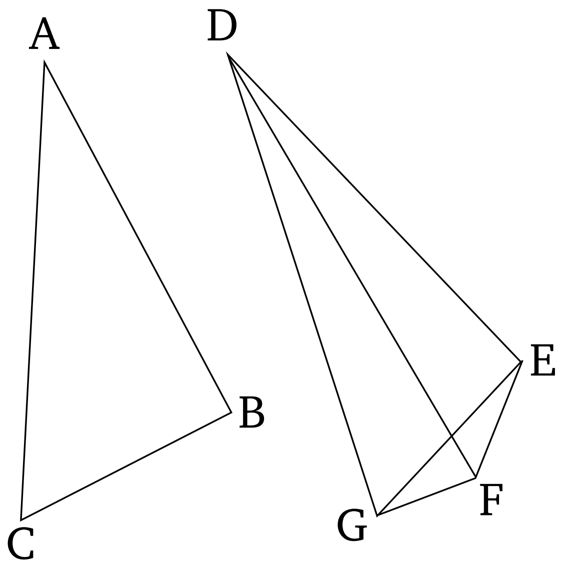
Proof. Euclid's proof omitted the degenerated cases that D is on CE, i.e., $\angle DCE$ is either 0 or π. In addition, when constructing $\triangle AFG$, Euclid didn't prove that such a triangle can be constructed from CD, CE, and DE, which can be done by applying Proposition 4.27. □

Euclid didn't prove that F can be constructed on either side of AB, which is useful for later proofs.

Proposition 4.33. *$\angle DCE$ is an angle. A and B are two distinct points on a line AB. X is a point not on AB. Then, there must exists a point F different from A, s.t., $\angle FAB = \angle DCE$, and F is either on AB or on the same side of AB with X.*

Proof. Similar to the previous proof. □

Proposition 24



If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively. (That is), AB (equal) to DE, and AC to DF. Let them also have the angle at A greater than the angle at D. I say that the base BC is also greater than the base EF.

For since angle BAC is greater than angle EDF, let (angle) EDG, equal to angle BAC, have been constructed at the point D on the straight-line DE [Prop. 1.23]. And let DG be made equal to either of AC or DF [Prop. 1.3], and let EG and FG have been joined.

Therefore, since AB is equal to DE and AC to DG, the two (straight-lines) BA, AC are equal to the two (straight-lines) ED, DG, respectively. Also the angle BAC is equal to the angle EDG. Thus, the base BC is equal to the base EG [Prop. 1.4]. Again, since DF is equal to DG, angle DGF is also equal to angle DFG [Prop. 1.5]. Thus, DFG (is) greater than EGF. Thus, EFG is much greater than EGF. And since triangle EFG has angle EFG greater than EGF, and the greater angle is subtended by the greater side [Prop. 1.19], side EG (is) thus also greater than EF. But EG (is) equal to BC. Thus, BC (is) also greater than EF.

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter). (Which is) the very thing it was required to show.

Commentary

Euclid did not cover the case where D and G are on different sides of EF (Fig. 4.5). Below we augment Euclid's proof to cover this case. The resulting proof is significantly more complicated than Euclid's proof.

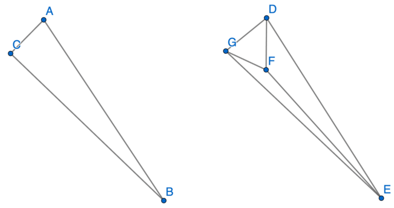


Figure 4.5: The case in Proposition 24 missed by Euclid.

Proposition 4.34. *$\triangle\ ABC$ and $\triangle\ DEF$ are two triangles s.t., $|AB| = |DE|$, $|AC| = |DF|$, and $\angle\ BAC\ >\ \angle\ EDF$. Then, $|BC|\ >\ |EF|$.*

Proof. We only prove the case missed by Euclid.

Let's construct $\triangle\ EDG$ s.t., $\angle\ EDG = \angle\ BAC$, $|DG| = |DF|$, and $|BC| = |EG|$ (Fig. 4.5). By Proposition 5, we have $\angle\ DGF = \angle\ DFG$; let's denote it by α. To prove $|BC|\ >\ |EF|$, we only need to prove $|EG|\ >\ |EF|$. By Proposition 19, this is further reduced to $\angle\ EFG\ >\ \angle\ EGF$. Let $x = \angle\ EFG$ and $y = \angle\ EGF$. We want to prove $x\ >\ y$.

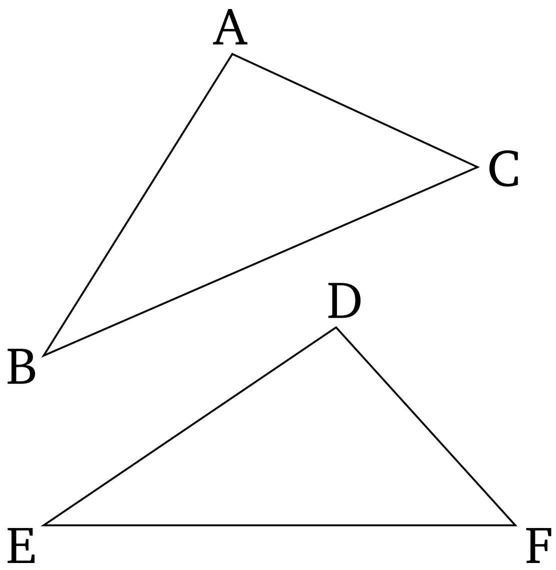
Note that $\angle\ DGE = \angle\ DGF + \angle\ EGF = \alpha + y$. $\angle\ DFE = 2\pi - \angle\ DFG - \angle\ EFG = 2\pi - \alpha - x$. Furthermore, Proposition 17 states that the sum of any two angles in a triangle must be smaller than π. Therefore, any angle in a triangle must also be smaller than π, i.e.,

$$\begin{aligned} \alpha + y &< \pi \\ 2\pi - \alpha - x &< \pi \end{aligned}$$

Simplifying these two inequalities leads to $x\ >\ y$. QED.

□

Proposition 25



If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).

Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF, respectively (That is), AB (equal) to DE, and AC to DF. And let the base BC be greater than the base EF. I say that angle BAC is also greater than EDF.

For if not, (BAC) is certainly either equal to, or less than, (EDF). In fact, BAC is not equal to EDF. For then the base BC would also have been equal to the base EF [Prop. 1.4]. But it is not. Thus, angle BAC is not equal to EDF. Neither, indeed, is BAC less than EDF. For then the base BC would also have been less than the base EF [Prop. 1.24]. But it is not. Thus, angle BAC is not less than EDF. But it was shown that (BAC is) not equal (to EDF) either. Thus, BAC is greater than EDF.

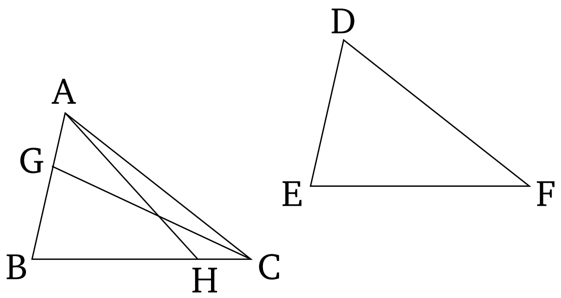
Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

Commentary

Proposition 4.35. *$\triangle$ ABC and $\triangle$ DEF are two triangles s.t. $|AB| = |DE|$, $|AC| = |DF|$, and $|BC| > |EF|$. Then, $\angle BAC > \angle EDF$.*

Proof. See the original proof by Euclid. $\square$

Proposition 26



If two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two (angles) DEF and EFD, respectively. (That is) ABC (equal) to DEF, and BCA to EFD. And let them also have one side equal to one side. First of all, the (side) by the equal angles. (That is) BC (equal) to EF. I say that they will have the remaining sides equal to the corresponding remaining sides. (That is) AB (equal) to DE, and AC to DF. And (they will have) the remaining angle (equal) to the remaining angle. (That is) BAC (equal) to EDF.

For if AB is unequal to DE then one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 1.3], and let GC have been joined.

Therefore, since BG is equal to DE, and BC to EF, the two (straight-lines) GB, BC are equal to the two (straight-lines) DE, EF, respectively. And angle GBC is equal to angle DEF. Thus, the base GC is equal to the base DF, and triangle GBC is equal to triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, GCB (is equal) to DFE. But, DFE was assumed (to be) equal to BCA. Thus, BCG is also equal to BCA, the lesser to the greater. The very thing (is) impossible. Thus, AB is not unequal to DE. Thus, (it is) equal. And BC is also equal to EF. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And angle ABC is equal to angle DEF. Thus, the base AC is equal to the base DF, and the remaining angle BAC is equal to the remaining angle EDF [Prop. 1.4].

But, again, let the sides subtending the equal angles be equal: for instance, (let) AB (be equal) to DE. Again, I say that the remaining sides will be equal to the remaining sides. (That is) AC (equal) to DF, and BC to EF. Furthermore, the remaining angle BAC is equal to the remaining angle EDF.

For if BC is unequal to EF then one of them is greater. If possible, let BC be greater. And let BH be made equal to EF [Prop. 1.3], and let AH have been joined. And since BH is equal to EF, and AB to DE, the two (straight-lines) AB, BH are equal to the two (straight-lines) DE, EF, respectively. And the angles they encompass (are also equal). Thus, the base AH is equal to the base DF, and the triangle ABH is equal to the triangle DEF, and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle BHA is equal to EFD. But, EFD is equal to BCA. So, in triangle AHC, the external angle BHA is equal to the internal and opposite angle BCA. The very thing (is) impossible [Prop. 1.16]. Thus, BC is not unequal to EF. Thus, (it is) equal. And AB is also equal to DE. So the two (straight-lines) AB, BC are equal to the two (straight-lines) DE, EF, respectively. And they encompass equal angles. Thus, the base AC is equal to the base DF, and triangle ABC (is) equal to triangle DEF, and the remaining angle BAC (is) equal to the remaining angle EDF [Prop. 1.4].

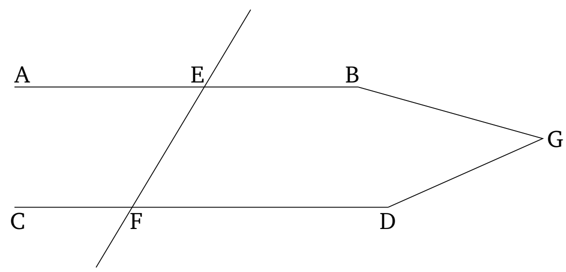
Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show.

Commentary

Proposition 4.36. *$\triangle\ ABC$ and $\triangle\ DEF$ are two triangles, s.t., $\angle\ ABC = \angle\ DEF$, $\angle\ BCA = \angle\ EFD$, and $|BC| = |EF|$ (or $|AB| = |DE|$). Then, $\triangle\ ABC \cong \triangle\ DEF$.*

Proof. See the original proof by Euclid. $\square$

Proposition 27



If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.

For let the straight-line EF, falling across the two straight-lines AB and CD, make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D, or (in the direction) of A and C [Def. 23]. Let them have been produced, and let them meet together in the direction of B and D at (point) G. So, for the triangle GEF, the external angle AEF is equal to the interior and opposite (angle) EFG. The very thing is impossible [Prop. 1.16]. Thus, being produced, AB and CD will not meet together in the direction of B and D. Similarly, it can be shown that neither (will they meet together) in (the direction of) A and C. But (straight-lines) meeting in neither direction are parallel [Def. 23]. Thus, AB and CD are parallel.

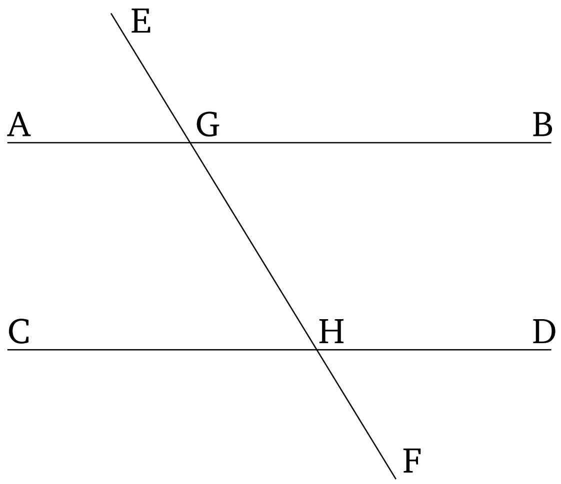
Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Commentary

Proposition 4.37. *A and E are two distcint points on the line AE. F and D are two distinct points on FD. E and F are two distinct points on EF. A and D are on different sides of EF. $\angle AEF = \angle EFD$. Then, AE and FD must be parallel.*

Proof. See the original proof by Euclid. □

Proposition 28



If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel to one another.

For let EF, falling across the two straight-lines AB and CD, make the external angle EGB equal to the internal and opposite angle GHD, or the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles. I say that AB is parallel to CD.

For since (in the first case) EGB is equal to GHD, but EGB is equal to AGH [Prop. 1.15], AGH is thus also equal to GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

Again, since (in the second case, the sum of) BGH and GHD is equal to two right-angles, and (the sum of) AGH and BGH is also equal to two right-angles [Prop. 1.13], (the sum of) AGH and BGH is thus equal to (the sum of) BGH and GHD. Let BGH have been subtracted from both. Thus, the remainder AGH is equal to the remainder GHD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

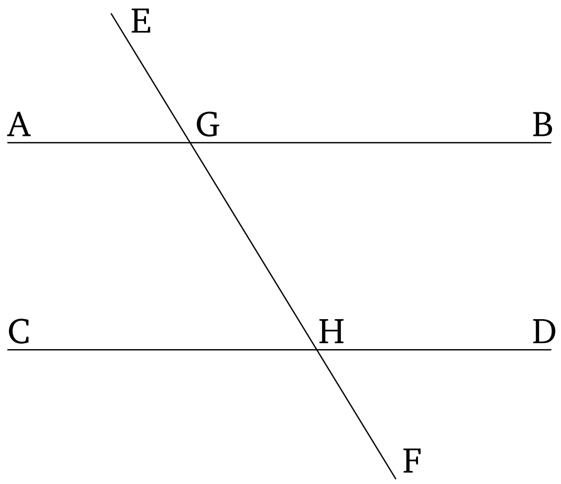
Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Commentary

Proposition 4.38. *A and B are two distcint points on AB. C and D are two distinct points on CD. E and F are two distinct points on EF. G is between A and B. H is between C and D. G is between E and H. H is between G and F. B and D are on the same side of EF. Either $\angle\, EGB = \angle\, GHD$ or $\angle\, BGH + \angle\, GHD = \pi$. Then, AB and CD must be parallel.*

Proof. See the original proof by Euclid. $\square$

Proposition 29



A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.

For let the straight-line EF fall across the parallel straight-lines AB and CD. I say that it makes the alternate angles, AGH and GHD, equal, the external angle EGB equal to the internal and opposite (angle) GHD, and the (sum of the) internal (angles) on the same side, BGH and GHD, equal to two right-angles.

For if AGH is unequal to GHD then one of them is greater. Let AGH be greater. Let BGH have been added to both. Thus, (the sum of) AGH and BGH is greater than (the sum of) BGH and GHD. But, (the sum of) AGH and BGH is equal to two right-angles [Prop 1.13]. Thus, (the sum of) BGH and GHD is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, AB and CD, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 23]. Thus, AGH is not unequal to GHD. Thus, (it is) equal. But, AGH is equal to EGB [Prop. 1.15]. And EGB is thus also equal to GHD. Let BGH be added to both. Thus, (the sum of) EGB and BGH is equal to (the sum of) BGH and GHD. But, (the sum of) EGB and BGH is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) BGH and GHD is also equal to two right-angles.

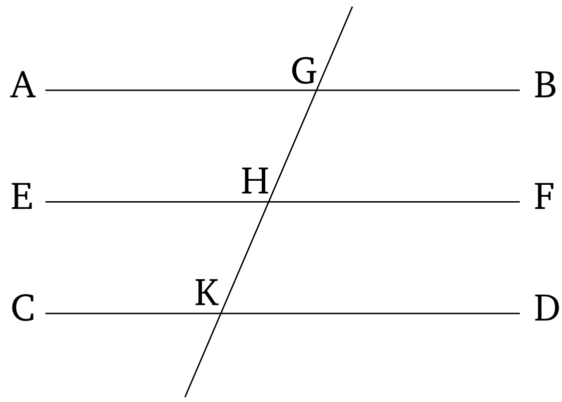
Thus, a straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.39. *If*

Proof. □

Proposition 30



(Straight-lines) parallel to the same straight-line are also parallel to one another.

Let each of the (straight-lines) AB and CD be parallel to EF. I say that AB is also parallel to CD.

For let the straight-line GK fall across (AB, CD, and EF).

And since the straight-line GK has fallen across the parallel straight-lines AB and EF, (angle) AGK (is) thus equal to GHF [Prop. 1.29]. Again, since the straight-line GK has fallen across the parallel straight-lines EF and CD, (angle) GHF is equal to GKD [Prop. 1.29]. But AGK was also shown (to be) equal to GHF. Thus, AGK is also equal to GKD. And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

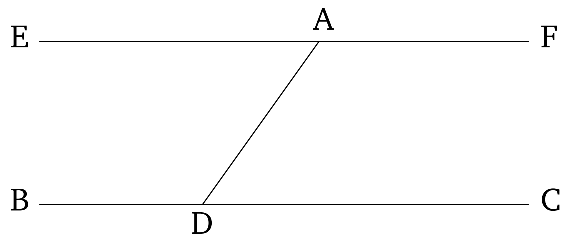
[Thus, (straight-lines) parallel to the same straight-line are also parallel to one another.] (Which is) the very thing it was required to show.

Commentary

Proposition 4.40. *If*

Proof. □

Proposition 31



To draw a straight-line parallel to a given straight-line, through a given point.

Let A be the given point, and BC the given straight-line. So it is required to draw a straight-line parallel to the straight-line BC, through the point A.

Let the point D have been taken a random on BC, and let AD have been joined. And let (angle) DAE, equal to angle ADC, have been constructed on the straight-line DA at the point A on it [Prop. 1.23]. And let the straight-line AF have been produced in a straight-line with EA.

And since the straight-line AD, (in) falling across the two straight-lines BC and EF, has made the alternate angles EAD and ADC equal to one another, EAF is thus parallel to BC [Prop. 1.27].

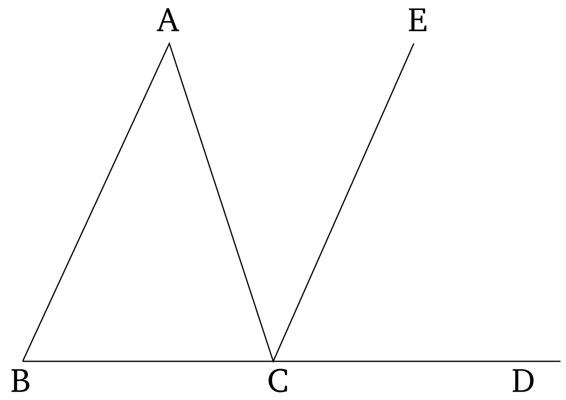
Thus, the straight-line EAF has been drawn parallel to the given straight-line BC, through the given point A. (Which is) the very thing it was required to do.

Commentary

Proposition 4.41. *If*

Proof. □

Proposition 32



In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D. I say that the external angle ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC, and the (sum of the) three internal angles of the triangle—ABC, BCA, and CAB—is equal to two right-angles.

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE, and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE, and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC. Thus, the whole angle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC.

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC, BCA, and CAB. But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB, CBA, and CAB is also equal to two right-angles.

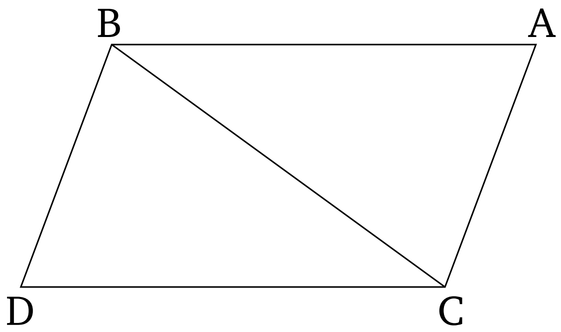
Thus, in any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

Commentary

Proposition 4.42. *If*

Proof. □

Proposition 33



Straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel.

Let AB and CD be equal and parallel (straight-lines), and let the straight-lines AC and BD join them on the same sides. I say that AC and BD are also equal and parallel.

Let BC have been joined. And since AB is parallel to CD, and BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. And since AB is equal to CD, and BC is common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, CB.† And the angle ABC is equal to the angle BCD. Thus, the base AC is equal to the base BD, and triangle ABC is equal to triangle DCB‡, and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides [Prop. 1.4]. Thus, angle ACB is equal to CBD. Also, since the straight-line BC, (in) falling across the two straight-lines AC and BD, has made the alternate angles (ACB and CBD) equal to one another, AC is thus parallel to BD [Prop. 1.27]. And (AC) was also shown (to be) equal to (BD).

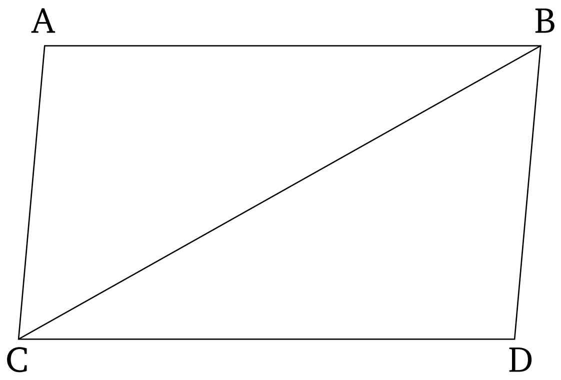
Thus, straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel. (Which is) the very thing it was required to show.

Commentary

Proposition 4.43. *If*

Proof. □

Proposition 34



In parallelogrammic figures the opposite sides and angles are equal to one another, and a diagonal cuts them in half.

Let $ACDB$ be a parallelogrammic figure, and BC its diagonal. I say that for parallelogram $ACDB$, the opposite sides and angles are equal to one another, and the diagonal BC cuts it in half.

For since AB is parallel to CD, and the straight-line BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. Again, since AC is parallel to BD, and BC has fallen across them, the alternate angles ACB and CBD are equal to one another [Prop. 1.29]. So ABC and BCD are two triangles having the two angles ABC and BCA equal to the two (angles) BCD and CBD, respectively, and one side equal to one side—the (one) by the equal angles and common to them, (namely) BC. Thus, they will also have the remaining sides equal to the corresponding remaining (sides), and the remaining angle (equal) to the remaining angle [Prop. 1.26]. Thus, side AB is equal to CD, and AC to BD. Furthermore, angle BAC is equal to CDB. And since angle ABC is equal to BCD, and CBD to ACB, the whole (angle) ABD is thus equal to the whole (angle) ACD. And BAC was also shown (to be) equal to CDB.

Thus, in parallelogrammic figures the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since AB is equal to CD, and BC (is) common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, $CB^{\dagger}$, respectively. And angle ABC is equal to angle BCD. Thus, the base AC (is) also equal to DB, and triangle ABC is equal to triangle BCD [Prop. 1.4].

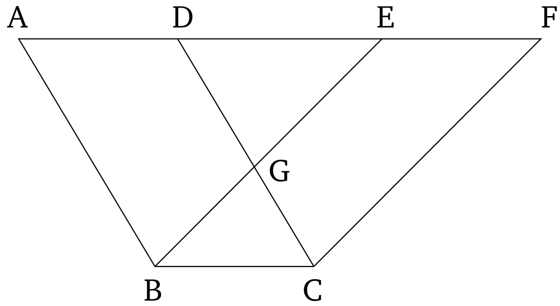
Thus, the diagonal BC cuts the parallelogram $ACDB^{\ddagger}$ in half. (Which is) the very thing it was required to show.

Commentary

Proposition 4.44. *If*

Proof. □

Proposition 35



Parallelograms which are on the same base and between the same parallels are equal† to one another.

Let $ABCD$ and $EBCF$ be parallelograms on the same base BC, and between the same parallels AF and BC. I say that $ABCD$ is equal to parallelogram $EBCF$.

For since $ABCD$ is a parallelogram, AD is equal to BC [Prop. 1.34]. So, for the same (reasons), EF is also equal to BC. So AD is also equal to EF. And DE is common. Thus, the whole (straight-line) AE is equal to the whole (straight-line) DF. And AB is also equal to DC. So the two (straight-lines) EA, AB are equal to the two (straight-lines) FD, DC, respectively. And angle FDC is equal to angle EAB, the external to the internal [Prop. 1.29]. Thus, the base EB is equal to the base FC, and triangle EAB will be equal to triangle DFC [Prop. 1.4]. Let DGE have been taken away from both. Thus, the remaining trapezium $ABGD$ is equal to the remaining trapezium $EGCF$. Let triangle GBC have been added to both. Thus, the whole parallelogram $ABCD$ is equal to the whole parallelogram $EBCF$.

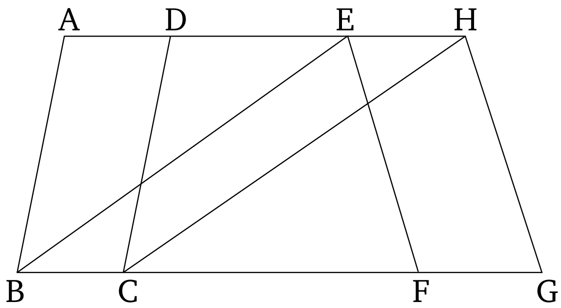
Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.45. *If*

Proof. □

Proposition 36



Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let $ABCD$ and $EFGH$ be parallelograms which are on the equal bases BC and FG, and (are) between the same parallels AH and BG. I say that the parallelogram $ABCD$ is equal to $EFGH$.

For let BE and CH have been joined. And since BC is equal to FG, but FG is equal to EH [Prop. 1.34], BC is thus equal to EH. And they are also parallel, and EB and HC join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, EB and HC are also equal and parallel]. Thus, $EBCH$ is a parallelogram [Prop. 1.34], and is equal to $ABCD$. For it has the same base, BC, as ($ABCD$), and is between the same parallels, BC and AH, as ($ABCD$) [Prop. 1.35]. So, for the same (reasons), $EFGH$ is also equal to the same (parallelogram) $EBCH$ [Prop. 1.34]. So that the parallelogram $ABCD$ is also equal to $EFGH$.

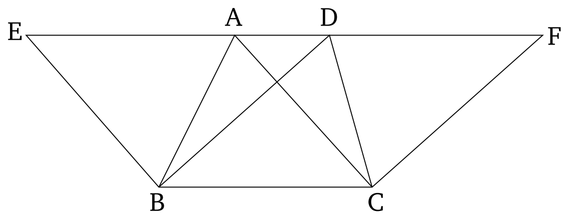
Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.46. *If*

Proof. □

Proposition 37



Triangles which are on the same base and between the same parallels are equal to one another.

Let ABC and DBC be triangles on the same base BC, and between the same parallels AD and BC. I say that triangle ABC is equal to triangle DBC.

Let AD have been produced in both directions to E and F, and let the (straight-line) BE have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) CF have been drawn through C parallel to BD [Prop. 1.31]. Thus, $EBCA$ and $DBCF$ are both parallelograms, and are equal. For they are on the same base BC, and between the same parallels BC and EF [Prop. 1.35]. And the triangle ABC is half of the parallelogram $EBCA$. For the diagonal AB cuts the latter in half [Prop. 1.34]. And the triangle DBC (is) half of the parallelogram $DBCF$. For the diagonal DC cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another.]† Thus, triangle ABC is equal to triangle DBC.

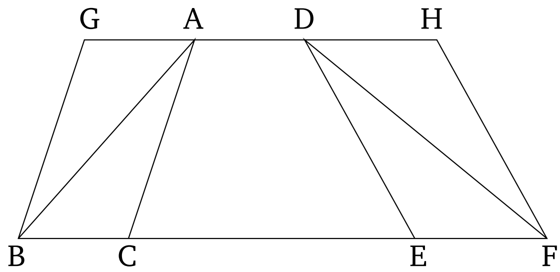
Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.47. *If*

Proof. □

Proposition 38



Triangles which are on equal bases and between the same parallels are equal to one another.

Let ABC and DEF be triangles on the equal bases BC and EF, and between the same parallels BF and AD. I say that triangle ABC is equal to triangle DEF.

For let AD have been produced in both directions to G and H, and let the (straight-line) BG have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) FH have been drawn through F parallel to DE [Prop. 1.31]. Thus, $GBCA$ and $DEFH$ are each parallelograms. And $GBCA$ is equal to $DEFH$. For they are on the equal bases BC and EF, and between the same parallels BF and GH [Prop. 1.36]. And triangle ABC is half of the parallelogram $GBCA$. For the diagonal AB cuts the latter in half [Prop. 1.34]. And triangle FED (is) half of parallelogram $DEFH$. For the diagonal DF cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle ABC is equal to triangle DEF.

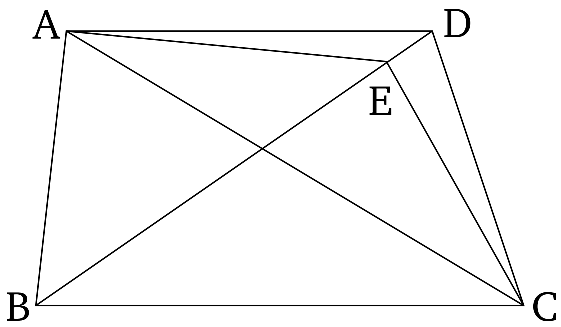
Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.48. *If*

Proof. □

Proposition 39



Equal triangles which are on the same base, and on the same side, are also between the same parallels.

Let ABC and DBC be equal triangles which are on the same base BC, and on the same side (of it). I say that they are also between the same parallels.

For let AD have been joined. I say that AD and BC are parallel.

For, if not, let AE have been drawn through point A parallel to the straight-line BC [Prop. 1.31], and let EC have been joined. Thus, triangle ABC is equal to triangle EBC. For it is on the same base as it, BC, and between the same parallels [Prop. 1.37]. But ABC is equal to DBC. Thus, DBC is also equal to EBC, the greater to the lesser. The very thing is impossible. Thus, AE is not parallel to BC. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BC.

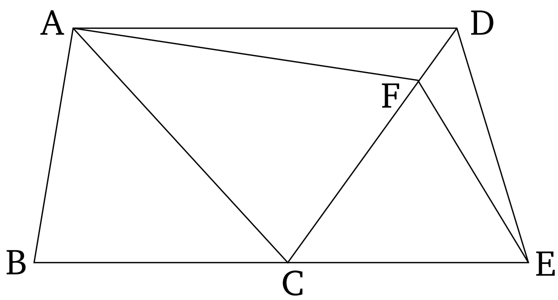
Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

Commentary

Proposition 4.49. *If*

Proof. □

Proposition 40



Equal triangles which are on equal bases, and on the same side, are also between the same parallels.

Let ABC and CDE be equal triangles on the equal bases BC and CE (respectively), and on the same side (of BE). I say that they are also between the same parallels.

For let AD have been joined. I say that AD is parallel to BE.

For if not, let AF have been drawn through A parallel to BE [Prop. 1.31], and let FE have been joined. Thus, triangle ABC is equal to triangle FCE. For they are on equal bases, BC and CE, and between the same parallels, BE and AF [Prop. 1.38]. But, triangle ABC is equal to [triangle] DCE. Thus, [triangle] DCE is also equal to triangle FCE, the greater to the lesser. The very thing is impossible. Thus, AF is not parallel to BE. Similarly, we can show that neither (is) any other (straight-line) than AD. Thus, AD is parallel to BE.

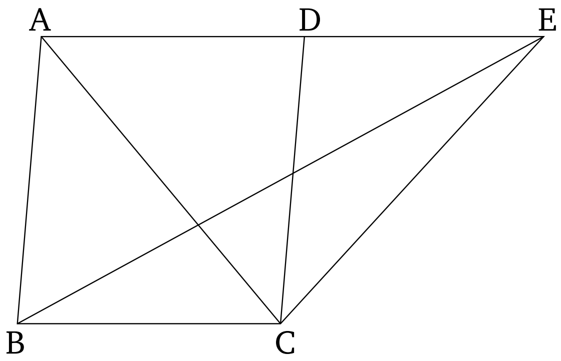
Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

Commentary

Proposition 4.50. *If*

Proof. □

Proposition 41



If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.

For let parallelogram $ABCD$ have the same base BC as triangle EBC, and let it be between the same parallels, BC and AE. I say that parallelogram $ABCD$ is double (the area) of triangle BEC.

For let AC have been joined. So triangle ABC is equal to triangle EBC. For it is on the same base, BC, as (EBC), and between the same parallels, BC and AE [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle ABC. For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle EBC.

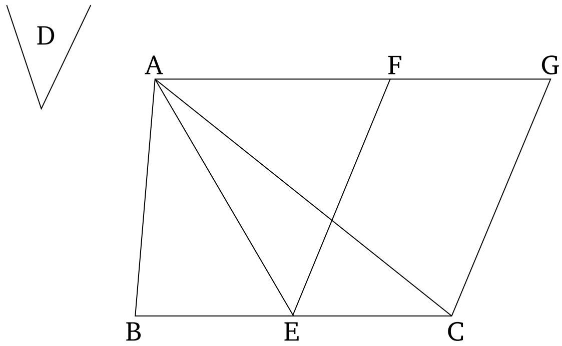
Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

Commentary

Proposition 4.51. *If*

Proof. □

Proposition 42



To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let ABC be the given triangle, and D the given rectilinear angle. So it is required to construct a parallelogram equal to triangle ABC in the rectilinear angle D.

Let BC have been cut in half at E [Prop. 1.10], and let AE have been joined. And let (angle) CEF, equal to angle D, have been constructed at the point E on the straight-line EC [Prop. 1.23]. And let AG have been drawn through A parallel to EC [Prop. 1.31], and let CG have been drawn through C parallel to EF [Prop. 1.31]. Thus, $FECG$ is a parallelogram. And since BE is equal to EC, triangle ABE is also equal to triangle AEC. For they are on the equal bases, BE and EC, and between the same parallels, BC and AG [Prop. 1.38]. Thus, triangle ABC is double (the area) of triangle AEC. And parallelogram $FECG$ is also double (the area) of triangle AEC. For it has the same base as (AEC), and is between the same parallels as (AEC) [Prop. 1.41]. Thus, parallelogram $FECG$ is equal to triangle ABC. ($FECG$) also has the angle CEF equal to the given (angle) D.

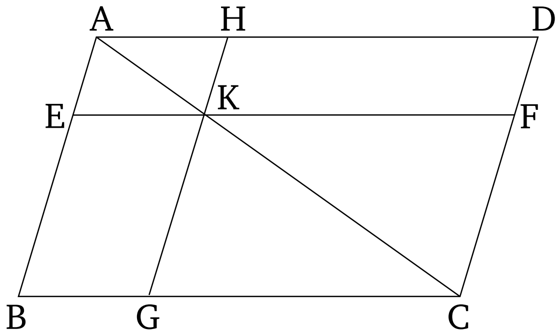
Thus, parallelogram $FECG$, equal to the given triangle ABC, has been constructed in the angle CEF, which is equal to D. (Which is) the very thing it was required to do.

Commentary

Proposition 4.52. *If*

Proof. □

Proposition 43



For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EH and FG be the parallelograms about AC, and BK and KD the so-called complements (about AC). I say that the complement BK is equal to the complement KD.

For since $ABCD$ is a parallelogram, and AC its diagonal, triangle ABC is equal to triangle ACD [Prop. 1.34]. Again, since EH is a parallelogram, and AK is its diagonal, triangle AEK is equal to triangle AHK [Prop. 1.34]. So, for the same (reasons), triangle KFC is also equal to (triangle) KGC. Therefore, since triangle AEK is equal to triangle AHK, and KFC to KGC, triangle AEK plus KGC is equal to triangle AHK plus KFC. And the whole triangle ABC is also equal to the whole (triangle) ADC. Thus, the remaining complement BK is equal to the remaining complement KD.

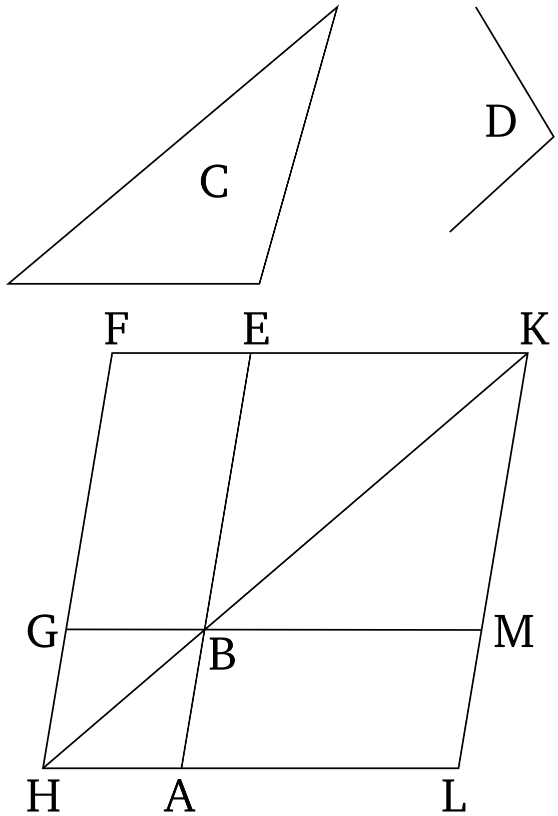
Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

Commentary

Proposition 4.53. *If*

Proof. □

Proposition 44



To apply a parallelogram equal to a given triangle to a given straight-line in a given rectilinear angle.

Let AB be the given straight-line, C the given triangle, and D the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle C to the given straight-line AB in an angle equal to (angle) D.

Let the parallelogram $BEFG$, equal to the triangle C, have been constructed in the angle EBG, which is equal to D [Prop. 1.42]. And let it have been placed so that BE is straight-on to AB.† And let FG have been drawn through to H, and let AH have been drawn through A parallel to either of BG or EF [Prop. 1.31], and let HB have been joined. And since the straight-line HF falls across the parallels AH and EF, the (sum of the) angles AHF and HFE is thus equal to two right-angles [Prop. 1.29]. Thus, (the sum of) BHG and GFE is less than two right-angles. And (straight-lines) produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, being produced, HB and FE will meet together. Let them have been produced, and let them meet together at K. And let KL have been drawn through point K parallel to either of EA or FH [Prop. 1.31]. And let HA and GB have been produced to points L and M (respectively). Thus, $HLKF$ is a parallelogram, and HK its diagonal. And AG and ME (are) parallelograms, and LB and BF the so-called complements, about HK. Thus, LB is equal to BF [Prop. 1.43]. But, BF is equal to triangle C. Thus, LB is also equal to C. Also, since angle GBE is equal to ABM [Prop. 1.15], but GBE is equal to D, ABM is thus also equal to angle D.

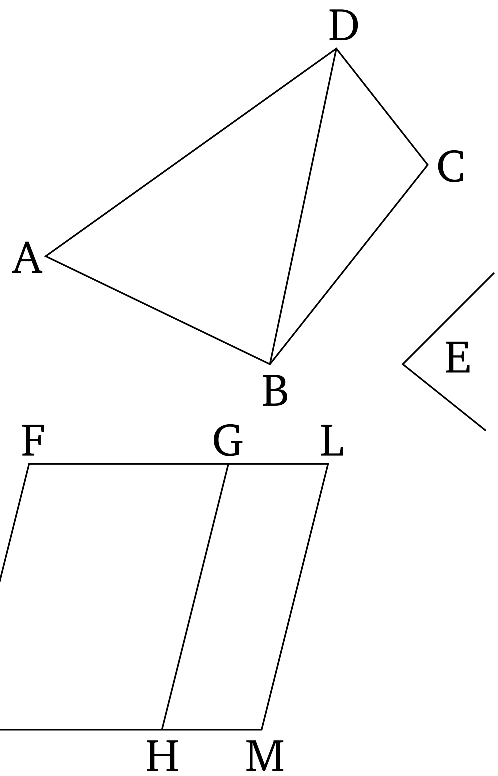
Thus, the parallelogram LB, equal to the given triangle C, has been applied to the given straight-line AB in the angle ABM, which is equal to D. (Which is) the very thing it was required to do.

Commentary

Proposition 4.54. *If*

Proof. □

Proposition 45



To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Let $ABCD$ be the given rectilinear figure,† and E the given rectilinear angle. So it is required to construct a parallelogram equal to the rectilinear figure $ABCD$ in the given angle E.

Let DB have been joined, and let the parallelogram FH, equal to the triangle ABD, have been constructed in the angle HKF, which is equal to E [Prop. 1.42]. And let the parallelogram GM, equal to the triangle DBC, have been applied to the straight-line GH in the angle GHM, which is equal to E [Prop. 1.44]. And since angle E is equal to each of (angles) HKF and GHM, (angle) HKF is thus also equal to GHM. Let KHG have been added to both. Thus, (the sum of) FKH and KHG is equal to (the sum of) KHG and GHM. But, (the sum of) FKH and KHG is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) KHG and GHM is also equal to two right-angles. So two straight-lines, KH and HM, not lying on the same side, make adjacent angles with some straight-line GH, at the point H on it, (whose sum is) equal to two right-angles. Thus, KH is straight-on to HM [Prop. 1.14]. And since the straight-line HG falls across the parallels KM and FG, the alternate angles MHG and HGF are equal to one another [Prop. 1.29]. Let HGL have been added to both. Thus, (the sum of) MHG and HGL is equal to (the sum of) HGF and HGL. But, (the sum of) MHG and HGL is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) HGF and HGL is also equal to two right-angles. Thus, FG is straight-on to GL [Prop. 1.14]. And since FK is equal and parallel to HG [Prop. 1.34], but also HG to ML [Prop. 1.34], KF is thus also equal and parallel to ML [Prop. 1.30]. And the straight-lines KM and FL join them. Thus, KM and FL are equal and parallel as well [Prop. 1.33]. Thus, $KFLM$ is a parallelogram. And since triangle ABD is equal to parallelogram FH, and DBC to GM, the whole rectilinear figure $ABCD$ is thus equal to the whole parallelogram $KFLM$.

Thus, the parallelogram $KFLM$, equal to the given rectilinear figure $ABCD$, has been constructed in the angle FKM, which is equal to the given (angle) E. (Which is) the very thing it was required to do.

Commentary

Proposition 4.55. *If*

Proof. □

Proposition 46

To describe a square on a given straight-line.

Let AB be the given straight-line. So it is required to describe a square on the straight-line AB.

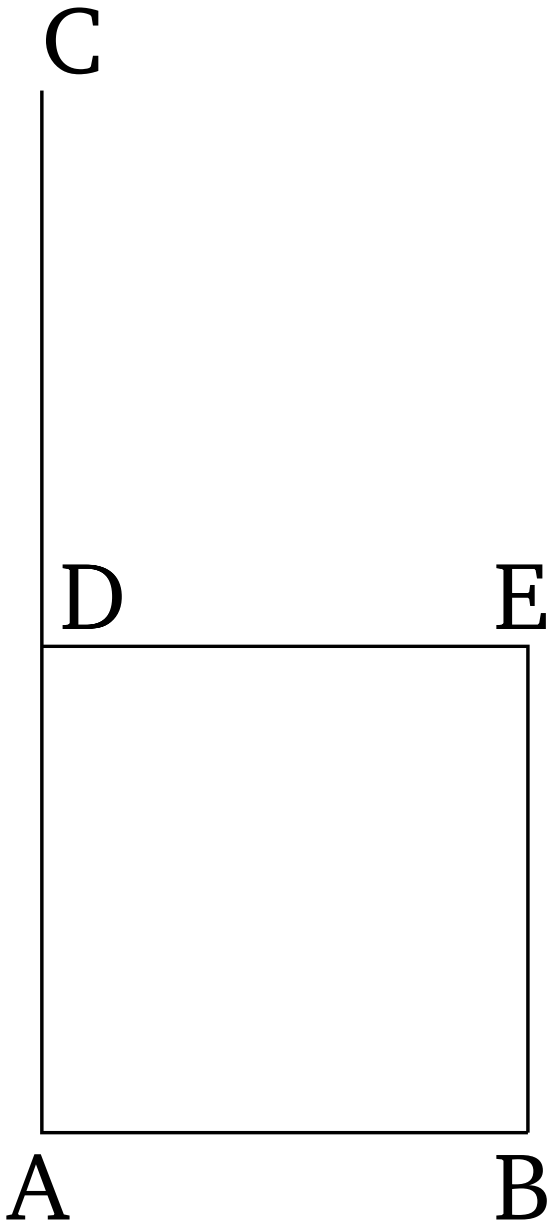
Let AC have been drawn at right-angles to the straight-line AB from the point A on it [Prop. 1.11], and let AD have been made equal to AB [Prop. 1.3]. And let DE have been drawn through point D parallel to AB [Prop. 1.31], and let BE have been drawn through point B parallel to AD [Prop. 1.31]. Thus, $ADEB$ is a parallelogram. Therefore, AB is equal to DE, and AD to BE [Prop. 1.34]. But, AB is equal to AD. Thus, the four (sides) BA, AD, DE, and EB are equal to one another. Thus, the parallelogram $ADEB$ is equilateral. So I say that (it is) also right-angled. For since the straight-line AD falls across the parallels AB and DE, the (sum of the) angles BAD and ADE is equal to two right-angles [Prop. 1.29]. But BAD (is a) right-angle. Thus, ADE (is) also a right-angle. And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles ABE and BED (are) also right-angles. Thus, $ADEB$ is right-angled. And it was also shown (to be) equilateral.

Thus, ($ADEB$) is a square [Def. 22]. And it is described on the straight-line AB. (Which is) the very thing it was required to do.

Commentary

Proposition 4.56. *If*

Proof. □



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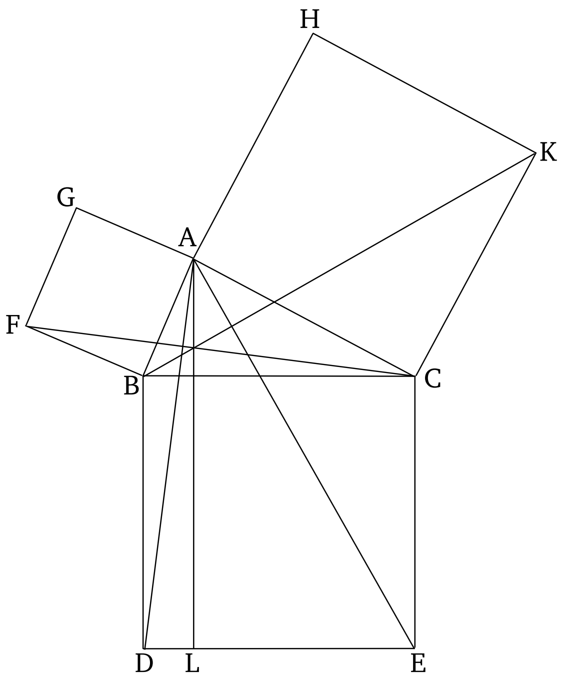
Proposition 47



In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let ABC be a right-angled triangle having the angle BACa right-angle. I say that the square on BC is equal to the (sum of the) squares on BA and AC.

For let the square $BDEC$ have been described on BC, and (the squares) GB and HC on AB and AC (respectively) [Prop. 1.46]. And let AL have been drawn through point A parallel to either of BD or CE [Prop. 1.31]. And let AD and FC have been joined. And since angles BAC and BAG are each right-angles, then two straight-lines AC and AG, not lying on the same side, make the adjacent angles with some straight-line BA, at the point A on it, (whose sum is) equal to two right-angles. Thus, CA is straight-on to AG [Prop. 1.14]. So, for the same (reasons), BA is also straight-on to AH. And since angle DBC is equal to FBA, for (they are) both right-angles, let ABC have been added to both. Thus, the whole (angle) DBA is equal to the whole (angle) FBC. And since DB is equal to BC, and FB to BA, the two (straight-lines) DB, BA are equal to the two (straight-lines) CB, BF,† respectively. And angle DBA (is) equal to angle FBC. Thus, the base AD [is] equal to the base FC, and the triangle ABD is equal to the triangle FBC [Prop. 1.4]. And parallelogram BL [is] double (the area) of triangle ABD. For they have the same base, BD, and are between the same parallels, BD and AL [Prop. 1.41]. And square GB is double (the area) of triangle FBC. For again they have the same base, FB, and are between the same parallels, FB and GC [Prop. 1.41]. [And the doubles of equal things are equal to one another.][‡] Thus, the parallelogram BL is also equal to the square GB. So, similarly, AE and BK being joined, the parallelogram CL can be shown (to be) equal to the square HC. Thus, the whole square $BDEC$ is equal to the (sum of the) two squares GB and HC. And the square $BDEC$ is described on BC, and the (squares) GB and HC on BA and AC (respectively). Thus, the square on the side BC is equal to the (sum of the) squares on the sides BA and AC.

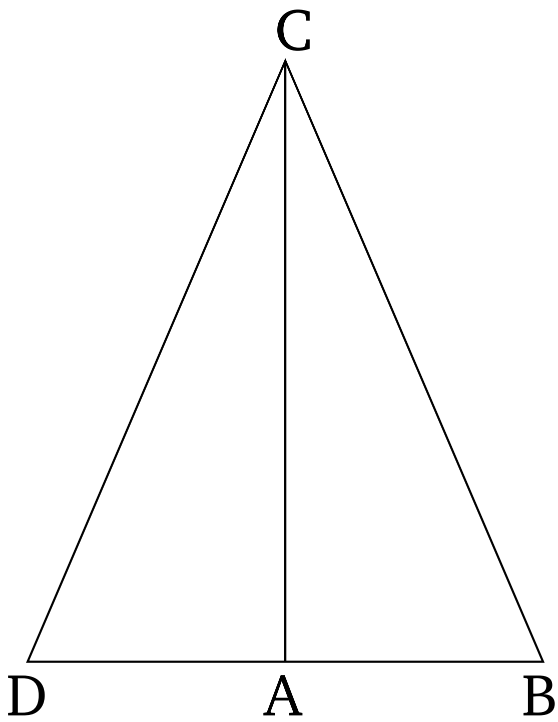
Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

Commentary

Proposition 4.57. *If*

Proof. □

Proposition 48



If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right-angle.

For let the square on one of the sides, BC, of triangle ABC be equal to the (sum of the) squares on the sides BA and AC. I say that angle BAC is a right-angle.

For let AD have been drawn from point A at right-angles to the straight-line AC [Prop. 1.11], and let AD have been made equal to BA [Prop. 1.3], and let DC have been joined. Since DA is equal to AB, the square on DA is thus also equal to the square on AB.[†] Let the square on AC have been added to both. Thus, the (sum of the) squares on DA and AC is equal to the (sum of the) squares on BA and AC. But, the (square) on DC is equal to the (sum of the squares) on DA and AC. For angle DAC is a right-angle [Prop. 1.47]. But, the (square) on BC is equal to (sum of the squares) on BA and AC. For (that) was assumed. Thus, the square on DC is equal to the square on BC. So side DC is also equal to (side) BC. And since DA is equal to

AB, and AC (is) common, the two (straight-lines) DA, AC are equal to the two (straight-lines) BA, AC. And the base DC is equal to the base BC. Thus, angle DAC [is] equal to angle BAC [Prop. 1.8]. But DAC is a right-angle. Thus, BAC is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show.

Commentary

Proposition 4.58. *If*

Proof. □