PHYBench 100 Problems Feedback Report

1 Problem Report

Among the 100 problems with detailed solutions published in PHYBench, there are 20 problems with incorrect answers and 5 problems that, although having correct answers, do not comply with benchmark standards.

1.1 Reference Answer Errors

1.1.1 Problem ID:55

Problem statement: In a vacuum, there is an infinitely long, uniformly charged straight line fixed in place, with a charge line density of λ. Additionally, there is a dust particle with mass m, which can be considered as an isotropic, uniform dielectric sphere with a volume V, and a relative permittivity ε_r. It is given that the volume V of the dielectric sphere is very small, the permittivity of vacuum is ε_0, and the following factors can be neglected: gravity, electromagnetic radiation caused by the motion of charges, and relativistic effects. Study the motion of the dust particle under the influence of the charged straight line: Find the force acting on the particle and provide its magnitude.

Final result provided by raw answer:

$$-\frac{3(\varepsilon_r - 1)V\lambda^2}{4\pi^2(\varepsilon_r + 2)r^3}$$

Reason for the error:

1. The problem asks for the magnitude of the force acting on the particle, but the reference answer includes a negative sign.

2. The final calculation step is missing ε_0.

Corrected final result:

$$\frac{3(\varepsilon_r - 1)V\lambda^2}{4\pi^2\varepsilon_0(\varepsilon_r + 2)r^3} \tag{1}$$

1.1.2 Problem ID:65

Problem statement: In the upper half-space, there is a uniform magnetic field with magnitude B directed vertically upwards, and the ground is sufficiently rough.

Now, there is a solid insulating sphere with uniformly distributed mass m and radius R . On its surface at the equator, there are 3 mutually orthogonal wires with negligible mass and thickness and a unit length resistance of λ. These wires are connected at the intersection point. The sphere is placed on the ground so that the circular surface formed by one of the wires is perpendicular to the direction of the magnetic field. It is known that the sphere's center has acquired a velocity v_0 in the horizontal plane, and its initial angular velocity is such that it is in a pure rolling state. What is the subsequent change in velocity?

Final result provided by raw answer:

$$\vec{v} = \vec{v_0} e^{-\frac{5\pi RB^2}{14m\lambda}t}$$

Reason for the error: The question asks for the "change in velocity," but the reference answer provides the final velocity instead of the change in velocity.

Corrected final result:

$$\vec{v} = \vec{v_0}\left(e^{-\frac{5\pi RB^2}{14m\lambda}t} - 1\right) \tag{2}$$

1.1.3 Problem ID:653

Problem statement: A massless insulating string of length α is fixed at one end, with a conductor ring of mass m and radius r ($r \ll \alpha$) suspended on the other end, forming a simple pendulum. Initially, the pendulum string is vertical, and the ring is just inside a sector of a uniform magnetic field. The center of the sector is at the suspension point, with its edges just tangent to the ring. The magnetic field is perpendicular to the plane of the ring and directed inward, with a magnetic flux density of B. If the ring is given an initial horizontal velocity (perpendicular to the magnetic field) with a magnitude of v, what is the minimum initial velocity v_0 required for the pendulum string to reach the horizontal direction? The gravitational acceleration is g, the magnetic field is sufficiently strong, and the torque due to the Ampere force is much greater than the torque due to gravity when the ring moves within the magnetic field. The ring is a superconducting ring with a self-inductance coefficient L.

Final result provided by raw answer:

$$\sqrt{2ga + \frac{\pi^2 B^2 r^4}{mL}}$$

Reason for the error: The problem uses the symbol α for length, but the reference answer uses the symbol a.

Corrected final result:

$$\sqrt{2g\alpha + \frac{\pi^2 B^2 r^4}{mL}}$$

1.1.4 Problem ID:204

Problem statement: In modern plasma physics experiments, two methods are commonly used to confine negatively charged particles. In the following discussion, relativistic effects and contributions such as delayed potentials are not considered.

In space, uniformly charged rings with a radius of R and a charge Q_0 are placed on planes $z = l$ and $z = -l$, respectively. Additionally, a particle with a charge of $-q$ and a mass of m is located at the origin of the coordinate system. The permittivity of vacuum is given as ε_0.

Find the angular frequency ω_z corresponding to the perturbation stability of the particle in the $\hat{z}$ direction.

Final result provided by raw answer:

$$\sqrt{\frac{Qq}{2\pi\varepsilon_0 m}\frac{R^2-2l^2}{(R^2+l^2)^{5/2}}}$$

Reason for the error: The problem uses the symbol Q_0, but the reference answer uses Q.

Corrected final result:

$$\sqrt{\frac{Q_0 q}{2\pi\varepsilon_0 m}\frac{R^2-2l^2}{(R^2+l^2)^{5/2}}} \tag{3}$$

1.1.5 Problem ID:259

Problem statement: In the inertial frame S, at time $t = 0$, four particles simultaneously start from the origin and move in the directions of $+x, -x, +y, -y$, respectively, with velocity v. Consider another inertial frame S', which moves relative to S along the positive x-axis with velocity u. At the initial moment, the two reference frames satisfy $t = t' = 0$, where t' represents the time in the S' reference frame, and at the initial moment, the origins of the two reference frames coincide. Derive the relationship between the area of the quadrilateral formed by connecting the four particles in reference frame S' and time t'. Consider the effects of special relativity, with the speed of light given as c.

Final result provided by raw answer:

$$\frac{2v^2t'^2(1-u^2/c^2)^{3/2}}{1-(uv/c)^2}$$

Reason for the error: The reference answer has incorrect dimensions; the denominator should be $(1 - (uv/c^2)^2)$.

Corrected final result:

$$\frac{2v^2t'^2(1-u^2/c^2)^{3/2}}{1-(uv/c^2)^2}$$

1.1.6 Problem ID:318

Problem statement: Consider a small cylindrical object with radius R and height h. Determine the expression for the force F acting on the cylinder when a sound wave passes through it. The axial direction of the cylinder is the direction of wave propagation. In the sound wave, the displacement of a particle from its equilibrium position is $\psi = A\cos kx \cos 2\pi f t$, the ambient pressure is P_0, and the adiabatic index of air is γ.

Final result provided by raw answer: The force on the cylinder will be

$$F = -\pi R^2(p(y+h) - p(y)) = -\pi R^2 h \frac{dp}{dy}$$

For a standing wave, we have

$$\psi = A\cos kx \cos 2\pi f t$$

Thus,

$$\Delta p = \gamma P_0 k A \sin kx \cos 2\pi f t$$

Therefore,

$$\frac{dp}{dx} = \gamma P_0 k^2 A \cos kx \cos 2\pi f t$$

Thus, the force is

$$F = -\pi R^2 h \gamma P_0 k^2 A \cos kx \cos 2\pi f t$$

Reason for the error: The answer does not clearly state important assumptions such as $kh \ll 1$; however, the reference answer made too many approximations and assumptions without explanation during the derivation process, resulting in a final force that always averages to zero over time, which is unreasonable.

Corrected final result:

$$\boxed{F = -\frac{1}{2}\pi R^2 h \gamma P_0 A^2 k^3 \sin(2kx)} \tag{4}$$

Reference derivation can be found in Appendix A.1, provided by LOCA.

1.1.7 Problem ID:10

Problem statement: The region in space where $x > 0$ and $y > 0$ is a vacuum, while the remaining region is a conductor. The surfaces of the conductor are the xOz plane and the yOz plane. A point charge q is fixed at the point (a, b, c) in the vacuum, and the system has reached electrostatic equilibrium. Find the magnitude of electric field intensity on the surface of the conductor at the xOz plane, $E(x, +0, z)$.

Final result provided by raw answer: The infinite conductor and the point at infinity are equipotential, U=0, with the boundary condition that the tangential electric field E_τ=0. By the uniqueness theorem of electrostatics, the induced charges on the conductor can be equivalently replaced by image charges to determine the contribution to the electric potential in the external space of the conductor. The top view of the image charges is shown below.

Therefore, the answer is $\vec{E}(x, +0, z) = \left(\frac{-2q}{4\pi\varepsilon_0}\frac{b}{r_1^3} + \frac{2q}{4\pi\varepsilon_0}\frac{b}{r_2^3}\right)\hat{y} = \frac{qb}{2\pi\varepsilon_0}\left[\left[(a+x)^2 + (z-c)^2 + b^2\right]^{\frac{-3}{2}} - \left[(x-a)^2 + (z-c)^2 + b^2\right]^{\frac{-3}{2}}\right]\hat{y}$, $\vec{E}(+0, y, z) = \left(-\frac{2q}{4\pi\varepsilon_0}\frac{a}{r_1^3} + \frac{2q}{4\pi\varepsilon_0}\frac{a}{r_2^3}\right)\hat{x} = \frac{qa}{2\pi\varepsilon_0}\left[\left[(y+b)^2 + \left(z-c\right)^2 + a^2\right]^{\frac{-3}{2}} - \left[\left(y-b\right)^2 + \left(z-c\right)^2 + a^2\right]^{\frac{-3}{2}}\right]\hat{x}$

The final answer is

$$\frac{qb}{2\pi\varepsilon_0}\left[((a+x)^2 + (z-c)^2 + b^2)^{-3/2} - ((x-a)^2 + (z-c)^2 + b^2)^{-3/2}\right]$$

Reason for the error: The problem asks for the "magnitude of electric field intensity", which means the magnitude of the electric field strength. The final result should be taken as an absolute value, but the reference answer does not take the absolute value, resulting in a difference of a negative sign from the correct result (note that the charge is located in vacuum at (a, b, c), so $a > 0$).

Corrected final result:

$$\boxed{|\vec{E}(x, +0, z)| = \frac{|q|b}{2\pi\varepsilon_0}\left[\frac{1}{((x-a)^2 + b^2 + (z-c)^2)^{3/2}} - \frac{1}{((x+a)^2 + b^2 + (z-c)^2)^{3/2}}\right]} \tag{5}$$

1.1.8 Problem ID:88

Problem statement: In a certain atom A, there are only two energy levels: the lower energy level A_0 is called the ground state, and the higher energy level $A^{\star}$ is called the excited state. The energy difference between the excited state and the ground state is E_0. When the atom is in the ground state, its rest mass is m_0; when it is in the excited state, due to its intrinsic instability, it will transition to the ground state while emitting photons externally. It is known that the probability of transition from the excited state to the ground state in unit time in the atom's rest reference frame is λ, and the probability of emitting photons in all directions is equal. At the initial moment, the total number of atoms is $N_0 (N_0 \gg 1)$, and the experimental reference frame S and the atom's proper reference frame S' are time-synchronized. All N_0 atoms are in the excited state and have a common velocity $\boldsymbol{v}$ in the $+\hat{\boldsymbol{x}}$ direction relative to the S frame.

Considering $m_0 c^2 \gg E_0$, at this time the recoil effect on the atom due to photon emission can be ignored. Try to solve the angular distribution of the light emission power $w(\theta)$ from the atom at time $\boldsymbol{t}$ in the laboratory reference frame S. (Calculate the emission power per unit solid angle in the direction of angle $\boldsymbol{\theta}$ relative to the $+\hat{\boldsymbol{x}}$ axis, rather than the received power at an infinite distance. The speed of light in vacuum is c (known). The answer should be expressed using $\lambda, N_0, E_0, v, \theta, \pi, t, c$. Please check and output the final answer.

Final result provided by raw answer: Consider the reference frame S':

$$h\nu_0 = E_0$$

Let the remaining number of atoms at time t' be N':

$$-\mathrm{d}N' = \lambda N' \mathrm{d}t'$$

Solving for N', we get:

$$N' = N_0 e^{-\lambda t'}$$

The total power emitted by photons:

$$P' = -\dot{N}' h\nu_0 = \lambda N_0 E_0 e^{-\lambda t'}$$

Consider the relativistic transformation of energy, momentum, and time between s' and s:

$$\mathrm{d}t = \gamma \mathrm{d}t'$$

$$\mathrm{d}E_{light} = \gamma(\mathrm{d}E'_{light} + v\mathrm{d}p'_{light}) = \gamma\mathrm{d}E'_{light}$$

Using the relations:

$$\mathrm{d}E_{light} = P\mathrm{d}t \ \mathrm{d}E'_{light} = P'\mathrm{d}t'$$

We obtain:

$$P = P' = \lambda N_0 E_0 e^{-\lambda\frac{\varepsilon}{\gamma}}$$

Where:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Considering that number density is Lorentz invariant, let the number of photons emitted in S, within $\theta \sim \theta + \mathrm{d}\theta$, be $\dot{n}$.

From the transformation of the photon energy and momentum, the relationship between $\boldsymbol{\theta'}$ and $\boldsymbol{\theta}$ can be expressed as:

$$\dot{n}\mathrm{d}t \times 2\pi \sin\theta \mathrm{d}\theta = \frac{\dot{N}'}{4\pi}\mathrm{d}t' \times 2\pi \sin\theta' \mathrm{d}\theta'$$

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

And the transformation of time:

$$\mathrm{d}t' = \sqrt{1-\beta^2}\mathrm{d}t$$

Substituting into the equations, we get:

$$\dot{n} = \frac{\dot{N}'}{4\pi} \cdot \frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta} \cdot \frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\dot{N}'}{4\pi} \cdot \frac{(1-\beta^2)^{\frac{3}{2}}}{(1-\beta\cos\theta)^2}$$

Considering the frequency transformation of photons:

$$\nu = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\nu_0$$

We further obtain the power distribution of the emitted light:

$$w = \frac{\mathrm{d}P}{\mathrm{d}\Omega} = \dot{n}h\nu = \frac{P}{4\pi}\frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} = \frac{\lambda N_0 E_0}{4\pi}\frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} \cdot e^{-\lambda\frac{4}{7}}$$

Rewriting β in terms of velocity: In the relativistic framework, $\beta = \frac{v}{c}$ (where v is the velocity and c is the speed of light in a vacuum). Replacing β and λ with velocity-related quantities, the formula becomes:

$$w = \frac{\mathrm{d}P}{\mathrm{d}\Omega} = \dot{n}h\nu = \frac{P}{4\pi}\frac{(1-(\frac{v}{c})^2)^2}{(1-\frac{v}{c}\cos\theta)^3} = \frac{\lambda N_0 E_0}{4\pi}\frac{(1-(\frac{v}{c})^2)^2}{(1-\frac{v}{c}\cos\theta)^3}\cdot e^{-\lambda\frac{4}{7}}$$

Reason for the error: The coefficient $\frac{4}{7}$ in the reference answer should be an error made by the problem setter during image-to-text conversion. It should be $\frac{t}{\gamma}$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Corrected final result:

$$\boxed{w(\theta,t) = \frac{\lambda N_0 E_0}{4\pi}\frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v}{c}\cos\theta\right)^3}e^{-\lambda t\sqrt{1-\frac{v^2}{c^2}}}} \tag{6}$$

Reference derivation can be found in Appendix A.2, provided by LOCA.

1.1.9 Problem ID:420

Problem statement: B and C are two smooth fixed pulleys with negligible size, positioned on the same horizontal line. A and D are two objects both with mass m, connected by a light and thin rope that passes over the fixed pulleys. Initially, the system is stationary, and the distances between AB and CD are both in the direction of gravity, where the distance between AB is x_0 and the distance between CD is L, which is sufficiently large such that D will not touch C within the time frame discussed in the problem. At this moment, ball A is pulled so that the line AB deviates from the vertical line by a small angle θ_0 (without changing the length of AB), and the system begins to move. Taking gravitational acceleration as $\boldsymbol{g}$, and assuming that A descends very slowly so that we can approximately consider the length of AB remains unchanged, A moves around B with a pendulum-like motion. Try to solve for the amplitude of the oscillation angle θ of A when $AB = x$.

Final result provided by raw answer:

Let the tension in the rope be T. According to Newton's second law for block D, $T - mg = m\ddot{x}$ (1). Using the given assumptions, write the relation of the pendulum's oscillation angle with time:

$$\theta_t = \theta\cos(\sqrt{\frac{g}{x}}t + \phi)$$

For A, write down the expression of Newton's second law:

$$mg - T + m\dot{\theta}_t^2 x = m\ddot{x}$$

Combining (2) and (3), we get:

$$mg - T + \tfrac{1}{2}mg\theta^2 = m\ddot{x}$$

Since the work done on A by the tension in the rope and the gravity of A is equal to the increment in A's kinetic energy, and A's kinetic energy can be separately written as the kinetic energy of vertical motion and the kinetic energy of oscillation (as the oscillation angle is small, the two can be considered independent):

$$(mg - T)\dot{x} = \frac{d}{dt}(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}mgx\theta^2)$$

Combining (1), (4), and (5), we get:

$$g\theta^2 = 4\ddot{x} \quad -\ddot{x}\dot{x} = \frac{d}{dt}(\frac{1}{2}\dot{x}^2 + 2x\ddot{x})$$

$\bar{\mathcal{F}}\psi/\bar{\mathcal{H}}\ddot{x}dx = \frac{1}{2}d\dot{x}^2$, simplifying the above equation, we get:

$$\dot{x}^2 + 2x\ddot{x} = const$$

Using the initial conditions at `t`=0:

$$x = x_0 \quad \ddot{x} = g\theta_0^2/4$$

We can rewrite equation (8) as:

$$\dot{x}^2 + 2x\ddot{x} = g\theta_0^2 x_0/2$$

Multiplying both sides of the above equation by `dx`, and using $\ddot{\mathbf{x}}dx = \frac{1}{2}d\dot{x}^2$, we get:

$$\dot{x}^2 dx + x d\dot{x}^2 = d(x\dot{x}^2) = \frac{1}{2}g\theta_0^2 x_0 dx$$

Integrating, we obtain:

$$\dot{x}^2 = \frac{1}{2}g\theta_0^2 x_0(x - x_0)/x$$

Differentiating both sides with respect to time, eliminating the first-order derivative of `x` with respect to `t`, we get:

$$\ddot{x} = \frac{1}{4}g\theta_0^2 \frac{x_0^2}{x^2}$$

Combining with equation (6), we can solve for the amplitude of the oscillation angle θ:

$$\theta = \theta_0 \frac{x_0}{x}$$

Reason for the error: The reference answer used many approximations (without explanation) in the process of deriving the equations of motion using Newtonian mechanics, and also used a large number of approximations in the subsequent derivation process. The final result is inconsistent with the result from LOCA. We conducted numerical simulations for verification, and ultimately proved that LOCA's answer is correct.

The numerical simulation code and results can be found in Appendix A.5.

Corrected final result: $\Theta(x) = \theta_0 \left(\frac{x_0}{x}\right)^{3/4}$

Reference derivation can be found in Appendix A.4, provided by LOCA.

1.1.10 Problem ID:654

Problem statement: The forty-second batch of Trisolarian immigrants arrived on Earth aboard a spaceship propelled by photons. The spaceship has a rest mass of M, and it uses the method of annihilating matter and antimatter to produce photons, which are then emitted directly backward to provide thrust. In the reference frame of the spaceship, the frequency of the photons emitted backward is u_0. For simplicity, we assume that the ẅindward sideöf the Trisolarian spaceship is a flat surface with an area S. During flight, dust floating in the universe may collide with it. We assume that the cosmic dust is composed of particles with rest mass m. For an observer on Earth: the spaceship moves at a constant speed of v_1, the particles in the cosmic dust have a speed of v_2 and move in the same direction as the spaceship, and the particles in the dust are uniformly distributed with a density of n. If, according to the Trisolarian beings on the spaceship, the cosmic dust undergoes a p̈erfectly elastic collisionüpon impact with the spaceship, what is the number N' of photons that need to be emitted backward per unit time in the reference frame of the spaceship to maintain its constant speed? Given are Planck's constant h, the speed of light in vacuum c, and relativistic effects need to be considered.

Final result provided by raw answer:

$$\frac{2nSmc(v_1 - v_2)^2}{(1 - \left(\frac{v_1}{c}\right)^2)\sqrt{1 - \left(\frac{v_2}{c}\right)^2} h\nu_0}$$

Reason for the error:

The problem uses the symbol u_0 for "the frequency of the photons emitted backward", but the reference answer uses ν_0.

Corrected final result:

$$\frac{2nSmc(v_1 - v_2)^2}{\left(1 - \left(\frac{v_1}{c}\right)^2\right)\sqrt{1 - \left(\frac{v_2}{c}\right)^2} hu_0}$$

1.1.11 Problem ID:121

Problem statement: Two spacecraft are traveling in a space medium that is flowing uniformly at a constant velocity u with respect to an inertial frame S. The spacecraft are moving through the medium with equal relative velocities v with respect to the medium. Neither of the velocities is known. Ultimately, the velocities of the two spacecraft with respect to the inertial frame S are v_1 and v_2, and the angle between the directions of these velocities is an acute angle α. These three quantities are given. The speed of light is c. Considering relativistic effects, determine the minimum possible value of u.

Final result provided by raw answer: First, let's derive a byproduct of velocity transformation. The velocity transformation is:

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2}$$

$$v'_{y,z} = v_{y,z} \cdot \frac{\sqrt{1 - u^2/c^2}}{1 - uv_x/c^2}$$

Hence, we can calculate:

$$\gamma = \frac{1}{\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)/c^2}} = \frac{1}{\sqrt{1 - (v_x'^2 + v_y'^2 + v_z'^2)/c^2}}$$

Obtain the transformation:

$$\gamma' = \gamma \cdot \frac{1 - uv_x/c^2}{\sqrt{1 - u^2/c^2}}$$

The minimum value of v can be calculated by first determining the relative velocity of two planes:

$$\gamma_{12} = \frac{1 - \beta_1\beta_2 \cos\alpha}{\sqrt{(1 - \beta_1^2)(1 - \beta_2^2)}}$$

Consider a system where two spaceships have equal but opposite velocities, this velocity $v = \beta c$ is what we seek. At this moment, the relative velocity of the two planes can be obtained:

Thus, jointly obtain:

$$\gamma_{12} = \frac{1 + \beta^2}{\sqrt{(1 - \beta^2)(1 - \beta^2)}} = \frac{1 + \beta^2}{1 - \beta^2}$$

$$v = c\sqrt{\frac{c^2 - v_1 v_2 \cos\alpha - \sqrt{(c^2 - v_1^2)(c^2 - v_2^2)}}{c^2 - v_1 v_2 \cos\alpha + \sqrt{(c^2 - v_1^2)(c^2 - v_2^2)}}}$$

The minimum value of $u = \beta_{\mathrm{u}} c$ can be set such that its magnitude and direction in the original reference system are between angles φ_1, φ_2 of both velocities. Therefore, after transformation, shared velocity:

$$= \gamma_i \frac{1 - \beta_u \beta_{\mathrm{icos}\varphi_\mathrm{i}}}{\sqrt{1-\beta_u^2}}$$

$$\gamma_1' = \gamma_2'$$

Obtain velocity:

$$\beta_u = \frac{\gamma_1 - \gamma_2}{\gamma_1 \beta_1 \cos\varphi_1 - \gamma_2 \beta_2 \cos\varphi_2}$$

The denominator can be expanded using the auxiliary angle formula to find extremes:

$$\cdots = (\gamma_1\beta_1 - \gamma_2\beta_2\cos\alpha)\cos\varphi_1 - \gamma_2\beta_2\sin\alpha\sin\varphi_1 \leq \sqrt{(\gamma_1\beta_1 - \gamma_2\beta_2\cos\alpha)^2 + (\gamma_2\beta_2\sin\alpha)^2}$$

Substitute to find the extreme value:

$$u = c^2 \frac{\left|\sqrt{c^2 - v_1^2} - \sqrt{c^2 - v_1^2}\right|}{\sqrt{c^2\left(v_1^2 + v_2^2\right) - 2v_1^2 v_2^2 - 2v_1 v_2 \cos\alpha\sqrt{(c^2 - v_1^2)(c^2 - v_2^2)}}}$$

Reason for the error:

A typographical error occurred during the derivation process, causing the second v_2 in the numerator of the final answer to be mistakenly written as v_1.

Corrected final result:

$$u = c^2 \frac{\left|\sqrt{c^2 - v_1^2} - \sqrt{c^2 - v_2^2}\right|}{\sqrt{c^2(v_1^2 + v_2^2) - 2v_1^2 v_2^2 - 2v_1 v_2 \cos\alpha\sqrt{(c^2 - v_1^2)(c^2 - v_2^2)}}}$$

Or in a more concise form using γ_1, γ_2:

$$u = \frac{c|\gamma_1 - \gamma_2|}{\sqrt{\gamma_1^2 + \gamma_2^2 - 2 - \frac{2}{c^2}\gamma_1\gamma_2 v_1 v_2 \cos\alpha}} \tag{7}$$

where $\gamma_1 = \frac{1}{\sqrt{1 - v_1^2/c^2}}$ and $\gamma_2 = \frac{1}{\sqrt{1 - v_2^2/c^2}}$.

1.1.12 Problem ID:221

Problem statement: A three-dimensional relativistic oscillator moves in a space filled with uniform "dust." During motion, "dust" continuously adheres to the oscillator, which is assumed to increase the rest mass of the sphere without altering its size. The collision is adiabatic, and the "dust" quickly replenishes the region the sphere just passed. The original length of the spring is vanishing, and the potential energy V can be expressed as $V = \frac{1}{2}kx^2$, where k is a known constant. The cross-sectional area of the sphere is A, the density of the "dust" is ρ, the initial rest mass of the sphere is m_0, and the speed of light in vacuum is c. Initially, the sphere performs uniform circular motion with a radius R_0. Find the time t required when the circular motion radius changes to R. Assume $R^2 \gg A$ and $m_0 \gg \rho AR$.

Final result provided by raw answer:

$$t = \frac{1}{\rho A}\sqrt{\frac{\left(\frac{kR_0^2}{m_0c^2} + \sqrt{\left(\frac{kR_0^2}{m_0c^2}\right)^2 + 4}\right) mR_0^4}{2k}} \left(\frac{k}{c^2}\left(-\frac{1}{R} + \frac{1}{R_0}\right) + \frac{4}{7}\frac{\left(\frac{kR_0^2}{m_0c^2} + \sqrt{\left(\frac{kR_0^2}{m_0c^2}\right)^2 + 4}\right) mR_0^4}{2}\left(\frac{1}{R^7} - \frac{1}{R_0^7}\right)\right) \tag{8}$$

Reason for the error: There are two typographical errors in the reference answer where the symbol m_0 was mistakenly written as m.

Corrected final result:

$$t = \frac{1}{\rho A}\sqrt{\frac{\left(\frac{kR_0^2}{m_0c^2} + \sqrt{\left(\frac{kR_0^2}{m_0c^2}\right)^2 + 4}\right) m_0R_0^4}{2k}} \left(\frac{k}{c^2}\left(-\frac{1}{R} + \frac{1}{R_0}\right) + \frac{4}{7}\frac{\left(\frac{kR_0^2}{m_0c^2} + \sqrt{\left(\frac{kR_0^2}{m_0c^2}\right)^2 + 4}\right) m_0R_0^4}{2}\left(\frac{1}{R^7} - \frac{1}{R_0^7}\right)\right) \tag{9}$$

1.1.13 Problem ID:636

Problem statement: Establish a Cartesian coordinate system Oxyz, with a hypothetical sphere of radius R at the origin. Place n rings of radius R along the meridional circles, all passing through the points (0, 0, R) and (0, 0, -R). The angle between any two adjacent rings is $\frac{\pi}{n}$, and each ring is uniformly charged with positive charge Q. The setup is stationary, and a negative charge $-q$ with mass m is performing circular motion on the equatorial plane at a distance r_0 from the center of the sphere ($r_0 \gg R$). Now, give the charge a radial disturbance, and attempt to find the difference between its radial oscillation period and angular revolution period. The vacuum permittivity is known as ϵ_0.

Final result provided by raw answer: Consider the case of a single circular loop, establishing a spherical coordinate system (r, α) with the loop axis as the polar axis. Then:

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{Qd\theta}{2\pi}\frac{1}{\sqrt{r^2+R^2-2Rr\sin\alpha\cos\Theta}} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{R^2}{r^3}\frac{1-3\cos^2\alpha}{2}\right)\#\left(\frac{1}{r}+\frac{R^2}{r^2}\right).$$

Next, consider the superposition of multiple circular loops. Based on geometric relations:

$$V_i = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{R^2}{r^3}\frac{1-3\sin^2\theta\cos^2\left(\varphi-\frac{\mathrm{i}\pi}{\mathrm{n}}\right)}{2}\right)$$

Summing results in:

$$V = \frac{nQ}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{R^2}{r^3}\frac{1-3\cos^2\theta}{4}\right)$$

On the equatorial plane, $\theta = \frac{\pi}{2}$

$$V = \frac{nQ}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{R^2}{4r^3}\right)$$

Assuming the angular momentum is $L = m\omega_\theta r_0^2$, the effective potential energy is

$$\mathrm{V_{eff}} = -\frac{nQq}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{R^2}{4r^3}\right)+\frac{L^2}{2mr^2}$$

The first derivative set to zero:

$$-\frac{nQq}{4\pi\varepsilon_0}\left(\frac{-1}{r^2}+\frac{3R^2}{4r^4}\right)-\frac{L^2}{mr^3}=0$$

Solving gives:

$$\omega_\theta^2 = \frac{nQq}{4\pi\varepsilon_0 m r_0}\left(\frac{1}{r_0^2}-\frac{3R^2}{4r_0^4}\right)$$

The second derivative:

$$-\frac{nQq}{4\pi\varepsilon_0}\left(\frac{2}{r^3}-\frac{3R^2}{r^5}\right)+\frac{3L^2}{mr^4}=m\omega_r^2$$

Solving gives:

$$\omega_r^2 = \frac{nQq}{4\pi\varepsilon_0 m r_0}\left(\frac{1}{r_0^2} + \frac{3R^2}{4r_0^4}\right)$$

$$T_r - T_{\oplus} = \frac{2\pi}{\omega^2}(\omega_\theta - \omega_r) = -\frac{3\pi R^2}{2r_0^2}\sqrt{\frac{4\pi\varepsilon_0 m r_0^3}{nQq}}$$

Reason for the error: The first step of the reference answer's derivation incorrectly wrote the multipole expansion formula for the electric potential, with the quadrupole moment coefficient differing by a factor of 1/2, leading to the final answer being off by a factor of 1/2.

Corrected final result:

$$-\frac{3\pi R^2}{4r_0^2}\sqrt{\frac{4\pi\epsilon_0 m r_0^3}{nQq}}$$

Reference derivation can be found in Appendix A.3, provided by LOCA.

1.1.14 Problem ID:332

Problem statement: There is a uniform thin spherical shell, with its center fixed on a horizontal axis and able to rotate freely around this axis. The spherical shell has a mass of M and a radius of R. There is also a uniform thin rod, with one end smoothly hinged to the axis at a distance d from the center of the sphere, and the other end resting on the spherical shell. The thin rod has a mass of m and a length of L ($d-R<L<\sqrt{d^2-R^2}$). Initially, the point of contact and the line connecting the sphere's center lie in the same vertical plane as the rod. Now, a small perturbation is applied, causing the system to begin rotating as a whole without relative sliding around the horizontal axis. When the entire system rotates by an angle φ, the spherical shell and the rod just begin to experience relative sliding. Determine the static friction coefficient μ between the spherical shell and the rod, expressed as a function of φ.

Final result provided by raw answer: Introducing simplified parameters

$$\beta=\cos^{-1}\frac{L^2+d^2-R^2}{2Ld}\gamma=\cos^{-1}\frac{R^2+d^2-L^2}{2Rd}$$

The coordinates of the contact point B are written as

$$\overrightarrow{OB}=(\cos\beta,\sin\beta\sin\varphi,\sin\beta\cos\varphi)L$$

Thus, the rod direction vector is written as

$$\hat{\boldsymbol{l}}=(\cos\beta,\sin\beta\sin\varphi,\sin\beta\cos\varphi)$$

Support force direction vector Assume the direction vector of the friction force is

$$\hat{\boldsymbol{n}}=(-\cos\gamma,\sin\gamma\sin\varphi,\sin\gamma\cos\varphi)$$

$$\hat{f}=\cos\theta\frac{\hat{l}\times\widehat{\boldsymbol{n}}}{\left|\hat{l}\times\widehat{\boldsymbol{n}}\right|}+\sin\theta\frac{\widehat{\boldsymbol{n}}\times(\hat{l}\times\widehat{\boldsymbol{n}})}{\left|\widehat{\boldsymbol{n}}\times(\hat{l}\times\widehat{\boldsymbol{n}})\right|}$$

That is

$$\hat{\boldsymbol{f}}=\cos\theta\,(0,-\cos\varphi,\sin\varphi)+\sin\theta\,(\sin\gamma,\cos\gamma\sin\varphi,\cos\gamma\cos\varphi)$$

$$(||\boldsymbol{\varphi}||\overset{\leq}{\sin}\theta|\leq\sin\gamma\sin\varphi(||\boldsymbol{\varphi}||,\cos\varphi|$$

For the spherical shell, by the angular momentum theorem

$$\frac{2}{3}MR^2\frac{d^2\varphi}{dt^2}=R\cos\theta\sin\gamma f$$

For the rod, by the angular momentum theorem

$$\begin{cases}\frac{1}{3}mL^2\sin\beta^2\frac{d^2\varphi}{dt^2}=\frac{1}{2}mgL\sin\beta\sin\varphi-L\cos\theta\sin\beta f & \triangledown\\ \frac{1}{3}mL^2\sin\beta\cos\beta(\frac{d\varphi}{dt})^2=\frac{1}{2}mgL\cos\beta\cos\varphi-L\sin(\beta+\gamma)N-L\sin\theta\cos(\beta+\gamma)f & \triangledown\end{cases}$$

From Equation ① It follows that

$$f\cos\theta = \frac{MmgR^2\sin\varphi}{2MR^2+mL^2\sin\beta^2} \qquad ⑭ \quad \frac{d^2\varphi}{dt^2} = \frac{3mgL\sin\beta\sin\varphi}{4MR^2+2mL^2\sin\beta^2} \qquad ⑮$$

From Equation ①

$$(\frac{d\varphi}{dt})^2 = \frac{3mgL\sin\beta(1-\cos\varphi)}{2MR^2+mL^2\sin\beta^2}$$

$$N = \frac{mg\left[MR^2(\cos\beta\cos\varphi - \tan\theta\sin\varphi\cos(\beta+\gamma)) + mL^2\sin\beta^2\cos\beta(\frac{3}{2}\cos\varphi-1)\right]}{\sin(\beta+\gamma)(2MR^2+mL^2\sin\beta^2)}$$

Thus

$$\mu = \frac{f}{N} = \frac{MR^2\sin\varphi\sin(\beta+\gamma)}{\cos\theta\left[MR^2(\cos\beta\cos\varphi - \tan\theta\sin\varphi\cos(\beta+\gamma)) + mL^2\sin\beta^2\cos\beta(\frac{3}{2}\cos\varphi-1)\right]}$$

To happen relative slipping, the above expression should take the minimum value

$$\mu = \frac{MR^2\sin\varphi\sin(\beta+\gamma)}{\sqrt{\cos\beta^2\left[MR^2\cos\varphi + mL^2\sin\beta^2(\frac{3}{2}\cos\varphi-1)\right]^2 + [MR^2\sin\varphi\cos(\beta+\gamma)]^2}}$$

Reason for the error: Due to the constraints, the friction force at the contact point cannot be in any arbitrary direction along the tangent plane of the sphere, because there is no tendency for relative motion in the tangent plane other than in the velocity direction (otherwise it would inevitably cause the rod to stretch or compress).

Corrected final result:

$$\mu = \frac{2MRd\sin\beta\sin\varphi}{\cos\beta\,[2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)]}, \cos\beta = \frac{d^2+L^2-R^2}{2dL}$$

Reference derivation can be found in Appendix A.6, provided by LOCA.

1.1.15 Problem ID:457

Problem statement: A regular solid uniform N-sided polygonal prism, with a mass of m and the distance from the center of its end face to a vertex as l, is resting on a horizontal table. The axis of the prism is horizontal and points forward. A constant horizontal force F acts on the center of the prism, perpendicular to its axis and large enough to cause the prism to start rotating. As a result, the prism will roll to the right while undergoing completely inelastic collisions with the table. It is known that the coefficient of static friction with the ground is sufficiently large. After a sufficiently long time, the angular velocity after each collision becomes constant. Find this angular velocity.

Final result provided by raw answer: For a regular N-sided polygon with the distance from its center to a vertex being l, the moment of inertia about the centroid is:

$$I_o = \frac{ml^2}{2}(\frac{1}{3}\sin^2\frac{\pi}{N} + \cos^2\frac{\pi}{N})$$

The moment of inertia about a vertex is:

$$I = \frac{ml^2}{2}(\frac{1}{3}\sin^2\frac{\pi}{N} + \cos^2\frac{\pi}{N}) + ml^2$$

After sufficient time has passed:

$$\frac{1}{2}I(\Omega^2 - \omega^2) = F \cdot 2l\sin\theta$$

$$I_o\Omega + m\Omega a\cos 2\theta a = \mathbf{I}\omega$$

Where:

$$\theta = \frac{\pi}{N}$$

After collision,

$$\omega = \sqrt{\frac{8F\sin\frac{\pi}{N}(9 - 5\tan^2\frac{\pi}{N})^2}{ml(2 + \frac{1}{3}\sin^2\frac{\pi}{N} + \cos^2\frac{\pi}{N})((9 + 7\tan^2\frac{\pi}{N})^2 - (9 - 5\tan^2\frac{\pi}{N})^2)}}$$

Reason for the error: 1. Too many steps skipped 2. Final equation solving error

Corrected final result:

$$\omega = \left|14\cos^2\left(\frac{\pi}{N}\right) - 5\right|\sqrt{\frac{F}{ml\sin\left(\frac{\pi}{N}\right)\left(7 + 2\cos^2\left(\frac{\pi}{N}\right)\right)\left(1 + 8\cos^2\left(\frac{\pi}{N}\right)\right)}}$$

Reference derivation can be found in Appendix A.7, provided by LOCA.

1.1.16 Problem ID:71

Problem statement: Given a particle with charge q and mass m moving in an electric field $\boldsymbol{E} = E_x\boldsymbol{x} + E_z\boldsymbol{z}$ and a magnetic field $\boldsymbol{B} = B\boldsymbol{z}$. The initial conditions are: position (x_0, y_0, z_0) and velocity $(v_\perp \cos\delta, v_\perp \sin\delta, v_z)$.

We know that a particle in a uniform magnetic field undergoes circular Larmor gyration, and the center of this rotation is called the guiding center. The drift velocity of this guiding center due to the electric field can be expressed using $\boldsymbol{E}, \boldsymbol{B}$ and their magnitudes.

Problem: We discuss the case where the magnetic field is uniform, but the electric field is non-uniform. For simplicity, we assume $\boldsymbol{E}$ is in the $\boldsymbol{x}$ direction and varies sinusoidally in the $\boldsymbol{y}$ direction.

$$\boldsymbol{E} \equiv E_0 \cos(ky)\hat{\boldsymbol{x}}$$

In reality, such a charge distribution can occur in a plasma during wave propagation. Find the corresponding guiding center drift velocity. It is known that the electric field is very weak, and the particle's initial position is (x_0, y_0, z_0). Approximate to the lowest order that can distinguish from the uniform electric field case.

Final result provided by raw answer: This problem has been modified; the original problem had four questions.

If there is an electric field present, we find that the motion of the particle will be a combination of two movements: the normal circular Larmor gyration and a drift towards the center of guidance. We can choose the $\boldsymbol{x}$ axis along the direction of $\boldsymbol{E}$, so $\boldsymbol{E}_y = 0$, and the velocity components related to transverse components can be treated separately. The equation of motion is

$$m\frac{dv}{dt} = q(\boldsymbol{E} + v \times \boldsymbol{B})$$

Its $\boldsymbol{z}$ component is

$$\frac{dv_z}{dt} = \frac{q}{m}E_z$$

Integrating yields

$$v_\tau = \frac{qE_z}{m}t + v_{z0}$$

This is a simple motion along the direction of $\boldsymbol{B}$. The transverse components of the previous equation are

$$\frac{dv_x}{dt} = \frac{q}{m}E_x \pm \omega_c v_y$$

$$\frac{dv_y}{dt} = 0 \pm \omega_c v_x$$

where $\omega_c \equiv \frac{|g|B}{m}$ is the Larmor gyration angular frequency. Differentiating these two equations gives

$$\ddot{v}_x = -\omega_c^2 v_x$$

$$\ddot{v}_y = -\omega_c^2 (v_y + \frac{E_x}{B})$$

We can rewrite the above equations as

$$\frac{d^2}{dt^2}(v_y + \frac{E_x}{B}) = -\omega_c^2 (v_y + \frac{E_x}{B})$$

Therefore, if we use $v_y + \frac{E_x}{B}$ instead of $\boldsymbol{v_y}$, the equation simplifies to the situation when the electric field is zero. Thus, the two equations can be replaced by

$$v_x = v_\perp e^{i(\omega_\mathrm{e} t + \delta)}$$

$$v_y = -i v_\perp e^{i(\omega_c t + \delta)} - \frac{E_x}{B}$$

where δ is the angle between the component of the particle's initial velocity in the $_{xy}$ plane $\boldsymbol{v}_\perp$ and the $\boldsymbol{x}$ axis (counter-clockwise direction). Further integration and taking the real part yields the equations of motion

$$x = x_0 + \frac{v_\perp}{\omega_c} \sin(\omega_c t + \delta)$$

$$y = y_0 - \frac{E_x}{B} t + \frac{v_\perp}{\omega_c} [1 - \cos(\omega_c t + \delta)]$$

$$z = \frac{qE_z}{2m} t^2 + v_{z0} t + z_0$$

From the equations of motion, it is observed that the Larmor motion of the particle is identical to the case without an electric field, but there is a drift superimposed in the $-y$ direction towards the center of guidance $\boldsymbol{v_{gc}}$ (for $E_z > 0$).

To obtain the general formula for $\boldsymbol{v_{gc}}$, we can solve the equation (1) using vector form. Since we already know that the term $m\frac{dv}{dt}$ only gives rise to circular motion with frequency ω_c, this term can be ignored in equation (1). Thus, equation (1) becomes

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0$$

By taking the cross product of $\boldsymbol{B}$ with the above expression, we get

$$\boldsymbol{E} \times \boldsymbol{B} = \boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{v}\boldsymbol{B}^2 - \boldsymbol{B}(\boldsymbol{v} \cdot \boldsymbol{B})$$

The transverse component of this equation is

$$v_{\perp qc} = E \times B/B^2 \equiv v_E$$

We define this transverse component as $\boldsymbol{v}_E$, which is the electric field drift towards the center of guidance.

The equation of motion for the particle is

$$m\frac{d\boldsymbol{v}}{dt} = q[\boldsymbol{E}(\boldsymbol{y}) + \boldsymbol{v} \times \boldsymbol{B}]$$

Decomposing into scalar form, we have

$$\dot{v}_x = \frac{qB}{m}v_y + \frac{q}{m}E_x(y)$$

$$\dot{v}_y = -\frac{qB}{m}v_x$$

Simplifying yields

$$\ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}_x(y)}{B}$$

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x(y)}{B}$$

Here, $\boldsymbol{E_x}(\boldsymbol{y})$ is the electric field at the particle's location. To compute this value, we need to know the particle's trajectory, which is precisely what we attempt to solve initially. If the electric field is weak, as a first approximation, we can estimate $E_x(y)$ using the undisturbed trajectory.

The trajectory in the absence of an electric field is given by

$$y = y_0 \pm r_L \cos \omega_c t$$

where $r_L = \frac{mv_\perp}{|q|B}$ is the Larmor gyration radius, and we obtain

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B} \cos k(y_0 \pm r_L \cos \omega_c t)$$

We seek a solution that is the sum of the gyration at ω_c and a stationary drift v_E. Since we are interested in expressing v_E, we can average over one cycle to eliminate the gyration motion. Thus, the equation yields

$$\bar{v}_x = 0$$

In the above equation, the average of the oscillating term $\ddot{v}_y$ is clearly zero, giving us

$$\overline{\ddot{v}}_y = 0 = -\omega_c^2 \overline{v}_y - \omega_c^2 \frac{E_0}{B} \overline{\cos k(y_0 \pm r_L \cos \omega_c t)}$$

Upon expanding the small approximation and averaging, we find

$$\overline{v}_y = -\frac{E_x(y_0)}{B}(1 - \frac{1}{4}k^2 r_L^2)$$

Thus, due to non-uniformity, the usual $\boldsymbol{E} \times \boldsymbol{B}$ drift is modified to

$$\boxed{v_E = \frac{E \times B}{B^2}(1 - \frac{1}{4}(k\frac{mv_\perp}{|q|B})^2)}$$

The correction term represents the finite Larmor radius effect under the sinusoidal electric field distribution.

Reason for the error: Incorrectly assumed the initial center position to be y_0

Corrected final result:

$$\mathbf{v}_{gc} = \frac{\mathbf{E}(Y_{gc,0}) \times \mathbf{B}}{B^2} \left(1 - \frac{1}{4} k^2 \left(\frac{mv_\perp}{|q|B} \right)^2 \right)$$

where the electric field is evaluated at the initial y-coordinate of the guiding center, $\mathbf{E}(Y_{gc,0}) = E_0 \cos(kY_{gc,0})\hat{\mathbf{x}}$, and this coordinate is given by $Y_{gc,0} = y_0 - \frac{mv_\perp \cos\delta}{qB}$. The total guiding center velocity is $\mathbf{V}_{gc} = \mathbf{v}_{gc} + v_z \hat{\mathbf{z}}$.

Reference derivation can be found in Appendix A.8, provided by LOCA.

1.1.17 Problem ID:139

Problem statement: Two smooth cylinders, each with a radius of r_1 but with weights P_1 and P_2 respectively, are placed in a smooth cylindrical groove with a radius of R. Find the tangent value of the angle φ when the two cylinders are in equilibrium. The angle φ is defined as the angle between the line connecting the center of the higher cylinder and the center of the cylindrical groove, and the horizontal line. Assume $\cos\beta = \frac{r}{R-r}$. Express the result in terms of β.

Final result provided by raw answer:

$$\varphi = \arctan \frac{P_2 - P_1 \cos(2\beta)}{P_1 \sin(2\beta)}$$

Reason for the error: The reference answer did not classify and discuss which cylinder is heavier between P_1 and P_2. The original answer is only valid for $P_1 > P_2$. The case where $P_1 < P_2$ should also be discussed, or a condition $P_1 > P_2$ should be added to the problem statement.

Corrected final result:

When $P_1 > P_2$: $\varphi = \arctan \frac{P_2 - P_1 \cos(2\beta)}{P_1 \sin(2\beta)}$
When $P_1 < P_2$: $\varphi = \arctan \frac{P_1 - P_2 \cos(2\beta)}{P_2 \sin(2\beta)}$

1.1.18 Problem ID:250

Problem statement: There is now an electrolyte with thickness L in the z direction, infinite in the x direction, and infinite in the y direction. The region where $y > 0$ is electrolyte 1, and the region where $y < 0$ is electrolyte 2. The conductivities of the two dielectrics are σ_1, σ_2, and the dielectric constants are $\varepsilon_1, \varepsilon_2$, respectively. On the xOz interface of the two dielectrics, two cylindrical holes with a radius R are drilled in the z direction, spaced $2D(D > R, R, D \ll L)$ apart, with centers located on the interface as long straight cylindrical holes. Two cylindrical bodies $\pm$ are inserted into the holes, with the type of the cylinders given by the problem text below.

The cylindrical bodies $\pm$ are metal electrodes filling the entire cylinder. Initially, the system is uncharged, and at $t = 0$, a power source with an electromotive force U and internal resistance r_0 is used to connect the electrodes. Find the relationship between the current through the power source and time, denoted as $i(t)$.

Final result provided by raw answer: Given the potential difference u, it can be seen:

$$\varphi_+ = u/2, \varphi_- = -u/2, \lambda = \frac{2\pi(\varepsilon_1 + \varepsilon_2)\varphi_+}{2\xi_+} = \frac{\pi(\varepsilon_1 + \varepsilon_2)u}{\text{arccosh(D/R)}}$$

Select a surface encapsulating the cylindrical surface and examine Gauss's theorem. For the positive electrode, it is easy to see:

$$\iint \vec{E} \cdot d\vec{S} = L \oint \vec{E} \cdot \hat{n} dl = \frac{\lambda L}{(\varepsilon_1 + \varepsilon_2)/2} = \frac{2\pi u L}{2\text{arccosh}(D/R)}$$

Since the above potential distribution is deemed directly applicable for the calculation of current, the total current flowing out of the positive electrode is:

$$I = \iint \sigma \vec{E} \cdot d\vec{S} = \frac{\sigma_1 + \sigma_2}{2} \times \frac{2\pi u L}{2\text{arccosh}(D/R)}$$

Given the current i passing through the power source, this current changes the net charge:

$$\frac{d(\lambda L)}{dt} = i - I = i - \frac{2\pi u L}{2\text{arccosh}(D/R)} = \frac{\pi(\varepsilon_1 + \varepsilon_2)L}{2\text{arccosh}(D/R)} \frac{du}{dt}$$

According to the loop voltage drop equation:

$$\begin{aligned}
&U = r_0 i + u \to u = U - r_0 i \\
&\to i - \frac{\pi(\sigma_1 + \sigma_2)L}{2\text{arccosh}(D/R)}(U - r_0 i) = -\frac{\pi(\varepsilon_1 + \varepsilon_2)L}{2\text{arccosh}(D/R)} r_0 \frac{di}{dt} \\
&\to \frac{di}{dt} = \frac{(\sigma_1 + \sigma_2)U}{r_0(\varepsilon_1 + \varepsilon_2)} - \left(\tfrac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} + \frac{2\text{arccosh}(D/R)}{\pi r_0 L(\varepsilon_1 + \varepsilon_2)} \right) i
\end{aligned}$$

At time $t = 0$, all current should preferentially enter the capacitor. At this time, the initial current is U/r_0, and this differential equation yields:

$$i(t) = \frac{U}{r_0\left(1 + \frac{2\text{arccosh}(D/R)}{\pi r_0 L(\sigma_1+\sigma_2)}\right)} \left\{2 + \frac{2\text{arccosh}(D/R)}{\pi r_0 L(\sigma_1 + \sigma_2)} - \exp\left[-\left(\frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} + \frac{2\text{arccosh}(D/R)}{\pi r_0 L(\varepsilon_1 + \sigma_2)}\right)\right]\right\},$$

Reason for the error: The final step of solving the differential equation was incorrect

Corrected final result:

$$\frac{UG}{2 + r_0 G}\left(1 + \frac{2}{r_0 G}\exp\left(-\frac{2 + r_0 G}{r_0 C}\right) t\right), \; G = \frac{\pi L(\sigma_1 + \sigma_2)}{\text{arccosh}\, D/R}, \; C = \frac{\pi L(\epsilon_1 + \epsilon_2)}{\text{arccosh}\, D/R}$$

Or in complete form:

$$\boxed{i(t) = \frac{U}{r_0 + \frac{2\,\text{arccosh}(D/R)}{\pi(\sigma_1+\sigma_2)L}} + \left(\frac{U}{r_0} - \frac{U}{r_0 + \frac{2\,\text{arccosh}(D/R)}{\pi(\sigma_1+\sigma_2)L}}\right)\exp\left[-\left(\frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} + \frac{2\,\text{arccosh}(D/R)}{\pi r_0(\varepsilon_1 + \varepsilon_2)L}\right) t\right]}$$

Reference derivation can be found in Appendix A.9, provided by LOCA.

1.1.19 Problem ID:450

Problem statement: Xiao Ming discovered an elliptical plate at home with semi-major and semi-minor axes of A and B, respectively. Using one focus F as the origin, a polar coordinate system was established such that the line connecting the focus and the vertex closest to the focus defines the polar axis direction. Through extremely precise measurements, it was found that the mass surface density satisfies the equation $\sigma = \sigma_0(1 + e\cos\varphi)^3$, where e is the eccentricity of the ellipse. When the plate's major axis is positioned vertically and released from rest on a sufficiently rough tabletop, the plate moves only within the plane it lies in. Find the angular acceleration β of the plate when the major axis becomes horizontal.

Final result provided by raw answer: At this time, the angular velocity is ω, the angular acceleration is β, and the acceleration of the center of mass is a_x, a_y.

The moment of inertia about the instantaneous center P is $I_P = I + mA^2$.

From the conservation of energy, the contact condition at point P gives the acceleration $a = \omega^2\rho = \omega^2\frac{A^2}{B}$.

Considering the rotational theorem about the instantaneous center: $I_P\vec{\beta} = \vec{M} - m\vec{r_c} \times \vec{a}$

We obtain:

$$I_P\beta = m\sqrt{A^2 - B^2}\left(g + \omega^2\frac{A^2}{B}\right)$$

$$\beta = 4A\sqrt{A^2 - B^2}g\frac{2A^4 + 2A^3\sqrt{A^2 - B^2} - A^3B + B^4}{(2A^3 + B^3)^2B}$$

Reason for the error: The final answer coefficient is incorrect.

Corrected final result:

$$\beta = \frac{2Ag\sqrt{A^2 - B^2}}{2A^3 + B^3}\left[1 + \frac{4A^3(A + \sqrt{A^2 - B^2} - B)}{B(2A^3 + B^3)}\right]$$

Reference derivation can be found in Appendix A.10, provided by LOCA.

1.1.20 Problem ID:649

Problem statement: In modern plasma physics experiments, negative particles are often constrained in two ways. In the following discussion, we do not consider relativistic effects or retarded potentials. Uniformly charged rings with radius R are placed on planes $z = l$ and $z = -l$ in space, respectively. The rings are perpendicular to the z axis, and each carries a charge of Q_0. A particle with charge $-q$ and mass m is placed at the origin of the coordinate system. Given that $Q_0, q > 0$, the vacuum permittivity is ϵ_0 and the vacuum permeability is μ_0. To keep the point charge stable in both the $\hat{z}$ and $\hat{r}$ directions, we consider rotating the two charged rings in the $\hat{z}$ direction with a constant angular velocity Ω. Given a small disturbance to the point charge at the origin, provide the minimum Ω required for the particle to remain stable in all directions.

Final result provided by raw answer: Analyzing the magnetic field at a small displacement z along the z-axis away from the coordinate origin

$$\vec{B_z} = \frac{\mu_0 Q \Omega R}{4\pi}\left(\frac{R}{[R^2+(l+z)^2]^{\frac{3}{2}}} + \frac{R}{[R^2+(l-z)^2]^{\frac{3}{2}}}\right)$$

yields

$$\vec{B_z} = \frac{\mu_0 Q \Omega}{2\pi}\frac{R^2}{(R^2+l^2)^{\frac{3}{2}}}\hat{z}$$

Observing that near the origin, $\vec{B}$ can be considered as a uniform field along the $\hat{z}$ direction, denoted as B_0, consider a particle starting from the origin. By the angular momentum theorem

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = -qB_0 r\dot{r}\vec{\varphi}$$

$$L + \frac{1}{2}qB_0 r^2 = 0$$

$$v_\varphi = -\frac{qB_0 r}{2m}$$

Substitute into the conservation of energy equation to obtain the system's effective potential energy

$$V_{eff} = [\frac{\mu_0^2 q^2 Q^2 \Omega^2}{32m\pi^2}\frac{R^4}{(R^2+l^2)^3} - \frac{qQ}{8\pi\varepsilon_0}\frac{(R^2-2l^2)}{(R^2+l^2)^{\frac{5}{2}}}]r^2 + \frac{qQ}{4\pi\varepsilon_0}\frac{(R^2-2l^2)}{(R^2+l^2)^{\frac{5}{2}}}z^2$$

For the system to be stable along $\hat{r}, \hat{z}$, the coefficients should both be greater than 0, solving gives

$$R^2 > 2l^2$$

$$\Omega > \frac{2\pi}{\mu_0 R^2}\sqrt{\frac{m(R^2-2l^2)(R^2+l^2)^{\frac{1}{2}}}{\pi\varepsilon_0 Q q}}$$

That is, the minimum value is

$$\frac{2\pi}{\mu_0 R^2}\sqrt{\frac{m(R^2-2l^2)(R^2+l^2)^{\frac{1}{2}}}{\pi\varepsilon_0 Qq}}$$

Reason for the error: 1. The definition of the "angle" vector in the angular momentum change formula is unclear. 2. The final result does not use the Q_0 symbol from the problem statement.

Corrected final result:

$$\frac{2\pi}{\mu_0 R^2}\sqrt{\frac{m(R^2-2l^2)(R^2+l^2)^{1/2}}{\pi\varepsilon_0 Q_0 q}}$$

1.2 Non-compliance with Benchmark Standards

The following are problems found during testing that do not comply with benchmark standards:

1.2.1 Problem ID:83

The problem asks to calculate the torque, but provides no indication regarding directionality. LLM responses sometimes include a negative sign and sometimes do not, but both should essentially be considered correct.

1.2.2 Problem ID:133

The problem asks for acceleration, and LLMs sometimes provide vector form answers, which should essentially be considered correct.

1.2.3 Problem ID:367

The problem does not specify whether to use $4\pi\epsilon_0$ or $1/k$, so LLM answers in both forms should be considered correct.

1.2.4 Problem ID:106

The problem does not clearly require the directionality of v (using the scalar symbol v rather than vector $\vec{v}$), so LLM answers giving the magnitude of velocity should be considered correct.

1.2.5 Problem ID:458

The problem uses μ in the viscous fluid viscosity formula, but uses η in the text description, so LLM answers using either η or μ symbols should be considered correct.

A Appendix

A.1 Problem ID:318

A.1.1 Problem Statement Explanation

This problem asks for the expression for the force F acting on a small cylindrical object of radius R and height h. The cylinder is placed in a medium where a sound wave is propagating. The cylinder's axis is aligned with the direction of wave propagation, which we define as the x-axis.

The given physical parameters are:

- R: The radius of the cylinder's circular faces.
- h: The height of the cylinder.
- $\psi(x,t) = A\cos(kx)\cos(2\pi ft)$: The displacement of particles in the medium from their equilibrium position. This mathematical form describes a one-dimensional standing wave.
- A: The amplitude of the particle displacement.
- k: The wavenumber of the sound wave.
- f: The frequency of the sound wave.
- P_0: The ambient pressure of the medium in the absence of the wave.
- γ: The adiabatic index of the medium (air).

Since the sound wave is oscillatory, the first-order pressure force will also be oscillatory with a time average of zero. The net, steady force arises from second-order effects and is known as the acoustic radiation force. We will calculate this time-averaged force, denoted by F. We assume the cylinder is located at a position x.

We will define the following auxiliary variables for the derivation:

- $\omega = 2\pi f$: The angular frequency of the sound wave.
- $S = \pi R^2$: The cross-sectional area of the cylinder.
- $V = Sh = \pi R^2 h$: The volume of the cylinder.
- ρ_0: The ambient density of the medium.
- c: The speed of sound in the medium.

We make the following assumptions:

1. The cylinder is "small," meaning its dimensions are much smaller than the sound wavelength $\lambda = 2\pi/k$. This is expressed as $kh \ll 1$ and $kR \ll 1$. This allows us to treat the cylinder as a point-like object within the sound field.
2. The cylinder is rigid (incompressible) and much denser than the surrounding medium (air). Its position is considered fixed for the force calculation.
3. The presence of the cylinder does not significantly distort the incident sound field. This allows us to use the properties of the unperturbed wave to calculate the force.

4. The compressions and rarefactions in the sound wave are adiabatic processes.

5. The Gor'kov potential, originally derived for a spherical particle, is assumed to be applicable to our small cylinder. This is a valid approximation when the particle dimensions (R, h) are much smaller than the wavelength, as the force primarily depends on the particle's volume and its bulk properties (compressibility and density) rather than its specific shape.

A.1.2 Step 1: Relate Pressure Perturbation to Particle Displacement

To find the force, we first need to determine the pressure variations in the sound wave. The pressure perturbation P_1 (the first-order deviation from ambient pressure P_0) can be related to the particle displacement ψ using the fundamental equations of fluid dynamics for an acoustic wave.

$$\boxed{\text{Linearized Equation of Continuity (1D): } \frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial x} = 0}$$

$$\boxed{\text{Adiabatic Pressure-Density Relation (Linearized): } P_1 = c^2 \rho_1} \tag{10}$$

$$\boxed{\text{Definition of Particle Velocity: } v_1 = \frac{\partial \psi}{\partial t}}$$

Here, v_1 is the particle velocity, P_1 is the pressure perturbation, and ρ_1 is the density perturbation, all considered first-order quantities.

We derive the relationship between P_1 and ψ as follows:

$$\begin{aligned}
\frac{\partial P_1}{\partial t} &= c^2 \frac{\partial \rho_1}{\partial t} && \text{(Time derivative of the adiabatic relation)} && (11) \\
&= c^2 \left(-\rho_0 \frac{\partial v_1}{\partial x} \right) && \text{(Substituting the continuity equation)} && (12) \\
&= -\rho_0 c^2 \frac{\partial}{\partial x} (v_1) && \text{(Rearranging terms)} && (13) \\
&= -\rho_0 c^2 \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial t} \right) && \text{(Substituting the definition of velocity)} && (14) \\
&= -\rho_0 c^2 \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial x} \right) && \text{(Swapping partial derivatives)} && (15)
\end{aligned}$$

Integrating with respect to time, we get $P_1(x,t) = -\rho_0 c^2 \frac{\partial \psi}{\partial x} + C(x)$. Since P_1 and ψ represent oscillating wave quantities, any time-independent term like $C(x)$ can be considered zero, as there should be no pressure perturbation in the absence of wave strain $(\partial \psi / \partial x = 0)$. Thus, we have our working relation:

$$P_1(x,t) = -\rho_0 c^2 \frac{\partial \psi}{\partial x} \tag{16}$$

A.1.3 Step 2: Calculate First-Order Field Variables

Using the given displacement field $\psi(x,t) = A\cos(kx)\cos(\omega t)$ (where $\omega = 2\pi f$), we can now find the explicit expressions for the first-order particle velocity v_1 and pressure perturbation P_1.

$$\begin{aligned}
v_1(x,t) &= \frac{\partial \psi(x,t)}{\partial t} = \frac{\partial}{\partial t}\left[A\cos(kx)\cos(\omega t)\right] \\
&= -A\omega\cos(kx)\sin(\omega t) && (17) \\
\frac{\partial \psi}{\partial x} &= \frac{\partial}{\partial x}\left[A\cos(kx)\cos(\omega t)\right] = -Ak\sin(kx)\cos(\omega t) && (18) \\
P_1(x,t) &= -\rho_0 c^2 \frac{\partial \psi}{\partial x} = -\rho_0 c^2\left[-Ak\sin(kx)\cos(\omega t)\right] \\
&= \rho_0 c^2 Ak\sin(kx)\cos(\omega t) && (19)
\end{aligned}$$

A.1.4 Step 3: The Acoustic Radiation Force and Gor'kov Potential

The time-averaged force on a small object in a sound field, the acoustic radiation force, is conservative and can be derived from an effective potential energy field, $U(x)$, known as the Gor'kov potential.

$$\boxed{\text{Force from Potential: } F = -\frac{dU}{dx}}$$

The general form of the Gor'kov potential for a small particle of volume V is given in terms of the time-averaged energy densities of the unperturbed sound field.

$$\boxed{\text{Gor'kov Potential (General Form): } U = V\left(f_1\langle E_p\rangle - f_2\langle E_k\rangle\right)}$$

where $\langle E_p\rangle$ is the time-averaged potential energy density and $\langle E_k\rangle$ is the time-averaged kinetic energy density. The dimensionless factors f_1 and f_2 are the monopole and dipole scattering coefficients, which depend on the material properties of the particle (subscript p) and the fluid medium (subscript f).

$$\boxed{f_1 = 1 - \frac{\kappa_p}{\kappa_f} \quad \text{and} \quad f_2 = \frac{2(\rho_p - \rho_f)}{2\rho_p + \rho_f}}$$

Here, κ is the compressibility ($\kappa = 1/K$, where K is the bulk modulus) and ρ is the density. For the problem of a solid cylinder in air, we assume the cylinder is rigid and dense.

1. **Rigid particle**: The compressibility of the solid particle is negligible compared to that of air, $\kappa_p \ll \kappa_f$, so $\kappa_p/\kappa_f \approx 0$.

2. **Dense particle**: The density of the solid particle is much greater than that of air, $\rho_p \gg \rho_f$, so $\rho_f/\rho_p \approx 0$.

Applying these assumptions to the factors f_1 and f_2:

$$f_1 = 1 - \frac{\kappa_p}{\kappa_f} \approx 1 - 0 = 1 \tag{20}$$

$$f_2 = \frac{2(\rho_p - \rho_f)}{2\rho_p + \rho_f} = \frac{2\rho_p(1 - \rho_f/\rho_p)}{2\rho_p(1 + \rho_f/(2\rho_p))} \approx \frac{2(1-0)}{2(1+0)} = 1 \tag{21}$$

Thus, for a small, rigid, dense object, the Gor'kov potential simplifies to:

$$U = V\left(\langle E_p\rangle - \langle E_k\rangle\right) \tag{22}$$

A.1.5 Step 4: Time-Averaged Kinetic and Potential Energy Densities

We now calculate the time-averaged energy densities using the first-order quantities from Step 2.

$$\boxed{\text{Acoustic Kinetic Energy Density: } E_k = \frac{1}{2}\rho_0 v_1^2}$$

$$\boxed{\text{Acoustic Potential Energy Density: } E_p = \frac{P_1^2}{2\rho_0 c^2}}$$

$$\boxed{\text{Time Average Identity: } \langle \sin^2(\omega t)\rangle = \langle \cos^2(\omega t)\rangle = \frac{1}{T}\int_0^T \sin^2(\omega t)dt = \frac{1}{2}}$$

First, we calculate the time-averaged kinetic energy density $\langle E_k\rangle$:

$$\begin{aligned}\langle E_k\rangle &= \left\langle \frac{1}{2}\rho_0 v_1^2 \right\rangle = \frac{1}{2}\rho_0 \left\langle (-A\omega\cos(kx)\sin(\omega t))^2 \right\rangle \\ &= \frac{1}{2}\rho_0 A^2\omega^2\cos^2(kx)\langle\sin^2(\omega t)\rangle \\ &= \frac{1}{4}\rho_0 A^2\omega^2\cos^2(kx)\end{aligned} \tag{23}$$

Next, we calculate the time-averaged potential energy density $\langle E_p\rangle$:

$$\begin{aligned}\langle E_p\rangle &= \left\langle \frac{P_1^2}{2\rho_0 c^2} \right\rangle = \frac{1}{2\rho_0 c^2}\left\langle (\rho_0 c^2 Ak\sin(kx)\cos(\omega t))^2 \right\rangle \\ &= \frac{(\rho_0 c^2 Ak)^2}{2\rho_0 c^2}\sin^2(kx)\langle\cos^2(\omega t)\rangle \\ &= \frac{1}{4}\rho_0 c^2 A^2 k^2 \sin^2(kx)\end{aligned} \tag{24}$$

A.1.6 Step 5: Derivation of the Radiation Potential

Now we substitute the expressions for the averaged energy densities into the simplified formula for the radiation potential U from Eq. (22).

$$\boxed{\text{Dispersion Relation for Sound: } \omega = ck}$$

$$\begin{aligned}
U(x) &= V\left(\langle E_p\rangle - \langle E_k\rangle\right) \\
&= V\left(\frac{1}{4}\rho_0 c^2 A^2 k^2 \sin^2(kx) - \frac{1}{4}\rho_0 A^2 \omega^2 \cos^2(kx)\right) && \text{(Substituting from eq. 24 and 23)} \quad (25) \\
&= V\left(\frac{1}{4}\rho_0 c^2 A^2 k^2 \sin^2(kx) - \frac{1}{4}\rho_0 A^2 (ck)^2 \cos^2(kx)\right) && \text{(Using the dispersion relation)} \\
&= \frac{1}{4}V\rho_0 c^2 A^2 k^2 \left(\sin^2(kx) - \cos^2(kx)\right) \\
&= -\frac{1}{4}V\rho_0 c^2 A^2 k^2 \left(\cos^2(kx) - \sin^2(kx)\right) \\
&= -\frac{1}{4}V\rho_0 c^2 A^2 k^2 \cos(2kx) && \text{(Using } \cos(2\theta) = \cos^2\theta - \sin^2\theta\text{)} \quad (26)
\end{aligned}$$

A.1.7 Step 5: Derivation of the Radiation Potential

Now we substitute the expressions for the averaged energy densities into the simplified formula for the radiation potential U from Eq. (22).

$$\boxed{\text{Dispersion Relation for Sound: } \omega = ck}$$

$$\begin{aligned}
U(x) &= V\left(\langle E_p\rangle - \langle E_k\rangle\right) \\
&= V\left(\frac{1}{4}\rho_0 c^2 A^2 k^2 \sin^2(kx) - \frac{1}{4}\rho_0 A^2 \omega^2 \cos^2(kx)\right) && \text{(Substituting from eq. 24 and 23)} \quad (27) \\
&= V\left(\frac{1}{4}\rho_0 c^2 A^2 k^2 \sin^2(kx) - \frac{1}{4}\rho_0 A^2 (ck)^2 \cos^2(kx)\right) && \text{(Using the dispersion relation)} \\
&= \frac{1}{4}V\rho_0 c^2 A^2 k^2 \left(\sin^2(kx) - \cos^2(kx)\right) \\
&= -\frac{1}{4}V\rho_0 c^2 A^2 k^2 \left(\cos^2(kx) - \sin^2(kx)\right) \\
&= -\frac{1}{4}V\rho_0 c^2 A^2 k^2 \cos(2kx) && \text{(Using } \cos(2\theta) = \cos^2\theta - \sin^2\theta\text{)} \quad (28)
\end{aligned}$$

A.1.8 Step 6: Calculation of the Force

Finally, we calculate the force F by taking the negative spatial derivative of the potential $U(x)$ and expressing the result in terms of the given parameters.

$$\boxed{\text{Sound Speed in an Ideal Gas: } c^2 = \frac{\gamma P_0}{\rho_0}}$$

$$\begin{aligned}
F(x) &= -\frac{dU}{dx} = -\frac{d}{dx}\left[-\frac{1}{4}V\rho_0 c^2 A^2 k^2 \cos(2kx)\right] \\
&= \frac{1}{4}V\rho_0 c^2 A^2 k^2 \frac{d}{dx}\left(\cos(2kx)\right) \\
&= \frac{1}{4}V\rho_0 c^2 A^2 k^2 \left(-2k\sin(2kx)\right) \\
&= -\frac{1}{2}V\rho_0 c^2 A^2 k^3 \sin(2kx)
\end{aligned} \tag{29}$$

Substituting $V = \pi R^2 h$ and $\rho_0 c^2 = \gamma P_0$:

$$\begin{aligned}
F &= -\frac{1}{2}(\pi R^2 h)(\gamma P_0)A^2 k^3 \sin(2kx) \\
&= -\frac{1}{2}\pi R^2 h\gamma P_0 A^2 k^3 \sin(2kx)
\end{aligned} \tag{30}$$

This result shows that the force on the rigid, dense cylinder depends on its position x in the standing wave. The force is directed towards the locations where the radiation potential $U(x)$ is minimum. The minima of $U(x) \propto -\cos(2kx)$ occur where $\cos(2kx)$ is maximum, i.e., at the pressure nodes ($kx = n\pi$, where n is an integer) of the standing wave.

A.1.9 Final Answer:

The expression for the time-averaged force F acting on the cylinder is:

$$\boxed{F = -\frac{1}{2}\pi R^2 h\gamma P_0 A^2 k^3 \sin(2kx)} \tag{31}$$

A.2 Problem ID:88

A.2.1 Problem Statement Explanation

This problem asks for the angular distribution of the light emission power, denoted by $w(\theta, t)$, from a collection of moving atoms as measured in the laboratory reference frame S.

The physical situation is as follows:

- **System:** A large number, N_0, of identical atoms of type A. These atoms have two energy levels: a ground state A_0 with rest mass m_0, and an excited state A^* with a higher energy.
- **Reference Frames:** The laboratory frame is denoted by S. The rest frame of the atoms is denoted by S'. The frame S' moves with a velocity $\mathbf{v} = v\hat{\mathbf{x}}$ relative to frame S.
- **Energy Levels:** The energy difference between the excited and ground states is E_0. An atom in the excited state can transition to the ground state by emitting a single photon. In the atom's rest frame S', the energy of this photon is E_0.
- **Decay Process:** In the atom's rest reference frame, S', the probability per unit time for an excited atom to decay is a constant, λ. The emission of photons is isotropic (equally probable in all directions) in this frame.
- **Initial Conditions:** At the initial time $t = 0$ in the lab frame S, all N_0 atoms are in the excited state A^*. The clocks in frame S and the atoms' rest frame S' are synchronized such that $t = t' = 0$.
- **Goal:** We need to find $w(\theta, t)$, which is the power emitted per unit solid angle in a direction specified by the angle θ with respect to the direction of motion (the $+\hat{\mathbf{x}}$ axis), at a time t as measured in the lab frame S.

Variables and Constants:

- N_0: Initial total number of atoms.
- E_0: Energy difference between the excited and ground states (photon energy in the rest frame).
- λ: Transition probability per unit time in the atom's rest frame S'.
- $\mathbf{v}$: Velocity of the atoms in the lab frame S, with magnitude v.
- c: Speed of light in vacuum.
- t: Time measured in the lab frame S.
- t': Time measured in the atom's rest frame S' (proper time).
- θ: Angle of photon emission relative to the $+\hat{\mathbf{x}}$ axis in frame S.
- θ': Angle of photon emission relative to the $+\hat{\mathbf{x}}'$ axis in frame S'.

- $\beta = v/c$: The speed of the atoms as a fraction of the speed of light.
- $\gamma = (1-\beta^2)^{-1/2}$: The Lorentz factor.

Assumptions:

1. $N_0 \gg 1$: The large number of atoms allows us to treat the decay process using a continuous differential equation.
2. $m_0c^2 \gg E_0$: The rest energy of an atom is much larger than the transition energy. This justifies ignoring the recoil of the atom upon photon emission, so the velocity $\mathbf{v}$ of the atom remains constant throughout the process.

A.2.2 Step 1: Decay Law and Photon Emission Rate in the Atom's Rest Frame (S')

In the rest frame of the atoms, S', the decay process is described by the law of radioactive decay. We first determine the number of excited atoms as a function of time t' and the rate of photon emission per unit solid angle.

Principles/Original Formulas/Assumptions: The number of atoms dN' that decay in an infinitesimal time interval dt' is proportional to the number of excited atoms $N'(t')$ present and the length of the time interval. This is the law of radioactive decay.

$$\boxed{\frac{dN'(t')}{dt'} = -\lambda N'(t')}$$

The emission of photons is isotropic in the rest frame S'. This means the number of photons emitted per unit time per unit solid angle is uniform over the total solid angle of 4π steradians.

Derivation: We solve the differential decay equation with the initial condition that at $t' = 0$, the number of excited atoms is $N'(0) = N_0$.

$$\begin{aligned} \int_{N_0}^{N'(t')} \frac{dN'}{N'} &= \int_0^{t'} -\lambda dt' \\ \ln\left(\frac{N'(t')}{N_0}\right) &= -\lambda t' \\ N'(t') &= N_0 e^{-\lambda t'} \end{aligned} \tag{32}$$

The total rate of decay, which is equal to the total number of photons emitted per unit time in S', is:

$$-\frac{dN'(t')}{dt'} = \lambda N'(t') = \lambda N_0 e^{-\lambda t'} \tag{33}$$

Due to the isotropic emission assumption, the number of photons emitted per unit time per unit solid angle in S' is this total rate divided by 4π:

$$\frac{d^2N_{ph}}{dt'd\Omega'} = \frac{1}{4\pi}\left(-\frac{dN'}{dt'}\right) = \frac{\lambda N_0}{4\pi}e^{-\lambda t'} \tag{34}$$

A.2.3 Step 2: Transformation of the Photon Emission Rate to the Lab Frame (S)

To find the power in the lab frame S, we first need to find the photon emission rate per unit solid angle in S, denoted $\frac{d^2N_{ph}}{dtd\Omega}$. This requires transforming the time interval dt' and the solid angle element $d\Omega'$ from frame S' to frame S.

Principles/Original Formulas/Assumptions: The number of photons d^2N_{ph} emitted into a corresponding solid angle element during a corresponding time interval is a Lorentz invariant quantity.

$$\boxed{d^2N_{ph} \text{ is invariant}}$$

The relationship between a time interval in the lab frame, dt, and the corresponding proper time interval in the moving frame, dt', is given by time dilation.

$$\boxed{dt = \gamma dt'}$$

The relationship between the emission angles in the two frames is given by the relativistic aberration of light formula.

$$\boxed{\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}}$$

Derivation: The invariance of d^2N_{ph} implies $d^2N_{ph} = \left(\frac{d^2N_{ph}}{dtd\Omega}\right)dtd\Omega = \left(\frac{d^2N_{ph}}{dt'd\Omega'}\right)dt'd\Omega'$. Rearranging gives the transformation for the rate:

$$\frac{d^2N_{ph}}{dtd\Omega} = \left(\frac{d^2N_{ph}}{dt'd\Omega'}\right)\frac{dt'}{dt}\frac{d\Omega'}{d\Omega} \tag{35}$$

From time dilation, we have $\frac{dt'}{dt} = \frac{1}{\gamma} = \sqrt{1-\beta^2}$. For the solid angle transformation, we have $d\Omega = 2\pi\sin\theta d\theta = -2\pi d(\cos\theta)$ and $d\Omega' = -2\pi d(\cos\theta')$. Thus, the ratio of solid angles is $\frac{d\Omega'}{d\Omega} = \frac{d(\cos\theta')}{d(\cos\theta)}$. Differentiating the aberration formula with respect to $\cos\theta$:

$$\begin{aligned}
\frac{d\Omega'}{d\Omega} = \frac{d(\cos\theta')}{d(\cos\theta)} &= \frac{d}{d(\cos\theta)}\left(\frac{\cos\theta - \beta}{1-\beta\cos\theta}\right) \\
&= \frac{1\cdot(1-\beta\cos\theta) - (\cos\theta - \beta)(-\beta)}{(1-\beta\cos\theta)^2} \\
&= \frac{1-\beta\cos\theta + \beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2} = \frac{1-\beta^2}{(1-\beta\cos\theta)^2}
\end{aligned} \tag{36}$$

Now, substitute the results from Eq. (34), time dilation, and Eq. (36) into Eq. (35):

$$
\begin{aligned}
\frac{d^2 N_{ph}}{dt d\Omega} &= \left(\frac{\lambda N_0}{4\pi} e^{-\lambda t'}\right)\left(\frac{1}{\gamma}\right)\left(\frac{1-\beta^2}{(1-\beta\cos\theta)^2}\right) \\
&= \frac{\lambda N_0}{4\pi} e^{-\lambda t'}\left(\sqrt{1-\beta^2}\right)\left(\frac{1-\beta^2}{(1-\beta\cos\theta)^2}\right) \\
&= \frac{\lambda N_0}{4\pi}\frac{(1-\beta^2)^{3/2}}{(1-\beta\cos\theta)^2} e^{-\lambda t'}
\end{aligned}
\tag{37}
$$

A.2.4 Step 3: Calculating the Angular Power Distribution w(, t)

The angular power distribution $w(\theta, t)$ is the energy emitted per unit time per unit solid angle in frame S. This is the product of the photon emission rate per solid angle (calculated in Step 2) and the energy of a single photon as measured in frame S.

Principles/Original Formulas/Assumptions: The angular power distribution is the product of the photon flux and the energy per photon.

$$
\boxed{w(\theta, t) = \frac{d^2 E_{emitted}}{dt d\Omega} = E_{ph}(\theta)\frac{d^2 N_{ph}}{dt d\Omega}}
$$

The energy of a photon in frame S, $E_{ph}(\theta)$, is related to its energy in the rest frame, E_0, by the relativistic Doppler effect.

$$
\boxed{E_{ph}(\theta) = E_0 \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}}
$$

Derivation: We multiply the photon flux in frame S (Eq. 37) by the energy of a photon in frame S.

$$
\begin{aligned}
w(\theta, t) &= \left(\frac{\lambda N_0}{4\pi}\frac{(1-\beta^2)^{3/2}}{(1-\beta\cos\theta)^2} e^{-\lambda t'}\right)\cdot\left(E_0 \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\right) \\
&= \frac{\lambda N_0 E_0}{4\pi}\frac{(1-\beta^2)^{3/2}(1-\beta^2)^{1/2}}{(1-\beta\cos\theta)^2(1-\beta\cos\theta)} e^{-\lambda t'} \\
&= \frac{\lambda N_0 E_0}{4\pi}\frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} e^{-\lambda t'}
\end{aligned}
\tag{38}
$$

A.2.5 Step 4: Final Expression in Lab Frame Variables

The expression for $w(\theta, t)$ in Eq. (38) depends on the time t' in the atom's rest frame. To get the final answer, we must express it in terms of the time t measured in the lab frame S.

Principles/Original Formulas/Assumptions: The proper time t' of the moving atoms is related to the time t in the lab frame by the time dilation formula. Since the atoms are at a fixed position in S' and the clocks are synchronized at the origin, the relation for finite time intervals starting at zero is:

$$\boxed{t' = \frac{t}{\gamma} = t\sqrt{1-\beta^2}}$$

Derivation: We substitute the expression for t' into Eq. (38).

$$\begin{aligned} w(\theta, t) &= \frac{\lambda N_0 E_0}{4\pi} \frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} \exp\left(-\lambda\frac{t}{\gamma}\right) \\ &= \frac{\lambda N_0 E_0}{4\pi} \frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3} e^{-\lambda t\sqrt{1-\beta^2}} \end{aligned} \tag{39}$$

Finally, we substitute $\beta = v/c$ to express the result in terms of the given variables.

$$w(\theta, t) = \frac{\lambda N_0 E_0}{4\pi} \frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v}{c}\cos\theta\right)^3} \exp\left(-\lambda t\sqrt{1-\frac{v^2}{c^2}}\right) \tag{40}$$

This is the final expression for the angular distribution of the light emission power at time t in the laboratory reference frame S.

A.2.6 Final Answer

The angular distribution of the light emission power $w(\theta, t)$ from the atom at time t in the laboratory reference frame S is given by:

$$\boxed{w(\theta, t) = \frac{\lambda N_0 E_0}{4\pi} \frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v}{c}\cos\theta\right)^3} e^{-\lambda t\sqrt{1-\frac{v^2}{c^2}}}} \tag{41}$$

A.3 Problem ID:636

A.3.1 Problem Statement Explanation

This problem describes a physical system and asks for a specific quantity related to the motion of a test charge within it.

System Configuration:

- A Cartesian coordinate system Oxyz is established.
- There are n rings, each with radius R and uniformly distributed positive charge Q.
- These n rings are arranged as meridional great circles on a hypothetical sphere of radius R centered at the origin. They all pass through the "poles" at (0, 0, R) and (0, 0, -R).

- The planes of these rings are distributed symmetrically around the z-axis, with the angle between the planes of any two adjacent rings being π/n.

Test Particle and its Motion:

- A particle of mass m and negative charge $-q$ (where q is a positive value) is introduced into the system.
- Initially, the particle is in a stable circular orbit of radius r_0 in the equatorial plane (the xy-plane, where $z = 0$).
- A key assumption is that the orbital radius is much larger than the ring radius, i.e., $r_0 \gg R$. This suggests that a multipole expansion of the electric potential is appropriate and that we can use approximations for $R/r_0 \ll 1$.
- The particle is then given a small radial disturbance, causing it to oscillate radially around its stable orbit while it continues to revolve.

Objective: The goal is to find the difference between the period of the radial oscillations, T_r, and the period of the angular revolution (the orbital period), T_θ.

Constants: The vacuum permittivity is denoted by ϵ_0.

A.3.2 Step 1: Electric Potential of a Single Charged Ring

We begin by finding the electric potential V_1 created by a single charged ring at a point far from the ring ($r \gg R$). For a charge distribution with axial symmetry, like a ring, the potential $V(r, \alpha)$ at a point (r, α) (in spherical coordinates relative to the ring's axis) can be expressed using a multipole expansion in terms of Legendre polynomials $P_l(\cos\alpha)$.

$$\boxed{V(r, \alpha) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{a_l}{r^{l+1}} P_l(\cos\alpha)}$$

Here, $a_l = \int (r')^l P_l(\cos\theta')\rho(\mathbf{r}')d\tau'$ are the multipole moments. For a thin ring of charge Q and radius R, the first few moments are:

- Monopole moment ($l = 0$): $a_0 = Q$ (the total charge).
- Dipole moment ($l = 1$): $a_1 = 0$ (due to the ring's symmetry).
- Quadrupole moment ($l = 2$): $a_2 = -QR^2/2$.
- Higher-order moments are negligible for $r \gg R$.

Keeping terms up to $l = 2$, and using $P_0(x) = 1$ and $P_2(x) = (3x^2 - 1)/2$, the potential is:

$$\begin{aligned} V_1(r,\alpha) &\approx \frac{1}{4\pi\epsilon_0}\left(\frac{a_0}{r}P_0(\cos\alpha) + \frac{a_2}{r^3}P_2(\cos\alpha)\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{(-QR^2/2)}{r^3}\frac{3\cos^2\alpha - 1}{2}\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{R^2}{4r^3}(3\cos^2\alpha - 1)\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{R^2}{4r^3}(1 - 3\cos^2\alpha)\right) \end{aligned} \tag{42}$$

A.3.3 Step 2: Total Electric Potential from n Rings on the Equatorial Plane

Now, we sum the potentials from all n rings at a point P on the equatorial plane. Let 'P' have cylindrical coordinates $(r, \phi, 0)$. Its position vector is $\mathbf{r} = r(\cos\phi, \sin\phi, 0)$. The i-th ring lies in a plane containing the z-axis, making an angle $\phi_i = i\pi/n$ with the xz-plane. The axis of this ring, $\mathbf{u}_i$, is normal to its plane and lies in the xy-plane.

$$\boxed{\mathbf{u}_i = (-\sin\phi_i, \cos\phi_i, 0)}$$

The angle α_i between the position vector $\mathbf{r}$ and the axis $\mathbf{u}_i$ is given by the dot product formula.

$$\boxed{\cos\alpha_i = \frac{\mathbf{r}\cdot\mathbf{u}_i}{|\mathbf{r}||\mathbf{u}_i|}}$$

$$\begin{aligned} \cos\alpha_i &= \frac{r(\cos\phi, \sin\phi, 0)\cdot(-\sin\phi_i, \cos\phi_i, 0)}{r\cdot 1} \\ &= -\cos\phi\sin\phi_i + \sin\phi\cos\phi_i = \sin(\phi - \phi_i) \end{aligned} \tag{43}$$

The total potential V_{total} is the sum of potentials from all n rings.

$$\begin{aligned} V_{total}(r,\phi) &= \sum_{i=1}^{n} V_i(r,\alpha_i) = \sum_{i=1}^{n}\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{R^2}{4r^3}(1 - 3\cos^2\alpha_i)\right) \\ &= \frac{nQ}{4\pi\epsilon_0 r} + \frac{QR^2}{16\pi\epsilon_0 r^3}\sum_{i=1}^{n}(1 - 3\cos^2\alpha_i) \\ &= \frac{nQ}{4\pi\epsilon_0 r} + \frac{QR^2}{16\pi\epsilon_0 r^3}\left(n - 3\sum_{i=1}^{n}\sin^2(\phi - \frac{i\pi}{n})\right) \end{aligned} \tag{44}$$

The sum $\sum_{i=1}^{n}\sin^2(\phi - \frac{i\pi}{n})$ can be evaluated using trigonometric identities.

$$\boxed{\sin^2(x) = \frac{1 - \cos(2x)}{2}}$$

$$\boxed{\sum_{k=1}^{n} \cos(A + k\Delta) = 0 \quad \text{if } n\Delta \text{ is a multiple of } 2\pi \text{ and } n > 1}$$

$$\begin{aligned}
\sum_{i=1}^{n} \sin^2(\phi - \frac{i\pi}{n}) &= \sum_{i=1}^{n} \frac{1}{2}\left(1 - \cos(2\phi - \frac{2i\pi}{n})\right) \\
&= \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n} \cos(2\phi - \frac{2i\pi}{n}) \\
&= \frac{n}{2} - 0 = \frac{n}{2}
\end{aligned} \tag{45}$$

Substituting this back into Eq. (44), we find the total potential is independent of ϕ:

$$\begin{aligned}
V_{total}(r) &= \frac{nQ}{4\pi\epsilon_0 r} + \frac{QR^2}{16\pi\epsilon_0 r^3}\left(n - 3\frac{n}{2}\right) = \frac{nQ}{4\pi\epsilon_0 r} - \frac{nQR^2}{32\pi\epsilon_0 r^3} \\
&= \frac{nQ}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{R^2}{8r^3}\right)
\end{aligned} \tag{46}$$

A.3.4 Step 3: Effective Potential Energy and Equilibrium Orbit

The electrostatic potential energy $U(r)$ of the charge $-q$ in the potential $V_{total}(r)$ is:

$$\boxed{U(r) = (-q)V_{total}(r)}$$

The effective potential energy $V_{eff}(r)$ for radial motion also includes the centrifugal potential energy term, where L is the conserved angular momentum.

$$\boxed{V_{eff}(r) = U(r) + \frac{L^2}{2mr^2}}$$

$$V_{eff}(r) = -q\left[\frac{nQ}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{R^2}{8r^3}\right)\right] + \frac{L^2}{2mr^2} = -\frac{nQq}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{R^2}{8r^3}\right) + \frac{L^2}{2mr^2} \tag{47}$$

A stable circular orbit exists at $r = r_0$ where the effective potential is at a minimum, i.e., $\frac{dV_{eff}}{dr} = 0$.

$$\begin{aligned}
\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} &= -\frac{nQq}{4\pi\epsilon_0}\left(-\frac{1}{r_0^2} + \frac{3R^2}{8r_0^4}\right) - \frac{L^2}{mr_0^3} = 0 \\
\frac{L^2}{mr_0^3} &= \frac{nQq}{4\pi\epsilon_0}\left(\frac{1}{r_0^2} - \frac{3R^2}{8r_0^4}\right) \\
L^2 &= \frac{nQqmr_0^3}{4\pi\epsilon_0}\left(\frac{1}{r_0^2} - \frac{3R^2}{8r_0^4}\right) = \frac{nQqmr_0}{4\pi\epsilon_0}\left(1 - \frac{3R^2}{8r_0^2}\right)
\end{aligned} \tag{48}$$

A.3.5 Step 4: Angular Revolution Frequency ω_θ

For a circular orbit, the angular momentum L is related to the angular frequency ω_θ by:

$$\boxed{L = mr_0^2\omega_\theta}$$

Squaring this and substituting the expression for L^2 from Eq. (48):

$$\begin{aligned} m^2r_0^4\omega_\theta^2 &= \frac{nQqmr_0}{4\pi\epsilon_0}\left(1 - \frac{3R^2}{8r_0^2}\right) \\ \omega_\theta^2 &= \frac{nQq}{4\pi\epsilon_0 mr_0^3}\left(1 - \frac{3R^2}{8r_0^2}\right) \end{aligned} \tag{49}$$

A.3.6 Step 5: Radial Oscillation Frequency ω_r

The frequency of small radial oscillations ω_r around the stable orbit r_0 is determined by the curvature of the effective potential at its minimum.

$$\boxed{m\omega_r^2 = \left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}}$$

First, we compute the second derivative of V_{eff}:

$$\begin{aligned} \frac{d^2V_{eff}}{dr^2} &= \frac{d}{dr}\left[-\frac{nQq}{4\pi\epsilon_0}\left(-\frac{1}{r^2} + \frac{3R^2}{8r^4}\right) - \frac{L^2}{mr^3}\right] \\ &= -\frac{nQq}{4\pi\epsilon_0}\left(\frac{2}{r^3} - \frac{12R^2}{8r^5}\right) + \frac{3L^2}{mr^4} \end{aligned} \tag{50}$$

Now, we evaluate at $r = r_0$ and substitute L^2 from Eq. (48):

$$\begin{aligned} m\omega_r^2 &= -\frac{nQq}{4\pi\epsilon_0}\left(\frac{2}{r_0^3} - \frac{3R^2}{2r_0^5}\right) + \frac{3}{mr_0^4}\left[\frac{nQqmr_0}{4\pi\epsilon_0}\left(1 - \frac{3R^2}{8r_0^2}\right)\right] \\ &= -\frac{nQq}{4\pi\epsilon_0}\left(\frac{2}{r_0^3} - \frac{3R^2}{2r_0^5}\right) + \frac{3nQq}{4\pi\epsilon_0 r_0^3}\left(1 - \frac{3R^2}{8r_0^2}\right) \\ &= \frac{nQq}{4\pi\epsilon_0 r_0^3}\left[-2 + \frac{3R^2}{2r_0^2} + 3 - \frac{9R^2}{8r_0^2}\right] \\ &= \frac{nQq}{4\pi\epsilon_0 r_0^3}\left[1 + \left(\frac{12-9}{8}\right)\frac{R^2}{r_0^2}\right] \\ &= \frac{nQq}{4\pi\epsilon_0 r_0^3}\left(1 + \frac{3R^2}{8r_0^2}\right) \end{aligned} \tag{51}$$

Thus, the square of the radial oscillation frequency is:

$$\omega_r^2 = \frac{nQq}{4\pi\epsilon_0 mr_0^3}\left(1 + \frac{3R^2}{8r_0^2}\right) \tag{52}$$

A.3.7 Step 6: Calculation of the Period Difference $T_r - T_\theta$

The periods are related to the angular frequencies by $T = 2\pi/\omega$. We need to calculate $T_r - T_\theta$.

$$\boxed{T_r - T_\theta = \frac{2\pi}{\omega_r} - \frac{2\pi}{\omega_\theta} = 2\pi\left(\frac{1}{\omega_r} - \frac{1}{\omega_\theta}\right)}$$

Let $\omega_0^2 = \frac{nQq}{4\pi\epsilon_0 m r_0^3}$. Since $r_0 \gg R$, the term R^2/r_0^2 is much smaller than 1. We can therefore use the binomial approximation.

$$\boxed{(1+x)^a \approx 1 + ax \quad \text{for } |x| \ll 1}$$

$$\omega_\theta = \omega_0\left(1 - \frac{3R^2}{8r_0^2}\right)^{1/2} \approx \omega_0\left(1 - \frac{3R^2}{16r_0^2}\right) \tag{53}$$

$$\omega_r = \omega_0\left(1 + \frac{3R^2}{8r_0^2}\right)^{1/2} \approx \omega_0\left(1 + \frac{3R^2}{16r_0^2}\right) \tag{54}$$

Now we find the reciprocals, again using the binomial approximation $(1+x)^{-1} \approx 1-x$:

$$\frac{1}{\omega_\theta} \approx \frac{1}{\omega_0}\left(1 - \frac{3R^2}{16r_0^2}\right)^{-1} \approx \frac{1}{\omega_0}\left(1 + \frac{3R^2}{16r_0^2}\right) \tag{55}$$

$$\frac{1}{\omega_r} \approx \frac{1}{\omega_0}\left(1 + \frac{3R^2}{16r_0^2}\right)^{-1} \approx \frac{1}{\omega_0}\left(1 - \frac{3R^2}{16r_0^2}\right) \tag{56}$$

Substituting these into the expression for the period difference:

$$\begin{aligned} T_r - T_\theta &\approx 2\pi\left[\frac{1}{\omega_0}\left(1 - \frac{3R^2}{16r_0^2}\right) - \frac{1}{\omega_0}\left(1 + \frac{3R^2}{16r_0^2}\right)\right] \\ &= \frac{2\pi}{\omega_0}\left(-\frac{3R^2}{16r_0^2} - \frac{3R^2}{16r_0^2}\right) = \frac{2\pi}{\omega_0}\left(-\frac{6R^2}{16r_0^2}\right) \\ &= -\frac{3\pi R^2}{4r_0^2\omega_0} \end{aligned} \tag{57}$$

Finally, we substitute the expression for $\omega_0 = \sqrt{\frac{nQq}{4\pi\epsilon_0 m r_0^3}}$:

$$T_r - T_\theta = -\frac{3\pi R^2}{4r_0^2}\sqrt{\frac{4\pi\epsilon_0 m r_0^3}{nQq}} \tag{58}$$

A.3.8 Final Answer

The difference between the radial oscillation period and the angular revolution period is:

$$\boxed{T_r - T_\theta = -\frac{3\pi R^2}{4r_0^2}\sqrt{\frac{4\pi\epsilon_0 m r_0^3}{nQq}}} \tag{59}$$

A.4 Problem ID:420

A.4.1 Problem Statement Explanation

This problem describes a mechanical system composed of two objects, A and D, each of mass m. They are connected by a single light, inextensible rope that runs over two smooth, fixed pulleys, B and C. The pulleys are positioned at the same horizontal level. Object A hangs from pulley B and can swing like a pendulum, while object D hangs from pulley C, acting as a counterweight.

The variables and constants are defined as follows:

- m: The mass of object A and object D.
- g: The acceleration due to gravity.
- $x(t)$: The length of the rope segment AB, which acts as the length of the pendulum. This is a dynamic variable.
- $\theta(t)$: The angle of the pendulum segment AB with respect to the vertical. This is a dynamic variable.
- x_0: The initial length of the pendulum, i.e., $x(0) = x_0$.
- θ_0: The initial angle of the pendulum, which is small.
- L: The initial length of the rope segment CD. It is stated to be sufficiently large to ensure object D does not interfere with pulley C.
- $\Theta(x)$: The amplitude of the angular oscillation, which depends on the pendulum length x.

The system is subject to the following initial conditions at time $t = 0$:

- The system is released from rest.
- The initial angle is $\theta(0) = \theta_0$. Since the system starts from rest, this is the initial amplitude of oscillation.
- The initial angular velocity is $\dot{\theta}(0) = 0$.
- The initial radial velocity is $\dot{x}(0) = 0$.

Key assumptions for the model are:

1. The pulleys B and C are smooth (frictionless) and their size is negligible (point-like).
2. The rope is massless and inextensible. Its total length is constant.

3. The initial angle θ_0 is small, and the angle $\theta(t)$ remains small throughout the motion. This allows for approximations such as $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$.

4. The problem states that "A descends very slowly". This is interpreted as an adiabatic process: the characteristic time scale for the change in the pendulum length x is much larger than the period of the pendulum's angular oscillation. This allows us to treat x as a slowly varying parameter when analyzing the θ motion.

The objective is to find the amplitude of the oscillation angle, Θ, as a function of the pendulum length x.

A.4.2 Step 1: Lagrangian of the System

To describe the dynamics of the system, we use the Lagrangian formalism. The Lagrangian $\mathcal{L}$ is defined as the difference between the total kinetic energy K and the total potential energy U.

$$\boxed{\mathcal{L} = K - U}$$

Derivation: We set up a Cartesian coordinate system with the origin at pulley B. The y-axis points vertically upwards, and the x-axis is horizontal. The horizontal line passing through pulleys B and C is the reference level for gravitational potential energy ($y = 0$).

- Object A (pendulum):
  - The position vector of A is $\vec{r}_A = (x\sin\theta, -x\cos\theta)$.
  - The velocity vector is $\vec{v}_A = \frac{d\vec{r}_A}{dt} = (\dot{x}\sin\theta + x\dot{\theta}\cos\theta, -\dot{x}\cos\theta + x\dot{\theta}\sin\theta)$.
  - The squared speed is $v_A^2 = \vec{v}_A \cdot \vec{v}_A = (\dot{x}\sin\theta + x\dot{\theta}\cos\theta)^2 + (-\dot{x}\cos\theta + x\dot{\theta}\sin\theta)^2 = \dot{x}^2(\sin^2\theta + \cos^2\theta) + x^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) = \dot{x}^2 + x^2\dot{\theta}^2$.
  - The kinetic energy of A is $K_A = \frac{1}{2}mv_A^2 = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2)$.
  - The potential energy of A is $U_A = mgy_A = -mgx\cos\theta$.

- Object D (counterweight):
  - Let the total length of the rope be L_{rope} and the distance between pulleys B and C be d_{BC}. The length of the segment CD is l_{CD}. Since the rope is inextensible, $x + d_{BC} + l_{CD} = L_{\text{rope}}$.
  - The vertical position of D is $y_D = -l_{CD} = -(L_{\text{rope}} - d_{BC} - x) = x - (L_{\text{rope}} - d_{BC})$.
  - The velocity of D is $v_D = \frac{dy_D}{dt} = \dot{x}$.
  - The kinetic energy of D is $K_D = \frac{1}{2}mv_D^2 = \frac{1}{2}m\dot{x}^2$.

- The potential energy of D is $U_D = mgy_D = mg(x - (L_{\text{rope}} - d_{BC}))$.

- Total Lagrangian:

  - The total kinetic energy is $K = K_A + K_D = \frac{1}{2}m(\dot{x}^2 + x^2\dot{\theta}^2) + \frac{1}{2}m\dot{x}^2 = m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2$.

  - The total potential energy is $U = U_A + U_D = -mgx\cos\theta + mgx - mg(L_{\text{rope}} - d_{BC})$.

  - We can drop the constant term $-mg(L_{\text{rope}} - d_{BC})$ as it does not affect the equations of motion.

  - The Lagrangian of the system is:

$$\begin{aligned}\mathcal{L} = K - U &= \left(m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2\right) - (-mgx\cos\theta + mgx) \\ &= m\dot{x}^2 + \frac{1}{2}mx^2\dot{\theta}^2 + mgx\cos\theta - mgx \end{aligned} \tag{60}$$

A.4.3 Step 2: Equations of Motion

The equations of motion for the generalized coordinates $q_i \in \{x, \theta\}$ are given by the Euler-Lagrange equations.

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

Derivation:

- **For the coordinate θ:**

  - The partial derivatives of the Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mx^2\dot{\theta} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \theta} = -mgx\sin\theta \tag{61}$$

  - Applying the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}(mx^2\dot{\theta}) - (-mgx\sin\theta) &= 0 \\ m(2x\dot{x}\dot{\theta} + x^2\ddot{\theta}) + mgx\sin\theta &= 0 \\ x\ddot{\theta} + 2\dot{x}\dot{\theta} + g\sin\theta &= 0 \end{aligned} \tag{62}$$

- **For the coordinate x:**

  - The partial derivatives of the Lagrangian are:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2m\dot{x} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial x} = mx\dot{\theta}^2 + mg\cos\theta - mg \tag{63}$$

– Applying the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt}(2m\dot{x}) - (mx\dot{\theta}^2 + mg\cos\theta - mg) &= 0 \\ 2m\ddot{x} - mx\dot{\theta}^2 - mg(\cos\theta - 1) &= 0 \end{aligned} \tag{64}$$

A.4.4 Step 3: Adiabatic Invariant of the Pendulum Oscillation

The problem assumes that x changes very slowly (adiabatic approximation). This means the parameters in the θ equation of motion vary slowly over one oscillation period. For such systems, an adiabatic invariant exists. We derive this invariant by analyzing the energy of the θ-oscillation.

$$\boxed{\text{Virial Theorem for Harmonic Oscillator: } \langle K \rangle = \langle U \rangle = \frac{1}{2}E_{\text{total}}}$$

Derivation: For small angles, we approximate $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$. The equation of motion for θ (62) becomes:

$$x\ddot{\theta} + 2\dot{x}\dot{\theta} + g\theta \approx 0 \tag{65}$$

This describes a harmonic oscillator with a slowly varying frequency $\omega(t) = \sqrt{g/x(t)}$ and a damping-like term $2\dot{x}\dot{\theta}$. We define the energy of this oscillation, E_θ, treating x as a slowly varying parameter.

$$E_\theta = K_\theta + U_\theta = \frac{1}{2}m(x\dot{\theta})^2 + mgx(1 - \cos\theta) \approx \frac{1}{2}mx^2\dot{\theta}^2 + \frac{1}{2}mgx\theta^2 \tag{66}$$

The rate of change of this energy is:

$$\frac{dE_\theta}{dt} \approx \frac{d}{dt}\left(\frac{1}{2}mx^2\dot{\theta}^2 + \frac{1}{2}mgx\theta^2\right) = mx\dot{x}\dot{\theta}^2 + mx^2\dot{\theta}\ddot{\theta} + \frac{1}{2}mg\dot{x}\theta^2 + mgx\theta\dot{\theta} \tag{67}$$

From the approximate EOM (65), we substitute $mx^2\ddot{\theta} \approx -2mx\dot{x}\dot{\theta} - mgx\theta$.

$$\begin{aligned}\frac{dE_\theta}{dt} &\approx mx\dot{x}\dot{\theta}^2 + \dot{\theta}(-2mx\dot{x}\dot{\theta} - mgx\theta) + \frac{1}{2}mg\dot{x}\theta^2 + mgx\theta\dot{\theta} \\ &= mx\dot{x}\dot{\theta}^2 - 2mx\dot{x}\dot{\theta}^2 - mgx\theta\dot{\theta} + \frac{1}{2}mg\dot{x}\theta^2 + mgx\theta\dot{\theta} \\ &= -mx\dot{x}\dot{\theta}^2 + \frac{1}{2}mg\dot{x}\theta^2 \end{aligned} \tag{68}$$

We now average this expression over one period of the fast θ-oscillation. Let $\langle\cdot\rangle$ denote the time average. Since x and $\dot{x}$ are slowly varying, we can treat them as constants during the averaging process.

$$\left\langle \frac{dE_\theta}{dt} \right\rangle \approx \frac{d\langle E_\theta\rangle}{dt} = \langle -mx\dot{x}\dot{\theta}^2 + \frac{1}{2}mg\dot{x}\theta^2\rangle \approx -mx\dot{x}\langle\dot{\theta}^2\rangle + \frac{1}{2}mg\dot{x}\langle\theta^2\rangle \tag{69}$$

We apply the Virial Theorem to the harmonic oscillator energy components. The total energy of the oscillation is $\langle E_\theta\rangle$. $\langle K_\theta\rangle = \langle \frac{1}{2}mx^2\dot{\theta}^2\rangle = \frac{1}{2}\langle E_\theta\rangle \implies \langle\dot{\theta}^2\rangle = \frac{\langle E_\theta\rangle}{mx^2}$. $\langle U_\theta\rangle = \langle \frac{1}{2}mgx\theta^2\rangle = \frac{1}{2}\langle E_\theta\rangle \implies \langle\theta^2\rangle = \frac{\langle E_\theta\rangle}{mgx}$. Substituting these averages back:

$$\begin{aligned}\frac{d\langle E_\theta\rangle}{dt} &\approx -mx\dot{x}\left(\frac{\langle E_\theta\rangle}{mx^2}\right) + \frac{1}{2}mg\dot{x}\left(\frac{\langle E_\theta\rangle}{mgx}\right)\\ &= -\frac{\dot{x}}{x}\langle E_\theta\rangle + \frac{\dot{x}}{2x}\langle E_\theta\rangle = -\frac{\dot{x}}{2x}\langle E_\theta\rangle\end{aligned} \tag{70}$$

This gives a differential equation for the slowly changing average energy $\langle E_\theta\rangle$:

$$\frac{1}{\langle E_\theta\rangle}\frac{d\langle E_\theta\rangle}{dt} = -\frac{1}{2x}\frac{dx}{dt} \implies \frac{d}{dt}(\ln\langle E_\theta\rangle) = \frac{d}{dt}(\ln x^{-1/2}) \tag{71}$$

Integrating gives $\ln\langle E_\theta\rangle = -\frac{1}{2}\ln x + \text{constant}$, which means $\langle E_\theta\rangle \propto x^{-1/2}$. The energy of the pendulum at its maximum angular displacement (amplitude) Θ is purely potential, $E_\theta = \frac{1}{2}mgx\Theta^2$. This is the total energy of one oscillation cycle.

$$\frac{1}{2}mgx\Theta^2 \propto x^{-1/2} \implies x\Theta^2 \propto x^{-1/2} \implies \Theta^2 \propto x^{-3/2} \tag{72}$$

This implies that the quantity $x^{3/2}\Theta^2$ is an adiabatic invariant.

$$x^{3/2}\Theta^2 = \text{constant} \tag{73}$$

A.4.5 Step 4: Amplitude as a Function of Pendulum Length

We use the adiabatic invariant derived in the previous step to relate the amplitude of oscillation to the pendulum length.

$$\boxed{x^{3/2}\Theta^2 = \text{constant}}$$

Derivation: We apply the invariant relation (73) to the initial state (length x_0, amplitude $\Theta(x_0)$) and a general state (length x, amplitude $\Theta(x)$).

$$x^{3/2}\Theta(x)^2 = x_0^{3/2}\Theta(x_0)^2 \tag{74}$$

The problem states that the system is released from rest at an angle θ_0. This means the initial velocity is zero, and the initial displacement is the maximum displacement for the first oscillation. Therefore, the initial amplitude is $\Theta(x_0) = \theta_0$. Substituting this into our relation:

$$\begin{aligned} x^{3/2}\Theta(x)^2 &= x_0^{3/2}\theta_0^2 \\ \Theta(x)^2 &= \theta_0^2\left(\frac{x_0}{x}\right)^{3/2} \\ \Theta(x) &= \theta_0\left(\frac{x_0}{x}\right)^{3/4} \end{aligned} \tag{75}$$

This expression gives the amplitude of the angle θ as a function of the pendulum length x.

A.4.6 Final Answer

The amplitude of the oscillation angle θ, denoted by Θ, when the pendulum length is x, is given by:

$$\boxed{\Theta(x) = \theta_0\left(\frac{x_0}{x}\right)^{3/4}} \tag{76}$$

A.5 Problem ID:420 (Numerical Simulation Verification Program)

To ensure the problem conditions: AB and CD distances are sufficiently large, A descends slowly enough, and oscillation amplitude is sufficiently small, the numerical simulation program sets constant initial conditions

$$m = 1.0, \tag{77}$$

$$g = 9.8, \tag{78}$$

$$x(t=0) = 5000000.0, \tag{79}$$

$$\theta(t=0) = 0.0, \tag{80}$$

$$\dot{x}(t=0) = 0, \tag{81}$$

$$\dot{\theta}(t=0) = 0.0004 \tag{82}$$

The time evolution plots of x, θ are shown in Figure 1. Next, we extract the extrema of $\theta(t)$, which reflect the amplitude of θ. We plot $x(t)\theta(t)$ and $x^{3/4}(t)\theta(t)$ curves : Figure 2 and Figure 3. Therefore, through numerical verification, the correct answer should be

$$\theta = \theta_0\frac{x_0^{3/4}}{x^{3/4}}$$

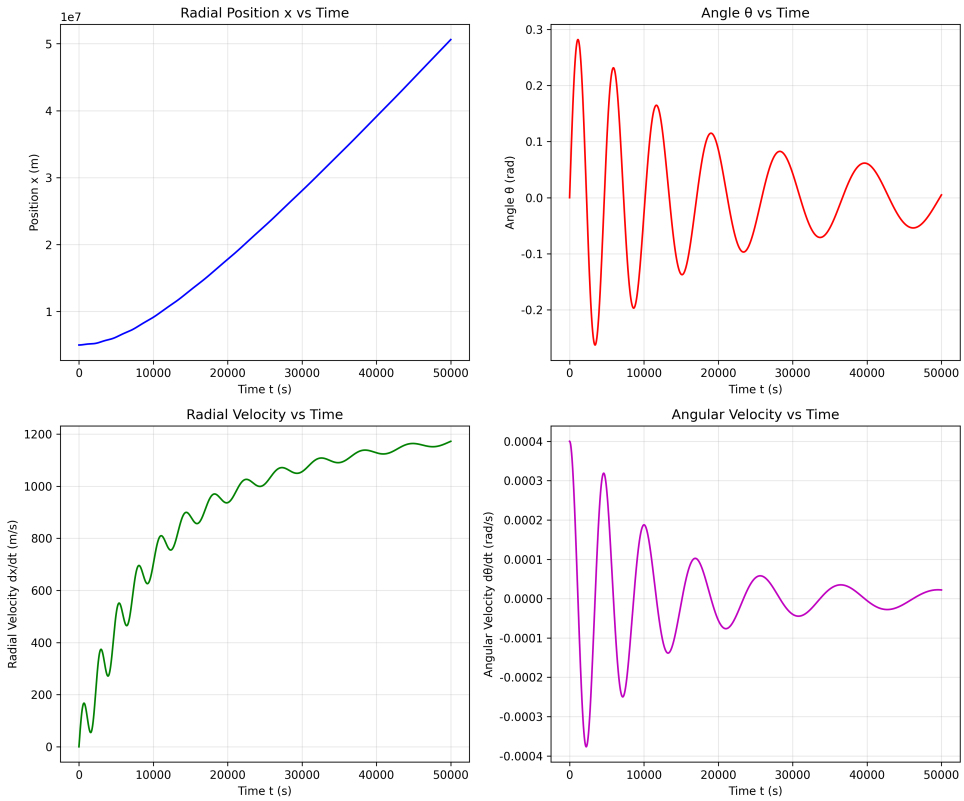

图 1: Numerical simulation results: System dynamics evolution

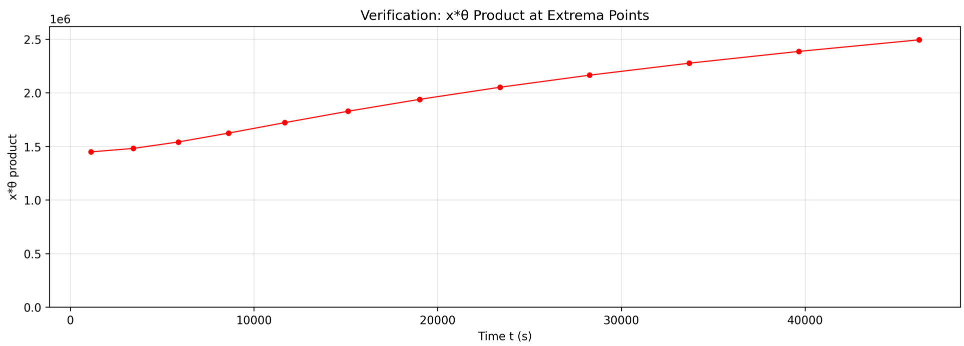

图 2: Numerical simulation results: $x(t)\dot{\theta}(t)$ vs time curve

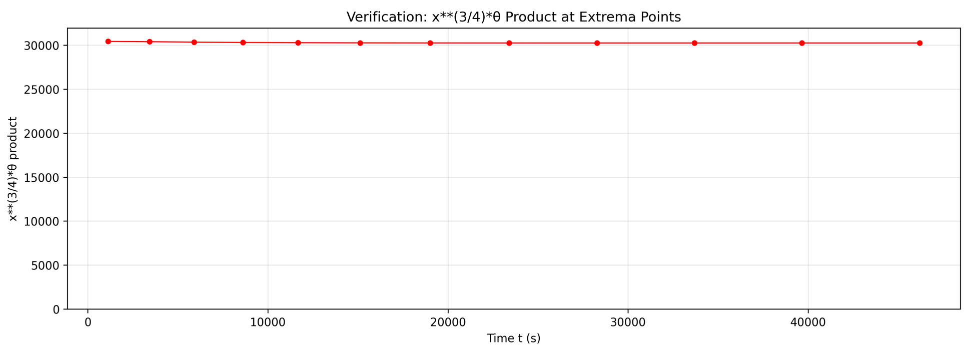

图 3: Numerical simulation results: $x^{3/4}(t)\dot{\theta}(t)$ vs time curve

```python
# %%
import sympy as sp

m, t, g = sp.symbols('m t g')

x = sp.Function('x')(t)
theta = sp.Function('theta')(t)

K = m * x.diff(t) ** 2 / 2 + m * x ** 2 * theta.diff(t) ** 2 / 2 + m *
    x.diff(t) ** 2 / 2
U = m * g * (x - x * sp.cos(theta))
L = K - U

# %%
lagrange_eq_x = sp.Eq(sp.diff(L, x), sp.diff(sp.diff(L, x.diff(t)), t))
lagrange_eq_theta = sp.Eq(sp.diff(L, theta), sp.diff(sp.diff(L,
    theta.diff(t)), t))
res = sp.solve([lagrange_eq_x, lagrange_eq_theta], (x.diff(t, 2),
    theta.diff(t, 2)))
x_ddot = res[x.diff(t, 2)]
theta_ddot = res[theta.diff(t, 2)]

const_value = {
    g: 9.8,
    m: 1.0,
}
init_value = {
    x: 5000000.0,
    theta: 0.0,
    x.diff(t): 0.0,
    theta.diff(t): 0.0004
}

# %%
# Import numerical computation and plotting libraries
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

# Convert symbolic expressions to numerical functions
# Substitute constant values
x_ddot_numeric = x_ddot.subs(const_value)
theta_ddot_numeric = theta_ddot.subs(const_value)

```

```
print("Second derivative of x:")
print(x_ddot_numeric)
print("\nSecond derivative of theta:")
print(theta_ddot_numeric)

# %%
# Convert sympy expressions to callable numerical functions
x_ddot_func = sp.lambdify((x, theta, x.diff(t), theta.diff(t)),
    x_ddot_numeric, 'numpy')
theta_ddot_func = sp.lambdify((x, theta, x.diff(t), theta.diff(t)),
    theta_ddot_numeric, 'numpy')

# Define the right-hand side function of the differential equation system
def system_ode(t_val, y):
    """
    Right-hand side function of the differential equation system
    y = [x, theta, x_dot, theta_dot]
    dy/dt = [x_dot, theta_dot, x_ddot, theta_ddot]
    """
    x_val, theta_val, x_dot, theta_dot = y

    # Calculate second derivatives
    x_ddot_val = x_ddot_func(x_val, theta_val, x_dot, theta_dot)
    theta_ddot_val = theta_ddot_func(x_val, theta_val, x_dot, theta_dot)

    return [x_dot, theta_dot, x_ddot_val, theta_ddot_val]

# %%
# Set initial conditions and time range with high precision
y0 = np.array([
    float(init_value[x]),           # x(0)
    float(init_value[theta]),       # theta(0)
    float(init_value[x.diff(t)]),   # x_dot(0)
    float(init_value[theta.diff(t)]) # theta_dot(0)
], dtype=np.float64)

# Time range with higher resolution
t_span = (0, 50000)  # From 0 to 50000 seconds
t_eval = np.linspace(0, 50000, 600000, dtype=np.float64)  # Higher
    resolution: 600k points

print(f"Initial conditions: x0={y0[0]}, theta0={y0[1]}, x_dot0={y0[2]},
    theta_dot0={y0[3]}")

# %%
```

```python
# High-precision numerical solution of differential equations
print("Starting high-precision numerical solution...")
try:
    sol = solve_ivp(system_ode, t_span, y0, t_eval=t_eval,
                    method='DOP853',  # Higher-order Runge-Kutta method (8th order)
                    rtol=1e-12,       # Very tight relative tolerance
                    atol=1e-15,       # Very tight absolute tolerance
                    max_step=1.0,     # Limit maximum step size
                    dense_output=True) # Enable dense output for better interpolation

    if sol.success:
        print("High-precision solution successful!")
        print(f"Solved for {len(sol.t)} time points")
        print(f"Number of function evaluations: {sol.nfev}")
    else:
        print("Solution failed:", sol.message)

except Exception as e:
    print(f"Error during solution: {e}")

# %%
# Extract results
t_result = sol.t
x_result = sol.y[0]
theta_result = sol.y[1]
x_dot_result = sol.y[2]
theta_dot_result = sol.y[3]

print(f"x range: [{np.min(x_result):.2e}, {np.max(x_result):.2e}]")
print(f"theta range: [{np.min(theta_result):.4f}, {np.max(theta_result):.4f}]")

# %%
# Create plots
plt.rcParams['font.sans-serif'] = ['DejaVu Sans']
plt.rcParams['axes.unicode_minus'] = False

fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(2, 2, figsize=(12, 10))

# Plot x(t)
ax1.plot(t_result, x_result, 'b-', linewidth=1.5)
ax1.set_xlabel('Time t (s)')
ax1.set_ylabel('Position x (m)')
```

```
ax1.set_title('Radial Position x vs Time')
ax1.grid(True, alpha=0.3)

# Plot theta(t)
ax2.plot(t_result, theta_result, 'r-', linewidth=1.5)
ax2.set_xlabel('Time t (s)')
ax2.set_ylabel('Angle   (rad)')
ax2.set_title('Angle   vs Time')
ax2.grid(True, alpha=0.3)

# Plot x_dot(t)
ax3.plot(t_result, x_dot_result, 'g-', linewidth=1.5)
ax3.set_xlabel('Time t (s)')
ax3.set_ylabel('Radial Velocity dx/dt (m/s)')
ax3.set_title('Radial Velocity vs Time')
ax3.grid(True, alpha=0.3)

# Plot theta_dot(t)
ax4.plot(t_result, theta_dot_result, 'm-', linewidth=1.5)
ax4.set_xlabel('Time t (s)')
ax4.set_ylabel('Angular Velocity d /dt (rad/s)')
ax4.set_title('Angular Velocity vs Time')
ax4.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('system_dynamics.png', dpi=300, bbox_inches='tight')
plt.close()
print("System dynamics plot saved as system_dynamics.png")

# %%
# Plot phase space diagrams
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5))

# Phase space plot for x (x vs x_dot)
ax1.plot(x_result, x_dot_result, 'b-', linewidth=1.5, alpha=0.7)
ax1.set_xlabel('Position x (m)')
ax1.set_ylabel('Radial Velocity dx/dt (m/s)')
ax1.set_title('Radial Motion Phase Space')
ax1.grid(True, alpha=0.3)

# Phase space plot for theta (theta vs theta_dot)
ax2.plot(theta_result, theta_dot_result, 'r-', linewidth=1.5, alpha=0.7)
ax2.set_xlabel('Angle   (rad)')
ax2.set_ylabel('Angular Velocity d /dt (rad/s)')
ax2.set_title('Angular Motion Phase Space')
```

```
ax2.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('phase_space.png', dpi=300, bbox_inches='tight')
plt.close()
print("Phase space plot saved as phase_space.png")

# %%
# Plot trajectory (polar to cartesian coordinates)
x_cart = x_result * np.cos(theta_result)
y_cart = x_result * np.sin(theta_result)

plt.figure(figsize=(10, 8))
plt.plot(x_cart, y_cart, 'b-', linewidth=1.5, alpha=0.7,
    label='Trajectory')
plt.plot(x_cart[0], y_cart[0], 'go', markersize=8, label='Start')
plt.plot(x_cart[-1], y_cart[-1], 'ro', markersize=8, label='End')
plt.xlabel('x coordinate (m)')
plt.ylabel('y coordinate (m)')
plt.title('Particle Trajectory in Cartesian Coordinates')
plt.legend()
plt.grid(True, alpha=0.3)
plt.axis('equal')
plt.savefig('trajectory.png', dpi=300, bbox_inches='tight')
plt.close()
print("Trajectory plot saved as trajectory.png")

print("Numerical solution and plotting completed!")

# %%
# Find extrema of theta(t) and verify x*theta constancy
from scipy.signal import find_peaks

# Find local maxima and minima of theta
peaks_max, _ = find_peaks(theta_result, height=None, distance=10)
peaks_min, _ = find_peaks(-theta_result, height=None, distance=10)

# Combine and sort all extrema points
all_extrema_indices = np.concatenate([peaks_max, peaks_min])
all_extrema_indices = np.sort(all_extrema_indices)

# Extract values at extrema
t_extrema = t_result[all_extrema_indices]
theta_extrema = theta_result[all_extrema_indices]
x_extrema = x_result[all_extrema_indices]
```

```python
x_theta_product = x_extrema * np.abs(theta_extrema)

print(f"Found {len(all_extrema_indices)} extrema points")

# Plot verification of x*theta constancy
plt.figure(figsize=(12, 8))

# Plot x*theta product vs time
plt.subplot(2, 1, 1)
plt.plot(t_extrema, x_theta_product, 'ro-', markersize=4, linewidth=1)
plt.xlabel('Time t (s)')
plt.ylabel('x*  product')
plt.title('Verification: x*  Product at Extrema Points')
plt.ylim(0, np.max(x_theta_product) * 1.05)  # Set y range from 0 to max
plt.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('x_theta_verification(human).png', dpi=300, bbox_inches='tight')
plt.close()


# Plot x**(3/4) * theta product vs time

x_theta_product = x_extrema ** 0.75 * np.abs(theta_extrema)


# Plot verification of x*theta constancy
plt.figure(figsize=(12, 8))

# Plot x*theta product vs time
plt.subplot(2, 1, 1)
plt.plot(t_extrema, x_theta_product, 'ro-', markersize=4, linewidth=1)
plt.xlabel('Time t (s)')
plt.ylabel('x**(3/4)*  product')
plt.title('Verification: x**(3/4)*  Product at Extrema Points')
plt.ylim(0, np.max(x_theta_product) * 1.05)  # Set y range from 0 to max
plt.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('x_theta_verification(gemini).png', dpi=300, bbox_inches='tight')
plt.close()
```

```
print(f"x*theta verification plot saved as x_theta_verification.png")
```

Listing 1: Numerical Simulation Verification Code

A.6 Problem ID:332

A.6.1 Problem Statement Explanation

This problem describes a mechanical system composed of a uniform thin spherical shell and a uniform thin rod. The system rotates as a single rigid body about a fixed horizontal axis until relative sliding occurs at a specific angle of rotation.

System Components and Geometry:

1. Spherical Shell: A uniform thin spherical shell with mass M and radius R. Its center, denoted as O, is fixed on a horizontal axis of rotation. The shell can rotate freely about this axis.

2. Thin Rod: A uniform thin rod with mass m and length L. One end, denoted as A, is smoothly hinged to the same horizontal axis at a distance d from the sphere's center O. The other end of the rod rests on the spherical shell at a contact point, B.

3. Geometric Configuration: The points O, A, and B form a triangle with constant side lengths $OA = d$, $OB = R$, and $AB = L$. The problem statement specifies $d-R < L < \sqrt{d^2 - R^2}$, which ensures a valid, non-degenerate triangle configuration where the rod is tangent to or rests on the sphere without passing through its center. We define the constant internal angle of this triangle at the hinge A as $\beta = \angle OAB$. From the Law of Cosines applied to $\triangle OAB$:

Coordinate System and Motion:

- We establish an inertial (lab) frame of reference. Let the horizontal axis of rotation be the x-axis. The center of the sphere O is at the origin $(0, 0, 0)$, and the hinge A is at $(d, 0, 0)$. The gravitational acceleration $\vec{g}$ acts in the negative z-direction, so $\vec{g} = -g\hat{k}$.

- Initially, the system is at rest with the plane containing $\triangle OAB$ being the vertical xz-plane, with the rod positioned above the x-axis. This corresponds to an unstable equilibrium position.

- The angle of rotation, φ, is the angle between the plane of $\triangle OAB$ and the vertical xz-plane, resulting from rotation about the x-axis. The initial state corresponds to $\varphi = 0$.

- A small perturbation causes the system to rotate as a single rigid body about the x-axis (no-slip condition). The angular velocity is $\omega = \dot{\varphi}$ and the angular acceleration is $\alpha = \ddot{\varphi}$.
- At a specific angle φ, the static friction force reaches its maximum value, and relative sliding between the rod and the sphere is imminent. The condition for impending slip is $f = \mu N$, where f is the magnitude of the static friction force, N is the magnitude of the normal force, and μ is the coefficient of static friction.

Objective: The goal is to determine the coefficient of static friction μ as a function of the angle φ at which sliding begins. This requires finding the magnitudes of the friction force f and the normal force N by analyzing the dynamics of the system at that instant.

A.6.2 Step 1: Kinematics of the Combined System

Since the rod and sphere rotate together as a single rigid body about the x-axis, we can determine the system's angular velocity and acceleration as a function of the rotation angle φ using the principle of conservation of energy.

Principles/Original Formulas/Assumptions: The total mechanical energy of the system is conserved because the hinge and contact forces (normal and static friction) do no work during the no-slip rotation.

$$\boxed{E = T + V = \text{constant}}$$

The total kinetic energy T is the sum of the rotational kinetic energies of the sphere and the rod about the x-axis.

$$\boxed{T = \frac{1}{2} I_{total} \omega^2 = \frac{1}{2} I_{total} \dot{\varphi}^2}$$

The moment of inertia of the thin spherical shell about a diameter (the x-axis) is:

$$\boxed{I_{sphere,x} = \frac{2}{3} M R^2}$$

The moment of inertia of the thin rod about the x-axis is found by integrating $r_\perp^2 dm$ over the rod, where $r_\perp(s) = s \sin\beta$ is the perpendicular distance of a mass element dm at a distance s from the hinge A.

$$\boxed{I_{rod,x} = \int_{rod} r_\perp^2 dm = \int_0^L (s \sin\beta)^2 \left(\frac{m}{L} ds\right) = \frac{1}{3} m (L \sin\beta)^2}$$

The gravitational potential energy V depends on the height of the center of mass (CM) of the system. The sphere's CM is at the origin, so its potential energy is constant (we can set it to zero). The rod's CM is at a distance $L/2$ from the hinge A. Its perpendicular distance from the x-axis is $r_{\perp,CM} = \frac{L}{2}\sin\beta$. Its height in the lab frame is $z_{CM} = r_{\perp,CM}\cos\varphi$.

$$\boxed{V(\varphi) = mgz_{CM,rod}(\varphi)}$$

The angular acceleration $\ddot{\varphi}$ is found from the net external torque about the x-axis.

$$\boxed{\tau_{ext,x} = I_{total}\ddot{\varphi}}$$

Derivation: The total moment of inertia of the system about the x-axis is:

$$I_{total} = I_{sphere,x} + I_{rod,x} = \frac{2}{3}MR^2 + \frac{1}{3}m(L\sin\beta)^2 \tag{83}$$

The potential energy of the rod's CM is:

$$V(\varphi) = mg\left(\frac{L}{2}\sin\beta\right)\cos\varphi = \frac{1}{2}mgL\sin\beta\cos\varphi \tag{84}$$

The system starts from rest ($\dot{\varphi} = 0$) at the top position ($\varphi = 0$). By conservation of energy, $T(\varphi) + V(\varphi) = T(0) + V(0)$.

$$\begin{aligned}
\frac{1}{2}I_{total}\dot{\varphi}^2 + \frac{1}{2}mgL\sin\beta\cos\varphi = 0 + \frac{1}{2}mgL\sin\beta\cos(0)& \\
I_{total}\dot{\varphi}^2 &= mgL\sin\beta(1-\cos\varphi) \\
\dot{\varphi}^2 &= \frac{mgL\sin\beta(1-\cos\varphi)}{I_{total}} = \frac{3mgL\sin\beta(1-\cos\varphi)}{2MR^2 + m(L\sin\beta)^2} \qquad (85)
\end{aligned}$$

The external torque about the x-axis is due to gravity on the rod: $\tau_{ext,x} = -\frac{dV}{d\varphi} = \frac{1}{2}mgL\sin\beta\sin\varphi$.

$$\begin{aligned}
\frac{1}{2}mgL\sin\beta\sin\varphi = I_{total}\ddot{\varphi}& \\
\ddot{\varphi} &= \frac{\frac{1}{2}mgL\sin\beta\sin\varphi}{\frac{1}{3}(2MR^2 + m(L\sin\beta)^2)} = \frac{3mgL\sin\beta\sin\varphi}{2(2MR^2 + m(L\sin\beta)^2)} \qquad (86)
\end{aligned}$$

A.6.3 Step 2: Dynamics of the Rod and Derivation of the Normal Force

To find the normal force N, we analyze the torques acting on the rod about the hinge A. We use Newton's second law for rotation, $\vec{\tau}_A = d\vec{L}_A/dt$, where all vectors are expressed in the inertial lab frame.

Principles/Original Formulas/Assumptions: The net torque on the rod about the hinge A is equal to the rate of change of its angular momentum about A.

$$\boxed{\vec{\tau}_{A,ext} = \frac{d\vec{L}_A}{dt}}$$

The angular momentum of the rod about A is given by $\vec{L}_A = \int_{rod} \vec{r}' \times (\vec{\omega} \times \vec{r}')dm$, where $\vec{r}'$ is the position vector from A to a mass element dm and $\vec{\omega}$ is the angular velocity of the system. The external torques on the rod about A are from gravity ($\vec{\tau}_{A,g}$), the normal force ($\vec{\tau}_{A,N}$), and the friction force ($\vec{\tau}_{A,f}$).

$$\boxed{\vec{\tau}_{A,ext} = \vec{\tau}_{A,g} + \vec{\tau}_{A,N} + \vec{\tau}_{A,f}}$$

Derivation: We project the torque equation onto the direction normal to the plane of $\triangle OAB$. Let this normal vector be $\hat{k}'$. In the lab frame, $\hat{k}' = (0, -\cos\varphi, \sin\varphi)$.

1. **Torque from Normal Force ($\vec{\tau}_{A,N}$):** The force $\vec{N}$ acts at B along the vector $\vec{r}_{O/B}$. The torque about A is $\vec{\tau}_{A,N} = \vec{r}_{B/A} \times \vec{N}$. This torque vector is perpendicular to the plane of $\triangle OAB$. A detailed calculation in the lab frame shows its magnitude is $\frac{NLd\sin\beta}{R}$ and its direction is along $\hat{k}'$. $(\vec{\tau}_{A,N})_{k'} = \vec{\tau}_{A,N} \cdot \hat{k}' = \frac{NLd\sin\beta}{R}$.

2. **Torque from Gravity ($\vec{\tau}_{A,g}$):** The gravitational force $m\vec{g} = (0, 0, -mg)$ acts at the rod's CM. The torque about A is $\vec{\tau}_{A,g} = \vec{r}_{CM/A} \times m\vec{g}$. The projection onto $\hat{k}'$ is $(\vec{\tau}_{A,g})_{k'} = -\frac{mgL}{2}\cos\beta\cos\varphi$.

3. **Torque from Friction ($\vec{\tau}_{A,f}$):** The friction force $\vec{f}$ is tangent to the sphere at B and perpendicular to the plane of $\triangle OAB$ (since impending slip is rotational). Thus $\vec{f}$ is parallel to $\hat{k}'$. The torque $\vec{\tau}_{A,f} = \vec{r}_{B/A} \times \vec{f}$ is therefore in the plane of $\triangle OAB$, so its projection onto $\hat{k}'$ is zero. $(\vec{\tau}_{A,f})_{k'} = 0$.

4. **Rate of Change of Angular Momentum ($d\vec{L}_A/dt$):** A full derivation in the lab frame yields the projection onto $\hat{k}'$ as $(d\vec{L}_A/dt) \cdot \hat{k}' = -\frac{1}{3}mL^2\dot{\varphi}^2\sin\beta\cos\beta$.

Combining these terms, the $\hat{k}'$ component of the torque equation is:

$$\begin{aligned}
(\vec{\tau}_{A,N})_{k'} + (\vec{\tau}_{A,g})_{k'} + (\vec{\tau}_{A,f})_{k'} &= \left(\frac{d\vec{L}_A}{dt}\right)_{k'} \\
\frac{NLd\sin\beta}{R} - \frac{mgL}{2}\cos\beta\cos\varphi + 0 &= -\frac{1}{3}mL^2\dot{\varphi}^2\sin\beta\cos\beta \qquad (87)
\end{aligned}$$

Solving for the normal force N:

$$\begin{aligned}
\frac{NLd\sin\beta}{R} &= \frac{mgL}{2}\cos\beta\cos\varphi - \frac{1}{3}mL^2\dot{\varphi}^2\sin\beta\cos\beta \\
\frac{Nd\sin\beta}{R} &= \frac{mg}{2}\cos\beta\cos\varphi - \frac{1}{3}mL\dot{\varphi}^2\sin\beta\cos\beta \\
N &= \frac{R}{d\sin\beta}\left(\frac{mg}{2}\cos\beta\cos\varphi - \frac{1}{3}mL\dot{\varphi}^2\sin\beta\cos\beta\right) \\
N &= \frac{mR\cos\beta}{d\sin\beta}\left(\frac{g}{2}\cos\varphi - \frac{L}{3}\dot{\varphi}^2\sin\beta\right) \qquad (88)
\end{aligned}$$

A.6.4 Step 3: Derivation of the Frictional Force

The friction force provides the torque that gives the sphere its angular acceleration $\ddot{\varphi}$. At the point of impending slip, this friction force f is purely tangential to the direction of motion.

Principles/Original Formulas/Assumptions: The rotational dynamics of the sphere about the x-axis is governed by:

$$\boxed{\tau_{sphere,x} = I_{sphere,x}\ddot{\varphi}}$$

The torque is provided by the friction force f acting at a lever arm equal to the radius of the circular path of the contact point B. This radius is $r_{\perp,B} = L\sin\beta$.

Derivation:

$$\tau_{sphere,x} = f \cdot r_{\perp,B} = f(L\sin\beta) \tag{89}$$

$$f(L\sin\beta) = I_{sphere,x}\ddot{\varphi} = \left(\frac{2}{3}MR^2\right)\ddot{\varphi}$$

$$f = \frac{2MR^2\ddot{\varphi}}{3L\sin\beta} \tag{90}$$

A.6.5 Step 4: Calculation of the Coefficient of Static Friction

At the point of impending slip, the coefficient of static friction μ is the ratio of the magnitude of the friction force f to the normal force N.

Principles/Original Formulas/Assumptions:

$$\boxed{\mu = \frac{f}{N}}$$

Derivation: First, substitute the expression for $\ddot{\varphi}$ (Eq. 86) into the formula for f (Eq. 90).

$$\begin{aligned} f &= \frac{2MR^2}{3L\sin\beta}\left(\frac{3mgL\sin\beta\sin\varphi}{2(2MR^2 + m(L\sin\beta)^2)}\right) \\ &= \frac{MR^2mg\sin\varphi}{2MR^2 + m(L\sin\beta)^2} \end{aligned} \tag{91}$$

Next, substitute the expression for $\dot{\varphi}^2$ (Eq. 85) into our corrected formula for N (Eq. 88).

$$
\begin{aligned}
N &= \frac{mR\cos\beta}{d\sin\beta}\left[\frac{g}{2}\cos\varphi - \frac{L\sin\beta}{3}\left(\frac{3mgL\sin\beta(1-\cos\varphi)}{2MR^2+m(L\sin\beta)^2}\right)\right] \\
&= \frac{mgR\cos\beta}{2d\sin\beta}\left[\cos\varphi - \frac{2(L\sin\beta)^2(1-\cos\varphi)}{2MR^2+m(L\sin\beta)^2}\right] \\
&= \frac{mgR\cos\beta}{2d\sin\beta}\left[\frac{\cos\varphi(2MR^2+m(L\sin\beta)^2) - 2m(L\sin\beta)^2(1-\cos\varphi)}{2MR^2+m(L\sin\beta)^2}\right] \\
&= \frac{mgR\cos\beta}{2d\sin\beta}\frac{2MR^2\cos\varphi + m(L\sin\beta)^2\cos\varphi - 2m(L\sin\beta)^2 + 2m(L\sin\beta)^2\cos\varphi}{2MR^2+m(L\sin\beta)^2} \\
&= \frac{mgR\cos\beta}{2d\sin\beta}\frac{2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)}{2MR^2+m(L\sin\beta)^2}
\end{aligned} \tag{92}
$$

Finally, we compute the ratio $\mu = f/N$. We assume conditions where $N > 0$.

$$
\begin{aligned}
\mu = \frac{f}{N} &= \frac{\frac{MR^2mg\sin\varphi}{2MR^2+m(L\sin\beta)^2}}{\frac{mgR\cos\beta}{2d\sin\beta}\frac{2MR^2\cos\varphi+m(L\sin\beta)^2(3\cos\varphi-2)}{2MR^2+m(L\sin\beta)^2}} \\
&= \frac{MR^2mg\sin\varphi}{1}\cdot\frac{2d\sin\beta}{mgR\cos\beta\left[2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)\right]} \\
&= \frac{2MR^2d\sin\beta\sin\varphi}{R\cos\beta\left[2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)\right]} \\
&= \frac{2MRd\sin\beta\sin\varphi}{\cos\beta\left[2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)\right]}
\end{aligned} \tag{93}
$$

A.6.6 Final Answer

The coefficient of static friction μ as a function of the angle φ is given by the following expression:

$$
\boxed{\mu = \frac{2MRd\sin\beta\sin\varphi}{\cos\beta\left[2MR^2\cos\varphi + m(L\sin\beta)^2(3\cos\varphi-2)\right]}} \tag{94}
$$

A.7 Problem ID: 457

A.7.1 Problem Statement Explanation

This problem asks for the steady-state angular velocity of a regular solid uniform N-sided polygonal prism rolling on a horizontal table. The prism is subject to a constant horizontal force that is large enough to sustain the motion.

The given physical quantities are:

- m: the mass of the prism.
- l: the distance from the center of the prism's end face to any of its vertices (the circumradius of the polygon).

- N: the number of sides of the polygonal base.
- F: a constant horizontal force applied to the center of mass (CM) of the prism, perpendicular to its axis and directed to the right.

The physical process is as follows:

1. The prism rolls to the right. The motion consists of a sequence of rotations about its bottom edges.
2. In each step, the prism pivots about a stationary edge, say P_1. It rotates until the adjacent face hits the table.
3. The collision with the table is completely inelastic. This means the new edge that hits the table, say P_2, becomes the new stationary pivot for the next phase of rotation.
4. This process of rotation and collision repeats.
5. After a sufficiently long time, the motion reaches a steady state. In this state, the angular velocity immediately after each collision is a constant value, ω. The angular velocity just before the next collision is also a constant value, Ω.

We define the following variables based on the geometry of the N-sided regular polygon, which is the cross-section of the prism:

$\theta = \frac{\pi}{N}$: Half the angle subtended by a side at the center of the polygon. The prism rotates by an angle of 2θ to move from one face to the next.

Overall assumptions:

- The coefficient of static friction is sufficiently large to prevent any slipping at the pivot edge during rotation.
- The collisions are instantaneous.
- The force F is large enough to sustain the rolling motion against the energy loss in collisions.
- The motion is planar. We analyze the 2D dynamics of the polygonal cross-section.

Our goal is to find the constant angular velocity ω just after each collision in the steady state.

A.7.2 Step 1: Geometric and Inertial Properties

First, we establish the moments of inertia required for the analysis. The rotation occurs in the plane of the polygonal cross-section. The moment of inertia of a uniform N-sided regular polygonal prism about its longitudinal axis through the center of mass (CM) is the same as that of a 2D lamina of the same mass. This is given by the standard formula:

$$\boxed{I_c = \frac{1}{6}ml^2\left(1 + 2\cos^2\left(\frac{\pi}{N}\right)\right)}$$

During the rolling motion, the prism pivots about one of its long edges. The axis of rotation is this edge. We use the parallel axis theorem to find the moment of inertia I_p about this pivot edge. The distance d from the center of mass to any pivot edge (a vertex of the polygon) is the circumradius l.

$$\boxed{I_p = I_c + md^2}$$

Applying this to our problem with $d = l$:

$$I_p = I_c + ml^2 \tag{95}$$

A.7.3 Step 2: Dynamics of Rotation Between Collisions

In the steady state, we analyze the motion between two consecutive collisions. Let ω be the angular velocity just after a collision (about pivot P_1) and Ω be the angular velocity just before the next collision (still about pivot P_1). We apply the work-energy theorem for this rotational phase.

$$\boxed{W_{\text{net}} = \Delta K}$$

The kinetic energy is purely rotational about the fixed pivot. The general formula for rotational kinetic energy is:

$$\boxed{K = \frac{1}{2}I\omega^2}$$

The net work W_{net} is the sum of the work done by the applied force F (W_F) and the work done by gravity (W_g). The change in kinetic energy is $\Delta K = K_f - K_i$, where $K_i = \frac{1}{2}I_p\omega^2$ is the initial kinetic energy and $K_f = \frac{1}{2}I_p\Omega^2$ is the final kinetic energy.

To roll to the right, the prism pivots on its rightmost edge, P_1. The CM rotates over this pivot. Let α be the angle of the line from the pivot P_1 to the CM with the upward vertical. The rotation phase starts with the CM to the left of the vertical ($\alpha_i = -\theta$) and ends with the CM to the right of the vertical ($\alpha_f = \theta$).

The work done by the constant horizontal force F is given by $W_F = F\Delta x_c$, where Δx_c is the horizontal displacement of the CM. The horizontal position of the CM relative to the pivot is $x_c = l\sin\alpha$.

$$\Delta x_c = x_{c,f} - x_{c,i} = l\sin(\theta) - l\sin(-\theta) = 2l\sin\theta \tag{96}$$

$$W_F = F \cdot \Delta x_c = F(2l\sin\theta) = 2Fl\sin\theta \tag{97}$$

The work done by gravity is $W_g = -mg\Delta y_c$, where Δy_c is the vertical displacement of the CM. The vertical position of the CM relative to the pivot is $y_c = l\cos\alpha$.

$$\Delta y_c = y_{c,f} - y_{c,i} = l\cos(\theta) - l\cos(-\theta) = l\cos\theta - l\cos\theta = 0 \tag{98}$$

$$W_g = -mg\Delta y_c = 0 \tag{99}$$

Applying the work-energy theorem:

$$\begin{aligned} W_F + W_g &= K_f - K_i \\ 2Fl\sin\theta &= \frac{1}{2}I_p\Omega^2 - \frac{1}{2}I_p\omega^2 \\ 4Fl\sin\theta &= I_p(\Omega^2 - \omega^2) \end{aligned} \tag{100}$$

A.7.4 Step 3: Collision Analysis

We analyze the instantaneous, completely inelastic collision. When the prism strikes the table, a new pivot P_2 is formed. During the brief collision, the impulsive force from the table at P_2 is much larger than the external force F and gravity mg. Thus, we can neglect the impulse of these external forces, and the angular momentum of the prism about the new pivot point P_2 is conserved.

$$\boxed{\vec{L}_{\text{before, about } P_2} = \vec{L}_{\text{after, about } P_2}}$$

After the collision, the prism rotates about the new stationary pivot P_2 with angular velocity ω. The angular momentum is:

$$L_{\text{after, about } P_2} = I_p\omega \tag{101}$$

Before the collision, the prism rotates about the old pivot P_1 with angular velocity Ω. The angular momentum about P_2 is found using the general formula for a rigid body in plane motion:

$$\boxed{\vec{L}_{\text{about a point P}} = I_c\vec{\Omega}_{\text{CM}} + \vec{r}_{\text{CM}/P} \times \vec{p}_{\text{CM}}}$$

Here, $\vec{p}_{\text{CM}} = m\vec{v}_c$ and the velocity of the CM is $\vec{v}_c = \vec{\Omega} \times \vec{r}_{c/P_1}$. The vectors are $\vec{r}_{c/P1}$ (from pivot P_1 to CM) and $\vec{r}_{c/P2}$ (from pivot P_2 to CM).

$$\begin{aligned}\vec{L}_{\text{before, about } P_2} &= I_c\vec{\Omega} + \vec{r}_{c/P_2} \times (m(\vec{\Omega} \times \vec{r}_{c/P_1})) \\ &= I_c\vec{\Omega} + m(\vec{r}_{c/P_2} \times (\vec{\Omega} \times \vec{r}_{c/P_1}))\end{aligned} \tag{102}$$

We use the vector triple product identity:

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})}$$

Applying this to Eq. (102):

$$\vec{L}_{\text{before, about } P_2} = I_c\vec{\Omega} + m(\vec{\Omega}(\vec{r}_{c/P_2} \cdot \vec{r}_{c/P_1}) - \vec{r}_{c/P_1}(\vec{r}_{c/P_2} \cdot \vec{\Omega})) \tag{103}$$

The vector $\vec{\Omega}$ is perpendicular to the plane of the polygon, while $\vec{r}_{c/P_1}$ and $\vec{r}_{c/P_2}$ lie within it, so their dot product with $\vec{\Omega}$ is zero. To find $\vec{r}_{c/P_1} \cdot \vec{r}_{c/P_2}$, we note that at the moment of impact, $\vec{r}_{c/P_1}$ and $\vec{r}_{c/P_2}$ are vectors of length l from the pivots to the CM. The angle between the vectors from the CM to the pivots P_1 and P_2 is 2θ. Thus, the angle between $\vec{r}_{c/P_1}$ and $\vec{r}_{c/P_2}$ is also 2θ. The dot product is $|\vec{r}_{c/P_1}||\vec{r}_{c/P_2}|\cos(2\theta) = l^2\cos(2\theta)$.

$$\vec{L}_{\text{before, about } P_2} = I_c\vec{\Omega} + m(\vec{\Omega}(l^2\cos(2\theta))) = (I_c + ml^2\cos(2\theta))\vec{\Omega} \tag{104}$$

Equating the magnitudes of the angular momentum from (101) and (104):

$$I_p\omega = (I_c + ml^2\cos(2\theta))\Omega \tag{105}$$

A.7.5 Step 4: Combining Equations to Find Steady-State Velocity

We now have a system of two equations, (100) and (105), for the two unknowns ω and Ω. We solve for ω. First, express Ω in terms of ω from (105):

$$\Omega = \frac{I_p}{I_c + ml^2\cos(2\theta)}\omega \tag{106}$$

Substitute this into the energy equation (100):

$$\begin{aligned}4Fl\sin\theta &= I_p(\Omega^2 - \omega^2) = I_p\left(\left(\frac{I_p}{I_c + ml^2\cos(2\theta)}\right)^2\omega^2 - \omega^2\right) \\ 4Fl\sin\theta &= I_p\omega^2\left(\frac{I_p^2}{(I_c + ml^2\cos(2\theta))^2} - 1\right) \\ 4Fl\sin\theta &= I_p\omega^2\frac{I_p^2 - (I_c + ml^2\cos(2\theta))^2}{(I_c + ml^2\cos(2\theta))^2} \\ \omega^2 &= \frac{4Fl\sin\theta(I_c + ml^2\cos(2\theta))^2}{I_p(I_p^2 - (I_c + ml^2\cos(2\theta))^2)}\end{aligned} \tag{107}$$

The denominator can be simplified using $I_p = I_c + ml^2$ and the difference of squares identity $a^2 - b^2 = (a-b)(a+b)$:

$$\begin{aligned}
&I_p(I_p - (I_c + ml^2\cos(2\theta)))(I_p + (I_c + ml^2\cos(2\theta))) \\
&= (I_c + ml^2)(I_c + ml^2 - I_c - ml^2\cos(2\theta))(I_c + ml^2 + I_c + ml^2\cos(2\theta)) \\
&= (I_c + ml^2)(ml^2(1 - \cos(2\theta)))(2I_c + ml^2(1 + \cos(2\theta))) \\
&= (I_c + ml^2)(2ml^2\sin^2\theta)(2I_c + 2ml^2\cos^2\theta) \\
&= 4ml^2\sin^2\theta(I_c + ml^2)(I_c + ml^2\cos^2\theta) \qquad (108)
\end{aligned}$$

Substituting (194) into (107):

$$\omega^2 = \frac{4Fl\sin\theta(I_c + ml^2\cos(2\theta))^2}{4ml^2\sin^2\theta(I_c + ml^2)(I_c + ml^2\cos^2\theta)}$$

$$\omega^2 = \frac{F(I_c + ml^2\cos(2\theta))^2}{ml\sin\theta(I_c + ml^2)(I_c + ml^2\cos^2\theta)} \qquad (109)$$

A.7.6 Step 5: Substituting Inertial Properties and Simplifying

To get the final answer, we substitute $I_c = \frac{1}{6}ml^2(1 + 2\cos^2\theta)$ into Eq. (109), where $\theta = \pi/N$. Let's evaluate each term involving I_c:

$$I_c + ml^2\cos(2\theta) = \frac{1}{6}ml^2(1 + 2\cos^2\theta) + ml^2(2\cos^2\theta - 1) = ml^2\left(\frac{1}{6} + \frac{1}{3}\cos^2\theta + 2\cos^2\theta - 1\right) = ml^2\left(\frac{7}{3} \qquad (110)$$

$$I_c + ml^2 = \frac{1}{6}ml^2(1 + 2\cos^2\theta) + ml^2 = ml^2\left(\frac{1}{6} + \frac{1}{3}\cos^2\theta + 1\right) = ml^2\left(\frac{7}{6} + \frac{1}{3}\cos^2\theta\right) \qquad (111)$$

$$I_c + ml^2\cos^2\theta = \frac{1}{6}ml^2(1 + 2\cos^2\theta) + ml^2\cos^2\theta = ml^2\left(\frac{1}{6} + \frac{1}{3}\cos^2\theta + \cos^2\theta\right) = ml^2\left(\frac{1}{6} + \frac{4}{3}\cos^2\theta\right) \qquad (112)$$

Substituting these into Eq. (109):

$$\omega^2 = \frac{F\left(ml^2\left(\frac{7}{3}\cos^2\theta - \frac{5}{6}\right)\right)^2}{ml\sin\theta\left(ml^2\left(\frac{7}{6} + \frac{1}{3}\cos^2\theta\right)\right)\left(ml^2\left(\frac{1}{6} + \frac{4}{3}\cos^2\theta\right)\right)}$$

$$\omega^2 = \frac{F(ml^2)^2\left(\frac{14\cos^2\theta - 5}{6}\right)^2}{ml\sin\theta(ml^2)^2\left(\frac{7 + 2\cos^2\theta}{6}\right)\left(\frac{1 + 8\cos^2\theta}{6}\right)}$$

$$\omega^2 = \frac{F(14\cos^2\theta - 5)^2/36}{ml\sin\theta(7 + 2\cos^2\theta)(1 + 8\cos^2\theta)/36}$$

$$\omega^2 = \frac{F(14\cos^2\theta - 5)^2}{ml\sin\theta(7 + 2\cos^2\theta)(1 + 8\cos^2\theta)} \qquad (113)$$

A.7.7 Final Answer

The steady-state angular velocity ω after each collision is given by taking the positive square root of Eq. (113), with $\theta = \pi/N$. The term $(14\cos^2(\pi/N) - 5)^2$ is always non-negative, so its square root is $|14\cos^2(\pi/N) - 5|$.

$$\boxed{\omega = \left|14\cos^2\left(\frac{\pi}{N}\right) - 5\right| \sqrt{\frac{F}{ml\sin\left(\frac{\pi}{N}\right)\left(7 + 2\cos^2\left(\frac{\pi}{N}\right)\right)\left(1 + 8\cos^2\left(\frac{\pi}{N}\right)\right)}}} \tag{114}$$

A.8 Problem ID: 71

A.8.1 Problem Statement Explanation

This problem asks for the guiding center drift velocity of a charged particle moving in a specified non-uniform electric field and a uniform magnetic field.

The physical system is defined by:

- **Particle properties**: A particle with mass m and charge q.
- **Electromagnetic fields**:
  - A uniform magnetic field directed along the z-axis: $\mathbf{B} = B\hat{\mathbf{z}}$.
  - A non-uniform, static electric field directed along the x-axis, with its magnitude varying sinusoidally in the y-direction: $\mathbf{E} = E_0 \cos(ky)\hat{\mathbf{x}}$.
- **Initial conditions (at $t = 0$)**:
  - Position: $\mathbf{r}(0) = \mathbf{r}_0 = (x_0, y_0, z_0)$.
  - Velocity: $\mathbf{v}(0) = \mathbf{v}_0 = (v_\perp \cos\delta, v_\perp \sin\delta, v_z)$. Here, $v_\perp$ is the magnitude of the initial velocity in the xy-plane, and v_z is the initial velocity along the z-axis.

The particle's motion can be decomposed into a fast gyration about a slowly moving point called the guiding center. The particle's position vector $\mathbf{r}(t)$ is expressed as the sum of the guiding center position $\mathbf{R}_{gc}(t)$ and the gyroradius vector $\boldsymbol{\rho}(t)$: $\mathbf{r}(t) = \mathbf{R}_{gc}(t) + \boldsymbol{\rho}(t)$.

The goal is to find the guiding center drift velocity, which is the velocity of the guiding center perpendicular to the magnetic field, denoted as $\mathbf{v}_{gc}$.

We are given the following assumptions:

1. The electric field is weak, implying the drift speed is much smaller than the gyration speed.
2. The Larmor radius r_L (the radius of gyration) is small compared to the spatial scale of the electric field's variation, i.e., $kr_L \ll 1$.

3. The final expression should be an approximation to the lowest order that distinguishes the drift from the case of a uniform electric field. This requires accounting for the Finite Larmor Radius (FLR) effect, which arises from averaging the non-uniform field over the particle's finite-sized orbit.

A.8.2 Step 1: Motion Parallel to the Magnetic Field

The motion of the charged particle is governed by the Lorentz force law. We first analyze the component of motion parallel to the magnetic field (in the z-direction).

Principles/Original Formulas/Assumptions: The equation of motion for a charged particle in electric and magnetic fields is the Lorentz force law.

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

Derivation: We write the z-component of the Lorentz force equation. Given $\mathbf{E} = E_0 \cos(ky)\hat{\mathbf{x}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, the z-component of the electric field is $E_z = 0$. The z-component of the magnetic force is $(\mathbf{v} \times \mathbf{B})_z = v_x B_y - v_y B_x = 0$, since $B_x = B_y = 0$.

$$\begin{aligned} m\frac{dv_z}{dt} &= q(E_z + (\mathbf{v} \times \mathbf{B})_z) \\ m\frac{dv_z}{dt} &= q(0 + 0) = 0 \end{aligned} \tag{115}$$

This implies that the velocity component parallel to the magnetic field is constant. The guiding center, representing the average position, must also move with this constant velocity along the z-axis.

$$v_z(t) = v_z(0) = v_z \quad \text{(constant)} \tag{116}$$

$$\mathbf{V}_{gc,\parallel} = v_z\hat{\mathbf{z}} \tag{117}$$

The drift we seek is the perpendicular component of the guiding center velocity, $\mathbf{v}_{gc}$.

A.8.3 Step 2: General Equation for Perpendicular Guiding Center Drift

To find the guiding center velocity, we average the equation of motion over one gyration period, T_c.

Principles/Original Formulas/Assumptions: The guiding center velocity $\mathbf{V}_{gc}$ is defined as the time average of the particle's velocity $\mathbf{v}$ over one gyration period.

$$\boxed{\mathbf{V}_{gc} = \langle \mathbf{v} \rangle \equiv \frac{1}{T_c}\int_0^{T_c} \mathbf{v}(t)dt}$$

For a steady drift superimposed on fast gyration, the change in velocity over one period is small, corresponding to the acceleration of the guiding center. In the lowest-order approximation for a steady drift, this acceleration is neglected.

$$\boxed{m\left\langle \frac{d\mathbf{v}}{dt} \right\rangle \approx 0}$$

The vector triple product identity is used for vector manipulations.

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})}$$

Derivation: Averaging the Lorentz force equation over one gyration period gives:

$$\begin{aligned}
m\left\langle \frac{d\mathbf{v}}{dt} \right\rangle &= q(\langle \mathbf{E} \rangle + \langle \mathbf{v} \times \mathbf{B} \rangle) \\
0 &\approx q(\langle \mathbf{E} \rangle + \langle \mathbf{v} \rangle \times \mathbf{B}) \quad \text{(using } \langle d\mathbf{v}/dt \rangle \approx 0 \text{ and uniform } \mathbf{B}\text{)} \\
\mathbf{V}_{gc} \times \mathbf{B} &= -\langle \mathbf{E} \rangle
\end{aligned} \tag{118}$$

We decompose the total guiding center velocity $\mathbf{V}_{gc} = \mathbf{v}_{gc} + \mathbf{V}_{gc,\|}$, where $\mathbf{v}_{gc}$ is the perpendicular drift velocity and $\mathbf{V}_{gc,\|}$ is the parallel velocity. Since $\mathbf{V}_{gc,\|}$ is parallel to $\mathbf{B}$, their cross product is zero.

$$\begin{aligned}
(\mathbf{v}_{gc} + \mathbf{V}_{gc,\|}) \times \mathbf{B} &= -\langle \mathbf{E} \rangle \\
\mathbf{v}_{gc} \times \mathbf{B} + \mathbf{0} &= -\langle \mathbf{E} \rangle
\end{aligned} \tag{119}$$

To solve for $\mathbf{v}_{gc}$, we take the cross product with $\mathbf{B}$ and use the vector triple product identity. By definition, $\mathbf{v}_{gc}$ is perpendicular to $\mathbf{B}$, so $\mathbf{v}_{gc} \cdot \mathbf{B} = 0$.

$$\begin{aligned}
(\mathbf{v}_{gc} \times \mathbf{B}) \times \mathbf{B} &= -\langle \mathbf{E} \rangle \times \mathbf{B} \\
\mathbf{B}(\mathbf{v}_{gc} \cdot \mathbf{B}) - \mathbf{v}_{gc}(\mathbf{B} \cdot \mathbf{B}) &= -\langle \mathbf{E} \rangle \times \mathbf{B} \\
0 - B^2 \mathbf{v}_{gc} &= -\langle \mathbf{E} \rangle \times \mathbf{B} \\
\mathbf{v}_{gc} &= \frac{\langle \mathbf{E} \rangle \times \mathbf{B}}{B^2}
\end{aligned} \tag{120}$$

A.8.4 Step 3: Initial Guiding Center Position

To evaluate $\langle \mathbf{E} \rangle$, we must average the electric field over the particle's orbit, which is centered at the guiding center position $\mathbf{R}_{gc}$. We first determine the initial guiding center position, $\mathbf{R}_{gc}(0)$.

Principles/Original Formulas/Assumptions: The particle position is decomposed into guiding center and gyroradius vectors.

$$\boxed{\mathbf{r}(t) = \mathbf{R}_{gc}(t) + \boldsymbol{\rho}(t)}$$

In the zeroth-order approximation (neglecting the weak E-field), the gyration velocity $\mathbf{v}_{gyr}$ is related to the gyroradius vector $\boldsymbol{\rho}$ and the magnetic field $\mathbf{B}$ by:

$$\boxed{\mathbf{v}_{gyr} = \frac{q}{m}(\boldsymbol{\rho} \times \mathbf{B})}$$

Derivation: To find the gyroradius vector $\boldsymbol{\rho}$ in terms of the gyration velocity $\mathbf{v}_{gyr}$, we take the cross product of the above relation with $\mathbf{B}$.

$$\mathbf{v}_{gyr} \times \mathbf{B} = \frac{q}{m}(\boldsymbol{\rho} \times \mathbf{B}) \times \mathbf{B} \tag{121}$$

Using the vector triple product identity $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C})$, we expand the right side. Let $\mathbf{A} = \boldsymbol{\rho}$, $\mathbf{B} = \mathbf{B}$, $\mathbf{C} = \mathbf{B}$.

$$(\boldsymbol{\rho} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B}(\boldsymbol{\rho} \cdot \mathbf{B}) - \boldsymbol{\rho}(\mathbf{B} \cdot \mathbf{B}) \tag{122}$$

Since the gyroradius vector $\boldsymbol{\rho}$ is perpendicular to the magnetic field $\mathbf{B}$, their dot product is zero: $\boldsymbol{\rho} \cdot \mathbf{B} = 0$.

$$(\boldsymbol{\rho} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B}(0) - \boldsymbol{\rho}(B^2) = -B^2\boldsymbol{\rho} \tag{123}$$

Substituting this back into Eq. (121):

$$\begin{aligned} \mathbf{v}_{gyr} \times \mathbf{B} &= \frac{q}{m}(-B^2\boldsymbol{\rho}) = -\frac{qB^2}{m}\boldsymbol{\rho} \\ \implies \boldsymbol{\rho} &= -\frac{m}{qB^2}(\mathbf{v}_{gyr} \times \mathbf{B}) = \frac{m}{qB^2}(\mathbf{B} \times \mathbf{v}_{gyr}) \end{aligned} \tag{124}$$

The initial guiding center position is $\mathbf{R}_{gc}(0) = \mathbf{r}(0) - \boldsymbol{\rho}(0)$. We approximate the initial gyration velocity with the total initial perpendicular velocity, $\mathbf{v}_{gyr}(0) \approx \mathbf{v}_\perp(0)$, since the drift velocity is small.

$$\mathbf{R}_{gc}(0) \approx \mathbf{r}_0 - \frac{m}{qB^2}(\mathbf{B} \times \mathbf{v}_\perp(0)) \tag{125}$$

Given $\mathbf{v}_\perp(0) = (v_\perp \cos\delta, v_\perp \sin\delta, 0)$ and $\mathbf{B} = (0, 0, B)$, we find the y-component of the initial guiding center position, $Y_{gc,0}$:

$$\mathbf{B} \times \mathbf{v}_\perp(0) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & B \\ v_\perp \cos\delta & v_\perp \sin\delta & 0 \end{vmatrix} = -Bv_\perp \sin\delta\hat{\mathbf{x}} + Bv_\perp \cos\delta\hat{\mathbf{y}} \tag{126}$$

$$Y_{gc,0} = y_0 - \left(\frac{m}{qB^2}(\mathbf{B} \times \mathbf{v}_\perp(0))\right)_y = y_0 - \frac{m}{qB^2}(Bv_\perp \cos\delta) = y_0 - \frac{mv_\perp \cos\delta}{qB} \tag{127}$$

A.8.5 Step 4: Calculation of the Time-Averaged Electric Field

We compute the average electric field $\langle \mathbf{E} \rangle$ experienced by the particle by averaging $\mathbf{E}(\mathbf{r}(t)) = \mathbf{E}(\mathbf{R}_{gc} + \boldsymbol{\rho}(t))$ over one gyration period.

Principles/Original Formulas/Assumptions: Since the Larmor radius is small ($kr_L \ll 1$), we can Taylor expand $\mathbf{E}$ around the guiding center position $\mathbf{R}_{gc}$.

$$\boxed{\mathbf{E}(\mathbf{R}_{gc} + \boldsymbol{\rho}) \approx \mathbf{E}(\mathbf{R}_{gc}) + (\boldsymbol{\rho} \cdot \nabla)\mathbf{E}|_{\mathbf{R}_{gc}} + \frac{1}{2}(\boldsymbol{\rho} \cdot \nabla)^2 \mathbf{E}|_{\mathbf{R}_{gc}}}$$

The time average of the gyroradius vector and its components over a period have the following properties for gyration in the xy-plane:

$$\boxed{\langle \boldsymbol{\rho} \rangle = 0}$$

$$\boxed{\langle \rho_i \rho_j \rangle = \frac{1}{2} r_L^2 \delta_{ij} \quad \text{for } i, j \in \{x, y\}, \text{ which implies } \langle \rho_x^2 \rangle = \langle \rho_y^2 \rangle = \frac{r_L^2}{2} \text{ and } \langle \rho_x \rho_y \rangle = 0}$$

Derivation: Averaging the Taylor expansion of the electric field:

$$\begin{aligned}
\langle \mathbf{E} \rangle &= \left\langle \mathbf{E}(\mathbf{R}_{gc}) + (\boldsymbol{\rho} \cdot \nabla)\mathbf{E}|_{\mathbf{R}_{gc}} + \frac{1}{2}(\boldsymbol{\rho} \cdot \nabla)^2 \mathbf{E}|_{\mathbf{R}_{gc}} + \dots \right\rangle \\
&\approx \mathbf{E}(\mathbf{R}_{gc}) + (\langle \boldsymbol{\rho} \rangle \cdot \nabla)\mathbf{E}|_{\mathbf{R}_{gc}} + \frac{1}{2} \left\langle (\rho_x \partial_x + \rho_y \partial_y)^2 \right\rangle \mathbf{E}|_{\mathbf{R}_{gc}} \\
&= \mathbf{E}(\mathbf{R}_{gc}) + \mathbf{0} + \frac{1}{2} \left(\langle \rho_x^2 \rangle \frac{\partial^2}{\partial x^2} + \langle \rho_y^2 \rangle \frac{\partial^2}{\partial y^2} + 2\langle \rho_x \rho_y \rangle \frac{\partial^2}{\partial x \partial y} \right) \mathbf{E}|_{\mathbf{R}_{gc}} \\
&= \mathbf{E}(\mathbf{R}_{gc}) + \frac{1}{2} \left(\frac{r_L^2}{2} \frac{\partial^2}{\partial x^2} + \frac{r_L^2}{2} \frac{\partial^2}{\partial y^2} \right) \mathbf{E}|_{\mathbf{R}_{gc}} \\
&= \mathbf{E}(\mathbf{R}_{gc}) + \frac{r_L^2}{4} \left(\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} \right) \Bigg|_{\mathbf{R}_{gc}} = \mathbf{E}(\mathbf{R}_{gc}) + \frac{r_L^2}{4} \nabla_\perp^2 \mathbf{E}|_{\mathbf{R}_{gc}}
\end{aligned} \tag{128}$$

For our specific electric field $\mathbf{E} = E_0 \cos(ky)\hat{\mathbf{x}}$, the perpendicular Laplacian is $\nabla_\perp^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial y^2} = \frac{\partial^2}{\partial y^2}(E_0 \cos(ky)\hat{\mathbf{x}}) = -k^2 E_0 \cos(ky)\hat{\mathbf{x}} = -k^2 \mathbf{E}$.

$$\langle \mathbf{E} \rangle \approx \mathbf{E}(\mathbf{R}_{gc}) + \frac{r_L^2}{4}(-k^2 \mathbf{E}(\mathbf{R}_{gc})) = \mathbf{E}(\mathbf{R}_{gc}) \left(1 - \frac{1}{4} k^2 r_L^2 \right) \tag{129}$$

A.8.6 Step 5: Final Expression for the Drift Velocity

We substitute the averaged electric field into the general expression for the drift velocity and use the definition of the Larmor radius.

Principles/Original Formulas/Assumptions: The Larmor radius r_L is the ratio of the perpendicular speed $v_\perp$ to the cyclotron angular frequency magnitude $|\omega_c| = |q|B/m$.

$$\boxed{r_L = \frac{v_\perp}{|\omega_c|} = \frac{m v_\perp}{|q| B}}$$

Derivation: Substitute the expression for $\langle \mathbf{E} \rangle$ from Eq. (129) into the general drift velocity formula from Eq. (120).

$$\begin{aligned}\mathbf{v}_{gc} &= \frac{\langle \mathbf{E} \rangle \times \mathbf{B}}{B^2} = \frac{\left[\mathbf{E}(\mathbf{R}_{gc})\left(1 - \frac{1}{4}k^2 r_L^2\right)\right] \times \mathbf{B}}{B^2} \\ &= \frac{\mathbf{E}(\mathbf{R}_{gc}) \times \mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right)\end{aligned} \tag{130}$$

Substituting the expression for the Larmor radius r_L:

$$\mathbf{v}_{gc} = \frac{\mathbf{E}(\mathbf{R}_{gc}) \times \mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2\left(\frac{mv_\perp}{|q|B}\right)^2\right) \tag{131}$$

This expression gives the drift velocity as a function of the guiding center position $\mathbf{R}_{gc}$. To find the initial drift velocity, we evaluate this expression at the initial guiding center position $\mathbf{R}_{gc}(0)$, specifically using its y-component $Y_{gc,0}$ from Eq. (127). The term $\frac{\mathbf{E}(\mathbf{R}_{gc}) \times \mathbf{B}}{B^2}$ is the standard $\mathbf{E} \times \mathbf{B}$ drift evaluated at the guiding center, and the factor in parentheses is the Finite Larmor Radius (FLR) correction, which is the required lowest-order correction distinguishing this case from a uniform field.

A.8.7 Final Answer

The guiding center drift velocity, which is the component of the guiding center velocity perpendicular to the magnetic field, is given by:

$$\boxed{\mathbf{v}_{gc} = \frac{\mathbf{E}(Y_{gc,0}) \times \mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2\left(\frac{mv_\perp}{|q|B}\right)^2\right)} \tag{132}$$

where the electric field is evaluated at the initial y-coordinate of the guiding center, $\mathbf{E}(Y_{gc,0}) = E_0 \cos(kY_{gc,0})\hat{\mathbf{x}}$, and this coordinate is given by $Y_{gc,0} = y_0 - \frac{mv_\perp \cos\delta}{qB}$. The total guiding center velocity is $\mathbf{V}_{gc} = \mathbf{v}_{gc} + v_z\hat{\mathbf{z}}$.

A.9 Problem ID:250

A.9.1 Problem Statement Explanation

The problem describes a physical system composed of two semi-infinite electrolytes separated by the xOz plane, which are connected to a power source at time $t = 0$. The goal is to find the current flowing from the power source as a function of time.

System Geometry and Materials:

- **Electrolyte 1**: Occupies the region $y > 0$. It is a conductive medium with electrical conductivity σ_1 and absolute electrical permittivity ε_1.

- **Electrolyte 2**: Occupies the region $y < 0$. It is a conductive medium with electrical conductivity σ_2 and absolute electrical permittivity ε_2.
- The system has a finite thickness L in the z-direction and is considered infinite in the x and y directions.
- **Electrodes**: Two long, parallel, cylindrical metal electrodes, denoted as + and −, are inserted into the system. They have a radius R and length L, and are oriented parallel to the z-axis. Their centers are located on the xOz interface, separated by a distance of $2D$.

Electrical Circuit:

- **Power Source**: At time $t = 0$, the electrodes are connected to a power source with a constant electromotive force (EMF) U and a constant internal resistance r_0.
- **Initial Condition**: The system is initially uncharged. This implies that the potential difference across the electrodes is zero at $t = 0$.

Assumptions:

- The geometric parameters satisfy $D > R$ and $\{R, D\} \ll L$. The second condition allows us to neglect end effects and treat the problem as two-dimensional in the xy-plane. All extensive quantities like capacitance and resistance will be proportional to the length L.
- The terms "dielectric constants" $\varepsilon_1, \varepsilon_2$ are interpreted as absolute permittivities.

Objective:

- Find the current $i(t)$ flowing from the power source as a function of time.

A.9.2 Step 1: Equivalent Circuit Model

The physical system of electrolytes and electrodes can be modeled as a lumped-element electrical circuit. The two electrodes immersed in the conductive electrolytes function simultaneously as a capacitor (storing charge) and a resistor (allowing current to flow through the medium).

Principles/Original Formulas/Assumptions: A physical system with properties of both energy storage (capacitive) and energy dissipation (resistive) between two terminals can be modeled by an equivalent circuit.

$$\boxed{\text{Distributed system} \rightarrow \text{Lumped-element model}}$$

Derivation:

- **Capacitance (C)**: Arises from the storage of electric charge on the surfaces of the electrodes when a potential difference is applied. The electrolytes act as the dielectric material.

- **Resistance (R_{elec})**: Arises from the flow of charge (current) through the conductive electrolytes under the influence of the electric field between the electrodes.

Since charge storage and current leakage occur between the same two electrodes, the equivalent capacitance C and the equivalent resistance R_{elec} of the electrolyte system are in parallel. This parallel combination is connected to the power source, which is modeled as an ideal EMF U in series with its internal resistance r_0. The total current from the source is $i(t)$.

A.9.3 Step 2: Derivation of the System's Capacitance

The total capacitance is the sum of the capacitances of the upper and lower halves of the system, which are treated as two capacitors in parallel.

Principles/Original Formulas/Assumptions: The capacitance per unit length, C'_{full}, between two parallel cylindrical conductors of radius R with centers separated by $2D$ in a uniform dielectric medium with permittivity ε is:

$$\boxed{C'_{full} = \frac{\pi\varepsilon}{\operatorname{arccosh}(D/R)}}$$

For a geometry with a plane of symmetry, the capacitance of each half is half that of the full system.

$$\boxed{C'_{half} = \frac{1}{2}C'_{full}}$$

The total capacitance of capacitors connected in parallel is the sum of their individual capacitances.

$$\boxed{C_{total} = \sum_i C_i}$$

Derivation: The system consists of two parallel capacitors: C_1 for the upper half-space ($y > 0$) with permittivity ε_1, and C_2 for the lower half-space ($y < 0$) with permittivity ε_2. The capacitance per unit length for each half is:

$$C'_1 = \frac{1}{2}\left(\frac{\pi\varepsilon_1}{\operatorname{arccosh}(D/R)}\right) = \frac{\pi\varepsilon_1}{2\operatorname{arccosh}(D/R)} \tag{133}$$

$$C'_2 = \frac{1}{2}\left(\frac{\pi\varepsilon_2}{\operatorname{arccosh}(D/R)}\right) = \frac{\pi\varepsilon_2}{2\operatorname{arccosh}(D/R)} \tag{134}$$

The total capacitance per unit length, C'_{total}, is the sum of these parallel contributions. The total capacitance C for the electrode length L is then $C'_{total} \cdot L$.

$$C'_{total} = C'_1 + C'_2 = \frac{\pi\varepsilon_1}{2\,\mathrm{arccosh}(D/R)} + \frac{\pi\varepsilon_2}{2\,\mathrm{arccosh}(D/R)} = \frac{\pi(\varepsilon_1+\varepsilon_2)}{2\,\mathrm{arccosh}(D/R)} \tag{135}$$

$$C = C'_{total} \cdot L = \frac{\pi(\varepsilon_1+\varepsilon_2)L}{2\,\mathrm{arccosh}(D/R)} \tag{136}$$

A.9.4 Step 3: Derivation of the System's Resistance

The total resistance of the electrolyte is determined by the two parallel paths for current flow through the upper and lower electrolytes.

Principles/Original Formulas/Assumptions: For a system with a given geometry in a homogeneous medium with uniform permittivity ε and conductivity σ, the product of its resistance R and capacitance C is a constant determined by the material properties.

$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$

The total resistance of resistors connected in parallel is given by the reciprocal of the sum of their reciprocals.

$$\boxed{\frac{1}{R_{total}} = \sum_i \frac{1}{R_i}}$$

Derivation: We apply the RC relation to each half of our system. Let R_1 and R_2 be the resistances of the upper and lower electrolytes, and $C_1 = C'_1 L$ and $C_2 = C'_2 L$ be their respective capacitances.

$$R_1 C_1 = \frac{\varepsilon_1}{\sigma_1} \implies R_1 = \frac{\varepsilon_1}{\sigma_1 C_1} = \frac{\varepsilon_1}{\sigma_1 (C'_1 L)} = \frac{\varepsilon_1}{\sigma_1 L}\frac{2\,\mathrm{arccosh}(D/R)}{\pi\varepsilon_1} = \frac{2\,\mathrm{arccosh}(D/R)}{\pi\sigma_1 L} \tag{137}$$

$$R_2 C_2 = \frac{\varepsilon_2}{\sigma_2} \implies R_2 = \frac{\varepsilon_2}{\sigma_2 C_2} = \frac{\varepsilon_2}{\sigma_2 (C'_2 L)} = \frac{\varepsilon_2}{\sigma_2 L}\frac{2\,\mathrm{arccosh}(D/R)}{\pi\varepsilon_2} = \frac{2\,\mathrm{arccosh}(D/R)}{\pi\sigma_2 L} \tag{138}$$

The current can flow through both electrolytes simultaneously, so these two resistances are in parallel. The total equivalent resistance of the electrolyte, R_{elec}, is:

$$\frac{1}{R_{elec}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{\pi\sigma_1 L}{2\,\mathrm{arccosh}(D/R)} + \frac{\pi\sigma_2 L}{2\,\mathrm{arccosh}(D/R)} = \frac{\pi(\sigma_1+\sigma_2)L}{2\,\mathrm{arccosh}(D/R)} \tag{139}$$

$$R_{elec} = \frac{2\,\mathrm{arccosh}(D/R)}{\pi(\sigma_1+\sigma_2)L} \tag{140}$$

A.9.5 Step 4: Circuit Analysis and Governing Differential Equation

Let $i(t)$ be the current from the source and $u(t)$ be the potential difference across the electrodes (the parallel $R_{elec}C$ branch).

Principles/Original Formulas/Assumptions: Kirchhoff's Voltage Law (KVL) for the circuit loop:

$$\boxed{U = \sum V_i}$$

Kirchhoff's Current Law (KCL) at a node:

$$\boxed{\sum I_{in} = \sum I_{out}}$$

The current-voltage relationship for a capacitor:

$$\boxed{i_C(t) = C\frac{du}{dt}}$$

Ohm's Law for a resistor:

$$\boxed{u(t) = i_R(t)R_{elec}}$$

Derivation: Applying KVL to the circuit loop gives:

$$U = i(t)r_0 + u(t) \tag{141}$$

Applying KCL at the node where the source current $i(t)$ splits into the capacitor current $i_C(t)$ and the resistor current $i_R(t)$:

$$i(t) = i_C(t) + i_R(t) = C\frac{du}{dt} + \frac{u(t)}{R_{elec}} \tag{142}$$

To obtain a differential equation solely for $i(t)$, we eliminate $u(t)$. From Eq. (141), we have $u(t) = U - i(t)r_0$. Differentiating this with respect to time gives $\frac{du}{dt} = -r_0\frac{di}{dt}$. Substituting these into Eq. (142):

$$\begin{aligned} i(t) &= C\left(-r_0\frac{di}{dt}\right) + \frac{U - i(t)r_0}{R_{elec}} \\ i(t) &= -Cr_0\frac{di}{dt} + \frac{U}{R_{elec}} - \frac{r_0}{R_{elec}}i(t) \\ Cr_0\frac{di}{dt} &= \frac{U}{R_{elec}} - i(t)\left(1 + \frac{r_0}{R_{elec}}\right) \\ \frac{di}{dt} + \left(\frac{1}{Cr_0} + \frac{1}{CR_{elec}}\right)i(t) &= \frac{U}{Cr_0R_{elec}} \end{aligned} \tag{143}$$

This is a first-order linear ordinary differential equation for the current $i(t)$.

A.9.6 Step 5: Solving the Differential Equation

We solve the ODE for $i(t)$ using the initial condition of the system.

Principles/Original Formulas/Assumptions: The general solution to a first-order linear ODE of the form $\frac{dy}{dt} + Py = Q$, where P and Q are constants, is:

$$\boxed{y(t) = \frac{Q}{P} + Ke^{-Pt}}$$

where K is an integration constant. The initial condition is that the system is uncharged, so the voltage across the capacitor is zero at $t = 0$.

$$\boxed{u(0) = 0}$$

Derivation: The ODE (143) is of the form $\frac{di}{dt} + Pi = Q$, with:

$$P = \frac{1}{Cr_0} + \frac{1}{CR_{elec}} = \frac{r_0 + R_{elec}}{Cr_0R_{elec}} \tag{144}$$

$$Q = \frac{U}{Cr_0R_{elec}} \tag{145}$$

The steady-state current ($t \to \infty$) is $i_{ss} = Q/P$:

$$i_{ss} = \frac{Q}{P} = \frac{U/(Cr_0R_{elec})}{(r_0 + R_{elec})/(Cr_0R_{elec})} = \frac{U}{r_0 + R_{elec}} \tag{146}$$

To find the integration constant K, we need the initial current $i(0)$. Using the initial condition $u(0) = 0$ in the KVL equation (141):

$$U = i(0)r_0 + u(0) \implies U = i(0)r_0 + 0 \implies i(0) = \frac{U}{r_0} \tag{147}$$

The general solution is $i(t) = i_{ss} + Ke^{-Pt}$. Applying the initial condition at $t = 0$:

$$\begin{aligned} i(0) &= i_{ss} + Ke^0 \\ \frac{U}{r_0} &= \frac{U}{r_0 + R_{elec}} + K \\ K &= \frac{U}{r_0} - \frac{U}{r_0 + R_{elec}} = U\left(\frac{(r_0 + R_{elec}) - r_0}{r_0(r_0 + R_{elec})}\right) = \frac{UR_{elec}}{r_0(r_0 + R_{elec})} \end{aligned} \tag{148}$$

Substituting i_{ss} and K back into the general solution gives $i(t)$:

$$i(t) = \frac{U}{r_0 + R_{elec}} + \left(\frac{U}{r_0} - \frac{U}{r_0 + R_{elec}}\right)\exp\left(-\frac{r_0 + R_{elec}}{Cr_0R_{elec}}t\right) \tag{149}$$

A.9.7 Step 6: Substituting Component Values into the Solution

We now substitute the expressions for the equivalent capacitance C from Eq. (136) and resistance R_{elec} from Eq. (140) into the general solution Eq. (149).

Principles/Original Formulas/Assumptions: This step involves direct substitution of previously derived results. No new principles are introduced.

Derivation: The steady-state and initial current terms depend on R_{elec}, which is given by Eq. (140). The exponent term contains the time constant $\tau = 1/P$. The coefficient P is:

$$P = \frac{r_0 + R_{elec}}{Cr_0R_{elec}} = \frac{1}{CR_{elec}} + \frac{1}{Cr_0} \tag{150}$$

Let's evaluate the terms in P. The product CR_{elec} is independent of the geometry:

$$CR_{elec} = \left(\frac{\pi(\varepsilon_1 + \varepsilon_2)L}{2\operatorname{arccosh}(D/R)}\right) \cdot \left(\frac{2\operatorname{arccosh}(D/R)}{\pi(\sigma_1 + \sigma_2)L}\right) = \frac{\varepsilon_1 + \varepsilon_2}{\sigma_1 + \sigma_2} \tag{151}$$

So, the first term in P is:

$$\frac{1}{CR_{elec}} = \frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} \tag{152}$$

The second term in P is:

$$\frac{1}{Cr_0} = \frac{1}{r_0}\left(\frac{2\operatorname{arccosh}(D/R)}{\pi(\varepsilon_1 + \varepsilon_2)L}\right) = \frac{2\operatorname{arccosh}(D/R)}{\pi r_0(\varepsilon_1 + \varepsilon_2)L} \tag{153}$$

Combining these gives the full expression for P:

$$P = \frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} + \frac{2\operatorname{arccosh}(D/R)}{\pi r_0(\varepsilon_1 + \varepsilon_2)L} \tag{154}$$

Substituting the expressions for R_{elec} (Eq. (140)) and P (Eq. (154)) into the solution for $i(t)$ (Eq. (149)) gives the final answer.

A.9.8 Final Answer

The relationship between the current through the power source and time, $i(t)$, is given by:

$$\boxed{i(t) = \frac{U}{r_0 + \frac{2\operatorname{arccosh}(D/R)}{\pi(\sigma_1+\sigma_2)L}} + \left(\frac{U}{r_0} - \frac{U}{r_0 + \frac{2\operatorname{arccosh}(D/R)}{\pi(\sigma_1+\sigma_2)L}}\right)\exp\left[-\left(\frac{\sigma_1 + \sigma_2}{\varepsilon_1 + \varepsilon_2} + \frac{2\operatorname{arccosh}(D/R)}{\pi r_0(\varepsilon_1 + \varepsilon_2)L}\right)t\right]} \tag{155}$$

A.10 Problem ID:450

A.10.1 Problem Statement Explanation

This problem asks for the angular acceleration, β, of a specific elliptical plate at the instant its major axis becomes horizontal. The plate is released from rest and moves in a vertical plane, rolling without slipping on a rough horizontal surface.

The physical situation and given parameters are as follows:

- **Geometric Properties of the Ellipse**:
  - Semi-major axis: A
  - Semi-minor axis: B
  - Distance from the geometric center to a focus: $c = \sqrt{A^2 - B^2}$
  - Eccentricity: $e = c/A$

– Semi-latus rectum: $p = A(1 - e^2) = B^2/A$

- **Mass Distribution**: The plate has a non-uniform surface mass density given by $\sigma(r, \varphi) = \sigma_0(1 + e\cos\varphi)^3$, where σ_0 is a constant.

- **Coordinate System for the Plate**: A polar coordinate system (r, φ) is defined with its origin at one focus F. The polar axis ($\varphi = 0$) points from this focus towards the closer vertex of the ellipse. In this system, the equation of the ellipse's boundary is $r(\varphi) = \frac{p}{1+e\cos\varphi}$.

- **Initial Conditions**: The plate is released from rest with its major axis vertical. For spontaneous motion to occur, it must be released from a position of unstable equilibrium. This corresponds to its center of mass (CM) being at the maximum possible height, which occurs when the vertex farthest from the focus F is in contact with the tabletop.

- **Motion and Constraints**: The plate moves in a vertical plane. The tabletop is "sufficiently rough," which implies the plate rolls without slipping.

- **Target Quantity**: We need to find the magnitude of the angular acceleration, β, at the instant the major axis is horizontal.

- **Laboratory Frame**: We define an inertial laboratory frame with a horizontal $\hat{i}$ axis along the tabletop and a vertical $\hat{j}$ axis pointing upwards. The plate rotates clockwise, so its angular velocity and acceleration vectors are $\vec{\omega} = -\omega\hat{k}$ and $\vec{\beta} = -\beta\hat{k}$ respectively, where ω and β are positive magnitudes.

A.10.2 Step 1: Calculate the Total Mass and Center of Mass

First, we determine the total mass m and locate the center of mass (CM) of the plate. The specific form of the mass density is designed to simplify these calculations.

Principles/Original Formulas/Assumptions: The total mass m is the integral of the surface mass density σ over the area S of the ellipse.

$$\boxed{m = \iint_S \sigma\, dA}$$

The position vector of the center of mass, $\vec{r}_{cm}$, is given by:

$$\boxed{\vec{r}_{cm} = \frac{1}{m}\iint_S \vec{r}\,\sigma\, dA}$$

The area element in polar coordinates is $dA = r\,dr\,d\varphi$.

Derivation: The total mass m is calculated by integrating the density over the area of the ellipse. The boundary of the ellipse is given by $r(\varphi) = \frac{p}{1+e\cos\varphi}$.

$$m = \int_0^{2\pi} \int_0^{r(\varphi)} \sigma_0(1 + e\cos\varphi)^3 \, r \, dr \, d\varphi \tag{156}$$

$$= \int_0^{2\pi} \sigma_0(1 + e\cos\varphi)^3 \left[\frac{r^2}{2}\right]_0^{r(\varphi)} d\varphi \tag{157}$$

$$= \frac{\sigma_0}{2} \int_0^{2\pi} (1 + e\cos\varphi)^3 \left(\frac{p}{1 + e\cos\varphi}\right)^2 d\varphi \tag{158}$$

$$= \frac{\sigma_0 p^2}{2} \int_0^{2\pi} (1 + e\cos\varphi) \, d\varphi = \frac{\sigma_0 p^2}{2} [\varphi + e\sin\varphi]_0^{2\pi} \tag{159}$$

$$= \pi\sigma_0 p^2 = \pi\sigma_0 \left(\frac{B^2}{A}\right)^2 = \frac{\pi\sigma_0 B^4}{A^2} \tag{160}$$

To find the center of mass, we calculate its coordinates in the focus-based polar frame. The position vector is $\vec{r} = r\cos\varphi\hat{\imath} + r\sin\varphi\hat{\jmath}$, where $\hat{\imath}$ is along the polar axis. The x-coordinate of the CM is:

$$x_{cm} = \frac{1}{m} \iint_S (r\cos\varphi) \, \sigma \, dA = \frac{1}{m} \int_0^{2\pi} \int_0^{r(\varphi)} (r\cos\varphi) \, \sigma_0(1 + e\cos\varphi)^3 \, r \, dr \, d\varphi \tag{161}$$

$$= \frac{\sigma_0}{m} \int_0^{2\pi} \cos\varphi(1 + e\cos\varphi)^3 \left[\frac{r^3}{3}\right]_0^{r(\varphi)} d\varphi \tag{162}$$

$$= \frac{\sigma_0 p^3}{3m} \int_0^{2\pi} \cos\varphi(1 + e\cos\varphi)^3 \left(\frac{1}{1 + e\cos\varphi}\right)^3 d\varphi \tag{163}$$

$$= \frac{\sigma_0 p^3}{3m} \int_0^{2\pi} \cos\varphi \, d\varphi = \frac{\sigma_0 p^3}{3m} [\sin\varphi]_0^{2\pi} = 0 \tag{164}$$

The y-coordinate of the CM is:

$$y_{cm} = \frac{1}{m} \iint_S (r\sin\varphi) \, \sigma \, dA = \frac{1}{m} \int_0^{2\pi} \int_0^{r(\varphi)} (r\sin\varphi) \, \sigma_0(1 + e\cos\varphi)^3 \, r \, dr \, d\varphi \tag{165}$$

$$= \frac{\sigma_0 p^3}{3m} \int_0^{2\pi} \sin\varphi(1 + e\cos\varphi)^3 \left(\frac{1}{1 + e\cos\varphi}\right)^3 d\varphi \tag{166}$$

$$= \frac{\sigma_0 p^3}{3m} \int_0^{2\pi} \sin\varphi \, d\varphi = \frac{\sigma_0 p^3}{3m} [-\cos\varphi]_0^{2\pi} = 0 \tag{167}$$

Since $x_{cm} = 0$ and $y_{cm} = 0$ in the chosen polar coordinate system, the center of mass (CM) is located precisely at the focus F.

A.10.3 Step 2: Calculate the Moment of Inertia about the Center of Mass

With the CM located at the focus F, we calculate the moment of inertia I_{cm} for rotation about an axis perpendicular to the plate and passing through the CM.

Principles/Original Formulas/Assumptions: The moment of inertia about an axis through the origin of the polar coordinates (which is the CM), perpendicular to the plane, is:

$$\boxed{I_{cm} = \iint_S r^2\, dm = \iint_S r^2(\sigma\, dA)}$$

A standard definite integral result is:

$$\boxed{\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for } |a| < 1}$$

Derivation:

$$I_{cm} = \int_0^{2\pi} \int_0^{r(\varphi)} r^2 \left(\sigma_0(1+e\cos\varphi)^3 r\right)\, dr\, d\varphi \tag{168}$$

$$= \sigma_0 \int_0^{2\pi} (1+e\cos\varphi)^3 \left[\frac{r^4}{4}\right]_0^{r(\varphi)} d\varphi \tag{169}$$

$$= \frac{\sigma_0}{4} \int_0^{2\pi} (1+e\cos\varphi)^3 \left(\frac{p}{1+e\cos\varphi}\right)^4 d\varphi = \frac{\sigma_0 p^4}{4} \int_0^{2\pi} \frac{d\varphi}{1+e\cos\varphi} \tag{170}$$

Using the standard integral with $a = e$, and the relation $\sqrt{1-e^2} = B/A$:

$$I_{cm} = \frac{\sigma_0 p^4}{4}\left(\frac{2\pi}{\sqrt{1-e^2}}\right) = \frac{\pi\sigma_0 p^4}{2\sqrt{1-e^2}} = \frac{\pi\sigma_0 (B^2/A)^4}{2(B/A)} = \frac{\pi\sigma_0 B^8/A^4}{2B/A} = \frac{\pi\sigma_0 B^7}{2A^3} \tag{171}$$

Expressing I_{cm} in terms of the total mass m using Eq. (160) ($\pi\sigma_0 = mA^2/B^4$):

$$I_{cm} = \left(\frac{mA^2}{B^4}\right)\frac{B^7}{2A^3} = \frac{mB^3}{2A} \tag{172}$$

A.10.4 Step 3: Energy Conservation and Angular Velocity

We find the angular velocity ω at the moment the major axis is horizontal using the conservation of mechanical energy.

Principles/Original Formulas/Assumptions: The total mechanical energy of the system is conserved as the gravitational force is conservative and the contact forces do no work for rolling without slipping.

$$\boxed{E_i = E_f \quad \text{or} \quad U_i + K_i = U_f + K_f}$$

The kinetic energy of a rigid body in planar motion is the sum of its translational and rotational kinetic energy.

$$\boxed{K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2}$$

The velocity of the center of mass for an object rolling without slipping is related to the angular velocity ω and the vector from the contact point P to the CM, $\vec{r}_{cm/P}$, as the velocity of the contact point is zero.

$$\boxed{\vec{v}_{cm} = \vec{\omega} \times \vec{r}_{cm/P}}$$

The gravitational potential energy is $U_g = mgh_{cm}$, where h_{cm} is the height of the center of mass above a reference level (the tabletop).

Derivation:

- **Initial State (i)**: The plate is released from rest ($K_i = 0$) from its unstable equilibrium position with the major axis vertical. The CM (focus F) is at its highest point. The contact point is the vertex farthest from F. The distance from F to this vertex is $r(\pi) = p/(1-e) = A(1+e) = A + c$. The initial height of the CM is $h_{cm,i} = A + c$. The initial potential energy is $U_i = mg(A + c)$.

- **Final State (f)**: The major axis is horizontal. The contact point is at an end of the minor axis. The height of the CM (focus F) above the tabletop is equal to the semi-minor axis, $h_{cm,f} = B$. The final potential energy is $U_f = mgB$. The plate is rotating with angular velocity ω. At this instant, let the contact point P be at the origin of our lab frame. The vector from P to the CM is $\vec{r}_{cm/P} = c\hat{i} + B\hat{j}$. The rotation is clockwise, so $\vec{\omega} = -\omega\hat{k}$. The velocity of the CM is $\vec{v}_{cm} = (-\omega\hat{k}) \times (c\hat{i} + B\hat{j}) = \omega B\hat{i} - \omega c\hat{j}$. The squared speed of the CM is $v_{cm}^2 = (\omega B)^2 + (-\omega c)^2 = \omega^2(B^2 + c^2) = \omega^2 A^2$. The final kinetic energy is $K_f = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}m(\omega^2 A^2) + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}(mA^2 + I_{cm})\omega^2$.

- **Energy Conservation**:

$$U_i + K_i = U_f + K_f \tag{173}$$

$$mg(A + c) + 0 = mgB + \frac{1}{2}(mA^2 + I_{cm})\omega^2 \tag{174}$$

$$\omega^2 = \frac{2mg(A + c - B)}{mA^2 + I_{cm}} \tag{175}$$

A.10.5 Step 4: Equations of Motion and Kinematics

We use the Newton-Euler equations for planar motion. The key is to correctly determine the acceleration of the material point at contact for a rolling ellipse.

Principles/Original Formulas/Assumptions: The rotational dynamics about the contact point P (a point on the body) is given by:

$$\boxed{\sum \vec{\tau}_P = I_{cm}\vec{\beta} + \vec{r}_{cm/P} \times (m\vec{a}_{cm})}$$

The kinematic relation between the acceleration of the CM and the acceleration of the material point at contact P:

$$\boxed{\vec{a}_{cm} = \vec{a}_{P,material} + \vec{\beta} \times \vec{r}_{cm/P} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{cm/P})}$$

For a body rolling on a fixed flat surface, the acceleration of the material point of the body instantaneously at the contact point P is directed along the normal to the surface, pointing into the body, with magnitude:

$$\boxed{a_{P,material} = \omega^2 \rho}$$

where ρ is the radius of curvature of the body at P. The radius of curvature of an ellipse at the end of the minor axis is:

$$\boxed{\rho = \frac{A^2}{B}}$$

Derivation: The external forces are gravity $\vec{F}_g = -mg\hat{j}$ acting at the CM, and the contact force at P. The torque about P is only due to gravity:

$$\vec{\tau}_P = \vec{r}_{cm/P} \times \vec{F}_g = (c\hat{i} + B\hat{j}) \times (-mg\hat{j}) = -mgc\hat{k} \tag{176}$$

The acceleration of the material point at contact P is directed vertically upwards ($\hat{j}$ direction):

$$\vec{a}_{P,material} = \omega^2 \rho \hat{j} = \omega^2 \frac{A^2}{B}\hat{j} \tag{177}$$

Now we find $\vec{a}_{cm}$ using the general kinematic relation, with $\vec{\beta} = -\beta\hat{k}$:

$$\vec{\beta} \times \vec{r}_{cm/P} = (-\beta\hat{k}) \times (c\hat{i} + B\hat{j}) = \beta B\hat{i} - \beta c\hat{j} \tag{178}$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{cm/P}) = -\omega^2 \vec{r}_{cm/P} = -\omega^2(c\hat{i} + B\hat{j}) \tag{179}$$

$$\vec{a}_{cm} = \vec{a}_{P,material} + \vec{\beta} \times \vec{r}_{cm/P} - \omega^2 \vec{r}_{cm/P} \tag{180}$$

$$= \left(\omega^2 \frac{A^2}{B}\right)\hat{j} + (\beta B\hat{i} - \beta c\hat{j}) - \omega^2(c\hat{i} + B\hat{j}) \tag{181}$$

$$= (\beta B - \omega^2 c)\hat{i} + \left(\omega^2 \frac{A^2}{B} - \beta c - \omega^2 B\right)\hat{j} \tag{182}$$

$$= (\beta B - \omega^2 c)\hat{i} + \left(\omega^2 \frac{A^2 - B^2}{B} - \beta c\right)\hat{j} = (\beta B - \omega^2 c)\hat{i} + \left(\omega^2 \frac{c^2}{B} - \beta c\right)\hat{j} \tag{183}$$

Next, we compute the cross product term for the torque equation:

$$\vec{r}_{cm/P} \times (m\vec{a}_{cm}) = m(c\hat{i} + B\hat{j}) \times \left[(\beta B - \omega^2 c)\hat{i} + \left(\omega^2 \frac{c^2}{B} - \beta c\right)\hat{j}\right] \tag{184}$$

$$= m\left[c\left(\omega^2 \frac{c^2}{B} - \beta c\right) - B(\beta B - \omega^2 c)\right]\hat{k} \tag{185}$$

$$= m\left[\frac{\omega^2 c^3}{B} - \beta c^2 - \beta B^2 + \omega^2 cB\right]\hat{k} \tag{186}$$

$$= m\left[\omega^2 c\left(\frac{c^2 + B^2}{B}\right) - \beta(c^2 + B^2)\right]\hat{k} = m\left[\omega^2 c\frac{A^2}{B} - \beta A^2\right]\hat{k} \tag{187}$$

Substituting into the torque equation $\vec{\tau}_P = I_{cm}\vec{\beta} + \vec{r}_{cm/P} \times (m\vec{a}_{cm})$:

$$-mgc\hat{k} = I_{cm}(-\beta\hat{k}) + m\left(\omega^2 c\frac{A^2}{B} - \beta A^2\right)\hat{k} \tag{188}$$

$$-mgc = -I_{cm}\beta + m\omega^2 c\frac{A^2}{B} - mA^2\beta \tag{189}$$

$$(I_{cm} + mA^2)\beta = mgc + m\omega^2 c\frac{A^2}{B} \tag{190}$$

A.10.6 Step 5: Final Calculation of Angular Acceleration

We now substitute the expressions for I_{cm} and ω^2 into Eq. (190) to find β.

Principles/Original Formulas/Assumptions: This step combines the results from previous steps. The key equations are:

$$\boxed{(I_{cm} + mA^2)\beta = mgc + m\omega^2 c\frac{A^2}{B} \quad \text{(from Eq. 190)}}$$

$$\boxed{\omega^2 = \frac{2mg(A + c - B)}{mA^2 + I_{cm}} \quad \text{(from Eq. 175)}}$$

$$\boxed{I_{cm} = \frac{mB^3}{2A} \quad \text{(from Eq. 172)}}$$

Derivation: From Eq. (190), we solve for β:

$$\beta = \frac{mgc + m\omega^2 c\frac{A^2}{B}}{I_{cm} + mA^2} = \frac{mgc}{I_{cm} + mA^2}\left(1 + \frac{\omega^2 A^2}{gB}\right) \tag{191}$$

Substitute ω^2 from Eq. (175):

$$\beta = \frac{mgc}{I_{cm} + mA^2}\left[1 + \frac{A^2}{gB}\left(\frac{2mg(A + c - B)}{mA^2 + I_{cm}}\right)\right] \tag{192}$$

$$\beta = \frac{mgc}{I_{cm} + mA^2}\left[1 + \frac{2mA^2(A + c - B)}{B(mA^2 + I_{cm})}\right] \tag{193}$$

Now, substitute $I_{cm} = \frac{mB^3}{2A}$ into the term $(mA^2 + I_{cm})$:

$$mA^2 + I_{cm} = mA^2 + m\frac{B^3}{2A} = m\left(A^2 + \frac{B^3}{2A}\right) = m\frac{2A^3 + B^3}{2A} \tag{194}$$

Substitute this into Eq. (193):

$$\beta = \frac{mgc}{m\frac{2A^3+B^3}{2A}}\left[1 + \frac{2mA^2(A + c - B)}{B\left(m\frac{2A^3+B^3}{2A}\right)}\right] \tag{195}$$

$$= \frac{2Agc}{2A^3 + B^3}\left[1 + \frac{4A^3(A + c - B)}{B(2A^3 + B^3)}\right] \tag{196}$$

Finally, substitute $c = \sqrt{A^2 - B^2}$ to get the expression in terms of A and B.

$$\beta = \frac{2Ag\sqrt{A^2 - B^2}}{2A^3 + B^3}\left[1 + \frac{4A^3(A + \sqrt{A^2 - B^2} - B)}{B(2A^3 + B^3)}\right] \tag{197}$$

A.10.7 Final Answer

The angular acceleration β of the plate when the major axis becomes horizontal is:

$$\boxed{\beta = \frac{2Ag\sqrt{A^2 - B^2}}{2A^3 + B^3}\left[1 + \frac{4A^3(A + \sqrt{A^2 - B^2} - B)}{B(2A^3 + B^3)}\right]} \tag{198}$$