PHYSICS 100 Problems Feedback Report

1 Problem Report

Among the 100 randomly selected PHYSICS problems with detailed solutions, 22 QA pairs are flawed: 21 have incorrect answers, and 1 has an ambiguous problem statement.

1.1 Reference Answer Errors

1.1.1 Problem ID: 4-28

Problem statement: The following elementary-particle reaction may be carried out on a proton target at rest in the laboratory:

$$K^- + p \to \pi^0 + \Lambda^0.$$

Find the special value of the incident K^- energy such that the Λ^0 can be produced at rest in the laboratory. Your answer should be expressed in terms of the rest masses m_{π^0}, m_{K^-}, m_p and m_{Λ^0}.

Final result provided by raw answer:

$$E_K = \frac{m_K^2 - m_\pi^2 + (m_\Lambda - m_p)^2}{2(m_\Lambda - m_p)} \tag{1}$$

Reason for the error: The reference answer does not fully comply with the symbols specified in the question.

Corrected final result:

$$E_{K^-} = \frac{m_{K^-}^2 - m_{\pi^0}^2 + (m_{\Lambda^0} - m_p)^2}{2(m_{\Lambda^0} - m_p)} \tag{2}$$

1.1.2 Problem ID: 2-35

Problem statement: Light from a monochromatic point source of wavelength λ is focused to a point image by a Fresnel half-period zone plate having 100 open odd half-period zones (1, 3, 5, ..., 199) with all even zones opaque. Compare the image dot intensity with that at the same point for the zone plate removed, and for a lens of the same focal length and diameter corresponding to 200 half-period zones of the zone plate. Assume the diameter of the opening is small compared to the distance from the source and the image.

Final result provided by raw answer:

$$I'/I = 4 \tag{3}$$

Reason for the error: The problem ultimately requires the relative ratios of three scenarios ("the image dot", "the zone plate removed", "diameter corresponding to 200 half-period zones"), but the reference answer only provides two.

Corrected final result:

$$160000 : 40000 : 1 \tag{4}$$

1.1.3 Problem ID: 1-39

Problem statement: When the sun is overhead, a flat white surface has a certain luminous flux. A lens of radius r and focal length f is now used to focus the sun's image on the sheet. How much greater is the flux in the image area? For a given r, what must f be so that the lens gives no increase in flux in the image? From the earth the diameter of the sun subtends about 0.01 radians. The only light in the image is through the lens.

Raw answer: The luminous flux reaching the earth is given by

$$\Phi = B\sigma' d\Omega$$

where B is the brightness of the sun, regarded as a Lambert radiator, σ' is the area illuminated on the earth, $d\Omega$ is the solid angle subtended by the sun at σ' and is given by

$$d\Omega = \pi\alpha^2,$$

α being the angular aperture of the sun, equal to about 0.01 radians.The luminous flux on the image area behind the lens is

$$\Phi' = \pi B\sigma' \sin^2 u' = \pi B\sigma' \left(\frac{r}{f}\right)^2$$

where u' is the semi-angular aperture of the lens, r and f are the radius and focal length of the lens respectively. Consider the ratio

$$\frac{\Phi'}{\Phi} = \frac{\pi B\sigma' \left(\frac{r}{f}\right)^2}{\pi B\sigma'(0.01)^2} = \frac{10^4 r^2}{f^2},$$

where we have taken the transparency of the lens glass to be unity. Thus the luminous flux on the image area is $10^4 r^2/f^2$ times that on a surface of the same area on the earth without the lens.For $\Phi'/\Phi \leq 1$, we have $f \geq 10^2 r$. Hence, for $f \geq 100r$ the lens will give no increase in flux in the image area.

Final result provided by raw answer:

$$f \geq 100r \tag{5}$$

Reason for the error: Angular radius should be used instead of diameter. Gemini and o3 provided answers with consistent results. For details, please refer to the appendix2.0.1

Corrected final result:

$$f \geq 200r \tag{6}$$

1.1.4 Problem ID: 1-75

Problem statement: The potential at a distance r from the axis of an infinite straight wire of radius a carrying a charge per unit length σ is given by

$$\frac{\sigma}{2\pi} \ln \frac{1}{r} + \text{const}.$$

This wire is placed at a distance $b \gg a$ from an infinite metal plane, whose potential is maintained at zero. Find the capacitance per unit length of the wire of this system.

Raw answer: In order that the potential of the metal plane is maintained at zero, we imagine that an infinite straight wire with linear charge density $-\sigma$ is symmetrically placed on the other side of the plane. Then the capacitance between the original wire and the metal plane is that between the two straight wires separated at $2b$.

The potential $\varphi(r)$ at a point between the two wires at distance r from the original wire (and at distance $2b - r$ from the image wire) is then

$$\varphi(r) = \frac{\sigma}{2\pi} \ln \frac{1}{r} - \frac{\sigma}{2\pi} \ln \frac{1}{2b - r}.$$

So the potential difference between the two wires is

$$V = \varphi(a) - \varphi(2b - a) = \frac{\sigma}{\pi} \ln \left(\frac{2b - a}{a} \right) \approx \frac{\sigma}{\pi} \ln \frac{2b}{a}.$$

Thus the capacitance of this system per unit length of the wire is

$$C = \frac{\sigma}{V} = \pi / \ln \frac{2b}{a}.$$

Reason for the error: The raw answer uses the method of images, but in the final step, it forgets that half of the potential difference between the wire and the image wire is actually the potential difference between the wire and the metal plane.

Corrected final result:

$$C = \frac{2\pi}{\ln \frac{2b}{a}}. \tag{7}$$

1.1.5 Problem ID: 1-18

Problem statement: A metal sphere of radius a is surrounded by a concentric metal sphere of inner radius b, where $b > a$. The space between the spheres is filled with a material whose electrical conductivity σ varies with the electric field strength E according to the relation $\sigma = KE$, where K is a constant. A potential difference V is maintained between the two spheres. What is the current between the spheres?

Raw answer: Since the current is

$$I = j \cdot S = \sigma E \cdot S = KE^2 \cdot S = KE^2 \cdot 4\pi r^2,$$

the electric field is

$$E = \frac{1}{r}\sqrt{\frac{I}{4\pi K}}$$

and the potential is

$$V = -\int_b^a E \cdot dr = -\int_b^a \sqrt{\frac{I}{4\pi K}}\frac{1}{r}dr = \sqrt{\frac{I}{4\pi K}} \ln\left(\frac{b}{a}\right).$$

Hence the current between the spheres is given by

$$I = 4\pi KV^2/\ln(b/a).$$

Reason for the error: The last step of the derivation is incorrect. A square on ln is missing.

Corrected final result:

$$C = \frac{4\pi KV^2}{(\ln(b/a))^2}. \tag{8}$$

1.1.6 Problem ID: 5-3

Problem statement: In the inertial frame of the fixed stars, a spaceship travels along the x-axis, with $x(t)$ being its position at time t. Of course, the velocity v and acceleration a in this frame are $v = \frac{dx}{dt}$ and $a = \frac{d^2x}{dt^2}$. Suppose the motion to be such that the acceleration as determined by the space passengers is constant in time. What this means is the following. At any instant we transform to an inertial frame in which the spaceship is momentarily at rest. Let g be the acceleration of the spaceship in that frame at that instant. Now suppose that g, so defined instant by instant, is a constant.

You are given the constant g. In the fixed star frame the spaceship starts with initial velocity $v = 0$ when $x = 0$. What is the distance x traveled when it has achieved a velocity v?

Allow for relativistic kinematics, so that v is not necessarily small compared with the speed of light c.

Final result provided by raw answer:

```
x = \frac{c^2}{g} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}
```

Reason for the error: The final answer provided does not include all the items, and even do not conform to LaTeX syntax.

Corrected final result:

$$x = \frac{c^2}{g}(1/\sqrt{1 - \frac{v^2}{c^2}} - 1). \tag{9}$$

1.1.7 Problem ID: 1-66

Problem statement: Pretend that the sun is surrounded by a dust cloud extending out at least as far as the radius of the earth. The sun produces the familiar potential $V = -GMm/r$, and the dust adds a small term $V = kr^2/2$. The earth revolves in a nearly circular ellipse of average radius r_0. The effect of the dust may cause the ellipse to precess slowly. Find an approximate expression (to first order in k) for the rate of precession and its sense compared to the direction of revolution.

Hint: Consider small oscillations about r_0.

Raw answer: In the equation for radial motion of a body under central force the effective potential is

$$U(r) = -\frac{GMm}{r} + \frac{kr^2}{2} + \frac{L^2}{2mr^2}.$$

The earth will move in a closed orbit of radius r_0 if $U(r_0)$ is an extreme value, i.e.

$$\left(\frac{dU(r)}{dr}\right)_{r=r_0} = 0,$$

or

$$\frac{GMm}{r_0^2} + kr_0 - \frac{L^2}{mr_0^3} = 0,$$

from which r_0 can be determined.

Expand $U(r)$ as a Taylor series:

$$U(r) = U(r_0) + \left(\frac{dU}{dr}\right)_{r=r_0}(r - r_0) + \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2 + \cdots$$
$$\approx U(r_0) + \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0}(r - r_0)^2,$$

as $\left(\frac{dU}{dr}\right)_{r=r_0} = 0$, retaining only the dominant terms. The energy equation can then be written with $r - r_0 = x$ as

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{d^2U}{dr^2}\right)_{r=r_0} x^2 = \text{constant}.$$

Differentiating with respect to time gives

$$m\ddot{x} + \left(\frac{d^2U}{dr^2}\right)_{r=r_0} x = 0.$$

Hence there are small oscillations about r_0 with angular frequency

$$\omega_r = \sqrt{\frac{U''(r_0)}{m}},$$

where

$$U''(r_0) = \frac{-2GMm}{r_0^3} + k + \frac{3L^2}{mr_0^4} = 3k + \frac{L^2}{mr_0^4},$$

using Eq. (1). For the near-circular orbit, $L = mr^2\omega_0$, and we find

$$U''(r) = 3k + m\omega_0^2,$$

ω_0 being the angular velocity of revolution around the sun. Thus to first order in k, we have

$$\omega_r = \sqrt{\omega_0^2 + \frac{3k}{m}} \approx \omega_0 + \frac{3k}{2m\omega_0},$$

and the rate of precession is

$$\omega_p = \omega_r - \omega_0 = \frac{3k}{2m\omega_0}.$$

As $\omega_r > \omega_0$, i.e. the period of radial oscillation is shorter than that of revolution, the direction of precession is opposite to that of rotation.

Reason for the error: The definition of the precession rate in the original answer is opposite to the standard definition.

Corrected final result:

$$-\frac{3k}{2m\omega_0}. \tag{10}$$

1.1.8 Problem ID: 3-48

Problem statement: Derive the Hamiltonian of a particle traveling with momentum $\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$ when it is placed in the fields defined by

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t},$$

$$\mathbf{H} = \nabla \times \mathbf{A}.$$

Final result provided by raw answer:

```
H = \sqrt{\left( \mathbf{p} - \frac{q\mathbf{A}}{c} \right)^2 c^2 + m_0^2
    c^4
```

Reason for the error: The final answer provided does not include all the items, and even do not conform to LaTeX syntax.

Corrected final result:

$$H = \sqrt{\left(\mathbf{p} - \frac{q\mathbf{A}}{c}\right)^2 c^2 + m_0^2 c^4} + q\Phi. \tag{11}$$

1.1.9 Problem ID: 1-44

Problem statement: A comet moves toward the sun with initial velocity v_0. The mass of the sun is M and its radius is R. Find the total cross section σ for striking the sun. Take the sun to be at rest and ignore all other bodies.

Final result provided by raw answer:

$$\sigma = \pi R^2 \left(1 + \frac{2GM}{V_0^2 R}\right) \tag{12}$$

Reason for the error: The symbol (V_0) used in the raw answer is inconsistent with the symbol (v_0) provided in the question.

Corrected final result:

$$\sigma = \pi R^2 \left(1 + \frac{2GM}{v_0^2 R}\right). \tag{13}$$

1.1.10 Problem ID: 2-72

Problem statement: Electromagnetic radiation following the Planck distribution fills a cavity of volume V. Initially ω_i is the frequency of the maximum of the curve of $u_i(\omega)$, the energy density per unit angular frequency versus ω. If the volume is expanded quasistatically to $2V$, what is the final peak frequency ω_f of the $u_f(\omega)$ distribution curve? The expansion is adiabatic.

Final result provided by raw answer:

$$\omega_f = \frac{\omega_i}{3\sqrt{2}} \tag{14}$$

Reason for the error: The raw answer inexplicably wrote $\sqrt[3]{2}$ as $3\sqrt{2}$.

Corrected final result:

$$\omega_f = \frac{\omega_i}{\sqrt[3]{2}}. \tag{15}$$

1.1.11 Problem ID: 1-80

Problem statement: The following measurements can be made on an elastic band:

(a) The change in temperature when the elastic band is stretched. (In case you have not tried this, hold the attached band with both hands, test the temperature by touching the band to your lips, stretch the band and check the temperature, relax the band and check the temperature once more).

(b) One end of the band is fixed, the other attached to weight W, and the frequency u of small vibrations is measured.

(c) With the weight at rest σQ is added, and the equilibrium length L is observed to change by δL.

Derive the equation by which you can predict the result of the last measurement from the results of the first two.

Final result provided by raw answer:

$$\delta L = \left(\frac{dL_0}{dT} - \frac{W}{\left(\frac{4\pi^2 W}{g} u^2\right)^2} \frac{d}{dT} \left(\frac{4\pi^2 W}{g} u^2 \right) \right) \frac{\delta Q}{C_W} \tag{16}$$

Reason for the error: The raw answer introduces a quantity that does not exist as described in the question in the final result (the heat capacity at constant weight C_W), and mix up isothermal elastic coefficient k_T in static state and adiabatic elastic coefficient k_S in oscillation. For the correct solution, please refer to the appendix 2.0.3.

Corrected final result:

$$\delta L = -\frac{(\partial T/\partial L)_S}{\frac{W}{g}(2\pi u)^2} \frac{\delta Q}{T}. \tag{17}$$

1.1.12 Problem ID: 1-125

Problem statement: Find the relation between the equilibrium radius r, the potential ϕ, and the excess of ambient pressure over internal pressure Δp of a charged soap bubble, assuming that surface tension can be neglected.

Final result provided by raw answer:

$$\Delta p = \frac{\phi^2}{4\pi r^2} \tag{18}$$

Reason for the error: The raw answer incorrectly calculates the free energy change of the bubble itself (γ phase), i.e., the change in electric field energy, and the final result given is twice that of the correct solution. For the correct solution, please refer to the appendix2.0.4.

Corrected final result:

$$\Delta p = \frac{\phi^2}{8\pi r^2}. \tag{19}$$

1.1.13 Problem ID: 1-3

Problem statement: A bimetallic strip of total thickness x is straight at temperature T. What is the radius of curvature of the strip, R, when it is heated to temperature $T+\Delta T$? The coefficients of linear expansion of the two metals are α_1 and α_2, respectively, with $\alpha_2 > \alpha_1$. You may assume that each metal has thickness $x/2$, and you may assume that $x \ll R$.

Raw answer: We assume that the initial length is l_0. After heating, the lengths of the mid-lines of the two metallic strips are respectively

$$l_1 = l_0(1 + \alpha_1 \Delta T),$$

$$l_2 = l_0(1 + \alpha_2 \Delta T).$$

Assuming that the radius of curvature is R, the subtending angle of the strip is θ, and the change of thickness is negligible, we have

$$l_2 = \left(R + \frac{x}{4}\right)\theta, \quad l_1 = \left(R - \frac{x}{4}\right)\theta,$$

$$l_2 - l_1 = \frac{x}{2}\theta = \frac{x}{2}\frac{l_1 + l_2}{2R} = \frac{x l_0}{4R}[2 + (\alpha_1 + \alpha_2)\Delta T]. \quad (3)$$

From (1) and (2) we obtain

$$l_2 - l_1 = l_0 \Delta T(\alpha_2 - \alpha_1), \quad (4)$$

(3) and (4) then give

$$R = \frac{x}{4}\frac{[2 + (\alpha_1 + \alpha_2)\Delta T]}{(\alpha_2 - \alpha_1)\Delta T}.$$

Reason for the error: The geometric relationship used at the beginning of the reference answer is incorrect. It ignores internal stress. In fact, these two formulas describe the length of a metal strip when it expands or contracts freely due to temperature changes. However, in a bimetallic strip, the two layers of metal are firmly bonded together. When the temperature rises, because of the different expansion coefficients ($\alpha_2 > \alpha_1$), the second layer of metal tends to expand more, while the first layer tends to expand less. For the correct solution, please refer to the appendix2.0.6.

Corrected final result:

$$R = \frac{2x}{3(\alpha_2 - \alpha_1)\Delta T}. \quad (20)$$

1.1.14 Problem ID: 19-4

Problem statement: A cavity containing a gas of electrons has a small hole, of area A, through which electrons can escape. External electrodes are so arranged that, if the potential energy of an electron inside the cavity is taken as zero, then its potential energy outside the cavity is V > 0. Thus, an electron will leave the leaky box IF it approaches the small hole with a kinetic energy larger than V. Estimate the electrical current carried by the escaping electrons assuming that

(i) a constant number density of electrons is maintained inside the cavity,

(ii) these electrons are in thermal equilibrium at a temperature τ and chemical potential μ,

(iii) interactions between the electrons can be neglected, and

(iv) $V - \mu > \tau$.

Final result provided by raw answer:

$$I = -\frac{4\pi e A m \tau^2}{h^3}\left(1 + \frac{V}{\tau}\right) e^{-(V-\mu)/\tau} \tag{21}$$

Reason for the error: The escape condition should be the normal kinetic energy $\epsilon_z > V$ not $\epsilon > V$ The correct solution can refer to the appendix2.0.5.

Corrected final result:

$$I = -\frac{4\pi e A m \tau^2}{h^3} e^{-(V-\mu)/\tau}. \tag{22}$$

1.1.15 Problem ID: 3-3025

Problem statement: The Hamiltonian for a spin-$\frac{1}{2}$ particle with charge $+e$ in an external magnetic field is

$$H = -\frac{ge}{2mc}\mathbf{s} \cdot \mathbf{B}.$$

Calculate the operator $d\mathbf{s}/dt$ if $\mathbf{B} = B\hat{y}$. What is $s_z(t)$ in matrix form?

Final result provided by raw answer:

$$s_z(t) = s_z(0)\cos\left(\frac{geB}{2mc}t\right) + s_x(0)\sin\left(\frac{geB}{2mc}t\right) \tag{23}$$

Reason for the error: The raw answer does not truly provide the matrix form required in the question.

Corrected final result:

$$s_z(t) = \frac{\hbar}{2}\begin{pmatrix} \cos\left(\frac{geB}{2mc}t\right) & \sin\left(\frac{geB}{2mc}t\right) \\ \sin\left(\frac{geB}{2mc}t\right) & -\cos\left(\frac{geB}{2mc}t\right) \end{pmatrix}. \tag{24}$$

1.1.16 Problem ID: 5003

Problem statement: A particle of mass m moves one-dimensionally in the oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. In the nonrelativistic limit, where the kinetic energy T and momentum p are related by $T = p^2/2m$, the ground state energy is well known to be $\frac{1}{2}\hbar\omega$.

Allow for relativistic corrections in the relation between T and p and compute the ground state level shift ΔE to order $\frac{1}{c^2}$ (c = speed of light).

Final result provided by raw answer:

$$\Delta E = -\frac{15}{32}\frac{(\hbar\omega)^2}{mc^2} \tag{25}$$

Reason for the error: Relativistic energy level correction of the one-dimensional harmonic oscillator. A classic problem. The integral in the raw answer is incorrect.

Corrected final result:

$$\Delta E = -\frac{3}{32}\frac{(\hbar\omega)^2}{mc^2}. \tag{26}$$

1.1.17 Problem ID: 5060

Problem statement: An atom is in a state of total electron spin S, total orbital angular momentum L, and total angular momentum J. (The nuclear spin can be ignored for this problem). The z component of the atomic total angular momentum is J_z. By how much does the energy of this atomic state change if a weak magnetic field of strength B is applied in the z direction? Assume that the interaction with the field is small compared with the fine structure interaction.

The answer should be given as an explicit expression in terms of the quantum numbers J, L, S, J_z and natural constants.

Final result provided by raw answer:

$$\Delta E = M_J \hbar\omega_l + \sum_{M_L} (M_J - M_L)\hbar\omega_l \left[\langle L M_L S, M_J - M_L | J M_J \rangle\right]^2 \tag{27}$$

Reason for the error: The problem is actually the general Zeeman effect. However, the raw answer did not provide a complete result.

Corrected final result:

$$\Delta E = \frac{eB}{2m_e c} J_z \left[1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}\right]. \tag{28}$$

(The specific form may vary slightly under different unit conventions, but the form of the Landé g-factor is fixed.)

1.1.18 Problem ID: 6014

Problem statement: In scattering from a potential $V(r)$, the wave function may be written as an incident plane wave plus an outgoing scattered wave: $\psi = e^{ikz} + v(r)$. Derive a differential equation for $v(r)$ in the first Born approximation.

Final result provided by raw answer:

$$(\nabla^2 + k^2)v(\mathbf{r}) = \frac{2m}{\hbar^2}V(\mathbf{r})e^{ikz} \tag{29}$$

Reason for the error: The potential described in the problem statement is a central potential $V(r)$, but the raw answer wrote it as a general potential $V(\boldsymbol{r})$. This is a small typo.

Corrected final result:

$$(\nabla^2 + k^2)v(\mathbf{r}) = \frac{2m}{\hbar^2}V(r)e^{ikz}. \tag{30}$$

1.1.19 Problem ID: 5054

Problem statement: A spin-$\frac{1}{2}$ particle of mass m moves in spherical harmonic oscillator potential $V = \frac{1}{2}m\omega^2 r^2$ and is subject to an interaction $\lambda\sigma \cdot \mathbf{r}$ (spin orbit forces are to be ignored). The net Hamiltonian is therefore

$$H = H_0 + H',$$

where

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 r^2, \quad H' = \lambda\sigma \cdot \mathbf{r}.$$

Compute the shift of the ground state energy through second order in the perturbation H

Final result provided by raw answer:

$$\frac{3}{2}\hbar\omega - \frac{3\lambda^2}{2m\omega^2} \tag{31}$$

Reason for the error: The question requires the energy level shift, but the raw answer provides the complete energy after the shift.

Corrected final result:

$$-\frac{3\lambda^2}{2m\omega^2}. \tag{32}$$

1.1.20 Problem ID: 1-1042

Problem statement: In one dimension, a particle of mass m is in the ground state of a potential which confines the particle to a small region of space. At time $t = 0$, the potential suddenly disappears, so that the particle is free for time $t > 0$. Give a formula for the probability per unit time that the particle arrives at time t at an observer who is a distance L away.

Final result provided by raw answer:

$$j = \frac{\hbar^2 L t}{\sqrt{\pi a^5 m^2}} \frac{1}{\left(1 + \frac{\hbar^2 t^2}{m^2 a^4}\right)^{3/2}} \exp\left[-\frac{L^2}{a^2} \frac{1}{1 + \left(\frac{\hbar t}{m a^2}\right)^2}\right] \tag{33}$$

Reason for the error: There is a calculation error in the final calculation of j in the reference answer. (This can be checked through dimensional analysis.)

Corrected final result:

$$j = \frac{\hbar^2 L t}{\sqrt{\pi} a^5 m^2} \frac{1}{\left(1 + \frac{\hbar^2 t^2}{m^2 a^4}\right)^{3/2}} \exp\left[-\frac{L^2}{a^2} \frac{1}{1 + \left(\frac{\hbar t}{m a^2}\right)^2}\right]. \tag{34}$$

1.1.21 Problem ID: 6010

Problem statement:

Consider the quantum-mechanical scattering problem in the presence of inelastic scattering. Suppose one can write the partial wave expansion of the scattering amplitude for the elastic channel in the form

$$f(k,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2ik} P_\ell(\cos\theta),$$

where $\delta_\ell(k)$ and $\eta_\ell(k)$ are real quantities with $0 \le \eta_\ell \le 1$, the wave number is denoted by $\mathbf{k}$, and θ is the scattering angle. For a given partial wave, obtain the lower and upper bounds for the elastic cross section $\sigma^{(\ell)}_{\text{elastic}}$ in terms of $\sigma^{(\ell)}_{\text{inelastic}}$.

Final result provided by raw answer:

$$\frac{(1-\eta_\ell)^2}{1-\eta_\ell^2}\,\sigma^{(\ell)}_{\text{inelastic}} \le \sigma^{(\ell)}_{\text{elastic}} \le \frac{(1+\eta_\ell)^2}{1-\eta_\ell^2}\,\sigma^{(\ell)}_{\text{inelastic}} \tag{35}$$

Reason for the error: The upper and lower bound expressions given in the raw answer still contain the variable η_ℓ, while the problem requires that the upper and lower bounds be expressed as functions of $\sigma^{(\ell)}_{\text{inelastic}}$. A given value of $\sigma^{(\ell)}_{\text{inelastic}}$ uniquely determines the value of η_ℓ, so the final answer should no longer contain η_ℓ. For the correct solution, please refer to Appendix 2.0.7.

Corrected final result:

$$\frac{\pi}{k^2}(2\ell+1)\left(1-\sqrt{1-\frac{\sigma^{(\ell)}_{\text{inelastic}}}{\frac{\pi}{k^2}(2\ell+1)}}\right)^2 \le \sigma^{(\ell)}_{\text{elastic}} \le \frac{\pi}{k^2}(2\ell+1)\left(1+\sqrt{1-\frac{\sigma^{(\ell)}_{\text{inelastic}}}{\frac{\pi}{k^2}(2\ell+1)}}\right)^2. \tag{36}$$

1.2 Ambiguous Problem Statement

1.2.1 Problem ID: 1-78

Problem statement: Suppose that the region $z > 0$ in three-dimensional space is filled with a linear dielectric material characterized by a dielectric constant ϵ_1, while the region $z < 0$ has a dielectric material ϵ_2. Fix a charge $-q$ at $(x, y, z) = (0, 0, a)$ and a charge q at $(0, 0, -a)$. What is the force one must exert on the negative charge to keep it at rest?

Raw answer: ### Solution: Consider first the simple case where a point charge q_1 is placed at $(0, 0, a)$. The method of images requires image charges q_1' at $(0, 0, -a)$ and q_1'' at $(0, 0, a)$. Then the potential (in Gaussian units) at a point (x, y, z) is given by

$$\varphi_1 = \frac{q_1}{\epsilon_1 r_1} + \frac{q_1'}{\epsilon_1 r_2}, \quad (z \geq 0), \quad \varphi_2 = \frac{q_1''}{\epsilon_2 r_1}, \quad (z < 0),$$

where

$$r_1 = \sqrt{x^2 + y^2 + (z - a)^2}, \quad r_2 = \sqrt{x^2 + y^2 + (z + a)^2}.$$

Applying the boundary conditions at $(x, y, 0)$:

$$\varphi_1 = \varphi_2, \quad \epsilon_1 \frac{\partial \varphi_1}{\partial z} = \epsilon_2 \frac{\partial \varphi_2}{\partial z},$$

we obtain

$$q_1' = q_1'' = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1(\epsilon_1 + \epsilon_2)} q_1.$$

Similarly, if a point charge q_2 is placed at $(0, 0, -a)$ inside the dielectric ϵ_2, its image charges will be q_2' at $(0, 0, a)$ and q_2'' at $(0, 0, -a)$ with magnitudes

$$q_2' = q_2'' = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2(s\epsilon_1 + \epsilon_2)} q_2.$$

When both q_1 and q_2 exist, the force on q_1 will be the resultant due to q_2, q_1' and q_1''. It follows that

$$F = \frac{q_1 q_1'}{4a^2 \varepsilon_1} + \frac{q_1 q_2}{4a^2 \varepsilon_2} + \frac{q_1 q_1''}{4a^2 \varepsilon_2} \qquad = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \cdot \frac{q^2}{4a^2} + \frac{q_1 q_2}{2(\varepsilon_1 + \varepsilon_2) a^2}.$$

In the present problem $q_1 = -q$, $q_2 = +q$, and one has

$$F = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1(\varepsilon_1 + \varepsilon_2)} \frac{q^2}{4a^2} - \frac{q^2}{2(\varepsilon_1 + \varepsilon_2) a^2} = -\frac{q^2}{4\varepsilon_1 a^2}.$$

Hence, a force $-F$ is required to keep on $-q$ at rest.

Final result provided by raw answer:

$$-\frac{q^2}{4\varepsilon_1 a^2} \tag{37}$$

Reason for the error: The problem is flawed as it does not specify that the Gaussian unit system should be used. (Note: The raw answer is suspected to have been generated by an LLM, due to the heading "### Solution") For the correct solution, please refer to the appendix.2.0.2

2 Appendix

2.0.1 Problem ID: 1-39

Refined Solution Problem statement: Explanation This problem asks us to analyze the concentration of sunlight by a converging lens. We need to address two questions: 1. By what factor does the illuminance (luminous flux per unit area) increase when a lens is used to form an image of the sun on a flat surface, compared to the illuminance without the lens? 2. What is the condition on the lens's focal length f for a given lens radius r such that there is no increase in illuminance?

The physical quantities involved are: * **Luminous Flux (Φ):** The total perceived power of light, measured in lumens (lm). * **Illuminance (E):** The luminous flux incident per unit area on a surface, measured in lux (lx), where 1 lx = 1 lm/m². * **Luminance (B):** The luminous intensity per unit projected area of an extended light source, like the sun. It is a measure of the brightness of the source, measured in candela per square meter (cd/m²) or lm/(sr · m²).

The given parameters are: * r : The radius of the lens. * f : The focal length of the lens. * The angular diameter of the sun as seen from Earth is $\theta = 0.01$ radians.

From the angular diameter, we define the angular radius of the sun, α , which is half the angular diameter: $\alpha = \theta$ / $2 = 0.01$ / $2 = 0.005$ radians.

We will make the following assumptions: * The sun is a distant object with a uniform, constant luminance B . * The sun is directly overhead, so its rays are incident normal to the flat surface and the lens. * The lens is ideal: it is thin and has a transmittance of 1 (no light is absorbed or reflected). * The angles involved are small, allowing for the approximation tan(x) x for angles x in radians.

Step 1: Illuminance on the Surface without the Lens First, we calculate the illuminance on the flat surface due to the sun without any lens. The illuminance from an extended source of uniform luminance B is the product of the luminance and the solid angle Ω subtended by the source, for normal incidence.

Principles/Original Formulas/Assumptions: The illuminance E on a surface normal to the direction of a source with uniform luminance B subtending a solid angle Ω is: $\boxed{E = B\Omega}$ The solid angle Ω of a cone with a small half-angle α (in radians) is approximated by: $\boxed{\Omega \approx \pi\alpha^2 \quad \text{(for small } \alpha)}$

Derivation: The sun's angular radius is $\alpha = 0.005$ rad. We use the small-angle approximation to find the solid angle Ω_{sun} subtended by the sun.

$$\Omega_{\text{sun}} \approx \pi\alpha^2 = \pi(0.005)^2 \tag{38}$$

The illuminance on the surface without the lens, E_{no_lens} , is then:

$$E_{\text{no lens}} = B\Omega_{\text{sun}} = B\pi\alpha^2 \tag{39}$$

Step 2: Image Formation and Size Before calculating the illuminance with the lens, we must first determine the location and size of the image of the sun formed by the lens.

Principles/Original Formulas/Assumptions: The thin lens equation relates the object distance s_o , image distance s_i , and focal length f :

$$\boxed{\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}}$$

The size of an image formed by a lens can be determined from the angular size of the object and the image distance. For a ray passing through the center of a thin lens, its direction is unchanged.

Derivation: The sun is a very distant object, so we can take the object distance s_o to be infinitely large ($s_o \to \infty$). Using the thin lens equation, we find the image distance s_i :

$$\begin{aligned} \frac{1}{\infty} + \frac{1}{s_i} &= \frac{1}{f} \\ 0 + \frac{1}{s_i} &= \frac{1}{f} \\ s_i &= f \end{aligned} \tag{40}$$

This shows that the image is formed at the focal plane of the lens. To find the image size, consider a ray from the top edge of the sun that passes through the center of the lens. This ray makes an angle α (the sun's angular radius) with the principal axis. The height of this ray at the image plane gives the radius of the sun's image, r_image . From trigonometry:

$$\tan(\alpha) = \frac{r_{\text{image}}}{s_i} = \frac{r_{\text{image}}}{f} \tag{41}$$

Since α is a small angle, we can use the approximation $\tan(\alpha) \quad \alpha$:

$$r_{\text{image}} \approx f\alpha \tag{42}$$

The area of this circular image is therefore:

$$A_{\text{image}} = \pi r_{\text{image}}^2 = \pi (f\alpha)^2 = \pi f^2 \alpha^2 \tag{43}$$

Step 3: Illuminance on the Surface with the Lens Now we can determine the illuminance of the sun's image. The total luminous flux collected by the lens is concentrated onto the image area.

Principles/Original Formulas/Assumptions: The total luminous flux $_collected$ entering the lens is the product of the incident illuminance E_{no_lens} and the area of the lens A_{lens} .

$$\boxed{\Phi_{\text{collected}} = E_{\text{no lens}} \cdot A_{\text{lens}}}$$

The illuminance on the image, E_{lens} , is the total collected flux divided by the image area A_{image} , assuming a lossless lens.

$$\boxed{E_{\text{lens}} = \frac{\Phi_{\text{collected}}}{A_{\text{image}}}}$$

Derivation: The area of the lens is $A_{lens} = r$. The flux collected by the lens is:

$$\Phi_{\text{collected}} = E_{\text{no lens}} \cdot A_{\text{lens}} = (B\pi\alpha^2)(\pi r^2) = B\pi^2\alpha^2 r^2 \tag{44}$$

Using the image area from eq. 43, the illuminance in the image area, E_{lens} , is:

$$E_{\text{lens}} = \frac{\Phi_{\text{collected}}}{A_{\text{image}}} = \frac{B\pi^2\alpha^2 r^2}{\pi f^2\alpha^2} = B\pi\left(\frac{r}{f}\right)^2 \tag{45}$$

Step 4: Ratio of Illuminances We can now find the factor by which the illuminance is increased by calculating the ratio $R = E_{lens}/E_{no_lens}$.

Principles/Original Formulas/Assumptions: The ratio of illuminances is the factor of increase we seek. $\boxed{R = \frac{E_{\text{lens}}}{E_{\text{no lens}}}}$

Derivation: Using the expressions for E_{lens} from eq. 45 and E_{no_lens} from eq. 39:

$$R = \frac{B\pi\left(\frac{r}{f}\right)^2}{B\pi\alpha^2} = \frac{(r/f)^2}{\alpha^2} = \left(\frac{r}{f\alpha}\right)^2 \tag{46}$$

Now, we substitute the numerical value for $\alpha = 0.005$:

$$R = \frac{1}{\alpha^2}\left(\frac{r}{f}\right)^2 = \frac{1}{(0.005)^2}\left(\frac{r}{f}\right)^2 = \frac{1}{(5\times10^{-3})^2}\left(\frac{r}{f}\right)^2 \tag{47}$$

Calculating the numerical factor:

$$\frac{1}{(5\times10^{-3})^2} = \frac{1}{25\times10^{-6}} = \frac{10^6}{25} = 4\times10^4 \tag{10}$$

So, the ratio of the illuminances is:

$$R = 4\times10^4\frac{r^2}{f^2} \tag{11}$$

Step 5: Condition for No Increase in Illuminance The condition for the lens to give ”no increase” in illuminance is that the ratio R must be less than or equal to one.

Principles/Original Formulas/Assumptions: The condition for no increase in illuminance is: $\boxed{R \leq 1}$

Derivation: We apply this condition to the result from eq. 11:

$$\begin{aligned} 4\times10^4\frac{r^2}{f^2} &\leq 1 \\ 4\times10^4 r^2 &\leq f^2 \\ \sqrt{4\times10^4 r^2} &\leq \sqrt{f^2} \\ (2\times10^2)r &\leq f \\ f &\geq 200r \end{aligned} \tag{12}$$

Thus, for the lens to provide no increase in illuminance, its focal length f must be at least 200 times its radius r .

Final Answer The factor by which the luminous flux per unit area (illuminance) is greater in the image area is:

$$\frac{E_{\text{lens}}}{E_{\text{no lens}}} = 4 \times 10^4 \left(\frac{r}{f}\right)^2 \tag{48}$$

For the lens to give no increase in illuminance in the image, the focal length f must satisfy the condition:

$$\boxed{f \geq 200r} \tag{49}$$

2.0.2 Problem ID: 1-78

Refined Solution Problem statement: Explanation The problem describes an electrostatic system consisting of two semi-infinite, linear, homogeneous dielectric media separated by the plane $z = 0$.

- **Region 1**: The half-space $z > 0$ is filled with a dielectric material with absolute permittivity ϵ_1. - **Region 2**: The half-space $z < 0$ is filled with a dielectric material with absolute permittivity ϵ_2. - The term "dielectric constant" in the Problem statement: is interpreted as the absolute permittivity ϵ.

Two point charges are fixed in this system: - A charge $q_A = -q$ is located at position $\vec{r}_A = (0, 0, a)$, where $a > 0$. This charge resides in Region 1. - A charge $q_B = +q$ is located at position $\vec{r}_B = (0, 0, -a)$. This charge resides in Region 2.

We use a standard Cartesian coordinate system (x, y, z), and the unit vector in the z-direction is denoted by $\hat{k}$.

The objective is to find the external force, $\vec{F}_{\text{ext}}$, that must be exerted on the charge $q_A = -q$ to keep it stationary. For the charge to be at rest, the net force acting on it must be zero. This means the external force must exactly counteract the total electric force, $\vec{F}_{\text{elec}}$, acting on the charge due to all other sources.

Step 1: General Approach using Superposition and Method of Images The problem involves point charges near a planar boundary between two linear dielectric media, which is a classic application of the method of images. Since the dielectrics are linear, we can use the principle of superposition to find the total electrostatic potential. We will solve for the potential created by each charge individually in the presence of the boundary and then sum the results. The electric force on charge q_A can then be determined from the resulting electric field.

The electrostatic potential Φ must satisfy the following boundary conditions at the interface $z = 0$, where Φ_1 is the potential in Region 1 ($z > 0$) and Φ_2 is the potential in Region 2 ($z < 0$):

$$\boxed{\text{Boundary Condition 1 (Continuity of Potential): } \Phi_1(x, y, 0) = \Phi_2(x, y, 0)}$$

$$\boxed{\text{Boundary Condition 2 (Continuity of Normal D-field): } \epsilon_1 \frac{\partial \Phi_1}{\partial z}\bigg|_{z=0} = \epsilon_2 \frac{\partial \Phi_2}{\partial z}\bigg|_{z=0}}$$

The condition for static equilibrium for charge q_A is:

$$\boxed{\text{Newton's First Law (Static Equilibrium): } \vec{F}_{\text{net}} = \vec{F}_{\text{elec}} + \vec{F}_{\text{ext}} = 0}$$

From this, the required external force is $\vec{F}_{\text{ext}} = -\vec{F}_{\text{elec}}$.

We decompose the problem into two sub-problems: 1. Find the potential due to charge q_A and the dielectric boundary. Let's call this potential φ_A. 2. Find the potential due to charge q_B and the dielectric boundary. Let's call this potential φ_B.

By superposition, the total potential is $\Phi = \varphi_A + \varphi_B$. The total potential in Region 1 is $\Phi_1 = \varphi_{A,1} + \varphi_{B,1}$.

Step 2: Potential due to Charge $q_A = -q$ First, we consider the sub-problem with only charge $q_A = -q$ at $(0, 0, a)$. We use the method of images to find the potential in both regions.

$\boxed{\text{Potential of a point charge Q in a uniform dielectric } \epsilon : \varphi(\vec{r}) = \frac{1}{4\pi\epsilon}\frac{Q}{|\vec{r} - \vec{r}_Q|}}$ **Derivation** - For Region 1 ($z > 0$), the potential $\varphi_{A,1}$ is modeled as the sum of the potential from the real charge q_A at $(0, 0, a)$ and an image charge q'_A at $(0, 0, -a)$, both considered to be in a uniform medium of permittivity ϵ_1. - For Region 2 ($z < 0$), the potential $\varphi_{A,2}$ is modeled as being produced by an image charge q''_A located at $(0, 0, a)$, in a uniform medium of permittivity ϵ_2.

Let $r_1 = \sqrt{x^2 + y^2 + (z - a)^2}$ be the distance from $(0, 0, a)$ and $r_2 = \sqrt{x^2 + y^2 + (z + a)^2}$ be the distance from $(0, 0, -a)$. The potentials are:

$$\varphi_{A,1}(x, y, z) = \frac{1}{4\pi\epsilon_1}\left(\frac{q_A}{r_1} + \frac{q'_A}{r_2}\right) \quad \text{for } z > 0 \tag{50}$$

$$\varphi_{A,2}(x, y, z) = \frac{1}{4\pi\epsilon_2}\frac{q''_A}{r_1} \quad \text{for } z < 0 \tag{51}$$

Applying the boundary conditions at $z = 0$ (where $r_1 = r_2 = \sqrt{x^2 + y^2 + a^2}$): 1. Continuity of potential: $\varphi_{A,1}(z = 0) = \varphi_{A,2}(z = 0) \implies \frac{1}{\epsilon_1}(q_A + q'_A) = \frac{1}{\epsilon_2}q''_A$. 2. Continuity of normal D-field: $\epsilon_1 \frac{\partial\varphi_{A,1}}{\partial z}\Big|_{z=0} = \epsilon_2 \frac{\partial\varphi_{A,2}}{\partial z}\Big|_{z=0}$. The derivatives are: $\frac{\partial}{\partial z}\left(\frac{1}{r_1}\right) = -\frac{z-a}{r_1^3}$, so at $z = 0$, it is $\frac{a}{r_1^3}$. $\frac{\partial}{\partial z}\left(\frac{1}{r_2}\right) = -\frac{z+a}{r_2^3}$, so at $z = 0$, it is $-\frac{a}{r_2^3}$. Thus, $\epsilon_1 \frac{1}{4\pi\epsilon_1}\left(\frac{q_A a}{r_1^3} - \frac{q'_A a}{r_2^3}\right)\Big|_{z=0} = \epsilon_2 \frac{1}{4\pi\epsilon_2}\left(\frac{q''_A a}{r_1^3}\right)\Big|_{z=0}$. This simplifies to $q_A - q'_A = q''_A$.

Solving the system of equations:

$$\begin{aligned}
\frac{1}{\epsilon_1}(q_A + q'_A) &= \frac{1}{\epsilon_2}(q_A - q'_A) \\
\epsilon_2(q_A + q'_A) &= \epsilon_1(q_A - q'_A) \\
q'_A(\epsilon_1 + \epsilon_2) &= q_A(\epsilon_1 - \epsilon_2) \\
q'_A &= \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q_A
\end{aligned} \tag{52}$$

Step 3: Potential due to Charge $q_B = +q$ Next, we consider the sub-problem with only charge $q_B = +q$ at $(0, 0, -a)$. We are interested in the potential this charge creates in Region 1, where charge q_A is located.

$\boxed{\text{Potential of a point charge Q in a uniform dielectric } \epsilon : \varphi(\vec{r}) = \frac{1}{4\pi\epsilon}\frac{Q}{|\vec{r} - \vec{r}_Q|}}$ **Derivation** - For Region 1 ($z > 0$), the potential $\varphi_{B,1}$ is modeled as being from an image charge q''_B at $(0, 0, -a)$, in a uniform medium of permittivity ϵ_1. - For Region 2 ($z < 0$), the potential $\varphi_{B,2}$ is modeled as the sum of potentials from the real charge q_B at $(0, 0, -a)$ and an image charge q'_B at $(0, 0, a)$, both in a uniform medium of permittivity ϵ_2.

The potentials are:

$$\varphi_{B,1}(x,y,z) = \frac{1}{4\pi\epsilon_1}\frac{q''_B}{r_2} \quad \text{for } z>0 \tag{53}$$

$$\varphi_{B,2}(x,y,z) = \frac{1}{4\pi\epsilon_2}\left(\frac{q_B}{r_2}+\frac{q'_B}{r_1}\right) \quad \text{for } z<0 \tag{54}$$

Applying the boundary conditions at $z=0$: 1. Continuity of potential: $\varphi_{B,1}(z=0) = \varphi_{B,2}(z=0) \implies \frac{1}{\epsilon_1}q''_B = \frac{1}{\epsilon_2}(q_B+q'_B)$. 2. Continuity of normal D-field: $\epsilon_1\frac{\partial\varphi_{B,1}}{\partial z}\Big|_{z=0} = \epsilon_2\frac{\partial\varphi_{B,2}}{\partial z}\Big|_{z=0}$. The derivatives at $z=0$ give: $\epsilon_1\frac{1}{4\pi\epsilon_1}\left(-\frac{q''_B a}{r_2^3}\right) = \epsilon_2\frac{1}{4\pi\epsilon_2}\left(-\frac{q_B a}{r_2^3}+\frac{q'_B a}{r_1^3}\right)$. This simplifies to $-q''_B = -q_B + q'_B$.

Solving this system for q''_B, which is the image charge needed for the potential in Region 1:

$$\begin{aligned} q'_B &= q_B - q''_B \\ \frac{1}{\epsilon_1}q''_B &= \frac{1}{\epsilon_2}(q_B+(q_B-q''_B)) = \frac{1}{\epsilon_2}(2q_B-q''_B) \\ \epsilon_2 q''_B &= \epsilon_1(2q_B-q''_B) \\ q''_B(\epsilon_1+\epsilon_2) &= 2\epsilon_1 q_B \\ q''_B &= \frac{2\epsilon_1}{\epsilon_1+\epsilon_2}q_B \end{aligned} \tag{55}$$

Step 4: Total Electric Force on Charge $q_A=-q$ By superposition, the total potential in Region 1 ($z>0$) is $\Phi_1 = \varphi_{A,1}+\varphi_{B,1}$. The force on charge q_A is due to the electric field created by all other sources. This field is derived from the part of the potential Φ_1 that is non-singular at the location of q_A.

$\boxed{\text{Coulomb's Law: } \vec{F}_{12} = \frac{1}{4\pi\epsilon}\frac{Q_1Q_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)}$ **Derivation** The total potential in Region 1 is:

$$\Phi_1(x,y,z) = \varphi_{A,1}+\varphi_{B,1} = \frac{1}{4\pi\epsilon_1}\left(\frac{q_A}{r_1}+\frac{q'_A}{r_2}\right)+\frac{1}{4\pi\epsilon_1}\frac{q''_B}{r_2} = \frac{1}{4\pi\epsilon_1}\left(\frac{q_A}{r_1}+\frac{q'_A+q''_B}{r_2}\right) \tag{56}$$

The term $\frac{1}{4\pi\epsilon_1}\frac{q_A}{r_1}$ is the potential of q_A itself and does not contribute to the force on q_A. The force is generated by the potential from all other sources, which in Region 1 is equivalent to the potential of an effective image charge $q_{\text{eff}} = q'_A + q''_B$ located at $(0,0,-a)$.

$$\begin{aligned} q_{\text{eff}} = q'_A + q''_B &= \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)q_A + \left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right)q_B \quad \text{(using eq. 52 and 55)} \\ &= \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)(-q) + \left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\right)(+q) \\ &= \frac{q}{\epsilon_1+\epsilon_2}\left(-(\epsilon_1-\epsilon_2)+2\epsilon_1\right) = \frac{q}{\epsilon_1+\epsilon_2}\left(-\epsilon_1+\epsilon_2+2\epsilon_1\right) = \frac{q(\epsilon_1+\epsilon_2)}{\epsilon_1+\epsilon_2} = q \end{aligned} \tag{57}$$

The electric force on $q_A=-q$ at $\vec{r}_A = a\hat{k}$ is the Coulomb force exerted by $q_{\text{eff}}=q$ at $\vec{r}_{\text{eff}} = -a\hat{k}$ in a uniform medium of permittivity ϵ_1. The vector from q_{eff} to q_A is $\vec{r}_A - \vec{r}_{\text{eff}} = 2a\hat{k}$.

$$\vec{F}_{\text{elec}} = \frac{1}{4\pi\epsilon_1}\frac{q_A\cdot q_{\text{eff}}}{|2a\hat{k}|^2}\hat{k} = \frac{1}{4\pi\epsilon_1}\frac{(-q)(q)}{(2a)^2}\hat{k} = -\frac{q^2}{16\pi\epsilon_1 a^2}\hat{k} \tag{58}$$

Step 5: Required External Force To keep the charge $q_A = -q$ at rest, the external force $\vec{F}_{\text{ext}}$ must balance the electric force $\vec{F}_{\text{elec}}$.

$\boxed{\vec{F}_{\text{net}} = \vec{F}_{\text{elec}} + \vec{F}_{\text{ext}} = 0}$ **Derivation** From the condition for static equilibrium, we find the required external force.

$$\begin{aligned}\vec{F}_{\text{ext}} &= -\vec{F}_{\text{elec}} \\ &= -\left(-\frac{q^2}{16\pi\epsilon_1 a^2}\hat{k}\right) \\ &= \frac{q^2}{16\pi\epsilon_1 a^2}\hat{k}\end{aligned} \tag{59}$$

Final Answer The force one must exert on the negative charge $-q$ at $(0, 0, a)$ to keep it at rest is:

$$\vec{F}_{\text{ext}} = \frac{q^2}{16\pi\epsilon_1 a^2}\hat{k} \tag{60}$$

2.0.3 Problem ID: 1-80

Problem statement: Explanation This problem asks us to derive a relationship between three different measurements performed on an elastic band. The goal is to predict the outcome of the third measurement using the results from the first two. The elastic band is treated as a one-dimensional thermodynamic system whose state can be described by variables such as temperature (T), entropy (S), length (L), and tension (W). The internal energy is denoted by U.

The three measurements are: 1. **(a) Adiabatic Stretching:** The band is stretched rapidly, which is an adiabatic process (constant entropy, S). The change in temperature, is measured for a corresponding change in length. This experiment provides an experimental value for the partial derivative $\left(\frac{\partial T}{\partial L}\right)_S$. 2. **(b) Adiabatic Vibrations:** A weight W is attached to one end of the band, and the other end is fixed. The frequency u of small vertical vibrations is measured. These vibrations are rapid and thus considered adiabatic. This measurement allows for the determination of the adiabatic spring constant, $k_S = \left(\frac{\partial W}{\partial L}\right)_S$. 3. **(c) Isobaric Heating:** The band is held in equilibrium under a constant tension force W. A small amount of heat, denoted as δQ (originally given as σQ), is added to the band. This is an isobaric process (constant tension, W). The resulting change in the equilibrium length, δL, is measured.

We will use the principles of thermodynamics to derive an equation for δL from measurement (c) in terms of quantities determined from measurements (a) and (b).

Assumptions: * The elastic band is a simple, one-dimensional thermodynamic system. * All processes are assumed to be reversible (quasi-static). * The changes in state variables ($\delta Q, \delta L$) are small enough that derivatives can be approximated by finite differences. * The mass of the weight is $m = W/g$, where g is the acceleration due to gravity.

Step 1: Thermodynamic Framework and a Maxwell Relation To connect the different physical processes, we begin with the fundamental laws of thermodynamics applied to an elastic system. The work done *on* the band to stretch it by an infinitesimal amount dL is WdL.

Principles/Original Formulas/Assumptions The first law of thermodynamics for a reversible process in a one-dimensional elastic system is:

$\boxed{dU = TdS + WdL}$ To change the natural variables from (S, L) to (S, W), which are more convenient for this problem, we perform a Legendre transformation by defining an enthalpy-like thermodynamic potential, H:

$\boxed{H = U - WL}$ For any exact differential of a function $z(x, y)$ written as $dz = M(x, y)dx + N(x, y)dy$, the equality of mixed partial derivatives (Euler's reciprocity relation) holds:

$\boxed{\left(\frac{\partial M}{\partial y}\right)_x = \left(\frac{\partial N}{\partial x}\right)_y}$

Derivation We find the total differential of the potential H:

$$\begin{aligned} dH &= d(U - WL) = dU - d(WL) \\ &= (TdS + WdL) - (WdL + LdW) \\ &= TdS - LdW \end{aligned} \tag{61}$$

From this exact differential, we see that H is a function of S and W, i.e., $H(S, W)$. We can identify its partial derivatives as:

$$T = \left(\frac{\partial H}{\partial S}\right)_W \quad \text{and} \quad -L = \left(\frac{\partial H}{\partial W}\right)_S \tag{62}$$

Applying the equality of mixed partial derivatives to the exact differential dH:

$$\begin{aligned} \frac{\partial}{\partial W}\left(\left(\frac{\partial H}{\partial S}\right)_W\right)_S &= \frac{\partial}{\partial S}\left(\left(\frac{\partial H}{\partial W}\right)_S\right)_W \\ \left(\frac{\partial T}{\partial W}\right)_S &= -\left(\frac{\partial L}{\partial S}\right)_W \end{aligned} \tag{63}$$

This is a Maxwell relation for the elastic system, which provides a crucial link between the adiabatic process variables (constant S) and the isobaric process variables (constant W).

Step 2: Analysis of the Isobaric Heating Process (Measurement c) Measurement (c) involves adding a small amount of heat δQ at constant tension W. We aim to express the resulting length change δL in terms of δQ.

Principles/Original Formulas/Assumptions For a reversible process, the infinitesimal heat added δQ is related to the change in entropy dS by the second law of thermodynamics:

$\boxed{\delta Q = TdS}$ For a state function of two variables, such as length $L(S, W)$, its total differential is given by:

$\boxed{dL = \left(\frac{\partial L}{\partial S}\right)_W dS + \left(\frac{\partial L}{\partial W}\right)_S dW}$

Derivation In measurement (c), the process occurs at constant weight, so the tension is constant, which means $dW = 0$. The change in length δL for this small process can be written as:

$$\delta L = \left(\frac{\partial L}{\partial S}\right)_W dS \tag{64}$$

From the second law, the entropy change dS corresponding to the heat added δQ is $dS = \delta Q/T$. Substituting this into Eq. 64 gives:

$$\delta L = \left(\frac{\partial L}{\partial S}\right)_W \frac{\delta Q}{T} \tag{65}$$

Step 3: Connecting the Measurements via the Maxwell Relation Now we combine the results from the previous steps to link the quantity measured in (c) to derivatives that can be determined from measurements (a) and (b).

Principles/Original Formulas/Assumptions The chain rule for partial derivatives allows for a change of variables in differentiation:

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{(\partial z/\partial w)_y}{(\partial x/\partial w)_y}}$$

Derivation We start with Eq. 65 and substitute the Maxwell relation from Eq. 63:

$$\delta L = -\left(\frac{\partial T}{\partial W}\right)_S \frac{\delta Q}{T} \tag{66}$$

The derivative $(\partial T/\partial W)_S$ is not directly measured. However, measurements (a) and (b) are adiabatic processes (constant S) where changes in length L are central. We can use the chain rule to express this derivative in terms of derivatives with respect to L at constant entropy S:

$$\left(\frac{\partial T}{\partial W}\right)_S = \frac{(\partial T/\partial L)_S}{(\partial W/\partial L)_S} \tag{67}$$

Substituting Eq. 67 into Eq. 66 yields our main theoretical result:

$$\delta L = -\frac{(\partial T/\partial L)_S}{(\partial W/\partial L)_S} \frac{\delta Q}{T} \tag{68}$$

This equation connects the result of measurement (c), δL, to two adiabatic derivatives, which we will now relate to measurements (a) and (b).

Step 4: Relating Theoretical Derivatives to Experimental Results Here, we interpret the results of measurements (a) and (b) to find expressions for the partial derivatives in Eq. 68.

Principles/Original Formulas/Assumptions For measurement (a), we use the approximation of a derivative by a finite difference for small changes:

$\boxed{\left(\frac{\partial y}{\partial x}\right) \approx \frac{\Delta y}{\Delta x}}$ For measurement (b), the system of a mass on a spring undergoes Simple Harmonic Motion (SHM). The angular frequency ω is related to the spring constant k and mass m by:

$\boxed{\omega = \sqrt{\frac{k}{m}}}$ The angular frequency ω is related to the ordinary frequency u by:

$\boxed{\omega = 2\pi u}$ The spring constant is defined as the change in restoring force per unit displacement. For the elastic band, this is the change in tension per unit length:
$\boxed{k = \frac{dW}{dL}}$
Derivation From measurement (a), the adiabatic stretching gives a temperature change for a length change:

$$\left(\frac{\partial T}{\partial L}\right)_S \tag{69}$$

From measurement (b), the small, rapid vibrations are adiabatic. The effective spring constant is therefore the adiabatic spring constant, k_S. By its definition:

$$k_S = \left(\frac{\partial W}{\partial L}\right)_S \tag{70}$$

The mass undergoing SHM is $m = W/g$. The angular frequency of oscillation is $\omega = 2\pi u$. Using the formula for angular frequency:

$$\begin{aligned} 2\pi u &= \sqrt{\frac{k_S}{m}} = \sqrt{\frac{k_S}{W/g}} \\ (2\pi u)^2 &= \frac{k_S g}{W} \\ k_S &= \frac{W}{g}(2\pi u)^2 \end{aligned} \tag{71}$$

Combining Eq. 70 and Eq. 71, we have the experimental value for the second derivative in Eq. 68:

$$\left(\frac{\partial W}{\partial L}\right)_S = \frac{W}{g}(2\pi u)^2 \tag{72}$$

Step 5: The Final Predictive Equation Finally, we substitute the experimental expressions for the derivatives into our central thermodynamic relation to obtain the predictive equation.

Derivation We take the central relation from Eq. 68:

$$\delta L = -\frac{(\partial T/\partial L)_S}{(\partial W/\partial L)_S} \frac{\delta Q}{T}$$

Now, we substitute the results from our analysis of the measurements, using Eq. 69 for the numerator and Eq. 72 for the denominator:

$$\delta L = -\frac{(\partial T/\partial L)_S}{\frac{W}{g}(2\pi u)^2} \frac{\delta Q}{T} \tag{73}$$

This equation allows the prediction of δL from measurement (c) using the data obtained from measurements (a) and (b). Experiment (a) shows that for a typical rubber band, stretching causes heating, so $(\partial T/\partial L)_S > 0$. Since all other quantities in the expression (W, g, u, T) are positive, the equation predicts $\delta L < 0$ when heat is added $(\delta Q > 0)$. This means a stretched rubber band contracts upon heating, a well-known counter-intuitive property explained by entropic elasticity.

Final Answer The equation by which you can predict the result of the last measurement from the results of the first two is:

$$\boxed{\delta L = -\frac{(\partial T/\partial L)_S}{\frac{W}{g}(2\pi u)^2}\frac{\delta Q}{T}} \tag{74}$$

where $(\partial T/\partial L)_S$ is the ratio of temperature change to length change from the adiabatic stretching in measurement (a), u is the vibration frequency from measurement (b), δQ is the heat added in measurement (c), W is the suspended weight, g is the acceleration due to gravity, and T is the absolute temperature of the band.

2.0.4 Problem ID: 1-125

Problem statement: Explanation The problem asks for the relationship between three physical quantities for a charged soap bubble at equilibrium: its radius r, the electric potential ϕ on its surface, and the excess pressure Δp. The excess pressure is defined as the ambient (outside) pressure minus the internal pressure, i.e., $\Delta p = p_{\text{out}} - p_{\text{in}}$. A key condition is that the effects of surface tension are to be neglected.

The system consists of a spherical bubble of radius r. It carries a total electric charge q which, for a conductor, is distributed uniformly on its surface. The bubble is in mechanical and thermal equilibrium with its surroundings. We will find the equilibrium condition by minimizing the total Helmholtz free energy of the system, which is the appropriate thermodynamic potential for a system at constant temperature and volume.

The variables involved are: - r: The equilibrium radius of the soap bubble. - q: The total electric charge on the bubble's surface. - ϕ: The electric potential at the surface of the bubble. - p_{in}: The pressure of the gas inside the bubble. - p_{out}: The pressure of the ambient gas outside the bubble. - $\Delta p = p_{\text{out}} - p_{\text{in}}$: The excess pressure. - ϵ_0: The permittivity of free space.

We make the following assumptions: 1. The bubble is a perfect, infinitesimally thin spherical shell. 2. The bubble and its surroundings are maintained at a constant temperature. 3. The total charge q on the bubble is constant. 4. The entire system (bubble + surrounding gas) is enclosed in a container of fixed total volume.

Step 1: Principle of Minimum Free Energy For a system at constant temperature and total volume, the condition for stable mechanical equilibrium is that its Helmholtz free energy F is at a minimum. This implies that for any small, virtual change in the system's configuration (in this case, a change δr in the bubble's radius), the first-order change in the total free energy must be zero.

$\boxed{\delta F_{\text{total}} = 0}$

The total free energy of the system is the sum of the free energies of the gas inside the bubble (F_{in}), the gas outside the bubble (F_{out}), and the electrostatic potential energy of the charged bubble itself (U_E). The electrostatic energy is a component of the system's internal energy, and therefore a component of its free energy ($F = U - TS$).

$$F_{\text{total}} = F_{\text{in}} + F_{\text{out}} + U_E \tag{75}$$

Applying the equilibrium condition to a virtual change δr in the radius:

$$\delta F_{\text{total}} = \delta F_{\text{in}} + \delta F_{\text{out}} + \delta U_E = 0 \tag{76}$$

Step 2: Calculating the Variations in Free Energy Components We now calculate the change in each component of the free energy due to a small increase in the radius, δr.

Principles/Original Formulas/Assumptions: The change in Helmholtz free energy for a fluid undergoing a volume change δV at constant temperature is derived from the thermodynamic identity $dF = -SdT - pdV$. At constant T, this becomes:

$\boxed{\delta F = -p\,\delta V}$ The volume of a sphere of radius r is:

$\boxed{V = \frac{4}{3}\pi r^3}$ The electrostatic potential energy of a spherical shell of radius r with a total charge q uniformly distributed on its surface is:

$\boxed{U_E = \frac{q^2}{8\pi\epsilon_0 r}}$

Derivation: First, we find the change in the bubble's internal volume, V_{in}, for a change in radius δr:

$$\delta V_{\text{in}} = \frac{dV_{\text{in}}}{dr}\delta r = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\delta r = 4\pi r^2 \delta r \tag{77}$$

The change in the free energy of the gas inside the bubble is:

$$\delta F_{\text{in}} = -p_{\text{in}}\delta V_{\text{in}} = -p_{\text{in}}(4\pi r^2 \delta r) \tag{78}$$

As per our assumption of a fixed total volume for the system, an increase in the bubble's volume must be compensated by an equal decrease in the volume of the surrounding gas. Thus, $\delta V_{\text{out}} = -\delta V_{\text{in}}$. The change in the free energy of the gas outside is:

$$\delta F_{\text{out}} = -p_{\text{out}}\delta V_{\text{out}} = -p_{\text{out}}(-\delta V_{\text{in}}) = p_{\text{out}}(4\pi r^2 \delta r) \tag{79}$$

Next, we find the change in the electrostatic potential energy. This calculation assumes the total charge q is constant.

$$\delta U_E = \frac{dU_E}{dr}\delta r = \frac{d}{dr}\left(\frac{q^2}{8\pi\epsilon_0 r}\right)\delta r = -\frac{q^2}{8\pi\epsilon_0 r^2}\delta r \tag{80}$$

Step 3: Establishing the Equilibrium Condition Equation We now substitute the expressions for the changes in free energy components into the equilibrium condition from Step 1.

Principles/Original Formulas/Assumptions: The equilibrium condition is:

$\boxed{\delta F_{\text{in}} + \delta F_{\text{out}} + \delta U_E = 0}$

Derivation: Using equations 78, 79, and 80 in equation 76:

$$-p_{\text{in}}(4\pi r^2 \delta r) + p_{\text{out}}(4\pi r^2 \delta r) - \frac{q^2}{8\pi\epsilon_0 r^2}\delta r = 0 \tag{81}$$

Since δr is an arbitrary non-zero variation, we can divide the entire equation by it:

$$(p_{\text{out}} - p_{\text{in}})4\pi r^2 - \frac{q^2}{8\pi\epsilon_0 r^2} = 0 \tag{82}$$

Using the problem's definition of excess pressure, $\Delta p = p_{\text{out}} - p_{\text{in}}$:

$$\Delta p \cdot 4\pi r^2 = \frac{q^2}{8\pi\epsilon_0 r^2} \tag{83}$$

Solving for Δp, we find its relation to the charge q and radius r:

$$\Delta p = \frac{q^2}{32\pi^2\epsilon_0 r^4} \tag{84}$$

This result has a clear physical interpretation. The equilibrium condition represents a balance of pressures. The term Δp is the net inward pressure. The right-hand side represents an outward electrostatic pressure, P_{elec}. For a conductor with surface charge density $\sigma = q/(4\pi r^2)$, this pressure is $P_{\text{elec}} = \sigma^2/(2\epsilon_0)$. Substituting the expression for σ, we get $P_{\text{elec}} = \frac{(q/4\pi r^2)^2}{2\epsilon_0} = \frac{q^2}{32\pi^2\epsilon_0 r^4}$. Thus, equation 84 is equivalent to the pressure balance condition $\Delta p = P_{\text{elec}}$.

Step 4: Expressing the Result in Terms of Potential ϕ The final step is to express the relationship in terms of the potential ϕ instead of the charge q, as requested by the problem.

Principles/Original Formulas/Assumptions: The electric potential on the surface of a conducting sphere of radius r with charge q is given by:

$$\boxed{\phi = \frac{q}{4\pi\epsilon_0 r}}$$

Derivation: From the potential formula, we can express the charge q in terms of the potential ϕ:

$$q = 4\pi\epsilon_0 r\phi \tag{85}$$

Substituting this expression for q into our result for Δp (eq. 84):

$$\begin{aligned}
\Delta p &= \frac{(4\pi\epsilon_0 r\phi)^2}{32\pi^2\epsilon_0 r^4} \\
&= \frac{16\pi^2\epsilon_0^2 r^2\phi^2}{32\pi^2\epsilon_0 r^4} \\
&= \left(\frac{16}{32}\right)\left(\frac{\pi^2}{\pi^2}\right)\left(\frac{\epsilon_0^2}{\epsilon_0}\right)\left(\frac{r^2}{r^4}\right)\phi^2 \\
&= \frac{1}{2}\cdot\epsilon_0\cdot\frac{1}{r^2}\cdot\phi^2 \\
\Delta p &= \frac{\epsilon_0\phi^2}{2r^2}
\end{aligned} \tag{86}$$

This equation provides the required relation between the equilibrium radius r, the surface potential ϕ, and the excess pressure Δp.

Final Answer The relation between the equilibrium radius r, the potential ϕ, and the excess pressure Δp of a charged soap bubble, neglecting surface tension, is:

$$\boxed{\Delta p = \frac{\epsilon_0\phi^2}{2r^2} = \frac{\phi^2}{8\pi r^2}} \tag{87}$$

2.0.5 Problem ID: 19-4

Problem statement: Explanation This problem asks for an estimate of the electrical current I produced by electrons escaping from a cavity through a small hole of area A.

Given Information and Variables: - **System**: A gas of non-interacting electrons is contained within a cavity. - **Equilibrium State**: The electron gas is in thermal equilibrium, described by: - : The temperature in energy units (i.e., $= k_B T$, where k_B is the Boltzmann constant and T is the absolute temperature). - : The chemical potential of the electron gas. - **Density**: The number density of electrons inside the cavity is maintained at a constant value. - **Escape Mechanism**: - A small hole of area A in the cavity wall allows electrons to escape. - An external potential creates a potential energy barrier $V > 0$ for electrons outside the cavity. The potential energy of an electron inside the cavity is taken as zero. - **Approximation**: The potential barrier is high relative to the thermal energy available to electrons at the chemical potential level, specifically $V - >>$.

Physical Model and Assumptions: 1. **Coordinate System**: We define a coordinate system where the hole lies in the xy-plane at $z = 0$. The interior of the cavity corresponds to $z < 0$ and the exterior to $z > 0$. The z-axis is normal to the surface of the hole. 2. **Potential Barrier**: The potential energy $U(\vec{r})$ is assumed to depend only on the z-coordinate, modeled as a step function: $U(z) = 0$ for $z < 0$ and $U(z) = V$ for $z > 0$. This potential gives rise to a force $\vec{F} = -\nabla U$ which has only a z-component, $F_z = -dU/dz$. Consequently, the momentum components parallel to the hole, p_x and p_y, are conserved as an electron passes through. 3. **Escape Condition**: For an electron to escape, it must overcome the potential barrier. By conservation of total energy, an electron with momentum $\vec{p}$ inside and $\vec{p}'$ outside must satisfy $\frac{p^2}{2m} = \frac{p'^2}{2m} + V$. Expanding the momenta, this is $\frac{p_x^2+p_y^2+p_z^2}{2m} = \frac{p_x'^2+p_y'^2+p_z'^2}{2m} + V$. Since $p_x' = p_x$ and $p_y' = p_y$, the energy conservation equation simplifies to $\frac{p_z^2}{2m} = \frac{p_z'^2}{2m} + V$. For the electron to successfully escape (i.e., to exist at $z > 0$ with a real momentum p_z'), we must have $\frac{p_z'^2}{2m} \geq 0$. This leads to the escape condition: the kinetic energy associated with the normal component of motion, $\epsilon_z = \frac{p_z^2}{2m}$, must be greater than the barrier height V. The problem states "larger than V", so we take $\epsilon_z > V$. 4. **Electron Gas Model**: The electrons are treated as a gas of non-interacting fermions. Their energy distribution is described by the Fermi-Dirac statistics. 5. **Current Definition**: The electric current I is the net rate of charge flow. Since each electron carries a charge of $-e$ (where e is the elementary charge), the current is related to the particle escape rate R by $I = -eR$.

Step 1: Electron Distribution in Phase Space The electrons inside the cavity form a non-interacting Fermi gas in thermal equilibrium. The number of electrons, dN, within a differential phase space volume $d^3r d^3p$ is given by the density of states in phase space multiplied by the Fermi-Dirac distribution function, which gives the probability of a state being occupied.

$\boxed{dN = \frac{2}{h^3} f(\epsilon) d^3r d^3p = \frac{2}{h^3} \frac{1}{e^{(\epsilon-\mu)/\tau} + 1} d^3r d^3p}$ Here, the factor of 2 accounts for the two possible spin states of an electron, h is Planck's constant, and $\epsilon = p^2/(2m)$ is the kinetic energy of an electron with momentum p. The number of electrons per unit volume with momentum in the range d^3p, denoted as dn, is obtained by dividing dN by the spatial volume element d^3r.

$$dn = \frac{dN}{d^3r} = \frac{2}{h^3}\frac{1}{e^{(\epsilon-\mu)/\tau}+1}d^3p \tag{88}$$

Step 2: Setting up the Particle Escape Rate The rate at which electrons escape is the total particle flux through the hole. The particle flux is found by integrating the normal component of velocity, v_z, over the distribution of all electrons that can escape. An electron with momentum $\vec{p}$ has a velocity component $v_z = p_z/m$ normal to the hole. The differential rate dR of electrons with momentum in d^3p escaping through area A is the number of such electrons that can cross the area per unit time.

$\boxed{dR = (v_z\, dn)\, A}$ To find the total escape rate R, we must integrate dR over all momenta corresponding to escaping electrons. The conditions for an electron to escape are: 1. It must be moving towards the hole: $p_z > 0$. 2. It must have sufficient kinetic energy in the z-direction to overcome the potential barrier V: $\epsilon_z = p_z^2/(2m) > V$, which implies $p_z > \sqrt{2mV}$. This second condition is stricter and thus incorporates the first. The momentum components parallel to the hole, p_x and p_y, can take any value from $-$ to $+$.

The total rate R is the integral of dR over the allowed momentum space volume, with $d^3p = dp_x dp_y dp_z$.

$$\begin{aligned} R &= \int_{\text{escape}} dR = \int_{\text{escape}} \frac{p_z}{m} A\, dn \\ &= A \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{\sqrt{2mV}}^{\infty} dp_z \left(\frac{p_z}{m}\right) \left(\frac{2}{h^3}\frac{1}{e^{(\epsilon-\mu)/\tau}+1}\right) \end{aligned} \tag{89}$$

where the total kinetic energy is $\epsilon = (p_x^2 + p_y^2 + p_z^2)/(2m)$.

Step 3: Applying the High-Barrier Approximation We are given the condition $V - >>$. For any escaping electron, its kinetic energy in the z-direction ϵ_z is greater than V. The total kinetic energy $\epsilon = \epsilon_x + \epsilon_y + \epsilon_z$ must therefore also be greater than V. This implies $\epsilon - > V -$. Since $V - >>$, it follows that $(\epsilon - \mu)/\tau$ is large and positive. Consequently, the exponential term $e^{(\epsilon-\mu)/\tau}$ in the denominator of the Fermi-Dirac distribution is much greater than 1. This allows us to approximate the Fermi-Dirac distribution with the Maxwell-Boltzmann distribution.

$\boxed{\frac{1}{e^{(\epsilon-\mu)/\tau}+1} \approx e^{-(\epsilon-\mu)/\tau} \quad \text{for} \quad (\epsilon-\mu)/\tau \gg 1}$ Applying this approximation to the rate integral in Eq. 89:

$$\begin{aligned} R &\approx \frac{2A}{mh^3} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{\sqrt{2mV}}^{\infty} dp_z\, p_z\, e^{-(\epsilon-\mu)/\tau} \\ &= \frac{2A}{mh^3} e^{\mu/\tau} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{\sqrt{2mV}}^{\infty} dp_z\, p_z\, e^{-(p_x^2+p_y^2+p_z^2)/(2m\tau)} \\ &= \frac{2A}{mh^3} e^{\mu/\tau} \left(\int_{\sqrt{2mV}}^{\infty} p_z e^{-p_z^2/(2m\tau)} dp_z\right) \left(\int_{-\infty}^{\infty} e^{-p_x^2/(2m\tau)} dp_x\right) \left(\int_{-\infty}^{\infty} e^{-p_y^2/(2m\tau)} dp_y\right) \end{aligned} \tag{90}$$

Step 4: Evaluating the Rate Integral The expression for R in Eq. 90 involves three independent integrals. The integrals over p_x and p_y are standard Gaussian integrals.

$\boxed{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}}$ For the p_x and p_y integrals, the parameter $\alpha = 1/(2m\tau)$.

$$\int_{-\infty}^{\infty} e^{-p_x^2/(2m\tau)} dp_x = \sqrt{2\pi m\tau} \tag{91}$$
$$\int_{-\infty}^{\infty} e^{-p_y^2/(2m\tau)} dp_y = \sqrt{2\pi m\tau} \tag{92}$$

The integral over p_z can be solved using a substitution. Let $u = p_z^2/(2m\tau)$. Then $du = (p_z/(m\tau))dp_z$, which gives $p_z dp_z = m\tau du$. The integration limits transform as follows: the lower limit $p_z = \sqrt{2mV}$ becomes $u = (2mV)/(2m\tau) = V/\tau$, and the upper limit $p_z = \infty$ becomes $u = \infty$.

$$\begin{aligned} \int_{\sqrt{2mV}}^{\infty} p_z e^{-p_z^2/(2m\tau)} dp_z &= \int_{V/\tau}^{\infty} m\tau e^{-u} du \\ &= m\tau \left[-e^{-u}\right]_{V/\tau}^{\infty} \\ &= m\tau(0 - (-e^{-V/\tau})) = m\tau e^{-V/\tau} \end{aligned} \tag{93}$$

Substituting the results from Eqs. 91, 92, and 93 into Eq. 90:

$$\begin{aligned} R &\approx \frac{2A}{mh^3} e^{\mu/\tau} \left(m\tau e^{-V/\tau}\right) \left(\sqrt{2\pi m\tau}\right) \left(\sqrt{2\pi m\tau}\right) \\ &= \frac{2A}{mh^3} e^{\mu/\tau} (m\tau e^{-V/\tau})(2\pi m\tau) \\ &= \frac{4\pi A m^2 \tau^2}{mh^3} e^{(\mu - V)/\tau} \\ &= \frac{4\pi A m\tau^2}{h^3} e^{-(V-\mu)/\tau} \end{aligned} \tag{94}$$

Step 5: Calculating the Electric Current The electric current I is the total charge passing through the hole per unit time. Since each escaping electron carries a charge of $-e$, the current is the particle rate R multiplied by $-e$.

$\boxed{I = (-e)R}$ Using the expression for R from Eq. 94, we find the current:

$$I \approx -e\left(\frac{4\pi A m\tau^2}{h^3} e^{-(V-\mu)/\tau}\right) = -\frac{4\pi e A m\tau^2}{h^3} e^{-(V-\mu)/\tau} \tag{95}$$

This result is a form of the Richardson-Dushman equation for thermionic emission, where the quantity $V-$ plays the role of the work function.

Final Answer The estimated electrical current carried by the escaping electrons is:

$$\boxed{I \approx -\frac{4\pi e A m\tau^2}{h^3} e^{-(V-\mu)/\tau}} \tag{96}$$

2.0.6 Problem ID: 1-3

Refined Solution Problem statement: Explanation This problem asks for the radius of curvature, R, of a bimetallic strip after it is heated by a temperature difference ΔT. The strip is initially straight at temperature T.

The given parameters are: - Total thickness of the strip: x. - The strip consists of two metal layers, each with a thickness of $x/2$. - The coefficients of linear thermal expansion for the two metals are α_1 and α_2. - It is specified that $\alpha_2 > \alpha_1$. - The temperature of the strip is uniformly increased by ΔT. - A key geometric assumption is that the thickness is much smaller than the radius of curvature, i.e., $x \ll R$.

The physical process is driven by differential thermal expansion. When heated, both metal layers expand. Since $\alpha_2 > \alpha_1$, the second metal layer will expand more than the first. Because the two layers are bonded together, this differential expansion induces internal stresses, creating a bending moment that causes the strip to curve. The metal with the higher expansion coefficient (α_2) will form the outer side of the curve (convex side), which has a longer arc length. The metal with the lower coefficient (α_1) will form the inner side (concave side).

We establish a coordinate system where the y-axis is perpendicular to the length of the strip, with the origin ($y = 0$) at the interface between the two metals. The metal with coefficient α_1 occupies the region $0 < y \le x/2$, and the metal with coefficient α_2 occupies the region $-x/2 \le y < 0$. Since $\alpha_2 > \alpha_1$, the strip will bend towards the metal with α_1. This means the center of curvature will be on the positive y-axis, and the radius of curvature R will be a positive value.

We make the following standard assumptions for this problem: 1. The Young's modulus, E, and the width of the strip, w, are the same for both metals, as these properties are not specified otherwise. 2. The principles of Euler-Bernoulli beam theory are applicable, which is justified by the condition $x \ll R$.

Step 1: Strain and Stress Distribution We first determine the distribution of strain and stress across the thickness of the strip after it has been heated and has bent. This analysis is based on the principles of beam theory and thermal expansion.

Principles/Original Formulas/Assumptions: The analysis relies on the following principles: 1. **Kinematic Assumption of Bending (Plane Sections Remain Plane)**: This core assumption of Euler-Bernoulli beam theory states that cross-sections that are initially plane and perpendicular to the beam's axis remain plane after bending. This leads to a linear distribution of the total longitudinal strain, $\epsilon(y)$, across the thickness of the beam. $\boxed{\epsilon(y) = \epsilon_0 - \frac{y}{R}}$ Here, y is the distance from the reference plane (the interface), ϵ_0 is the strain at this plane ($y = 0$), and R is the radius of curvature. The sign convention is consistent with our coordinate system where the center of curvature is at a large positive y. 2. **Strain Superposition**: The total strain $\epsilon(y)$ is the linear sum of the mechanical (elastic) strain, $\epsilon_M(y)$, and the thermal strain, $\epsilon_T(y)$. $\boxed{\epsilon(y) = \epsilon_M(y) + \epsilon_T(y)}$ 3. **Linear Thermal Expansion**: The thermal strain is proportional to the temperature change ΔT. $\boxed{\epsilon_T(y) = \alpha(y)\Delta T}$ where $\alpha(y)$ is the coefficient of linear expansion, which depends on the material at position y. 4. **Hooke's Law**: For a linearly elastic material, the longitudinal stress $\sigma(y)$ is proportional to the mechanical strain. $\boxed{\sigma(y) = E\epsilon_M(y)}$ where E is the Young's modulus, assumed to be the same for both metals.

Derivation: By combining these principles, we can derive an expression for the stress distribution $\sigma(y)$. From the strain superposition principle, the mechanical strain is $\epsilon_M(y) = \epsilon(y) - \epsilon_T(y)$. Substituting this into Hooke's Law gives:

$$\begin{aligned} \sigma(y) &= E\left(\epsilon(y) - \epsilon_T(y)\right) \\ &= E\left(\left(\epsilon_0 - \frac{y}{R}\right) - \alpha(y)\Delta T\right) \end{aligned} \tag{97}$$

The coefficient of thermal expansion, $\alpha(y)$, is a piecewise function based on our coordinate system:

$$\alpha(y) = \begin{cases} \alpha_1 & \text{for } 0 < y \leq x/2 \\ \alpha_2 & \text{for } -x/2 \leq y < 0 \end{cases} \tag{98}$$

Step 2: Applying Force Equilibrium Condition Since the bimetallic strip is not subjected to any external axial forces, the net internal force integrated over any cross-section must be zero. This condition of static equilibrium allows us to determine the unknown strain at the interface, ϵ_0.

Principles/Original Formulas/Assumptions: The condition for zero net axial force on a cross-section with area A is: $\boxed{\int_A \sigma(y) dA = 0}$

Derivation: Let the width of the strip be w. The differential area element is $dA = w\,dy$. We integrate the stress expression from eq. 97 over the entire cross-section, from $y = -x/2$ to $y = x/2$.

$$\begin{aligned} \int_{-x/2}^{x/2} \sigma(y) w\,dy &= 0 \\ wE \int_{-x/2}^{x/2} \left(\epsilon_0 - \frac{y}{R} - \alpha(y)\Delta T\right) dy &= 0 \\ \epsilon_0 \int_{-x/2}^{x/2} dy - \frac{1}{R} \int_{-x/2}^{x/2} y\,dy - \Delta T \int_{-x/2}^{x/2} \alpha(y)\,dy &= 0 \end{aligned} \tag{99}$$

We evaluate each integral separately:

$$\int_{-x/2}^{x/2} dy = [y]_{-x/2}^{x/2} = \frac{x}{2} - \left(-\frac{x}{2}\right) = x \tag{100}$$

$$\int_{-x/2}^{x/2} y\,dy = \left[\frac{y^2}{2}\right]_{-x/2}^{x/2} = \frac{(x/2)^2}{2} - \frac{(-x/2)^2}{2} = 0 \tag{101}$$

$$\int_{-x/2}^{x/2} \alpha(y)\,dy = \int_{-x/2}^{0} \alpha_2\,dy + \int_{0}^{x/2} \alpha_1\,dy = \alpha_2[y]_{-x/2}^{0} + \alpha_1[y]_0^{x/2} = \alpha_2\frac{x}{2} + \alpha_1\frac{x}{2} = \frac{x}{2}(\alpha_1 + \alpha_2) \tag{102}$$

Substituting these results into the force equilibrium equation 99:

$$\begin{aligned} \epsilon_0(x) - \frac{1}{R}(0) - \Delta T\left(\frac{x}{2}(\alpha_1 + \alpha_2)\right) &= 0 \\ \epsilon_0 x &= \frac{x}{2}(\alpha_1 + \alpha_2)\Delta T \\ \epsilon_0 &= \frac{(\alpha_1 + \alpha_2)\Delta T}{2} \end{aligned} \tag{103}$$

This expression gives the strain at the interface, which is the average of the free thermal strains of the two metals.

Step 3: Applying Moment Equilibrium Condition Similarly, because there are no external bending moments applied to the strip, the net internal bending moment about any axis within the cross-section must also be zero. This second equilibrium condition allows us to solve for the radius of curvature, R.

Principles/Original Formulas/Assumptions: The condition for zero net internal bending moment about the axis at the interface ($y = 0$) is: $\boxed{\int_A y\,\sigma(y) dA = 0}$

Derivation: We set up the moment integral using the stress distribution from eq. 97 and integrate over the cross-section.

$$\int_{-x/2}^{x/2} y\,\sigma(y) w\,dy = 0$$

$$wE \int_{-x/2}^{x/2} y\left(\epsilon_0 - \frac{y}{R} - \alpha(y)\Delta T\right) dy = 0$$

$$\epsilon_0 \int_{-x/2}^{x/2} y\,dy - \frac{1}{R}\int_{-x/2}^{x/2} y^2\,dy - \Delta T \int_{-x/2}^{x/2} y\alpha(y)\,dy = 0 \tag{104}$$

The first integral is zero, as calculated in the previous step. We evaluate the remaining two integrals:

$$\int_{-x/2}^{x/2} y^2\,dy = \left[\frac{y^3}{3}\right]_{-x/2}^{x/2} = \frac{(x/2)^3}{3} - \frac{(-x/2)^3}{3} = 2\frac{x^3}{24} = \frac{x^3}{12} \tag{105}$$

$$\begin{aligned}
\int_{-x/2}^{x/2} y\alpha(y)\,dy &= \int_{-x/2}^{0} y\alpha_2\,dy + \int_{0}^{x/2} y\alpha_1\,dy \\
&= \alpha_2\left[\frac{y^2}{2}\right]_{-x/2}^{0} + \alpha_1\left[\frac{y^2}{2}\right]_{0}^{x/2} \\
&= \alpha_2\left(0 - \frac{(-x/2)^2}{2}\right) + \alpha_1\left(\frac{(x/2)^2}{2} - 0\right) \\
&= -\alpha_2\frac{x^2}{8} + \alpha_1\frac{x^2}{8} = \frac{x^2}{8}(\alpha_1 - \alpha_2)
\end{aligned} \tag{106}$$

Substituting these results into the moment equilibrium equation 104:

$$\begin{aligned}
\epsilon_0(0) - \frac{1}{R}\left(\frac{x^3}{12}\right) - \Delta T\left(\frac{x^2}{8}(\alpha_1 - \alpha_2)\right) &= 0 \\
-\frac{x^3}{12R} &= \frac{x^2\Delta T}{8}(\alpha_1 - \alpha_2) \\
\frac{x}{12R} &= -\frac{\Delta T}{8}(\alpha_1 - \alpha_2) \\
\frac{x}{12R} &= \frac{\Delta T}{8}(\alpha_2 - \alpha_1) \\
R &= \frac{8x}{12(\alpha_2 - \alpha_1)\Delta T} \\
R &= \frac{2x}{3(\alpha_2 - \alpha_1)\Delta T}
\end{aligned} \tag{107}$$

The resulting expression for R is positive, which is consistent with our initial physical reasoning that the strip bends towards the material with the lower coefficient of thermal expansion (α_1), placing the center of curvature at a positive y value.

Final Answer The radius of curvature of the bimetallic strip, R, when it is heated by a temperature difference ΔT is given by:

$$\boxed{R = \frac{2x}{3(\alpha_2 - \alpha_1)\Delta T}} \tag{108}$$

2.0.7 Problem ID: 6010

Problem statement: Explanation This problem deals with quantum-mechanical scattering in a scenario where both elastic and inelastic processes can occur for a given partial wave. The scattering amplitude for the elastic channel, $f(k, \theta)$, is given by its partial wave expansion:

$$f(k, \theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2ik} P_\ell(\cos\theta)$$

This expression describes the scattering of a particle with wave number k at a scattering angle θ. The process is characterized by several parameters: - ℓ: The angular momentum quantum number, which labels the partial waves. - $\delta_\ell(k)$: A real quantity known as the phase shift for the ℓ-th partial wave. - $\eta_\ell(k)$: A real quantity known as the inelasticity factor for the ℓ-th partial wave, which is constrained by $0 \leq \eta_\ell \leq 1$.

The complex quantity $S_\ell = \eta_\ell e^{2i\delta_\ell}$ is the S-matrix element for the ℓ-th partial wave. The condition $0 \leq \eta_\ell \leq 1$ is equivalent to $|S_\ell| \leq 1$, which is a consequence of probability conservation (the outgoing flux cannot exceed the incoming flux). If $\eta_\ell = 1$, scattering in the ℓ-th partial wave is purely elastic. If $\eta_\ell < 1$, there is absorption or loss of flux to inelastic channels.

The objective is to find the lower and upper bounds for the elastic cross section of a specific partial wave, $\sigma^{(\ell)}_{\text{elastic}}$, as a function of the corresponding inelastic cross section for that same partial wave, $\sigma^{(\ell)}_{\text{inelastic}}$.

Step 1: Formulate the Partial Wave Cross Sections The partial wave cross sections can be expressed in terms of the S-matrix element S_ℓ. These are standard results in scattering theory, derivable from the partial wave expansion of the scattering amplitude and the optical theorem. We will use these formulas to express the cross sections in terms of the parameters η_ℓ and δ_ℓ.

Principles/Original Formulas/Assumptions The S-matrix element for the ℓ-th partial wave is given by: $\boxed{S_\ell = \eta_\ell e^{2i\delta_\ell}}$ The partial elastic cross section, $\sigma^{(\ell)}_{\text{elastic}}$, is given by: $\boxed{\sigma^{(\ell)}_{\text{elastic}} = \frac{\pi}{k^2}(2\ell + 1)|1 - S_\ell|^2}$ The partial inelastic cross section, $\sigma^{(\ell)}_{\text{inelastic}}$, is given by: $\boxed{\sigma^{(\ell)}_{\text{inelastic}} = \frac{\pi}{k^2}(2\ell + 1)(1 - |S_\ell|^2)}$

Derivation First, we express the inelastic cross section in terms of η_ℓ. We calculate $|S_\ell|^2$:

$$|S_\ell|^2 = |\eta_\ell e^{2i\delta_\ell}|^2 = \eta_\ell^2 |e^{2i\delta_\ell}|^2 = \eta_\ell^2 \tag{109}$$

Substituting this into the formula for $\sigma^{(\ell)}_{\text{inelastic}}$ gives:

$$\sigma^{(\ell)}_{\text{inelastic}} = \frac{\pi}{k^2}(2\ell + 1)(1 - \eta_\ell^2) \tag{110}$$

Next, we express the elastic cross section in terms of η_ℓ and δ_ℓ. We calculate the term $|1 - S_\ell|^2$:

$$\begin{aligned}
|1 - S_\ell|^2 &= |1 - \eta_\ell e^{2i\delta_\ell}|^2 \\
&= |1 - \eta_\ell(\cos(2\delta_\ell) + i\sin(2\delta_\ell))|^2 \\
&= |(1 - \eta_\ell \cos(2\delta_\ell)) - i(\eta_\ell \sin(2\delta_\ell))|^2 \\
&= (1 - \eta_\ell \cos(2\delta_\ell))^2 + (-\eta_\ell \sin(2\delta_\ell))^2 \\
&= 1 - 2\eta_\ell \cos(2\delta_\ell) + \eta_\ell^2 \cos^2(2\delta_\ell) + \eta_\ell^2 \sin^2(2\delta_\ell) \\
&= 1 - 2\eta_\ell \cos(2\delta_\ell) + \eta_\ell^2(\cos^2(2\delta_\ell) + \sin^2(2\delta_\ell)) \\
&= 1 + \eta_\ell^2 - 2\eta_\ell \cos(2\delta_\ell) \qquad (111)
\end{aligned}$$

Substituting this result into the formula for $\sigma^{(\ell)}_{\text{elastic}}$ yields:

$$\sigma^{(\ell)}_{\text{elastic}} = \frac{\pi}{k^2}(2\ell + 1)(1 + \eta_\ell^2 - 2\eta_\ell \cos(2\delta_\ell)) \qquad (112)$$

Step 2: Express Elastic Cross Section in Terms of Inelastic Cross Section Our goal is to find the bounds on $\sigma^{(\ell)}_{\text{elastic}}$ for a fixed value of $\sigma^{(\ell)}_{\text{inelastic}}$. To do this, we first express $\sigma^{(\ell)}_{\text{elastic}}$ as a function of $\sigma^{(\ell)}_{\text{inelastic}}$ and the remaining free parameter, δ_ℓ.

Principles/Original Formulas/Assumptions We will use the expressions for the cross sections derived in the previous step. No new principles are needed here.

Derivation Let's introduce a constant geometric factor $C_\ell = \frac{\pi}{k^2}(2\ell+1)$. The cross section formulas from Step 1 become:

$$\sigma^{(\ell)}_{\text{inelastic}} = C_\ell(1 - \eta_\ell^2) \qquad (113)$$

$$\sigma^{(\ell)}_{\text{elastic}} = C_\ell(1 + \eta_\ell^2 - 2\eta_\ell \cos(2\delta_\ell)) \qquad (114)$$

For a given $\sigma^{(\ell)}_{\text{inelastic}}$, the value of η_ℓ is fixed. From eq. 113, we can express η_ℓ^2 and η_ℓ in terms of $\sigma^{(\ell)}_{\text{inelastic}}$:

$$\eta_\ell^2 = 1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} \qquad (115)$$

$$\eta_\ell = \sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \quad (\text{since } 0 \le \eta_\ell \le 1) \qquad (116)$$

Note that for η_ℓ to be a real number, we must have $1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} \ge 0$, which implies $\sigma^{(\ell)}_{\text{inelastic}} \le C_\ell$. Now, we substitute these expressions for η_ℓ^2 and η_ℓ back into the equation for $\sigma^{(\ell)}_{\text{elastic}}$ (eq. 114):

$$\begin{aligned}
\sigma^{(\ell)}_{\text{elastic}} &= C_\ell\left(1 + \left(1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}\right) - 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\cos(2\delta_\ell)\right) \\
&= C_\ell\left(2 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} - 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\cos(2\delta_\ell)\right) \\
&= 2C_\ell - \sigma^{(\ell)}_{\text{inelastic}} - 2C_\ell\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\cos(2\delta_\ell) \qquad (117)
\end{aligned}$$

This equation expresses $\sigma^{(\ell)}_{\text{elastic}}$ in terms of the given $\sigma^{(\ell)}_{\text{inelastic}}$ and the variable phase shift δ_ℓ.

Step 3: Determine the Bounds on the Elastic Cross Section The expression for $\sigma^{(\ell)}_{\text{elastic}}$ in eq. 117 depends on the phase shift δ_ℓ only through the term $\cos(2\delta_\ell)$. By considering the full range of possible values for $\cos(2\delta_\ell)$, we can find the minimum and maximum possible values for $\sigma^{(\ell)}_{\text{elastic}}$.

Principles/Original Formulas/Assumptions Since the phase shift δ_ℓ is a real quantity, the cosine of $2\delta_\ell$ is bounded: $\boxed{-1 \leq \cos(2\delta_\ell) \leq 1}$

Derivation The expression for the elastic cross section is:

$$\sigma^{(\ell)}_{\text{elastic}} = 2C_\ell - \sigma^{(\ell)}_{\text{inelastic}} - 2C_\ell\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\cos(2\delta_\ell)$$

The term that varies is $-2C_\ell\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\cos(2\delta_\ell)$. Since $C_\ell > 0$ and the square root term is non-negative, the entire expression for $\sigma^{(\ell)}_{\text{elastic}}$ is maximized when $\cos(2\delta_\ell)$ is minimized, and vice-versa.

The upper bound, $\sigma^{(\ell)}_{\text{elastic, max}}$, occurs when $\cos(2\delta_\ell) = -1$:

$$\sigma^{(\ell)}_{\text{elastic, max}} = 2C_\ell - \sigma^{(\ell)}_{\text{inelastic}} + 2C_\ell\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \tag{10}$$

The lower bound, $\sigma^{(\ell)}_{\text{elastic, min}}$, occurs when $\cos(2\delta_\ell) = 1$:

$$\sigma^{(\ell)}_{\text{elastic, min}} = 2C_\ell - \sigma^{(\ell)}_{\text{inelastic}} - 2C_\ell\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \tag{11}$$

Step 4: Simplify the Bound Expressions The expressions for the upper and lower bounds can be simplified by recognizing them as perfect squares.

Principles/Original Formulas/Assumptions We utilize the algebraic identity for perfect squares: $\boxed{(a \pm b)^2 = a^2 \pm 2ab + b^2}$

Derivation Let's simplify the lower bound expression from eq. 11. We can factor out C_ℓ:

$$\begin{aligned}
\sigma^{(\ell)}_{\text{elastic, min}} &= C_\ell\left(2 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} - 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\right) \\
&= C_\ell\left(1 + \left(1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}\right) - 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\right) \\
&= C_\ell\left(1^2 - 2(1)\left(\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\right) + \left(\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\right)^2\right) \\
&= C_\ell\left(1 - \sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}}\right)^2 \qquad (12)
\end{aligned}$$

Similarly, for the upper bound expression from eq. 10:

$$\begin{aligned}
\sigma^{(\ell)}_{\text{elastic, max}} &= C_\ell \left(2 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} + 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right) \\
&= C_\ell \left(1 + \left(1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell} \right) + 2\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right) \\
&= C_\ell \left(1^2 + 2(1) \left(\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right) + \left(\sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right)^2 \right) \\
&= C_\ell \left(1 + \sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right)^2 \qquad (13)
\end{aligned}$$

These simplified expressions give the final bounds for the elastic cross section.

Final Answer For a given partial wave ℓ and a given inelastic cross section $\sigma^{(\ell)}_{\text{inelastic}}$, the elastic cross section $\sigma^{(\ell)}_{\text{elastic}}$ is bounded by:

$$\boxed{C_\ell \left(1 - \sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right)^2 \leq \sigma^{(\ell)}_{\text{elastic}} \leq C_\ell \left(1 + \sqrt{1 - \frac{\sigma^{(\ell)}_{\text{inelastic}}}{C_\ell}} \right)^2} \qquad (118)$$

where C_ℓ is a geometric factor defined as:

$$C_\ell = \frac{\pi}{k^2}(2\ell + 1) \qquad (119)$$

The constant C_ℓ has a physical significance. From the relation $\sigma^{(\ell)}_{\text{inelastic}} = C_\ell(1 - \eta_\ell^2)$ and the fact that $\eta_\ell^2 \geq 0$, it is clear that $\sigma^{(\ell)}_{\text{inelastic}} \leq C_\ell$. Thus, C_ℓ represents the maximum possible inelastic cross section for the ℓ-th partial wave, which occurs when $\eta_\ell = 0$ (total absorption).