PDE type		Software	Solver	#Mesh
Burgers	1d-C	Comsol	BDF,MUMPS	1000
	2d-C	Comsol	BDF,MUMPS	24912
Poisson	2d-C	Comsol	MUMPS	40000
	2d-CG	Comsol	MUMPS	22420
	3d-CG	Comsol	MUMPS	371024
	2d-MS	Comsol	MUMPS	24912
Heat	all	Comsol	BDF,MUMPS	24912
Naiver-Stokes	2d-C	Comsol	PARDISO	10000
	2d-CG	Comsol	PARDISO	39294
	2d-LT	Comsol	BDF,PARDISO	43250
Wave	1d-C		Analytical	
	2d-CG	Comsol	Generalized alpha, MUMPS	24912
	2d-MS	Comsol	Generalized alpha, MUMPS	25140
Chaotic	GS	Chebfun	ETDRK4	1000
	KS	Chebfun	ETDRK4	1000
High dim	all		Analytical	
Inverse problems	all		Analytical	

Table 1: Details of the solver for reference data.

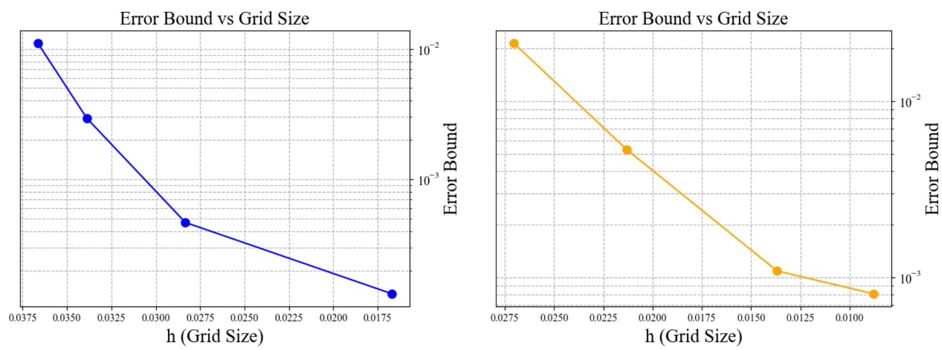

Figure 1: Richardson extrapolation and error bound analysis for Poisson3d-CG(left) and NS2d-CG(right).

Discussion about the reference solutions and Mesh Convergence Study.

Given that our benchmark includes various types of PDEs, we generated the reference data using different types of numerical solvers, including the FEM solvers in COMSOL, Chebfun. For different PDEs, selecting the appropriate numerical solver allows for higher precision results. The types of solvers used, mesh sizes, parameters, and convergence accuracies are detailed in Table 1. For these problems, highly optimized numerical solvers can achieve solutions with very high accuracy and theoretical guarantees. However, the choice of mesh discretization and solver type is highly dependent on the PDE type and parameters.

We conducted a mesh convergence study for the Poisson3d and NS2d-CG equations using grids with varying spacing. To estimate the error bound, we employed Richardson extrapolation, a technique that leverages the solutions from different grid sizes to predict the solution's behavior as the grid is further refined. The principle behind Richardson extrapolation is that the error in the numerical solution decreases predictably with grid refinement. If the error reduces as a power of the grid size h, the extrapolated solution $u_{\text{extrapolated}}$ can be calculated as:

$$u_{\text{extrapolated}} = \frac{h_2^p u_{h_1} - h_1^p u_{h_2}}{h_2^p - h_1^p}, \tag{1}$$

where u_{h_1} and u_{h_2} are the solutions obtained on grids with sizes h_1 and h_2 respectively, and p is the theoretical convergence rate. The error bound can then be estimated by comparing the extrapolated solution with the finer grid solution u_{h_2} as follows:

$$\text{Error Bound} = \frac{||u_{\text{extrapolated}} - u_{h_2}||}{||u_{\text{extrapolated}}||}. \tag{2}$$

We obtained the following Figure showing the error bound as a function of grid size. As we refined the grid, the differences between the solutions decreased, and the error bound rapidly decreased with smaller grid sizes. The error for the reference data we used was below 0.1%, indicating that it is highly reliable and can serve as a valid reference for the PINN solutions.