A Auxiliary Proofs

A.1 Proof of Theorem 1

Proof. Using Fenchel-Rockafellar's duality theorem, the dual of (7) can be written as

$$\max_{f,g,\gamma\geq 0} \int_{\mathcal{X}} f(x)d\alpha(x) + g^\top \beta - \sum_{j,k,j\neq k} \gamma_{jk}\lambda_j \tag{22}$$
$$s.t. \quad \left(1+\sum_{k\neq j}\gamma_{jk}\right)c(x,j) - \sum_{k\neq j}\gamma_{ky}c(x,k)\frac{\beta_k}{\beta_j}$$
$$- f(x) - g_j \geq 0 \quad \forall x\in\mathcal{X}, y_j\in\mathcal{Y}$$

Fixing $g\in\mathbb{R}^n$ and $\gamma\in\mathbb{R}^{n(n-1)}$, we can check using first order conditions that the optimal $f(x)$ has the closed form expression:

$$\min_{j\in[n]} \bar{g}_{\gamma,c}(x,y_j) := \left(1+\sum_{k\neq j}\gamma_{jk}\right)c(x,j) - \sum_{k\neq j}\gamma_{kj}c(x,k)\frac{\beta_k}{\beta_j} - g_j$$

Using this, the infinite dimensional optimization problem in (22) can be transformed to a finite dimensional optimization problem:

$$\max_{g,\gamma\geq 0} \mathcal{E}(g,\gamma) := \int_{\mathcal{X}} \min_{j\in[n]} \bar{g}_{\gamma,c}(x,y_j)\, d\alpha(x) + g^\top\beta - \sum_{j,k,j\neq k}\gamma_{jk}\lambda_j \tag{23}$$

Alternatively, we can adapt the Laguerre cell notation in (2) to (23):

$$\mathcal{E}(g,\gamma) = \sum_{i\in[n]}\int_{\mathbb{L}_{y_i}(g,\gamma)} \bar{g}_{\gamma,c}(x,y_i)d\alpha(x) + g^\top\beta - \sum_{j,k,j\neq k}\gamma_{jk}\lambda_j$$

where $\mathbb{L}_{y_i}(g,\gamma) = \left\{x\in\mathcal{X} : y_i = \arg\min_{y_j} \bar{g}_{\gamma,c}(x,y_j)\right\}$. □

A.2 Proof of Theorem 2

Proof. We prove the result via uniform convergence:

$$\begin{aligned}
&\mathcal{E}(g^*,\gamma^*) - \mathcal{E}(\hat{g}_S,\hat{\gamma}_S)\\
=&\mathcal{E}(g^*,\gamma^*) - \mathcal{E}_S(\hat{g}_S,\hat{\gamma}_S) + \mathcal{E}_S(\hat{g}_S,\hat{\gamma}_S) - \mathcal{E}(\hat{g}_S,\hat{\gamma}_S)\\
\leq&\mathcal{E}(g^*,\gamma^*) - \mathcal{E}_S(g^*,\gamma^*) + \mathcal{E}_S(\hat{g}_S,\hat{\gamma}_S) - \mathcal{E}(\hat{g}_S,\hat{\gamma}_S)\\
\leq&\sup_{g,\gamma}\left(\mathcal{E}(g,\gamma) - \mathcal{E}_S(g,\gamma)\right) + \sup_{g,\gamma}\left(\mathcal{E}_S(g,\gamma) - \mathcal{E}(g,\gamma)\right)\\
\leq&2\sup_{g,\gamma}|\mathcal{E}(g,\gamma) - \mathcal{E}_S(g,\gamma)|
\end{aligned} \tag{24}$$

Clearly, it suffices to show that $\mathcal{E}_S(\cdot)$ converges uniformly to $\mathcal{E}(\cdot)$. For a given g,γ, the dual objective function and its' empirical version can be written as

$$\mathcal{E}(g,\gamma) = \mathbb{E}_\alpha[f(X)], \quad \mathcal{E}_S(g,\gamma) = \frac{1}{m}\sum_{t=1}^m f(X^t).$$

Then we can rewrite the supremum in (24) as:

$$\sup_{g,\gamma}|\mathcal{E}(g,\gamma) - \mathcal{E}_S(g,\gamma)| = \sup_{f\in F}\left|\mathbb{E}_\alpha[f(X)] - \frac{1}{m}\sum_{X\in S} f(X)\right| \tag{25}$$

Since $|f(X)| \leq (R\bar{x} + R)$ for all $f \in F, X \in \mathcal{X}$, it follows from Theorem 26.5 in [26] that with probability $1 - \delta$,

$$\sup_{f \in F} \mathbb{E}_{\alpha}[f(X)] - \frac{1}{m} \sum_{X \in S} f(X) \leq 2\mathbb{E}_S \left[\text{Rad}_m(F \circ S)\right] + (R\bar{x} + R)\sqrt{\frac{2\log(2/\delta)}{m}} \tag{26}$$

and the same also holds by replacing F with $-F$. Here

$$\text{Rad}_m(F \circ S) := \mathbb{E}_{\sigma}\left[\frac{1}{m} \sup_f \sum_{j=1}^{m} \sigma_j f(X_j)\right]$$

is the standard definition of Rademacher complexity of the set $F \circ S$. Since σ_i are $i.i.d.$ Rademacher random variables, it is easy to see that $\text{Rad}_m(F \circ S) = \text{Rad}_m(-F \circ S)$. Therefore we can use a union bound to obtain that with probability $1 - \delta$,

$$\sup_{f \in F} \left| \mathbb{E}_{\alpha}[f(X)] - \frac{1}{m} \sum_{X \in S} f(X) \right| \leq 2\mathbb{E}_S \left[\text{Rad}_m(F \circ S)\right] + (R\bar{x} + R)\sqrt{\frac{2\log(4/\delta)}{m}} \tag{27}$$

It remains to bound the Rademacher complexity of the $F \circ S$. To do so, we use tools from learning theory, and give the following bound on the fat-shattering dimension ([8]) of the hypothesis class F.

Lemma 1. *Under Assumption 1, F has ζ-fat-shattering dimension of at most $\frac{c_0(R\bar{x}+R)^2}{\zeta^2} n \log(n)$, where c_0 is some universal constant.*

The proof of Lemma 1 can be found in the Appendix. The above bound on the fat-shattering dimension can be used to bound the covering number (see Definition 27.1 of [26]) of $F \circ S$. Theorem 1 from [22] states that

$$\mathcal{N}(\delta, F, ||\cdot||_2) \leq \left(\frac{2B}{\delta}\right)^{c_1 \text{fat}_{c_2\delta}(F)} \tag{28}$$

where B is a uniform bound on the absolute value of any $f \in F$. Let $B = (R\bar{x} + R)$, we have that

$$\begin{aligned}
&\text{Rad}_m(F \circ S) \\
\leq &\inf_{\delta' > 0} \left\{ 4\delta' + 12 \int_{\delta'}^{B} \sqrt{\frac{\log \mathcal{N}(\delta, F, ||\cdot||_2)}{m}} d\delta \right\} \\
\leq &\inf_{\delta' > 0} \left\{ 4\delta' + 12 \frac{\sqrt{c_1 c_0}}{c_2} B \sqrt{\frac{n \log n}{m}} \int_{\delta'}^{B} \sqrt{\log\left(\frac{2B}{\delta}\right)} d\delta \right\} \\
= &c' \sqrt{\frac{n \log n (\log m)^3}{m}}
\end{aligned}$$

Where we used Dudley's chaining integral [28, 16], Lemma 1 and (28), and setting $\delta' = \frac{1}{\sqrt{m}}$ respectively. Plugging the above back to (27) and (24), we see that with probability $1 - \delta$,

$$\mathcal{E}(g^*, \gamma^*) - \mathcal{E}(\hat{g}_S, \hat{\gamma}_S) \leq c' \left(\sqrt{\frac{n \log n (\log m)^3}{m}} + \sqrt{\frac{1 \log \frac{1}{\delta}}{m}} \right).$$

Conversely, ignoring the log terms, m needs to be at most on the order of $\tilde{O}\left(\frac{n}{\epsilon^2}\right)$ in order for $\mathcal{E}(g^*, \gamma^*) - \mathcal{E}(\hat{g}_S, \hat{\gamma}_S)$ to be bounded by ϵ with high probability.

□

Proof of Lemma 1

Proof. Theorem 3 in [20] shows that $\text{fat}_{\zeta}(F_{min}) \leq \frac{c_0(R\bar{x}+R)^2}{\zeta^2} n \log n$. Since the shattering dimension is monotone in the size of the set, we are done.

□

A.3 Experimental Setup

For the artificial data, the value utility vectors are generated from $X = [1, 0.7] - Z \begin{bmatrix} 0.2, & 0 \\ 0.8, & 0.4 \end{bmatrix}$ where $Z \sim Unif(0,1) \times Unif(0,1)$. For finding the optimal allocation policy on the artificial data, we used Algorithm 1 with $T = 2 \cdot 10^5$. For simulator data, we used $T = 2.5 \cdot 10^6$. To generate Figure 2 we sampled 6000 points from the distribution and plotted them, colored by the allocation. For Figure 4, for each m we ran 16 trials, sampling a different set of m data points as our training data per trial. All experiments are run on a 2019, 6-core Macbook Pro laptop. The simulator code is open sourced by [21] at `https://github.com/duncanmcelfresh/blood-matching-simulations`, and also included in the supplementary material.