Acknowledgments

We thank Leo Dorst and Steven de Keninck for insightful discussions about Clifford (geometric) algebras and their applications as well as pointing us to the relevant literature. Further, we thank Robert-Jan Schlimbach and Jayesh Gupta for their help with computational infrastructure and scaling up the experiments. This work used the Dutch national e-infrastructure with the support of the SURF Cooperative using grant no. EINF-5757 as well as the DAS-5 and DAS-6 clusters [BEdL+16].

Broader Impact Statement

The advantages of more effective equivariant graph neural networks could be vast, particularly in the physical sciences. Improved equivariant neural networks could revolutionize fields like molecular biology, astrophysics, and materials science by accurately predicting system behaviors under various transformations. They could further enable more precise simulations of physical systems. Such new knowledge could drive forward scientific discovery, enabling innovations in healthcare, materials engineering, and our understanding of the universe. However, a significant drawback of relying heavily on advanced equivariant neural networks in physical sciences is the potential for unchecked errors or biases to propagate when these systems are not thoroughly cross-validated using various methods. This could potentially lead to flawed scientific theories or dangerous real-world applications.

Reproducibility Statement

We have released the code to reproduce the experimental results, including hyperparameters, network architectures, and so on, at https://github.com/DavidRuhe/clifford-group-equivariant-neural-networks.

Computational Resources

The volumetric quantities and regression experiments were carried out on 1×11 GB *NVIDIA GeForce GTX 1080 Ti* and 1×11 GB *NVIDIA GeForce GTX 2080 Ti* instances. The n-body experiment ran on 1×24 GB *NVIDIA GeForce RTX 3090* and 1×24 GB *NVIDIA RTX A5000* nodes. Finally, the top tagging experiment was conducted on 4×40 GB *NVIDIA Tesla A100 Ampere* instances.

Licenses

We would like to thank the scientific software development community, without whom this work would not be possible.

This work made use of `Python` (PSF License Agreement), `NumPy` [VDWCV11] (BSD-3 Clause), `PyTorch` [PGM+19] (BSD-3 Clause), `CUDA` (proprietary license), *Weights and Biases* [Bie20] (MIT), *Scikit-Learn* [PVG+11] (BSD-3 Clause), *Seaborn* [Was21] (BSD-3 Clause), *Matplotlib* [Hun07] (PSF), [VGO+20] (BSD-3).

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Contents

A Glossary

Notation	Meaning
$\mathrm{O(n)}$	The orthogonal group acting on an n-dimensional vector space.
$\mathrm{SO}(n)$	The special orthogonal group acting on an n-dimensional vector space.
$\mathrm{E}(n)$	The Euclidean group acting on an n-dimensional vector space.
$\mathrm{SE}(n)$	The special Euclidean group acting on an n-dimensional vector space.
$\mathrm{GL}(n)$	The general linear group acting on an n-dimensional vector space.
$\mathrm{O}(V, \mathfrak{q})$	The orthogonal group of vector space V with respect to a quadratic form $\mathfrak{q}$.
$\mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$	The orthogonal group of vector space V with respect to a quadratic form $\mathfrak{q}$ that acts as the identity on $\mathcal{R}$.
V	A vector space.
$\mathcal{R}$	The radical vector subspace of V such that for any $f \in \mathcal{R}$, $\mathfrak{b}(f, v) = 0$ for all $v \in V$.
$\mathbb{F}$	A field.
$\mathbb{R}$	The real numbers.
$\mathbb{N}$	The natural numbers.
$\mathbb{Z}$	The integers.
$[n]$	The set of integers $\{1, \ldots, n\}$.
$\mathfrak{q}$	A quadratic form, $\mathfrak{q} : V \to \mathbb{F}$.
$\mathfrak{b}$	A bilinear form, $\mathfrak{b} : V \times V \to \mathbb{F}$.
$\mathrm{Cl}(V, \mathfrak{q})$	The Clifford algebra over a vector space V with quadratic form $\mathfrak{q}$.
$\mathrm{Cl}^{\times}(V, \mathfrak{q})$	The group of invertible Clifford algebra elements.
$\mathrm{Cl}^{[\times]}(V, \mathfrak{q})$	The group of invertible parity-homogeneous Clifford algebra elements. $\mathrm{Cl}^{[\times]}(V, \mathfrak{q}) := \{w \in \mathrm{Cl}^{\times}(V, \mathfrak{q}) \mid \eta(w) \in \{\pm 1\}\}$.
$\mathrm{Cl}^{[0]}(V, \mathfrak{q})$	The even subalgebra of the Clifford algebra. $\mathrm{Cl}^{[0]}(V, \mathfrak{q}) := \bigoplus_{m \text{ even}}^{n} \mathrm{Cl}^{(m)}(V, \mathfrak{q})$.
$\mathrm{Cl}^{[1]}(V, \mathfrak{q})$	The odd part of the Clifford algebra. $\mathrm{Cl}^{[1]}(V, \mathfrak{q}) := \bigoplus_{m \text{ odd}}^{n} \mathrm{Cl}^{(m)}(V, \mathfrak{q})$.
$\mathrm{Cl}^{(m)}(V, \mathfrak{q})$	The grade-m subspace of the Clifford algebra.
$(\_)^{(m)}$	Grade projection, $(\_)^{(m)} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}^{(m)}(V, \mathfrak{q})$.
ζ	Projection onto the zero grade, $\zeta : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}^{(0)}(V, \mathfrak{q})$.
e_i	A basis vector, $e_i \in V$.
f_i	A basis vector of $\mathcal{R}$.
e_A	A Clifford basis element (product of basis vectors) with $e_A \in \mathrm{Cl}(V, \mathfrak{q})$, $A \subseteq \{1, \ldots, n\}$.
$\bar{\mathfrak{b}}$	The extended bilinear form on the Clifford algebra, $\bar{\mathfrak{b}} : \mathrm{Cl}(V, \mathfrak{q}) \times \mathrm{Cl}(V, \mathfrak{q}) \to \mathbb{F}$.
$\bar{\mathfrak{q}}$	The extended quadratic form on the Clifford algebra, $\bar{\mathfrak{q}} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathbb{F}$.
α	Clifford main involution $\alpha : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q})$.
β	Main Clifford anti-involution, also known as *reversion*, $\beta : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q})$.
γ	Clifford conjugation $\gamma : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q})$.
η	Coboundary of α. For $w \in \mathrm{Cl}^{\times}(V, \mathfrak{q})$, $\eta(w) \in \{\pm 1\}$ if and only if w is parity-homogeneous.
$\rho(w)$	The (adjusted) twisted conjugation, used as the action of the Clifford group. $\rho(w) : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q})$, $w \in \mathrm{Cl}^{\times}(V, \mathfrak{q})$.
$\Gamma(V, \mathfrak{q})$	The Clifford group of $\mathrm{Cl}(V, \mathfrak{q})$.

Notation	Meaning
$\Gamma^{[0]}(V, \mathfrak{q})$	The special Clifford group. It excludes orientation-reversing (odd) elements. $\Gamma^{[0]}(V, \mathfrak{q}) := \Gamma(V, \mathfrak{q}) \cap \mathrm{Cl}^{[0]}(V, \mathfrak{q})$.
$\bigwedge(\mathcal{R})$	The radical subalgebra of $\mathrm{Cl}(V, \mathfrak{q})$. I.e., those elements that have zero $\bar{\mathfrak{q}}$. It is equal to the exterior algebra of $\mathcal{R}$.
$\bigwedge^{[i]}(\mathcal{R})$	The even or odd ($i \in \{0, 1\}$) subalgebra of $\bigwedge(\mathcal{R})$. $\bigwedge^{[i]}(\mathcal{R}) := \bigwedge(\mathcal{R}) \cap \mathrm{Cl}^{[i]}(V, \mathfrak{q})$.
$\bigwedge^{(\geq 1)}(\mathcal{R})$	The subalgebra of $\bigwedge(\mathcal{R})$ with grade greater than or equal to one. $\bigwedge^{(\geq 1)}(\mathcal{R}) := \mathrm{span}\{f_1 \dots f_k \mid k \geq 1, f_l \in \mathcal{R}\}$.
$\bigwedge^{\times}(\mathcal{R})$	The group of invertible elements of $\bigwedge(\mathcal{R})$. $\bigwedge^{\times}(\mathcal{R}) = \mathbb{F}^{\times} + \bigwedge^{(\geq 1)}(\mathcal{R})$.
$\bigwedge^{[\times]}(\mathcal{R})$	The group of invertible elements of $\bigwedge(\mathcal{R})$ with even grades. $\bigwedge^{[\times]}(\mathcal{R}) := \mathbb{F}^{\times} + \mathrm{span}\{f_1 \dots f_k \mid k \geq 2 \text{ even}, f_l \in \mathcal{R}\}$.
$\bigwedge^{*}(\mathcal{R})$	Same as $\bigwedge^{\times}(\mathcal{R})$, but with $\mathbb{F}^{\times}$ set to 1.
$\bigwedge^{[*]}(\mathcal{R})$	Same as $\bigwedge^{[\times]}(\mathcal{R})$, but with $\mathbb{F}^{\times}$ set to 1.
SN	Spinor norm, $\mathrm{SN} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathbb{F}$.
CN	Clifford norm, $\mathrm{CN} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathbb{F}$.

B Supplementary Material: Introduction

Existing literature on Clifford algebra uses varying notations, conventions, and focuses on several different applications. We gathered previous definitions and included independently derived results to achieve the desired outcomes for this work and to provide proofs for the most general cases, including, e.g., potential degeneracy of the metric. As such, this appendix acts as a comprehensive resource on quadratic spaces, orthogonal groups, Clifford algebras, their constructions, and specific groups represented within Clifford algebras.

We start in Appendix C with a primer to quadratic spaces and the orthogonal group. In particular, we specify how the orthogonal group is defined on a quadratic space, and investigate its action. We pay special attention to the case of spaces with degenerate quadratic forms, and what we call "radical-preserving" orthogonal automorphisms. The section concludes with the presentation of the Cartan-Dieudonné theorem.

In Appendix D, we introduce the definition of the Clifford algebra as a quotient of the tensor algebra. We investigate several key properties of the Clifford algebra. Then, we introduce its parity grading, which in turn is used to prove that the dimension of the algebra is 2^n, where n is the dimension of the vector space on which it is defined. This allows us to construct an algebra basis. Subsequently, we extend the quadratic and bilinear forms of the vector space to their Clifford algebra counterparts. This then leads to the construction of an orthogonal basis for the algebra. Furthermore, the multivector grading of the algebra is shown to be basis independent, leading to the orthogonal sum decomposition into the usual subspaces commonly referred to as scalars, vectors, bivectors, and so on. Finally, we investigate some additional properties of the algebra, such as its center, its radical subalgebra, and its twisted center.

In Appendix E we introduce, motivate, and adjust the twisted conjugation map. We show that it comprises an algebra automorphism, thereby respecting the algebra's vector space and also its multiplicative properties. Moreover, it can serve as a representation of the Clifford algebra's group of invertible elements. We then identify what we call the Clifford group, whose action under the twisted conjugation respects also the multivector decomposition. We investigate properties of the Clifford group and its action. Specifically, we show that its action yields a radical-preserving orthogonal automorphism. Moreover, it acts orthogonally on each individual subspace.

Finally, after introducing the Spinor and Clifford norm, the section concludes with the pursuit of a general definition for the Pin and Spin group, which are used in practice more often than the Clifford group. This, however, turns out to be somewhat problematic when generalizing to fields beyond $\mathbb{R}$. We motivate several choices, and outline their (dis)advantages. For $\mathbb{R}$, we finally settle on definitions that are compatible with existing literature.

C Quadratic Vector Spaces and the Orthogonal Group

We provide a short introduction to quadratic spaces and define the orthogonal group of a (non-definite) quadratic space. We use these definitions in our analysis of how the Clifford group relates to the orthogonal group.

We will always denote with $\mathbb{F}$ a *field* of a characteristic different from 2, $\mathrm{char}(\mathbb{F}) \neq 2$. Sometimes we will specialize to the real numbers $\mathbb{F} = \mathbb{R}$. Let V be a *vector space* over $\mathbb{F}$ of finite dimension $\dim_{\mathbb{F}} V = n$. We will follow [Ser12].

Definition C.1 (Quadratic forms and quadratic vector spaces)**.** *A map* $\mathfrak{q} : V \to \mathbb{F}$ *will be called a* quadratic form *of* V *if for all* $c \in \mathbb{F}$ *and* $v \in V$*:*

$$\mathfrak{q}(c \cdot v) = c^2 \cdot \mathfrak{q}(v), \tag{17}$$

and if:

$$\mathfrak{b}(v_1, v_2) := \frac{1}{2}\left(\mathfrak{q}(v_1 + v_2) - \mathfrak{q}(v_1) - \mathfrak{q}(v_2)\right), \tag{18}$$

is a bilinear form over $\mathbb{F}$ *in* $v_1, v_2 \in V$*, i.e., it is separately* $\mathbb{F}$*-linear in each of the arguments* v_1 *and* v_2 *when the other one is fixed.*

The tuple $(V, \mathfrak{q})$ *will then be called a* quadratic (vector) space.

Remark C.2.

1. *Note that we explicitely do not make assumptions about the non-degeneracy of* $\mathfrak{b}$*. Even the extreme case with constant* $\mathfrak{q} = 0$ *is allowed and of interest.*
2. *Further note, that* $\mathfrak{b}$ *will automatically be a symmetric bilinear form.*

Definition C.3 (The radical subspace)**.** *Now consider the quadratic space* $(V, \mathfrak{q})$*. We then call the subspace:*

$$\mathcal{R} := \{f \in V \,|\, \forall v \in V.\, \mathfrak{b}(f, v) = 0\}, \tag{19}$$

the radical subspace *of* $(V, \mathfrak{q})$.

Remark C.4.

1. *The radical subspace of* $(V, \mathfrak{q})$ *is the biggest subspace of* V *where* $\mathfrak{q}$ *is degenerate. Note that this space is orthogonal to all other subspaces of* V*.*
2. *If* W *is any complementary subspace of* $\mathcal{R}$ *in* V*, so that* $V = \mathcal{R} \oplus^{\perp} W$*, then* $\mathfrak{q}$ *restricted to* W *is non-degenerate.*

Definition C.5 (Orthogonal basis)**.** *A basis* $e_1, \ldots, e_n$ *of* V *is called* orthogonal basis *of* V *if for all* $i \neq j$ *we have:*

$$\mathfrak{b}(e_i, e_j) = 0. \tag{20}$$

It is called an orthonormal basis *if, in addition,* $\mathfrak{q}(e_i) \in \{-1, 0, +1\}$ *for all* $i = 1, \ldots, n$.

Remark C.6. *Note that every quadratic space* $(V, \mathfrak{q})$ *has an orthogonal basis by [Ser12] p. 30 Thm. 1, but not necessarily a orthonormal basis. However, if* $\mathbb{F} = \mathbb{R}$ *then* $(V, \mathfrak{q})$ *has an orthonormal basis by Sylvester's law of inertia, see [Syl52].*

Definition C.7 (The orthogonal group)**.** *For a quadratic space* $(V, \mathfrak{q})$ *we define the* orthogonal group *of* $(V, \mathfrak{q})$ *as follows:*

$$\mathrm{O}(\mathfrak{q}) := \mathrm{O}(V, \mathfrak{q}) := \left\{\Phi : V \to V \,\middle|\, \Phi \ \mathbb{F}\text{-linear automorphism}^{8},\ \text{s.t.}\ \forall v \in V.\, \mathfrak{q}(\Phi(v)) = \mathfrak{q}(v)\right\}. \tag{21}$$

If $(V, \mathfrak{q}) = \mathbb{R}^{(p,q,r)}$*, i.e. if* $\mathbb{F} = \mathbb{R}$ *and* $V = \mathbb{R}^n$ *and* $\mathfrak{q}$ *has the signature* (p, q, r) *with* $p + q + r = n$ *then we define the* group of orthogonal matrices of signature (p, q, r) *as follows:*

$$\mathrm{O}(p, q, r) := \left\{O \in \mathrm{GL}(n) \,\middle|\, O^{\top} \Delta_{(p,q,r)} O = \Delta_{(p,q,r)}\right\}, \tag{22}$$

where we used the $(n \times n)$*-diagonal signature matrix:*

$$\Delta_{(p,q,r)} := \mathrm{diag}(\underbrace{+1, \ldots, +1}_{p\text{-times}}, \underbrace{-1, \ldots, -1}_{q\text{-times}}, \underbrace{0, \ldots, 0}_{r\text{-times}}). \tag{23}$$

Theorem C.8 (See [YCo20]). *Let $(V, \mathfrak{q})$ be a finite dimensional quadratic space over a field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) \neq 2$. Let $\mathcal{R} \subseteq V$ be the radical subspace, $r := \dim \mathcal{R}$, and, $W \subseteq V$ a complementary subspace, $m := \dim W$. Then we get an isomorphism:*

$$\mathrm{O}(V, \mathfrak{q}) \cong \begin{pmatrix} \mathrm{O}(W, \mathfrak{q}|_W) & 0_{m\times r} \\ \mathrm{M}(r,m) & \mathrm{GL}(r) \end{pmatrix} = \left\{ \begin{pmatrix} O & 0_{m\times r} \\ M & G \end{pmatrix} \middle| O \in \mathrm{O}(W, \mathfrak{q}|_W), M \in \mathrm{M}(r,m), G \in \mathrm{GL}(r) \right\}, \tag{24}$$

where $\mathrm{M}(r,m) := \mathbb{F}^{r\times m}$ is the additive group of all $(r \times m)$-matrices with coefficients in $\mathbb{F}$ and where $\mathrm{GL}(r)$ is the multiplicative group of all invertible $(r \times r)$-matrices with coefficients in $\mathbb{F}$.

Proof. Let $e_1, \dots, e_m$ be an orthogonal basis for W and $f_1, \dots, f_r$ be a basis for $\mathcal{R}$, then the associated bilinear form $\mathfrak{b}$ of $\mathfrak{q}$ has the following matrix representation:

$$\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \tag{25}$$

with an invertible diagonal $(m \times m)$-matrix Q. For the matrix of any orthogonal automorphism Φ of V we get the necessary and sufficient condition:

$$\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} A^\top & C^\top \\ B^\top & D^\top \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{26}$$

$$= \begin{pmatrix} A^\top Q A & A^\top Q B \\ B^\top Q A & B^\top Q B \end{pmatrix}. \tag{27}$$

This is equivalent to the conditions:

$$Q = A^\top Q A, \qquad 0 = A^\top Q B, \qquad 0 = B^\top Q B. \tag{28}$$

This shows that A is the matrix of an orthogonal automorphism of W (w.r.t. $\mathfrak{b}|_W$), which, since Q is invertible, is also invertible. The second equation then shows that necessarily $B = 0$. Since the whole matrix needs to be invertible also D must be invertible. Furthermore, there are no constraints on C.

If all those conditions are satisfied the whole matrix satisfies the orthogonality constraints from above. □

Corollary C.9. *For $\mathbb{R}^{(p,q,r)}$ we get:*

$$\mathrm{O}(p,q,r) \cong \begin{pmatrix} \mathrm{O}(p,q) & 0_{(p+q)\times r} \\ \mathrm{M}(r, p+q) & \mathrm{GL}(r) \end{pmatrix}, \tag{29}$$

where $\mathrm{O}(p,q) := \mathrm{O}(p,q,0)$.

Remark C.10. *Note that the composition $\Phi_1 \circ \Phi_2$ of orthogonal automorphisms of $(V, \mathfrak{q})$ corresponds to the matrix multiplication as follows:*

$$\begin{pmatrix} O_1 & 0 \\ M_1 & G_1 \end{pmatrix} \begin{pmatrix} O_2 & 0 \\ M_2 & G_2 \end{pmatrix} = \begin{pmatrix} O_1 O_2 & 0 \\ M_1 O_2 + G_1 M_2 & G_1 G_2 \end{pmatrix}. \tag{30}$$

Definition C.11 (Radical preserving orthogonal automorphisms). *For a quadratic space $(V, \mathfrak{q})$ with radical subspace $\mathcal{R} \subseteq V$ we define the group of* radical preserving orthogonal automorphisms *as follows:*

$$\mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}) := \{\Phi \in \mathrm{O}(V, \mathfrak{q}) \mid \Phi|_{\mathcal{R}} = \mathrm{id}_{\mathcal{R}}\}. \tag{31}$$

If $(V, \mathfrak{q}) = \mathbb{R}^{(p,q,r)}$, i.e. if $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^n$ and $\mathfrak{q}$ has the signature (p,q,r) with $p+q+r = n$ then we define the group of radical preserving orthogonal matrices of signature (p,q,r) *as follows:*

$$\mathrm{O}_{\mathcal{R}}(p,q,r) := \{O \in \mathrm{O}(p,q,r) \mid \text{bottom right corner of } O = I_r\} = \begin{pmatrix} \mathrm{O}(p,q) & 0_{(p+q)\times r} \\ \mathrm{M}(r, p+q) & I_r \end{pmatrix}, \tag{32}$$

where I_r is the $(r \times r)$-identity matrix.

Remark C.12. *Note that the matrix representation of* $\Phi \in \mathrm{O}_{\mathcal{R}}(\mathfrak{q})$ *w.r.t. an orthogonal basis like in the proof of theorem C.8 is of the form:*

$$\begin{pmatrix} O & 0_{m\times r} \\ M & I_r \end{pmatrix}, \tag{33}$$

with $O \in \mathrm{O}(W, \mathfrak{q}|_W)$*,* $M \in \mathrm{M}(r, m)$ *and where* I_r *is the* $(r \times r)$*-identity matrix.*

The composition $\Phi_1 \circ \Phi_2$ *of* Φ_1 *and* $\Phi_2 \in \mathrm{O}_{\mathcal{R}}(\mathfrak{q})$ *is then given by the corresponding matrix multiplication:*

$$\begin{pmatrix} O_1 & 0 \\ M_1 & I_r \end{pmatrix} \begin{pmatrix} O_2 & 0 \\ M_2 & I_r \end{pmatrix} = \begin{pmatrix} O_1 O_2 & 0 \\ M_1 O_2 + M_2 & I_r \end{pmatrix}. \tag{34}$$

By observing the left column (the only part that does not transform trivially), we see that $\mathrm{O}(\mathfrak{q}|_W)$ *acts on* $\mathrm{M}(r, m)$ *just by matrix multiplication from the right (in the corresponding basis):*

$$(O_1, M_1) \cdot (O_2, M_2) = (O_1 O_2, M_1 O_2 + M_2). \tag{35}$$

This immediately shows that we can write $\mathrm{O}_{\mathcal{R}}(\mathfrak{q})$ *as the semi-direct product:*

$$\mathrm{O}_{\mathcal{R}}(\mathfrak{q}) \cong \mathrm{O}(\mathfrak{q}|_W) \ltimes \mathrm{M}(r, m). \tag{36}$$

In the special case of $\mathbb{R}^{(p,q,r)}$ *we get:*

$$\mathrm{O}_{\mathcal{R}}(p, q, r) \cong \mathrm{O}(p, q) \ltimes \mathrm{M}(r, p + q). \tag{37}$$

We conclude this chapter by citing the Theorem of Cartan and Dieudonné about the structure of orthogonal groups in the non-degenerate case, but still for arbitrary fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$.

Theorem C.13 (Theorem of Cartan-Dieudonné, see [Art57] Thm. 3.20)**.** *Let* $(V, \mathfrak{q})$ *be a* non-degenerate *quadratic space of finite dimension* $\dim V = n < \infty$ *over a field* $\mathbb{F}$ *of* $\mathrm{char}(\mathbb{F}) \neq 2$*. Then every element* $g \in \mathrm{O}(V, \mathfrak{q})$ *can be written as:*

$$g = r_1 \circ \cdots \circ r_k, \tag{38}$$

with $1 \leq k \leq n$*, where* r_i *are reflections w.r.t. non-singular hyperplanes.*

D The Clifford Algebra and Typical Constructions

In this section, we provide the required definitions, constructions, and derivations leading to the results stated in the main paper. We start with a general introduction to the Clifford algebra.

D.1 The Clifford Algebra and its Universal Property

We follow [LS09, Cru90].

Let $(V, \mathfrak{q})$ be a finite dimensional *quadratic space* over a field $\mathbb{F}$ (also denoted an $\mathbb{F}$-vector space) with $\mathrm{char}(\mathbb{F}) \neq 2$. We abbreviate the corresponding *bilinear form* $\mathfrak{b}$ on vectors $v_1, v_2 \in V$ as follows:

$$\mathfrak{b}(v_1, v_2) := \frac{1}{2}\left(\mathfrak{q}(v_1 + v_2) - \mathfrak{q}(v_1) - \mathfrak{q}(v_2)\right). \tag{39}$$

Definition D.1. *To define the* Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ *we first consider the* tensor algebra *of* V*:*

$$\mathrm{T}(V) := \bigoplus_{m=0}^{\infty} V^{\otimes m} = \mathrm{span}\left\{v_1 \otimes \cdots \otimes v_m \,|\, m \geq 0, v_i \in V\right\}, \tag{40}$$

$$V^{\otimes m} := \underbrace{V \otimes \cdots \otimes V}_{m\text{-times}}, \qquad V^{\otimes 0} := \mathbb{F}, \tag{41}$$

and the following two-sided ideal[9]*:*

$$I(\mathfrak{q}) := \left\langle v \otimes v - \mathfrak{q}(v) \cdot 1_{\mathrm{T}(V)} \,\middle|\, v \in V \right\rangle. \tag{42}$$

Then we define the Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ *as the following quotient:*

$$\mathrm{Cl}(V, \mathfrak{q}) := \mathrm{T}(V)/I(\mathfrak{q}). \tag{43}$$

In words, we identify the square of a vector with its quadratic form. We also denote the canonical algebra quotient map as:

$$\pi : \mathrm{T}(V) \to \mathrm{Cl}(V, \mathfrak{q}). \tag{44}$$

Remark D.2.

1. *It is not easy to see, but always true, that* $\dim \mathrm{Cl}(V, \mathfrak{q}) = 2^n$*, where* $n := \dim V$*, see Theorem D.15.*
2. *If* $e_1, \ldots, e_n$ *is any basis of* $(V, \mathfrak{q})$ *then* $(e_A)_{A \subseteq [n]}$ *is a basis for* $\mathrm{Cl}(V, \mathfrak{q})$*, where we put for a subset* $A \subseteq [n] := \{1, \ldots, n\}$*:*

$$e_A := \prod_{i \in A}^{<} e_i, \qquad e_\emptyset := 1_{\mathrm{Cl}(V,\mathfrak{q})}. \tag{45}$$

where the product is taken in increasing order of the indices $i \in A$*, see Corollary D.16.*

3. *If* $e_1, \ldots, e_n$ *is any* orthogonal *basis of* $(V, \mathfrak{q})$*, then one can even show that* $(e_A)_{A \subseteq [n]}$ *is an* orthogonal *basis for* $\mathrm{Cl}(V, \mathfrak{q})$ *w.r.t. an extension of the bilinear form* $\mathfrak{b}$ *from* V *to* $\mathrm{Cl}(V, \mathfrak{q})$*, see Theorem D.26.*

Lemma D.3 (The fundamental identity)**.** *Note that for* $v_1, v_2 \in V$*, we always have the fundamental identity in* $\mathrm{Cl}(V, \mathfrak{q})$*:*

$$v_1 v_2 + v_2 v_1 = 2\mathfrak{b}(v_1, v_2). \tag{46}$$

Proof. By definition of the Clifford algebra we have the identities:

$$\begin{aligned}
\mathfrak{q}(v_1) + v_1 v_2 + v_2 v_1 + \mathfrak{q}(v_2) &= v_1 v_1 + v_1 v_2 + v_2 v_1 + v_2 v_2 && (47)\\
&= (v_1 + v_2)(v_1 + v_2) && (48)\\
&= \mathfrak{q}(v_1 + v_2) && (49)\\
&= \mathfrak{b}(v_1 + v_2, v_1 + v_2) && (50)\\
&= \mathfrak{b}(v_1, v_1) + \mathfrak{b}(v_1, v_2) + \mathfrak{b}(v_2, v_1) + \mathfrak{b}(v_2, v_2) && (51)\\
&= \mathfrak{q}(v_1) + 2\mathfrak{b}(v_1, v_2) + \mathfrak{q}(v_2). && (52)
\end{aligned}$$

Substracting $\mathfrak{q}(v_1) + \mathfrak{q}(v_2)$ on both sides gives the claim. □

[9] The ideal ensures that for all elements containing $v \otimes v$ (i.e., linear combinations, multiplications on the left, and multiplications on the right), $v \otimes v$ is identified with $\mathfrak{q}(v)$; e.g. $x \otimes v \otimes v \otimes y \sim x \otimes (\mathfrak{q}(v) \cdot 1_{\mathrm{T}(V)}) \otimes y$ for every $x, y \in \mathrm{T}(V)$.

Further, the Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ is fully characterized by the following property.

Theorem D.4 (The universal property of the Clifford algebra)**.** *For every $\mathbb{F}$-algebra*[10] *(an algebra over field $\mathbb{F}$) $\mathcal{A}$ and every $\mathbb{F}$-linear map $f : V \to \mathcal{A}$ such that for all $v \in V$ we have:*

$$f(v)^2 = \mathfrak{q}(v) \cdot 1_{\mathcal{A}}, \tag{53}$$

there exists a unique $\mathbb{F}$-algebra homomorphism[11] *$\bar{f} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathcal{A}$ such that $\bar{f}(v) = f(v)$ for all $v \in V$.*

More explicitely, if f satisfies equation 53 and $x \in \mathrm{Cl}(V, \mathfrak{q})$. Then we can take any *representation of x of the following form:*

$$x = c_0 + \sum_{i \in I} c_i \cdot v_{i,1} \cdots v_{i,k_i}, \tag{54}$$

with finite index sets I and $k_i \in \mathbb{N}$ and coefficients $c_0, c_i \in \mathbb{F}$ and vectors $v_{i,j} \in V$, $j = 1, \dots, k_i$, $i \in I$, and, then we can compute $\bar{f}(x)$ by the following formula (without ambiguity):

$$\bar{f}(x) = c_0 \cdot 1_{\mathcal{A}} + \sum_{i \in I} c_i \cdot f(v_{i,1}) \cdots f(v_{i,k_i}). \tag{55}$$

In the following, we will often denote $\bar{f}$ again with f without further indication.

D.2 The Multivector Filtration and the Grade

Note that the Clifford algebra is a filtered algebra.

Definition D.5. *We define the* multivector filtration[12] *of* $\mathrm{Cl}(V, \mathfrak{q})$ *for* grade $m \in \mathbb{N}_0$ *as follows:*

$$\mathrm{Cl}^{(\leq m)}(V, \mathfrak{q}) := \pi\left(\mathrm{T}^{(\leq m)}(V)\right), \qquad \mathrm{T}^{(\leq m)}(V) := \bigoplus_{l=0}^{m} V^{\otimes l}. \tag{56}$$

Remark D.6. *Note that we really get a filtration on the space* $\mathrm{Cl}(V, \mathfrak{q})$*:*

$$\mathbb{F} = \mathrm{Cl}^{(\leq 0)}(V, \mathfrak{q}) \subseteq \mathrm{Cl}^{(\leq 1)}(V, \mathfrak{q}) \subseteq \mathrm{Cl}^{(\leq 2)}(V, \mathfrak{q}) \subseteq \ldots \subseteq \mathrm{Cl}^{(\leq n)}(V, \mathfrak{q}) = \mathrm{Cl}(V, \mathfrak{q}). \tag{57}$$

Furthermore, note that this is compatible with the algebra structure of $\mathrm{Cl}(V, \mathfrak{q})$*, i.e. for $i, j \geq 0$ we get:*

$$x \in \mathrm{Cl}^{(\leq i)}(V, \mathfrak{q}) \quad \wedge \quad y \in \mathrm{Cl}^{(\leq j)}(V, \mathfrak{q}) \qquad \Longrightarrow \qquad xy \in \mathrm{Cl}^{(\leq i+j)}(V, \mathfrak{q}). \tag{58}$$

Together with the equality: $\pi(\mathrm{T}^{(\leq m)}(V)) = \mathrm{Cl}^{(\leq m)}(V, \mathfrak{q})$ for all m, we see that the natural map:

$$\pi : \mathrm{T}(V) \to \mathrm{Cl}(V, \mathfrak{q}), \tag{59}$$

is a surjective homomorphism of filtered algebras. Indeed, since $\mathrm{Cl}(V, \mathfrak{q})$ is a well-defined algebra, since we modded out a two-sided ideal, π is clearly a homomorphism of filtered algebras. As a quotient map π is automatically surjective.

Definition D.7 (The grade of an element)**.** *For $x \in \mathrm{Cl}(V, \mathfrak{q}) \setminus \{0\}$ we define its* grade *through the following condition:*

$$\operatorname{grd} x := k \qquad \textit{such that} \qquad x \in \mathrm{Cl}^{(\leq k)}(V, \mathfrak{q}) \setminus \mathrm{Cl}^{(\leq k-1)}(V, \mathfrak{q}), \tag{60}$$

$$\operatorname{grd} 0 := -\infty. \tag{61}$$

[10] For the purpose of this text an algebra is always considered to be associative and unital (containing an identity element), but not necessarily commutative.

[11] An algebra homomorphism is both linear and multiplicative.

[12] A filtration $\mathcal{F}$ is an indexed family $(A_i)_{i \in I}$ (I is an ordered index set) of subsets of an algebraic structure A such that for $i \leq j : A_i \subseteq A_j$.

D.3 The Parity Grading

In this section, we introduce the parity grading of the Clifford algebra. We will use the parity grading to later construct the (adjusted) twisted conjugation map, which will be used as a group action on the Clifford algebra.

Definition D.8 (The main involution). *The linear map:*

$$\alpha: V \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha(v) := -v, \tag{62}$$

satisfies $(-v)^2 = v^2 = \mathfrak{q}(v)$. The universal property of $\mathrm{Cl}(V, \mathfrak{q})$ thus extends α to a unique algebra homomorphism:

$$\alpha: \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha\left(c_0 + \sum_{i \in I} c_i \cdot v_{i,1} \cdots v_{i,k_i}\right) \tag{63}$$

$$= c_0 + \sum_{i \in I} c_i \cdot \alpha(v_{i,1}) \cdots \alpha(v_{i,k_i}) \tag{64}$$

$$= c_0 + \sum_{i \in I} (-1)^{k_i} \cdot c_i \cdot v_{i,1} \cdots v_{i,k_i}, \tag{65}$$

for any finite sum representation with $v_{i,j} \in V$ and $c_i \in \mathbb{F}$. This extension α will be called the main involution[13] *of $\mathrm{Cl}(V, \mathfrak{q})$.*

Definition D.9 (Parity grading). *The main involution α of $\mathrm{Cl}(V, \mathfrak{q})$ now defines the* parity grading *of $\mathrm{Cl}(V, \mathfrak{q})$ via the following* homogeneous parts*:*

$$\mathrm{Cl}^{[0]}(V, \mathfrak{q}) := \{x \in \mathrm{Cl}(V, \mathfrak{q}) \mid \alpha(x) = x\}, \tag{66}$$

$$\mathrm{Cl}^{[1]}(V, \mathfrak{q}) := \{x \in \mathrm{Cl}(V, \mathfrak{q}) \mid \alpha(x) = -x\}. \tag{67}$$

With this we get the direct sum decomposition:

$$\mathrm{Cl}(V, \mathfrak{q}) = \mathrm{Cl}^{[0]}(V, \mathfrak{q}) \oplus \mathrm{Cl}^{[1]}(V, \mathfrak{q}), \tag{68}$$

$$x = x^{[0]} + x^{[1]}, \qquad x^{[0]} := \frac{1}{2}(x + \alpha(x)), \quad x^{[1]} := \frac{1}{2}(x - \alpha(x)), \tag{69}$$

with the homogeneous parts $x^{[0]} \in \mathrm{Cl}^{[0]}(V, \mathfrak{q})$ and $x^{[1]} \in \mathrm{Cl}^{[1]}(V, \mathfrak{q})$.

We define the parity *of an (homogeneous) element $x \in \mathrm{Cl}(V, \mathfrak{q})$ as follows:*

$$\mathrm{prt}(x) := \begin{cases} 0 & \text{if } x \in \mathrm{Cl}^{[0]}(V, \mathfrak{q}), \\ 1 & \text{if } x \in \mathrm{Cl}^{[1]}(V, \mathfrak{q}). \end{cases} \tag{70}$$

Definition D.10 ($\mathbb{Z}/2\mathbb{Z}$-graded algebras). *An $\mathbb{F}$-algebra $(\mathcal{A}, +, \cdot)$ together with a direct sum decomposition of sub-vector spaces:*

$$\mathcal{A} = \mathcal{A}^{[0]} \oplus \mathcal{A}^{[1]}, \tag{71}$$

is called a $\mathbb{Z}/2\mathbb{Z}$-graded algebra *if for every $i, j \in \mathbb{Z}/2\mathbb{Z}$ we always have:*

$$x \in \mathcal{A}^{[i]} \quad \wedge \quad y \in \mathcal{A}^{[j]} \qquad \Longrightarrow \qquad x \cdot y \in \mathcal{A}^{[i+j]}. \tag{72}$$

Note that $[i + j]$ is meant here to be computed modulo 2. The $\mathbb{Z}/2\mathbb{Z}$-grade of an (homogeneous) element $x \in \mathcal{A}$ will also be called the parity *of x:*

$$\mathrm{prt}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{A}^{[0]}, \\ 1 & \text{if } x \in \mathcal{A}^{[1]}. \end{cases} \tag{73}$$

The above requirement then implies for homogeneous elements $x, y \in \mathcal{A}$ the relation:

$$\mathrm{prt}(x \cdot y) = \mathrm{prt}(x) + \mathrm{prt}(y) \mod 2. \tag{74}$$

We can now summarize the results of this section as follows:

Theorem D.11. *The Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra in its parity grading.*

[13] An involution is a map that is its own inverse.

D.4 The Dimension of the Clifford Algebra

In this subsection we determine the dimension of the Clifford algebra, which allows us to construct bases for the Clifford algebra.

In the following, we again let $\mathbb{F}$ be any field of $\mathrm{char}(\mathbb{F}) \neq 2$.

Definition/Lemma D.12 (The twisted tensor product of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebras)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{Z}/2\mathbb{Z}$-graded algebras over $\mathbb{F}$. Then their* twisted tensor product *$\mathcal{A}\hat{\otimes}\mathcal{B}$ is defined via the usual tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathbb{F}$-vector spaces, but where the product is defined on homogeneous elements $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ via:*

$$(a_1\hat{\otimes}b_1) \cdot (a_2\hat{\otimes}b_2) := (-1)^{\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_2)}(a_1a_2)\hat{\otimes}(b_1b_2). \tag{75}$$

This turns $\mathcal{A}\hat{\otimes}\mathcal{B}$ also into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra over $\mathbb{F}$ with the $\mathbb{Z}/2\mathbb{Z}$-grading:

$$(\mathcal{A}\hat{\otimes}\mathcal{B})^{[0]} := \left(\mathcal{A}^{[0]} \otimes \mathcal{B}^{[0]}\right) \oplus \left(\mathcal{A}^{[1]} \otimes \mathcal{B}^{[1]}\right), \tag{76}$$

$$(\mathcal{A}\hat{\otimes}\mathcal{B})^{[1]} := \left(\mathcal{A}^{[0]} \otimes \mathcal{B}^{[1]}\right) \oplus \left(\mathcal{A}^{[1]} \otimes \mathcal{B}^{[0]}\right). \tag{77}$$

In particular, if $a \in \mathcal{A}$, $b \in \mathcal{B}$ are homogeneous elements then we have:

$$\mathrm{prt}_{\mathcal{A}\hat{\otimes}\mathcal{B}}(a\hat{\otimes}b) = \mathrm{prt}_{\mathcal{A}}(a) + \mathrm{prt}_{\mathcal{B}}(b) \mod 2. \tag{78}$$

Proof. By definition we already know that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is an $\mathbb{F}$-vector space. So we only need to investigate the multiplication and $\mathbb{Z}/2\mathbb{Z}$-grading.

For homogenous elements $a_1, a_2, a_3 \in \mathcal{A}$, $b_1, b_2, b_3 \in \mathcal{B}$ we have:

$$\begin{aligned}
&\left((a_1\hat{\otimes}b_1) \cdot (a_2\hat{\otimes}b_2)\right) \cdot (a_3\hat{\otimes}b_3) && (79)\\
&= (-1)^{\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_2)} \left((a_1a_2)\hat{\otimes}(b_1b_2)\right) \cdot (a_3\hat{\otimes}b_3) && (80)\\
&= (-1)^{\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_2)} \cdot (-1)^{\mathrm{prt}(b_1b_2)\cdot\mathrm{prt}(a_3)}(a_1a_2a_3)\hat{\otimes}(b_1b_2b_3) && (81)\\
&= (-1)^{\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_2)+\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_3)+\mathrm{prt}(b_2)\cdot\mathrm{prt}(a_3)}(a_1a_2a_3)\hat{\otimes}(b_1b_2b_3) && (82)\\
&= (-1)^{\mathrm{prt}(b_1)\cdot\mathrm{prt}(a_2a_3)} \cdot (-1)^{\mathrm{prt}(b_2)\cdot\mathrm{prt}(a_3)}(a_1a_2a_3)\hat{\otimes}(b_1b_2b_3) && (83)\\
&= (-1)^{\mathrm{prt}(b_2)\cdot\mathrm{prt}(a_3)}(a_1\hat{\otimes}b_1) \cdot \left((a_2a_3)\hat{\otimes}(b_2b_3)\right) && (84)\\
&= (a_1\hat{\otimes}b_1) \cdot \left((a_2\hat{\otimes}b_2) \cdot (a_3\hat{\otimes}b_3)\right). && (85)
\end{aligned}$$

This shows associativity of multiplication on homogeneous elements, which extends by linearity to general elements. The distributive law is clear.

To check that we have a $\mathbb{Z}/2\mathbb{Z}$-grading, note that for homogeneous elements $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ we have:

$$\begin{aligned}
\mathrm{prt}_{\mathcal{A}\hat{\otimes}\mathcal{B}}\left((a_1\hat{\otimes}b_1) \cdot (a_2\hat{\otimes}b_2)\right) &= \mathrm{prt}_{\mathcal{A}\hat{\otimes}\mathcal{B}}\left(((-1)^{\mathrm{prt}(a_2)\cdot\mathrm{prt}(b_1)}a_1a_2)\hat{\otimes}(b_1b_2)\right) && (86)\\
&= \mathrm{prt}_{\mathcal{A}}(a_1a_2) + \mathrm{prt}_{\mathcal{B}}(b_1b_2) && (87)\\
&= \mathrm{prt}_{\mathcal{A}}(a_1) + \mathrm{prt}_{\mathcal{A}}(a_2) + \mathrm{prt}_{\mathcal{B}}(b_1) + \mathrm{prt}_{\mathcal{B}}(b_2) && (88)\\
&= \mathrm{prt}_{\mathcal{A}\hat{\otimes}\mathcal{B}}\left(a_1\hat{\otimes}b_1\right) + \mathrm{prt}_{\mathcal{A}\hat{\otimes}\mathcal{B}}\left(a_2\hat{\otimes}b_2\right) \mod 2. && (89)
\end{aligned}$$

The general case follows by linear combinations. □

Remark D.13 (The universal property of the twisted tensor product of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebras)**.** *Let $\mathcal{A}_1, \mathcal{A}_2$, $\mathcal{B}$ be $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebras. Consider $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra homomorphisms:*

$$\psi_1 : \mathcal{A}_1 \to \mathcal{B}, \qquad \psi_2 : \mathcal{A}_2 \to \mathcal{B}, \tag{90}$$

such that for all homogeneous elements $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ we have:

$$\psi_1(a_1) \cdot \psi_2(a_2) = (-1)^{\mathrm{prt}(a_1)\cdot\mathrm{prt}(a_2)} \cdot \psi_2(a_2) \cdot \psi_1(a_1). \tag{91}$$

Then there exists a unique $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra homomorphism:

$$\psi : \mathcal{A}_1\hat{\otimes}\mathcal{A}_2 \to \mathcal{B}, \tag{92}$$

such that:

$$\psi \circ \phi_1 = \psi_1, \qquad \psi \circ \phi_2 = \psi_2, \tag{93}$$

where ϕ_1, ϕ_2 are the following $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra homomorphisms:

$$\phi_1 : \mathcal{A}_1 \to \mathcal{A}_1 \hat{\otimes} \mathcal{A}_2, \qquad a_1 \mapsto a_1 \hat{\otimes} 1, \tag{94}$$

$$\phi_2 : \mathcal{A}_2 \to \mathcal{A}_1 \hat{\otimes} \mathcal{A}_2, \qquad a_2 \mapsto 1 \hat{\otimes} a_2, \tag{95}$$

which satisfy for all homogeneous elements $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$:

$$\phi_1(a_1) \cdot \phi_2(a_2) = (-1)^{\mathrm{prt}(a_1) \cdot \mathrm{prt}(a_2)} \cdot \phi_2(a_2) \cdot \phi_1(a_1). \tag{96}$$

Furthermore, $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ together with ϕ_1, ϕ_2 is uniquely characterized as a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra by the above property (when considering all possible such $\mathcal{B}$ and ψ_1, ψ_2).

Proposition D.14. *Let $(V, \mathfrak{q})$ be a finite dimensional quadratic vector space over $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, with an orthogonal sum decomposition:*

$$(V, \mathfrak{q}) = (V_1, \mathfrak{q}_1) \oplus (V_2, \mathfrak{q}_2). \tag{97}$$

Then the inclusion maps: $\mathrm{Cl}(V_i, \mathfrak{q}_i) \to \mathrm{Cl}(V, \mathfrak{q})$ induce an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebras:

$$\mathrm{Cl}(V, \mathfrak{q}) \cong \mathrm{Cl}(V_1, \mathfrak{q}_1) \hat{\otimes} \mathrm{Cl}(V_2, \mathfrak{q}_2). \tag{98}$$

In particular:

$$\dim \mathrm{Cl}(V, \mathfrak{q}) = \dim \mathrm{Cl}(V_1, \mathfrak{q}_1) \cdot \dim \mathrm{Cl}(V_2, \mathfrak{q}_2). \tag{99}$$

Proof. First consider the canonical inclusion maps, $l = 1, 2$:

$$\phi_l : \mathrm{Cl}(V_l, \mathfrak{q}_l) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad x_l \mapsto x_l. \tag{100}$$

These satisfy for all (parity) homogeneous elements $x_l \in \mathrm{Cl}(V_l, \mathfrak{q}_l)$, $l = 1, 2$, the condition:

$$\phi_1(x_1)\phi_2(x_2) = x_1 x_2 \overset{!}{=} (-1)^{\mathrm{prt}(x_1) \cdot \mathrm{prt}(x_2)} \cdot x_2 x_1 = (-1)^{\mathrm{prt}(x_1) \cdot \mathrm{prt}(x_2)} \cdot \phi_2(x_2)\phi_1(x_1). \tag{101}$$

Indeed, we can by linearity reduce to the case that x_1 and x_2 are products of vectors $v_1 \in V_1$ and $v_2 \in V_2$, resp. Since V_1 and V_2 are orthogonal to each other by assumption, for those elements we get by the identities D.3:

$$v_1 v_2 = -v_2 v_1 + 2 \underbrace{\mathfrak{b}(v_1, v_2)}_{=0} = -v_2 v_1. \tag{102}$$

This shows the condition above for x_1 and x_2.

We then define the $\mathbb{F}$-bilinear map:

$$\mathrm{Cl}(V_1, \mathfrak{q}_1) \times \mathrm{Cl}(V_2, \mathfrak{q}_2) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad (x_1, x_2) \mapsto x_1 x_2, \tag{103}$$

which thus factorizes through the $\mathbb{F}$-linear map:

$$\phi : \mathrm{Cl}(V_1, \mathfrak{q}_1) \hat{\otimes} \mathrm{Cl}(V_2, \mathfrak{q}_2) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad x_1 \hat{\otimes} x_2 \mapsto x_1 x_2. \tag{104}$$

We now show that ϕ also respects multiplication. For this let $x_1, y_1 \in \mathrm{Cl}(V_1, \mathfrak{q}_1)$ and $x_2, y_2 \in \mathrm{Cl}(V_2, \mathfrak{q}_2)$ (parity) homogeneous elements. We then get:

$$\phi\left((x_1 \hat{\otimes} x_2) \cdot (y_1 \hat{\otimes} y_2)\right) \tag{105}$$

$$= \phi\left((-1)^{\mathrm{prt}(x_2) \cdot \mathrm{prt}(y_1)} \cdot (x_1 y_1) \hat{\otimes} (x_2 y_2)\right) \tag{106}$$

$$= (-1)^{\mathrm{prt}(x_2) \cdot \mathrm{prt}(y_1)} \cdot x_1 y_1 x_2 y_2 \tag{107}$$

$$= x_1 x_2 y_1 y_2 \tag{108}$$

$$= \phi(x_1 \hat{\otimes} x_2) \cdot \phi(y_1 \hat{\otimes} y_2). \tag{109}$$

This shows the multiplicativity of ϕ on homogeneous elements. The general case follows by the $\mathbb{F}$-linearity of ϕ. Note that ϕ also respects the parity grading.

Now consider the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra $\mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2)$ and $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}$-algebra homomorphisms:

$$\psi_1 : \mathrm{Cl}(V_1,\mathfrak{q}_1) \to \mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2), \qquad x_1 \mapsto x_1\hat{\otimes}1, \tag{110}$$
$$\psi_2 : \mathrm{Cl}(V_2,\mathfrak{q}_2) \to \mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2), \qquad x_2 \mapsto 1\hat{\otimes}x_2. \tag{111}$$

Note that for all homogeneous elements $x_l \in \mathrm{Cl}(V_l,\mathfrak{q}_l)$, $l = 1,2$, we have:

$$\begin{aligned}\psi_2(x_2)\cdot\psi_1(x_1) &= (1\hat{\otimes}x_2)\cdot(x_1\hat{\otimes}1) &(112)\\ &= (-1)^{\mathrm{prt}(x_1)\cdot\mathrm{prt}(x_2)}\cdot x_1\hat{\otimes}x_2 &(113)\\ &= (-1)^{\mathrm{prt}(x_1)\cdot\mathrm{prt}(x_2)}\cdot(x_1\hat{\otimes}1)\cdot(1\hat{\otimes}x_2) &(114)\\ &= (-1)^{\mathrm{prt}(x_1)\cdot\mathrm{prt}(x_2)}\cdot\psi_1(x_1)\cdot\psi_2(x_2). &(115)\end{aligned}$$

We then define the $\mathbb{F}$-linear map:

$$\psi : V = V_1\oplus V_2 \to \mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2), \quad v = v_1+v_2 \mapsto \psi_1(v_1)+\psi_2(v_2) =: \psi(v). \tag{116}$$

Note that we have:

$$\begin{aligned}\psi(v)^2 &= (\psi_1(v_1)+\psi_2(v_2))\cdot(\psi_1(v_1)+\psi_2(v_2)) &(117)\\ &= \psi_1(v_1)^2+\psi_2(v_2)^2+\psi_1(v_1)\cdot\psi_2(v_2)+\psi_2(v_2)\cdot\psi_1(v_1) &(118)\\ &= \psi_1(v_1)^2+\psi_2(v_2)^2+\underbrace{\psi_1(v_1)\cdot\psi_2(v_2)+(-1)^{\mathrm{prt}(v_1)\cdot\mathrm{prt}(v_2)}\psi_1(v_1)\cdot\psi_2(v_2)}_{=0} &(119)\\ &= \psi_1(v_1^2)+\psi_2(v_2^2) &(120)\\ &= \mathfrak{q}_1(v_1)\cdot\psi_1(1_{\mathrm{Cl}(V_1,\mathfrak{q}_1)})+\mathfrak{q}_2(v_2)\cdot\psi_2(1_{\mathrm{Cl}(V_2,\mathfrak{q}_2)}) &(121)\\ &= \mathfrak{q}_1(v_1)\cdot 1_{\mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2)}+\mathfrak{q}_2(v_2)\cdot 1_{\mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2)} &(122)\\ &= (\mathfrak{q}_1(v_1)+\mathfrak{q}_2(v_2))\cdot 1_{\mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2)} &(123)\\ &= \mathfrak{q}(v)\cdot 1_{\mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2)}. &(124)\end{aligned}$$

By the universal property of the Clifford algebra ψ uniquely extends to an $\mathbb{F}$-algebra homomorphism:

$$\psi : \mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\,\mathrm{Cl}(V_2,\mathfrak{q}_2), \tag{125}$$
$$v_{i_1}\cdots v_{i_k} \mapsto (\psi_1(v_{i_1,1})+\psi_2(v_{i_1,2}))\cdots(\psi_1(v_{i_k,1})+\psi_2(v_{i_k,2})). \tag{126}$$

One can see from this, by explicit calculation, that ψ also respects the $\mathbb{Z}/2\mathbb{Z}$-grading. Furthermore, we see that $\psi\circ\phi_l = \psi_l$ for $l = 1,2$, and, also, $\phi\circ\psi_l = \phi_l$ for $l = 1,2$.

One easily sees that ϕ and ψ are inverse to each other and the claim follows. □

Theorem D.15 (The dimension of the Clifford algebra)**.** *Let* $(V,\mathfrak{q})$ *be a finite dimensional quadratic vector space over* $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, $n := \dim V < \infty$. *Then we have:*

$$\dim \mathrm{Cl}(V,\mathfrak{q}) = 2^n. \tag{127}$$

Proof. Let $e_1,\dots,e_n$ be an orthogonal basis of $(V,\mathfrak{q})$ and $V_i := \mathrm{span}(e_i) \subseteq V$, $\mathfrak{q}_i := \mathfrak{q}|_{V_i}$. Then we get the orthogonal sum decomposition:

$$(V,\mathfrak{q}) = (V_1,\mathfrak{q}_1)\oplus\cdots\oplus(V_n,\mathfrak{q}_n), \tag{128}$$

and thus by Proposition D.14 the isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras:

$$\mathrm{Cl}(V,\mathfrak{q}) \cong \mathrm{Cl}(V_1,\mathfrak{q}_1)\hat{\otimes}\cdots\hat{\otimes}\,\mathrm{Cl}(V_n,\mathfrak{q}_n), \tag{129}$$

and thus:

$$\dim \mathrm{Cl}(V,\mathfrak{q}) = \dim \mathrm{Cl}(V_1,\mathfrak{q}_1)\cdots\dim \mathrm{Cl}(V_n,\mathfrak{q}_n), \tag{130}$$

Since $\dim V_i = 1$ and thus:

$$\dim \mathrm{Cl}(V_i,\mathfrak{q}_i) = \dim(\mathbb{F}\oplus V_i) = 2, \tag{131}$$

for $i = 1,\dots,n$, we get the claim:

$$\dim \mathrm{Cl}(V,\mathfrak{q}) = 2^n. \tag{132}$$

□

Corollary D.16 (Bases for the Clifford algebra)**.** *Let $(V,\mathfrak{q})$ be a finite dimensional quadratic vector space over $\mathbb{F}$, $\operatorname{char}(\mathbb{F}) \neq 2$, $n := \dim V < \infty$. Let $e_1,\dots,e_n$ be any basis of V, then $(e_A)_{A\subseteq[n]}$ is a basis for $\operatorname{Cl}(V,\mathfrak{q})$, where we put for a subset $A \subseteq [n] := \{1,\dots,n\}$:*

$$e_A := \prod_{i\in A}^{<} e_i, \qquad\qquad e_\emptyset := 1_{\operatorname{Cl}(V,\mathfrak{q})}. \tag{133}$$

where the product is taken in increasing order of the indices $i \in A$.

Proof. Since $\operatorname{Cl}(V,\mathfrak{q}) = \operatorname{T}(V)/I(\mathfrak{q})$ and

$$\operatorname{T}(V) = \operatorname{span}\left\{e_{i_1}\otimes\cdots\otimes e_{i_m} \,|\, m \geq 0, i_j \in [n], j \in [m]\right\}, \tag{134}$$

we see that:

$$\operatorname{Cl}(V,\mathfrak{q}) = \operatorname{span}\left\{e_{i_1}\cdots e_{i_m} \,|\, m \geq 0, i_j \in [n], j \in [m]\right\}. \tag{135}$$

By several applications of the fundamental identities D.3:

$$e_{i_k}e_{i_l} = -e_{i_l}e_{i_k} + 2\mathfrak{b}(e_{i_k}, e_{i_l}), \qquad\qquad e_{i_k}e_{i_k} = \mathfrak{q}(e_{i_k}), \tag{136}$$

we can turn products $e_{i_1}\cdots e_{i_m}$ into sums of smaller products if some of the occuring indices agree, say $i_k = i_l$. Furthermore, we can also use those identities to turn the indices in increasing order. This then shows:

$$\operatorname{Cl}(V,\mathfrak{q}) = \operatorname{span}\left\{e_A \,|\, A \subseteq [n]\right\}. \tag{137}$$

Since $\#\{A \subseteq [n]\} = 2^n$ and $\dim \operatorname{Cl}(V,\mathfrak{q}) = 2^n$ by Theorem D.15 we see that $\{e_A \,|\, A \subseteq [n]\}$ must already be a basis for $\operatorname{Cl}(V,\mathfrak{q})$. □

D.5 Extending the Quadratic Form to the Clifford Algebra

We provide the extension of the quadratic form from the vector space to the Clifford algebra, which will lead to the constrution of an *orthogonal* basis of the Clifford algebra.

Definition D.17 (The opposite algebra of an algebra)**.** *Let $(\mathcal{A},+,\cdot)$ be an algebra. The* opposite algebra $(\mathcal{A}^{\mathrm{op}},+,\bullet)$ *is defined to consist of the same underlying vector space $(\mathcal{A}^{\mathrm{op}},+) = (\mathcal{A},+)$, but where the multiplication is reversed in comparison to $\mathcal{A}$, i.e. for $x,y \in \mathcal{A}^{\mathrm{op}}$ we have:*

$$x \bullet y := y \cdot x. \tag{138}$$

Note that this really turns $(\mathcal{A}^{\mathrm{op}},+,\bullet)$ into an algebra.

Definition D.18 (The main anti-involution of the Clifford algebra)**.** *Consider the following linear map:*

$$\beta:\ V \to \operatorname{Cl}(V,\mathfrak{q})^{\mathrm{op}}, \qquad\qquad v \mapsto v, \tag{139}$$

which also satisfies $v \bullet v = vv = \mathfrak{q}(v)$. By the universal property of the Clifford algebra we get a unique extension to an algebra homomorphism:

$$\beta:\ \operatorname{Cl}(V,\mathfrak{q}) \to \operatorname{Cl}(V,\mathfrak{q})^{\mathrm{op}}, \qquad \beta\left(c_0 + \sum_{i\in I} c_i\cdot v_{i,1}\cdots v_{i,k_i}\right) \tag{140}$$

$$= c_0 + \sum_{i\in I} c_i \cdot v_{i,1}\bullet\cdots\bullet v_{i,k_i} \tag{141}$$

$$= c_0 + \sum_{i\in I} c_i \cdot v_{i,k_i}\cdots v_{i,1}, \tag{142}$$

for any finite sum representation with $v_{i,j} \in V$ and $c_i \in \mathbb{F}$. We call β the main anti-involution *of* $\operatorname{Cl}(V,\mathfrak{q})$.

Definition D.19 (The combined anti-involution of the Clifford algebra)**.** *The* combined anti-involution *or* Clifford conjugation *of* $\operatorname{Cl}(V,\mathfrak{q})$ *is defined to be the $\mathbb{F}$-algebra homomorphism:*

$$\gamma:\ \operatorname{Cl}(V,\mathfrak{q}) \to \operatorname{Cl}(V,\mathfrak{q})^{\mathrm{op}}, \qquad\qquad \gamma(x) := \beta(\alpha(x)). \tag{143}$$

More explicitely, it is given by the formula:

$$\gamma\left(c_0 + \sum_{i\in I} c_i\cdot v_{i,1}\cdots v_{i,k_i}\right) = c_0 + \sum_{i\in I} (-1)^{k_i}\cdot c_i\cdot v_{i,k_i}\cdots v_{i,1}, \tag{144}$$

for any finite sum representation with $v_{i,j} \in V$ and $c_i \in \mathbb{F}$.

Remark D.20. *If $e_1, \dots, e_n \in V$ is an orthogonal basis for $(V, \mathfrak{q})$. Then we have for $A \subseteq [n]$:*

$$\alpha(e_A) = (-1)^{|A|} e_A, \qquad \beta(e_A) = (-1)^{\binom{|A|}{2}} e_A, \qquad \gamma(e_A) = (-1)^{\binom{|A|+1}{2}} e_A. \tag{145}$$

Recall the definition of a trace of a linear map:

Definition D.21 (The trace of an endomorphism)**.** *Let $\mathcal{Y}$ be a vector space over a field $\mathbb{F}$ of dimension $\dim \mathcal{Y} = m < \infty$ and $\Phi : \mathcal{Y} \to \mathcal{Y}$ a vector space endomorphism*[14]*. Let $B = \{b_1, \dots, b_m\}$ be a basis for $\mathcal{Y}$ and $B^* = \{b_1^*, \dots, b_m^*\}$ be the corresponding dual basis of $\mathcal{Y}^*$, defined via: $b_j^*(b_i) := \delta_{i,j}$. Let $A = (a_{i,j})_{\substack{i=1,\dots,m,\\ j=1,\dots,m}}$ be the matrix representation of Φ w.r.t. B:*

$$\forall j \in [m]. \qquad \Phi(b_j) = \sum_{i=1}^{m} a_{i,j} b_i. \tag{146}$$

Then the trace *of Φ is defined via:*

$$\operatorname{Tr}(\Phi) := \sum_{j=1}^{m} a_{j,j} = \sum_{j=1}^{m} b_j^*(\Phi(b_j)) \in \mathbb{F}. \tag{147}$$

It is a well known fact that $\operatorname{Tr}(\Phi)$ is not dependent on the initial choice of the basis B. Furthermore, Tr is a well-defined $\mathbb{F}$-linear map (homomorphism of vector spaces):

$$\operatorname{Tr}: \operatorname{End}_{\mathbb{F}}(\mathcal{Y}) \to \mathbb{F}, \qquad \Phi \mapsto \operatorname{Tr}(\Phi). \tag{148}$$

We now want to define the projection of $x \in \operatorname{Cl}(V, \mathfrak{q})$ onto its zero component $x^{(0)} \in \mathbb{F}$ in a basis independent way.

Definition D.22 (The projection onto the zero component)**.** *We define the $\mathbb{F}$-linear map:*

$$\zeta : \operatorname{Cl}(V, \mathfrak{q}) \to \mathbb{F}, \qquad \zeta(x) := 2^{-n} \operatorname{Tr}(x), \tag{149}$$

where $n := \dim V$ and $\operatorname{Tr}(x) := \operatorname{Tr}(L_x)$, where L_x is the endomorphism of $\operatorname{Cl}(V, \mathfrak{q})$ given by left multiplication with x:

$$L_x : \operatorname{Cl}(V, \mathfrak{q}) \to \operatorname{Cl}(V, \mathfrak{q}), \qquad y \mapsto L_x(y) := xy. \tag{150}$$

We call ζ the projection onto the zero component. *We also often write for $x \in \operatorname{Cl}(V, \mathfrak{q})$:*

$$x^{(0)} := \zeta(x). \tag{151}$$

The name is justified by following property:

Lemma D.23. *Let $e_1, \dots, e_n$ be a fixed orthogonal basis of $(V, \mathfrak{q})$. Then we know that $(e_A)_{A \subseteq [n]}$ is a basis for $\operatorname{Cl}(V, \mathfrak{q})$. So we can write every $x \in \operatorname{Cl}(V, \mathfrak{q})$ as:*

$$x = \sum_{A \subseteq [n]} x_A \cdot e_A, \tag{152}$$

with $x_A \in \mathbb{F}$, $A \subseteq [n]$. The claim is now that we have:

$$\zeta(x) \stackrel{!}{=} x_\emptyset. \tag{153}$$

Proof. By the linearity of the trace we only need to investigate $\operatorname{Tr}(e_A)$ for $A \subseteq [n]$. For $A = \emptyset$, we have $e_\emptyset = 1$ and we get:

$$\operatorname{Tr}(1) = \sum_{B \subseteq [n]} e_B^*(1 \cdot e_B) = \sum_{B \subseteq [n]} 1 = 2^n, \tag{154}$$

[14] A map from a mathematical object space to itself.

which shows: $\zeta(1) = 2^{-n}\,\mathrm{Tr}(1) = 1$. Now, consider $A \subseteq [n]$ with $A \neq \emptyset$. Let $\triangle$ denote the symmetric difference of two sets. Further, we can write $\pm$ to refrain from distinguishing between signs, which will not affect the result.

$$\begin{aligned}
\mathrm{Tr}(e_A) &= \sum_{B\subseteq[n]} e_B^*(e_A e_B) && (155)\\
&= \sum_{B\subseteq[n]} \pm \prod_{i\in A\cap B} \mathfrak{q}(e_i)\cdot e_B^*(e_{A\triangle B}) && (156)\\
&= \sum_{B\subseteq[n]} \pm \prod_{i\in A\cap B} \mathfrak{q}(e_i)\cdot \delta_{B,A\triangle B} && (157)\\
&= \sum_{B\subseteq[n]} \pm \prod_{i\in A\cap B} \mathfrak{q}(e_i)\cdot \delta_{\emptyset,A} && (158)\\
&= 0, && (159)
\end{aligned}$$

where the third equality follows from the fact that $B = A\triangle B$ holds if and only if $A = \emptyset$, regardless of B. However, $A = \emptyset$ was ruled out by assumption, so then in the last equality we always have $\delta_{\emptyset,A} = 0$. So for $A \neq \emptyset$ we have: $\zeta(e_A) = 2^{-n}\,\mathrm{Tr}(e_A) = 0$. Altogether we get:

$$\zeta(e_A) = \delta_{A,\emptyset} = \begin{cases} 1, & \text{if } A = \emptyset, \\ 0, & \text{else.} \end{cases} \tag{160}$$

With this and linearity we get:

$$\zeta(x) = \zeta\left(\sum_{A\subseteq[n]} x_A \cdot e_A\right) = \sum_{A\subseteq[n]} x_A \cdot \zeta(e_A) = \sum_{A\subseteq[n]} x_A \cdot \delta_{A,\emptyset} = x_\emptyset. \tag{161}$$

This shows the claim. □

Definition D.24 (The bilinear form on the Clifford algebra)**.** *For our quadratic $\mathbb{F}$-vector space $(V,\mathfrak{q})$ with corresponding bilinear form $\mathfrak{b}$ and corresponding Clifford algebra $\mathrm{Cl}(V,\mathfrak{q})$ we now define the following $\mathbb{F}$-bilinear form on $\mathrm{Cl}(V,\mathfrak{q})$:*

$$\bar{\mathfrak{b}}:\ \mathrm{Cl}(V,\mathfrak{q}) \times \mathrm{Cl}(V,\mathfrak{q}) \to \mathbb{F}, \qquad \bar{\mathfrak{b}}(x,y) := \zeta(\beta(x)y). \tag{162}$$

We also define the corresponding quadratic form on $\mathrm{Cl}(V,\mathfrak{q})$ via:

$$\bar{\mathfrak{q}}:\ \mathrm{Cl}(V,\mathfrak{q}) \to \mathbb{F}, \qquad \bar{\mathfrak{q}}(x) := \bar{\mathfrak{b}}(x,x) = \zeta(\beta(x)x). \tag{163}$$

We will see below that $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{q}}$ will agree with $\mathfrak{b}$ and $\mathfrak{q}$, resp., when they are restricted to V. From that point on we will denote $\bar{\mathfrak{b}}$ just by $\mathfrak{b}$, and $\bar{\mathfrak{q}}$ with $\mathfrak{q}$, resp., without (much) ambiguity.

Lemma D.25. *For $v, w \in V$ we have:*

$$\bar{\mathfrak{b}}(v,w) = \mathfrak{b}(v,w). \tag{164}$$

Proof. We pick an orthogonal basis $e_1, \dots, e_n$ for V and write:

$$v = \sum_{i=1}^{n} a_i \cdot e_i, \qquad w = \sum_{j=1}^{n} c_j \cdot e_j. \tag{165}$$

We then get by linearity:

$$\bar{\mathfrak{b}}(v,w) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_j \cdot \bar{\mathfrak{b}}(e_i, e_j) \tag{166}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_j \cdot \zeta(\beta(e_i)e_j) \tag{167}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_j \cdot \zeta(e_i e_j) \tag{168}$$
$$= \sum_{i\neq j} a_i c_j \cdot \underbrace{\zeta(e_i e_j)}_{=0} + \sum_{i=j} a_i c_j \cdot \zeta(e_i e_j) \tag{169}$$
$$= \sum_{i=1}^{n} a_i c_i \cdot \mathfrak{q}(e_i) \cdot \underbrace{\zeta(1)}_{=1} \tag{170}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_j \cdot \overbrace{\mathfrak{b}(e_i, e_j)}^{\mathfrak{q}(e_i)\cdot\delta_{i,j}=} \tag{171}$$
$$= \mathfrak{b}\left(\sum_{i=1}^{n} a_i \cdot e_i, \sum_{j=1}^{n} c_j \cdot e_j\right) \tag{172}$$
$$= \mathfrak{b}(v,w). \tag{173}$$

This shows the claim. □

Theorem D.26. *Let $e_1, \dots, e_n$ be an orthogonal basis for $(V, \mathfrak{q})$ then $(e_A)_{A\subseteq[n]}$ is an orthogonal basis for $\mathrm{Cl}(V, \mathfrak{q})$ w.r.t. the induced bilinear form $\bar{\mathfrak{b}}$. Furthermore, for $x, y \in \mathrm{Cl}(V, \mathfrak{q})$ of the form:*

$$x = \sum_{A\subseteq[n]} x_A \cdot e_A, \qquad y = \sum_{A\subseteq[n]} y_A \cdot e_A, \tag{174}$$

with $x_A, y_A \in \mathbb{F}$ we get:

$$\bar{\mathfrak{b}}(x,y) = \sum_{A\subseteq[n]} x_A \cdot y_A \cdot \prod_{i\in A} \mathfrak{q}(e_i), \qquad \bar{\mathfrak{q}}(x) = \sum_{A\subseteq[n]} x_A^2 \cdot \prod_{i\in A} \mathfrak{q}(e_i). \tag{175}$$

Note that: $\bar{\mathfrak{q}}(e_\emptyset) = \bar{\mathfrak{q}}(1) = 1$.

Proof. We already know that $(e_A)_{A\subseteq[n]}$ is a basis for $\mathrm{Cl}(V, \mathfrak{q})$. So we only need to check the orthogonality condition. First note that for $e_C = e_{i_1} \cdots e_{i_r}$ we get:

$$\bar{\mathfrak{q}}(e_C) = \zeta(\beta(e_C)e_C) = \zeta(e_{i_r} \cdots e_{i_1} \cdot e_{i_1} \cdots e_{i_r}) = \mathfrak{q}(e_{i_1}) \cdots \mathfrak{q}(e_{i_r}). \tag{176}$$

Now let $A, B \subseteq [n]$ with $A \neq B$, i.e. with $A \triangle B \neq \emptyset$. We then get:

$$\bar{\mathfrak{b}}(e_A, e_B) = \zeta(\beta(e_A)e_B) = \pm \prod_{i\in A\cap B} \mathfrak{q}(e_i) \cdot \zeta(e_{A\triangle B}) = 0. \tag{177}$$

This shows that $(e_A)_{A\subseteq[n]}$ is an orthogonal basis for $\mathrm{Cl}(V,\mathfrak{q})$. For $x, y \in \mathrm{Cl}(V,\mathfrak{q})$ from above we get:

$$\bar{\mathfrak{b}}(x,y) = \bar{\mathfrak{b}}\left(\sum_{A\subseteq[n]} x_A \cdot e_A, \sum_{B\subseteq[n]} y_B \cdot e_B\right) \tag{178}$$

$$= \sum_{A\subseteq[n]} \sum_{B\subseteq[n]} x_A \cdot y_B \cdot \bar{\mathfrak{b}}(e_A, e_B) \tag{179}$$

$$= \sum_{A\subseteq[n]} \sum_{B\subseteq[n]} x_A \cdot y_B \cdot \bar{\mathfrak{q}}(e_A) \cdot \delta_{A,B} \tag{180}$$

$$= \sum_{A\subseteq[n]} x_A \cdot y_A \cdot \bar{\mathfrak{q}}(e_A) \tag{181}$$

$$= \sum_{A\subseteq[n]} x_A \cdot y_A \cdot \prod_{i\in A} \mathfrak{q}(e_i). \tag{182}$$

This shows the claim. □

D.6 The Multivector Grading

Now that we have an orthogonal basis for the algebra, we show that the Clifford algebra allows a vector space grading that is independent of the chosen orthogonal basis.

Let $(V,\mathfrak{q})$ be a quadratic space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and $\dim V = n < \infty$.

In the following we present a technical proof that works for fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. An alternative, simpler and more structured proof, but for the more restrictive case of $\mathrm{char}(\mathbb{F}) = 0$, can be found in Theorem D.37 later.

Theorem D.27 (The multivector grading of the Clifford algebra). *Let $e_1, \dots, e_n$ be an orthogonal basis of $(V,\mathfrak{q})$. Then for every $m = 0, \dots, n$ we define the following sub-vector space of $\mathrm{Cl}(V,\mathfrak{q})$:*

$$\mathrm{Cl}^{(m)}(V,\mathfrak{q}) := \mathrm{span}\left\{e_{i_1}\cdots e_{i_m} \,|\, 1 \leq i_1 < \cdots < i_m \leq n\right\} \tag{183}$$

$$= \mathrm{span}\left\{e_A \,|\, A \subseteq [n], |A| = m\right\}, \tag{184}$$

where $\mathrm{Cl}^{(0)}(V,\mathfrak{q}) := \mathbb{F}$.

Then the sub-vector spaces $\mathrm{Cl}^{(m)}(V,\mathfrak{q})$, $m = 0, \dots, n$, are independent of the choice of the orthogonal basis, i.e. if $b_1, \dots, b_n$ is another orthogonal basis of $(V,\mathfrak{q})$, then:

$$\mathrm{Cl}^{(m)}(V,\mathfrak{q}) = \mathrm{span}\left\{b_{i_1}\cdots b_{i_m} \,|\, 1 \leq i_1 < \cdots < i_m \leq n\right\}. \tag{185}$$

Proof. First note that by the orthogonality and the fundamental relation of the Clifford algebra we have for all $i \neq j$:

$$e_i e_j = -e_j e_i, \qquad\qquad b_i b_j = -b_j b_i. \tag{186}$$

We now abbreviate:

$$B^{(m)} := \mathrm{span}\left\{b_{j_1}\cdots b_{j_m} \,|\, 1 \leq j_1 < \cdots < j_m \leq n\right\}, \tag{187}$$

and note that:

$$\mathrm{Cl}^{(m)}(V,\mathfrak{q}) = \mathrm{span}\left\{e_{i_1}\cdots e_{i_m} \,|\, 1 \leq i_1 < \cdots < i_m \leq n\right\} \tag{188}$$

$$= \mathrm{span}\left\{e_{i_1}\cdots e_{i_m} \,|\, 1 \leq i_1, \dots, i_m \leq n, \forall s \neq t.\, i_s \neq i_t\right\}. \tag{189}$$

We want to show that for $1 \leq j_1 < \cdots < j_m \leq n$ we have that:

$$b_{j_1}\cdots b_{j_m} \in \mathrm{Cl}^{(m)}(V,\mathfrak{q}). \tag{190}$$

Since we have two bases we can write an orthogonal change of basis:

$$b_j = \sum_{i=1}^{n} a_{i,j} e_i \in V = \mathrm{Cl}^{(1)}(V,\mathfrak{q}). \tag{191}$$

Using this, we can now write the above product as the sum of two terms:

$$b_{j_1}\cdots b_{j_m} = \sum_{i_1,\dots,i_m} a_{i_1,j_1}\cdots a_{i_m,j_m}\cdot e_{i_1}\cdots e_{i_m} \tag{192}$$

$$= \sum_{\substack{i_1,\dots,i_m \\ \forall s\neq t.\, i_s\neq i_t}} a_{i_1,j_1}\cdots a_{i_m,j_m}\cdot e_{i_1}\cdots e_{i_m} + \sum_{\substack{i_1,\dots,i_m \\ \exists s\neq t.\, i_s= i_t}} a_{i_1,j_1}\cdots a_{i_m,j_m}\cdot e_{i_1}\cdots e_{i_m}. \tag{193}$$

Our claim is equivalent to the vanishing of the second term. Note that the above equation for b_j already shows the claim for $m = 1$. The case $m = 0$ is trivial.

We now prove the claim for $m = 2$ by hand before doing induction after. Recall that $j_1 \neq j_2$:

$$b_{j_1}b_{j_2} = \sum_{\substack{i_1,i_2 \\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2}\cdot e_{i_1}e_{i_2} + \sum_{i=1}^{n} a_{i,j_1}a_{i,j_2}\cdot e_i e_i \tag{194}$$

$$= \sum_{\substack{i_1,i_2 \\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2}\cdot e_{i_1}e_{i_2} + \sum_{i=1}^{n} a_{i,j_1}\underbrace{a_{i,j_2}\cdot \mathfrak{q}(e_i)}_{=\mathfrak{b}(e_i,b_{j_2})} \tag{195}$$

$$= \sum_{\substack{i_1,i_2 \\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2}\cdot e_{i_1}e_{i_2} + \underbrace{\mathfrak{b}(b_{j_1},b_{j_2})}_{=0} \tag{196}$$

$$= \sum_{\substack{i_1,i_2 \\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2}\cdot e_{i_1}e_{i_2} \tag{197}$$

$$\in \mathrm{Cl}^{(2)}(V,\mathfrak{q}). \tag{198}$$

This shows the claim for $m = 2$.

By way of induction we now assume that we have shown the claim until some $m \geq 2$, i.e. we have:

$$b_{j_1}\cdots b_{j_m} = \sum_{\substack{i_1,\dots,i_m \\ \forall k\neq l.\, i_k\neq i_l}} a_{i_1,j_1}\cdots a_{i_m,j_m} e_{i_1}\cdots e_{i_m} \in \mathrm{Cl}^{(m)}(V,\mathfrak{q}). \tag{199}$$

Now consider another b_j with $j := j_{m+1} \neq j_k$, $k = 1,\dots,m$. We then get:

$$b_{j_1} \cdots b_{j_m} b_j = \left(\sum_{\substack{i_1,\ldots,i_m \\ \forall k \neq l.\ i_k \neq i_l}} a_{i_1,j_1} \cdots a_{i_m,j_m} e_{i_1} \cdots e_{i_m} \right) \left(\sum_{i=1}^{n} a_{i,j} e_i \right) \tag{200}$$

$$= \sum_{i=1}^{n} \sum_{\substack{i_1,\ldots,i_m \\ \forall k \neq l.\ i_k \neq i_l}} a_{i_1,j_1} \cdots a_{i_m,j_m} a_{i,j} e_{i_1} \cdots e_{i_m} e_i \tag{201}$$

$$= \sum_{i=1}^{n} \sum_{\substack{i_1,\ldots,i_m \\ \forall k \neq l.\ i_k \neq i_l \\ i \notin \{i_1,\ldots,i_m\}}} a_{i_1,j_1} \cdots a_{i_m,j_m} a_{i,j} e_{i_1} \cdots e_{i_m} e_i \tag{202}$$

$$+ \sum_{i=1}^{n} \sum_{\substack{i_1,\ldots,i_m \\ \forall k \neq l.\ i_k \neq i_l \\ i \in \{i_1,\ldots,i_m\}}} a_{i_1,j_1} \cdots a_{i_m,j_m} a_{i,j} e_{i_1} \cdots e_{i_m} e_i \tag{203}$$

$$= \sum_{\substack{i_1,\ldots,i_m,i_{m+1} \\ \forall k \neq l.\ i_k \neq i_l}} a_{i_1,j_1} \cdots a_{i_m,j_m} a_{i_{m+1},j_{m+1}} e_{i_1} \cdots e_{i_m} e_{i_{m+1}} \tag{204}$$

$$+ \sum_{i=1}^{n} \sum_{s=1}^{m} \sum_{\substack{i_1,\ldots,i_m \\ \forall k \neq l.\ i_k \neq i_l \\ i_s = i}} a_{i_1,j_1} \cdots a_{i_m,j_m} a_{i,j} e_{i_1} \cdots e_{i_m} e_i. \tag{205}$$

We have to show that the last term vanishes. The last term can be written as:

$$\sum_{i=1}^{n}\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l\\ i_s=i}} a_{i_1,j_1}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots e_{i_m}e_i \tag{206}$$

$$=\sum_{i=1}^{n}\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l\\ i_s=i}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots e_{i_s}\cdots e_{i_m}e_i \tag{207}$$

$$=\sum_{i=1}^{n}\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l\\ i_s=i}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}e_{i_s}e_i\cdot(-1)^{m-s} \tag{208}$$

$$=\sum_{i=1}^{n}\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l\\ i_s=i}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_i) \tag{209}$$

$$=\sum_{s=1}^{m}\sum_{i=1}^{n}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l\\ i_s=i}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_i) \tag{210}$$

$$=\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i_s,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_s}) \tag{211}$$

$$=\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}}\ \sum_{\substack{i_s=1\\ i_s\notin\{i_1,..,\cancel{i_s},..,i_m\}}}^{n} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i_s,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_s}) \tag{212}$$

$$=\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}}\sum_{i_s=1}^{n} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i_s,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_s}) \tag{213}$$

$$-\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}}\ \sum_{i_s\in\{i_1,..,\cancel{i_s},..,i_m\}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i_s,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_s}) \tag{214}$$

$$=\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}} a_{i_1,j_1}\cdots \cancel{a_{i_s,j_s}}\cdots a_{i_m,j_m}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\underbrace{\sum_{i_s=1}^{n} a_{i_s,j_s}a_{i_s,j}\mathfrak{q}(e_{i_s})}_{=\mathfrak{b}(b_j,b_{j_s})=0} \tag{215}$$

$$-\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}}\ \sum_{i_s\in\{i_1,..,\cancel{i_s},..,i_m\}} a_{i_1,j_1}\cdots a_{i_s,j_s}\cdots a_{i_m,j_m}a_{i_s,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_s}) \tag{216}$$

$$=-\sum_{s=1}^{m}\sum_{\substack{i_1,\ldots,\cancel{i_s},\ldots,i_m\\ \forall k\neq l.\ i_k\neq i_l}}\ \sum_{i\in\{i_1,..,\cancel{i_s},..,i_m\}} a_{i_1,j_1}\cdots a_{i,j_s}\cdots a_{i_m,j_m}a_{i,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_i) \tag{217}$$

$$=-\sum_{s=1}^{m}\sum_{\substack{t=1\\ t\neq s}}^{m}\sum_{\substack{i_1,..,\cancel{i_s},..,i_t,..,i_m\\ \forall k\neq l.\ i_k\neq i_l}} a_{i_1,j_1}\cdots a_{i_t,j_s}\cdots a_{i_m,j_m}a_{i_t,j}\cdot e_{i_1}\cdots \cancel{e_{i_s}}\cdots e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_t}). \tag{218}$$

Note that the positional index t can occure before or after the positional index s. Depending of its position the elements e_{i_t} will appear before or after the element e_{i_s} in the product. We look at both cases separately and suppress the dots in between for readability.

First consider $t > s$:

$$\sum_{s=1}^{m} \sum_{\substack{t=1 \\ t>s}}^{m} \sum_{\substack{i_1,..,\cancel{i_s},..,i_t,..,i_m \\ \forall k \neq l.\, i_k \neq i_l}} a_{i_1,j_1} \cdot a_{i_t,j_s} \cdot a_{i_t,j_t} \cdot a_{i_m,j_m} a_{i_t,j} \cdot e_{i_1} \cdot \cancel{e_{i_s}} \cdot e_{i_t} \cdot e_{i_m} \cdot (-1)^{m-s} \mathfrak{q}(e_{i_t}) \tag{219}$$

$$= \sum_{s=1}^{m} \sum_{\substack{t=1 \\ t>s}}^{m} \sum_{\substack{i_1,..,\cancel{i_s},..,i_t,..,i_m \\ \forall k \neq l.\, i_k \neq i_l}} a_{i_1,j_1} \cdot a_{i_t,j_s} \cdot a_{i_t,j_t} \cdot a_{i_m,j_m} a_{i_t,j} \cdot (-1)^{t-2} e_{i_t} \cdot e_{i_1} \cdot \cancel{e_{i_s}} \cdot \cancel{e_{i_t}} \cdot e_{i_m} \cdot (-1)^{m-s} \mathfrak{q}(e_{i_t}) \tag{220}$$

$$= \sum_{s=1}^{m} \sum_{\substack{t=1 \\ t>s}}^{m} \sum_{i_t=1}^{n} a_{i_t,j_s} \cdot a_{i_t,j_t} \cdot a_{i_t,j} (-1)^{m-s} \mathfrak{q}(e_{i_t}) (-1)^{t} e_{i_t} \cdot \pi_{s,t}(i_t) \tag{221}$$

$$\text{with} \quad \pi_{s,t}(i) := \sum_{\substack{i_1,..,\cancel{i_s},..,\cancel{i_t},..,i_m \\ \forall k \neq l.\, i_k \neq i_l \\ \forall k.\, i_k \neq i}} a_{i_1,j_1} \cdot \cancel{a_{i_t,j_s}} \cdot \cancel{a_{i_t,j_t}} \cdot a_{i_m,j_m} \cdot e_{i_1} \cdot \cancel{e_{i_s}} \cdot \cancel{e_{i_t}} \cdot e_{i_m} \tag{222}$$

$$= \sum_{s=1}^{m} \sum_{\substack{t=1 \\ t>s}}^{m} \sum_{i=1}^{n} (-1)^{m+s+t} a_{i,j_s} \cdot a_{i,j_t} \cdot a_{i,j} \cdot \mathfrak{q}(e_i) \cdot e_i \cdot \pi_{s,t}(i) \tag{223}$$

$$= \sum_{s=1}^{m} \sum_{\substack{t=1 \\ t>s}}^{m} y(s,t), \tag{224}$$

$$\text{with} \quad y(s,t) := \sum_{i=1}^{n} (-1)^{m+s+t} a_{i,j_s} \cdot a_{i,j_t} \cdot a_{i,j} \cdot \mathfrak{q}(e_i) \cdot e_i \cdot \pi_{s,t}(i). \tag{225}$$

It is important to note that for all $s \neq t$ we have:

$$y(s,t) = y(t,s). \tag{226}$$

Now consider $t < s$:

$$\sum_{s=1}^{m}\sum_{\substack{t=1\\t<s}}^{m}\sum_{\substack{i_1,.,i_t,.,\cancel{i_s},.,i_m\\\forall k\neq l.\ i_k\neq i_l}} a_{i_1,j_1}\cdot a_{i_t,j_t}\cdot a_{i_t,j_s}\cdot a_{i_m,j_m}a_{i_t,j}\cdot e_{i_1}\cdot e_{i_t}\cdot \cancel{e_{i_s}}\cdot e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_t}) \tag{227}$$

$$= \sum_{s=1}^{m}\sum_{\substack{t=1\\t<s}}^{m}\sum_{\substack{i_1,.,i_t,.,\cancel{i_s},.,i_m\\\forall k\neq l.\ i_k\neq i_l}} a_{i_1,j_1}\cdot a_{i_t,j_t}\cdot a_{i_t,j_s}\cdot a_{i_m,j_m}a_{i_t,j}\cdot(-1)^{t-1}e_{i_t}\cdot e_{i_1}\cdot \cancel{e_{i_t}}\cdot \cancel{e_{i_s}}\cdot e_{i_m}\cdot(-1)^{m-s}\mathfrak{q}(e_{i_t}) \tag{228}$$

$$= \sum_{s=1}^{m}\sum_{\substack{t=1\\t<s}}^{m}\sum_{i_t=1}^{n} a_{i_t,j_s}\cdot a_{i_t,j_t}\cdot a_{i_t,j}(-1)^{m-s}\mathfrak{q}(e_{i_t})(-1)^{t-1}e_{i_t}\cdot\pi_{s,t}(i_t) \tag{229}$$

$$= \sum_{s=1}^{m}\sum_{\substack{t=1\\t<s}}^{m}\sum_{i=1}^{n}(-1)^{m+s+t+1}a_{i,j_s}\cdot a_{i,j_t}\cdot a_{i,j}\cdot\mathfrak{q}(e_i)\cdot e_i\cdot\pi_{s,t}(i) \tag{230}$$

$$= -\sum_{s=1}^{m}\sum_{\substack{t=1\\t<s}}^{m} y(s,t) \tag{231}$$

$$= -\sum_{t=1}^{m}\sum_{\substack{s=1\\s<t}}^{m} y(t,s) \tag{232}$$

$$= -\sum_{t=1}^{m}\sum_{\substack{s=1\\s<t}}^{m} y(s,t) \tag{233}$$

$$= -\sum_{s=1}^{m}\sum_{\substack{t=1\\t>s}}^{m} y(s,t). \tag{234}$$

In total we see that both terms appear with a different sign and thus cancel out. This shows the claim. □

Corollary D.28. *Let $e_1,\dots,e_n$ and $b_1,\dots,b_n$ be two orthogonal bases of $(V,\mathfrak{q})$ with basis transition matrix $C=(c_{i,j})_{i\in[n],j\in[n]}$:*

$$\forall j\in[n].\qquad b_j=\sum_{i\in[n]} c_{i,j}\cdot e_i. \tag{235}$$

Then C is invertible and we have the following matrix relations:

$$\operatorname{diag}(\mathfrak{q}(b_1),\dots,\mathfrak{q}(b_n)) = C^\top\operatorname{diag}(\mathfrak{q}(e_1),\dots,\mathfrak{q}(e_n))C. \tag{236}$$

Furthermore, for every subset $J\subseteq[n]$ we have the formula:

$$b_J=\sum_{\substack{I\subseteq[n]\\|I|=|J|}}\det C_{I,J}\cdot e_I, \tag{237}$$

with the submatrix: $C_{I,J}=(c_{i,j})_{i\in I,j\in J}$.

Proof. For $j, l \in [n]$ we have:

$$\operatorname{diag}(\mathfrak{q}(b_1), \ldots, \mathfrak{q}(b_n))_{j,l} = \mathfrak{q}(b_j) \cdot \delta_{j,l} \tag{238}$$

$$= \mathfrak{b}(b_j, b_l) \tag{239}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} c_{i,j} \cdot c_{k,l} \cdot \mathfrak{b}(e_i, e_k) \tag{240}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} c_{i,j} \cdot c_{k,l} \cdot \mathfrak{q}(e_i) \cdot \delta_{i,k} \tag{241}$$

$$= \left(C^\top \operatorname{diag}(\mathfrak{q}(e_1), \ldots, \mathfrak{q}(e_n))C\right)_{j,l}. \tag{242}$$

This shows the matrix identity:

$$\operatorname{diag}(\mathfrak{q}(b_1), \ldots, \mathfrak{q}(b_n)) = C^\top \operatorname{diag}(\mathfrak{q}(e_1), \ldots, \mathfrak{q}(e_n))C. \tag{243}$$

Furthermore, we have the following identites:

$$b_J = b_{j_1} \cdots b_{j_m} \tag{244}$$

$$= \left(\sum_{i_1 \in [n]} c_{i_1,j_1} \cdot e_{i_1}\right) \cdots \left(\sum_{i_m \in [n]} c_{i_m,j_m} \cdot e_{i_m}\right) \tag{245}$$

$$= \sum_{i_1 \in [n], \ldots, i_m \in [n]} (c_{i_1,j_1} \cdots c_{i_m,j_m}) \cdot (e_{i_1} \cdots e_{i_m}) \tag{246}$$

$$\overset{D.27}{=} \sum_{\substack{i_1 \in [n], \ldots, i_m \in [n] \\ |\{i_1, \ldots, i_m\}| = m}} (c_{i_1,j_1} \cdots c_{i_m,j_m}) \cdot (e_{i_1} \cdots e_{i_m}) \tag{247}$$

$$= \sum_{\substack{i_1 \in [n], \ldots, i_m \in [n] \\ |\{i_1, \ldots, i_m\}| = m}} \operatorname{sgn}(i_1, \ldots, i_m) \cdot (c_{i_1,j_1} \cdots c_{i_m,j_m}) \cdot e_{\{i_1, \ldots, i_m\}} \tag{248}$$

$$= \sum_{\substack{i_1, \ldots, i_m \in [n] \\ i_1 < \cdots < i_m}} \left(\sum_{\sigma \in \mathrm{S}_m} \operatorname{sgn}(\sigma) \cdot c_{i_{\sigma(1)},j_1} \cdots c_{i_{\sigma(m)},j_m}\right) \cdot e_{\{i_1, \ldots, i_m\}} \tag{249}$$

$$= \sum_{\substack{I \subseteq [n] \\ |I| = m}} \det C_{I,J} \cdot e_I. \tag{250}$$

This shows the claim. □

Notation D.29. *For $m \notin \{0, \ldots, n\}$ it is sometimes convenient to put:*

$$\mathrm{Cl}^{(m)}(V, \mathfrak{q}) := 0. \tag{251}$$

Corollary D.30. *We have the following orthogonal sum decomposition (w.r.t. $\bar{\mathfrak{q}}$) of the Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ into its $\mathbb{F}$-vector spaces of multivector components:*

$$\mathrm{Cl}(V, \mathfrak{q}) = \bigoplus_{m=0}^{n} \mathrm{Cl}^{(m)}(V, \mathfrak{q}), \tag{252}$$

which is independent of the choice of orthogonal basis of $(V, \mathfrak{q})$. Also note that for all $m = 0, \ldots, n$:

$$\dim \mathrm{Cl}^{(m)}(V, \mathfrak{q}) = \binom{n}{m}. \tag{253}$$

Definition D.31. *We call an element $x \in \mathrm{Cl}^{(m)}(V, \mathfrak{q})$ an m-*multivector *or an element of* $\mathrm{Cl}(V, \mathfrak{q})$ *of* pure grade m*. For $x \in \mathrm{Cl}(V, \mathfrak{q})$ we have a decomposition:*

$$x = x^{(0)} + x^{(1)} + \cdots + x^{(n)}, \tag{254}$$

with $x^{(m)} \in \mathrm{Cl}^{(m)}(V, \mathfrak{q})$, $m = 0, \ldots, n$. We call $x^{(m)}$ the grade-m-component *of x.*

Remark D.32. *Note that the multivector grading of* $\mathrm{Cl}(V, \mathfrak{q})$ *is only a grading of* $\mathbb{F}$*-vector spaces, but not of* $\mathbb{F}$*-algebras. The reason is that multiplication can make the grade drop. For instance, for* $v \in V = \mathrm{Cl}^{(1)}(V, \mathfrak{q})$ *we have* $vv = \mathfrak{q}(v) \in \mathrm{Cl}^{(0)}(V, \mathfrak{q})$*, while a grading for algebras would require that* $vv \in \mathrm{Cl}^{(2)}(V, \mathfrak{q})$*, which is here not the case.*

Remark D.33 (Parity grading and multivector filtation in terms of multivector grading)**.**

1. *We clearly have:*

$$\mathrm{Cl}^{[0]}(V, \mathfrak{q}) = \bigoplus_{\substack{m=0,\dots,n \\ m \text{ even}}} \mathrm{Cl}^{(m)}(V, \mathfrak{q}), \qquad \mathrm{Cl}^{[1]}(V, \mathfrak{q}) = \bigoplus_{\substack{m=1,\dots,n \\ m \text{ odd}}} \mathrm{Cl}^{(m)}(V, \mathfrak{q}). \tag{255}$$

2. *It is also clear that for general* m *we have:*

$$\mathrm{Cl}^{(\leq m)}(V, \mathfrak{q}) = \bigoplus_{l=0}^{m} \mathrm{Cl}^{(m)}(V, \mathfrak{q}) \quad \subseteq \quad \mathrm{Cl}(V, \mathfrak{q}). \tag{256}$$

A simpler proof of Theorem D.27 can be obtained if we assume that $\mathrm{char}(\mathbb{F}) = 0$. We would then argue as follows.

Definition D.34 (Antisymmetrization)**.** *For* $m \in \mathbb{N}$ *and* $x_1, \dots, x_m \in \mathrm{Cl}(V, \mathfrak{q})$ *we define their* antisymmetization *as:*

$$[x_1; \dots; x_m] := \frac{1}{m!} \sum_{\sigma \in \mathrm{S}_m} \mathrm{sgn}(\sigma) \cdot x_{\sigma(1)} \cdots x_{\sigma(m)}, \tag{257}$$

where S_m *denotes the group of all permuations of* $[m]$*. Note that, due to the division by* $m!$ *we need that* $\mathrm{char}(\mathbb{F}) = 0$ *if we want to accommodate arbitrary* $m \in \mathbb{N}$*.*

Lemma D.35. *Let* $e_1, \dots, e_n$ *be an orthogonal basis of* $(V, \mathfrak{q})$ *and* $x_1, \dots, x_m \in \mathrm{Cl}(V, \mathfrak{q})$*. Then we have:*

1. $[x_1; \dots; x_m]$ *is linear in each of its arguments (if the other arguments are fixed).*
2. $[x_1; \dots; x_k; \dots; x_l; \dots; x_m] = -[x_1; \dots; x_l; \dots; x_k; \dots; x_m]$.
3. $[x_1; \dots; x_k; \dots; x_l; \dots; x_m] = 0$ *if* $x_k = x_l$.
4. $[e_{i_1}; \dots; e_{i_m}] = e_{i_1} \cdots e_{i_m}$ *if* $|\{i_1, \dots, i_m\}| = m$ *(i.e. if all indices are different).*

Definition D.36 (Multivector grading - alternative, basis independent definition)**.** *For* $m = 0, \dots, n$ *we (re-)define:*

$$\mathrm{Cl}^{(m)}(V, \mathfrak{q}) := \mathrm{span}\left\{[v_1; \dots; v_m] \,|\, v_1, \dots, v_m \in V\right\}. \tag{258}$$

Theorem D.37. *Let* $e_1, \dots, e_n$ *be an orthogonal basis of* $(V, \mathfrak{q})$*,* $\mathrm{char}(\mathbb{F}) = 0$*. Then for every* $m = 0, \dots, n$ *we have the equality:*

$$\mathrm{Cl}^{(m)}(V, \mathfrak{q}) = \mathrm{span}\left\{e_A \,|\, A \subseteq [n], |A| = m\right\}. \tag{259}$$

Note that the rhs is seemingly dependent of the choice of the orthogonal basis while the lhs is defined in a basis independent way.

Proof. We can write every $v_k \in V$ as a linear combination of its basis vectors:

$$v_k = \sum_{j_k \in [n]} c_{k,j_k} \cdot e_{j_k}. \tag{260}$$

With this we get:

$$[v_1;\dots;v_m] = \sum_{j_1,\dots,j_m\in[n]} \prod_{k\in[m]} c_{k,j_k}\cdot[e_{j_1};\dots;e_{j_m}] \tag{261}$$

$$= \sum_{\substack{j_1,\dots,j_m\in[n]\\ |\{j_1,\dots,j_m\}|=m}} \prod_{k\in[m]} c_{k,j_k}\cdot[e_{j_1};\dots;e_{j_m}] \tag{262}$$

$$= \sum_{\substack{j_1,\dots,j_m\in[n]\\ |\{j_1,\dots,j_m\}|=m}} \prod_{k\in[m]} c_{k,j_k}\cdot e_{j_1}\cdots e_{j_m} \tag{263}$$

$$= \sum_{\substack{j_1,\dots,j_m\in[n]\\ |\{j_1,\dots,j_m\}|=m}} \pm\prod_{k\in[m]} c_{k,j_k}\cdot e_{\{j_1,\dots,j_m\}} \tag{264}$$

$$\in \operatorname{span}\{e_A \mid A\subseteq[n], |A|=m\}. \tag{265}$$

This shows the inclusion:

$$\mathrm{Cl}^{(m)}(V,\mathfrak{q}) \subseteq \operatorname{span}\{e_A \mid A\subseteq[n], |A|=m\}. \tag{266}$$

The reverse inclusion is also clear as:

$$e_A = [e_{j_1};\dots;e_{j_m}] \in \mathrm{Cl}^{(m)}(V,\mathfrak{q}), \tag{267}$$

where $A=\{j_1,\dots,j_m\}$ and $|A|=m$. This shows the equality of both sets. □

D.7 The Radical Subalgebra of the Clifford Algebra

Again, let $(V,\mathfrak{q})$ be a quadratic vector space of finite dimensions $\dim V = n < \infty$ over a field $\mathbb{F}$ of $\mathrm{char}(\mathbb{F})\neq 2$. Let $\mathfrak{b}$ the corresponding bilinear form of $\mathfrak{q}$.

Notation D.38. *We denote the group of the* invertible elements *of* $\mathrm{Cl}(V,\mathfrak{q})$*:*

$$\mathrm{Cl}^{\times}(V,\mathfrak{q}) := \{x\in\mathrm{Cl}(V,\mathfrak{q}) \mid \exists y\in\mathrm{Cl}(V,\mathfrak{q}).\, xy=yx=1\}. \tag{268}$$

Let $\mathcal{R}\subseteq V$ be V's radical subspace. Recall:

$$\mathcal{R} := \{f\in V \mid \forall v\in V.\, \mathfrak{b}(f,v)=0\}. \tag{269}$$

Definition D.39 (The radical subalgebra)**.** *We define the* radical subalgebra *of* $\mathrm{Cl}(V,\mathfrak{q})$ *to be:*

$$\textstyle\bigwedge(\mathcal{R}) := \operatorname{span}\{1, f_1\cdots f_k \mid k\in\mathbb{N}_0, f_l\in\mathcal{R}, l=1,\dots,k\} \subseteq \mathrm{Cl}(V,\mathfrak{q}). \tag{270}$$

Note that $\mathfrak{q}|_{\mathcal{R}}=0$ *and that* $\bigwedge(\mathcal{R})$ *coincides with* $\mathrm{Cl}(\mathcal{R},\mathfrak{q}|_{\mathcal{R}})$.

Notation D.40. *We make the following further abbreviations:*

$$\textstyle\bigwedge^{[i]}(\mathcal{R}) := \bigwedge(\mathcal{R})\cap\mathrm{Cl}^{[i]}(V,\mathfrak{q}), \tag{271}$$

$$\textstyle\bigwedge^{(\geq 1)}(\mathcal{R}) := \operatorname{span}\{f_1\cdots f_k \mid k\geq 1, f_l\in\mathcal{R}, l=1,\dots,k\}, \tag{272}$$

$$\textstyle\bigwedge^{\times}(\mathcal{R}) := \mathbb{F}^{\times} + \bigwedge^{(\geq 1)}(\mathcal{R}), \tag{273}$$

$$\textstyle\bigwedge^{[\times]}(\mathcal{R}) := \mathbb{F}^{\times} + \operatorname{span}\{f_1\cdots f_k \mid k\geq 2 \textit{ even}, f_l\in\mathcal{R}, l=1,\dots,k\}, \tag{274}$$

$$\textstyle\bigwedge^{*}(\mathcal{R}) := 1 + \bigwedge^{(\geq 1)}(\mathcal{R}), \tag{275}$$

$$\textstyle\bigwedge^{[*]}(\mathcal{R}) := 1 + \operatorname{span}\{f_1\cdots f_k \mid k\geq 2 \textit{ even}, f_l\in\mathcal{R}, l=1,\dots,k\}. \tag{276}$$

Here, $\mathbb{F}^{\times}$ denotes the set of invertible elements of $\mathbb{F}$.

Lemma D.41. *1. For every* $h\in\bigwedge^{(\geq 1)}(\mathcal{R})$ *there exists a* $k\geq 0$ *such that:*

$$h^{k+1}=0. \tag{277}$$

In particular, no $h\in\bigwedge^{(\geq 1)}(\mathcal{R})$ *is ever invertible.*

2. *Every* $y \in \bigwedge^{\times}(\mathcal{R}) = \bigwedge(\mathcal{R}) \setminus \bigwedge^{(\geq 1)}(\mathcal{R})$ *is invertible. Its inverse is given by:*

$$y^{-1} = c^{-1}\left(1 - h + h^2 - \cdots + (-1)^k h^k\right), \tag{278}$$

where we write: $y = c \cdot (1+h)$ *with* $c \in \mathbb{F}^{\times}$, $h \in \bigwedge^{(\geq 1)}(\mathcal{R})$, *and* k *is such that* $h^{k+1} = 0$.

3. *In particular, we get:*

$$\bigwedge^{\times}(\mathcal{R}) = \bigwedge(\mathcal{R}) \cap \mathrm{Cl}^{\times}(V, \mathfrak{q}). \tag{279}$$

Proof. Items 1 and 3 are clear. For item 2 we refer to Example E.21. □

Lemma D.42 (Twisted commutation relationships)**.** 1. *For every* $f \in \mathcal{R}$ *and* $v \in V$ *we have the anticommutation relationship:*

$$fv = -vf + 2\underbrace{\mathfrak{b}(f, v)}_{=0} = -vf. \tag{280}$$

2. *For every* $f \in \mathcal{R}$ *and* $x \in \mathrm{Cl}(V, \mathfrak{q})$ *we get the following twisted commutation relationship:*

$$\alpha(x)f = (x^{[0]} - x^{[1]})f = f(x^{[0]} + x^{[1]}) = fx. \tag{281}$$

3. *For every* $y \in \bigwedge(\mathcal{R})$ *and every* $x \in \mathrm{Cl}^{[0]}(V, \mathfrak{q})$ *we get:*

$$xy = yx. \tag{282}$$

4. *For every* $y \in \bigwedge(\mathcal{R})$ *and* $v \in V$ *we get:*

$$\alpha(y)v = (y^{[0]} - y^{[1]})v = v(y^{[0]} + y^{[1]}) = vy. \tag{283}$$

5. *For every* $y \in \bigwedge^{[0]}(\mathcal{R})$ *(of even parity) and every* $x \in \mathrm{Cl}(V, \mathfrak{q})$ *we get:*

$$yx = xy. \tag{284}$$

Remark D.43. *A direct consequence from Lemma D.42 is that the even parity parts:* $\bigwedge^{[0]}(\mathcal{R})$, $\bigwedge^{[\times]}(\mathcal{R})$ *and* $\bigwedge^{[*]}(\mathcal{R})$ *all lie in the center of* $\mathrm{Cl}(V, \mathfrak{q})$*, which we denote by* $\mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$*:*

$$\bigwedge^{[*]}(\mathcal{R}) \subseteq \bigwedge^{[\times]}(\mathcal{R}) \subseteq \bigwedge^{[0]}(\mathcal{R}) \subseteq \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q})), \tag{285}$$

i.e. every $y \in \bigwedge^{[0]}(\mathcal{R})$ *(of even parity) commutes with every* $x \in \mathrm{Cl}(V, \mathfrak{q})$*. For more detail we refer to Theorem D.47.*

In the following we will study the center of the Clifford algebra $\mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$ more carefully.

D.8 The Center of the Clifford Algebra

In the following final subsections, we study additional properties of the Clifford algebra that will aid us in studying group representations and actions on the algebra in the upcoming sections.

Let $(V, \mathfrak{q})$ be a quadratic space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and $\dim V = n < \infty$.

Definition D.44 (The center of an algebra)**.** *The* center *of an algebra* $\mathcal{A}$ *is defined to be:*

$$\mathfrak{Z}(\mathcal{A}) := \{z \in \mathcal{A} \,|\, \forall x \in \mathcal{A}.\, xz = zx\}. \tag{286}$$

Lemma D.45. *Let* $e_1, \ldots, e_n$ *be an orthogonal basis of* $(V, \mathfrak{q})$*. For* $A \subseteq [n] := \{1, \ldots, n\}$ *let* $e_A := \prod_{i \in A}^{<} e_i$ *be the product in* $\mathrm{Cl}(V, \mathfrak{q})$ *in increasing index order,* $e_\emptyset := 1$*. Then we get for two subsets* $A, B \subseteq [n]$*:*

$$e_A e_B = (-1)^{|A| \cdot |B| - |A \cap B|} \cdot e_B e_A. \tag{287}$$

In particular, for $j \notin A$ we get:

$$e_A e_j = (-1)^{|A|} \cdot e_j e_A, \tag{288}$$

and:

$$e_A e_j - e_j e_A = ((-1)^{|A|} - 1) \cdot e_j e_A = (-1)^t ((-1)^{|A|+1} + 1) e_{A \dot\cup \{j\}}, \tag{289}$$

where t is the position of j in the ordered set $A \dot\cup \{j\}$.

For $i \in A$ we get:

$$e_A e_i = (-1)^{|A|-1} \cdot e_i e_A = (-1)^{|A|-s} \mathfrak{q}(e_i) e_{A \setminus \{i\}}, \tag{290}$$

and:

$$e_A e_i - e_i e_A = (-1)^s ((-1)^{|A|} + 1) \mathfrak{q}(e_i) e_{A \setminus \{i\}}, \tag{291}$$

where s is the position of i in the ordered set A.

Proof. Let $B := \{j_1, \dots, j_{|B|}\} \subseteq [n]$.

$$e_A e_B = \prod_{i \in A}^{<} e_i \prod_{j \in B}^{<} e_j \tag{292}$$

$$= (-1)^{|A| - \mathbb{1}[j_1 \in A]} e_{j_1} \prod_{i \in A}^{<} e_i \prod_{j \in B \setminus j_1}^{<} e_j \tag{293}$$

$$= (-1)^{|B||A| - \sum_{j \in B} \mathbb{1}[j \in A]} e_B e_A \tag{294}$$

$$= (-1)^{|B||A| - |A \cap B|} e_B e_A \tag{295}$$

□

For the other two identities, we similarly make use of the fundamental Clifford identity.

Lemma D.46. *Let $(V, \mathfrak{q})$ be a quadratic space with $\dim V = n < \infty$. Let $e_1, \dots, e_n$ be an orthogonal basis for $(V, \mathfrak{q})$. For $x \in \mathrm{Cl}(V, \mathfrak{q})$ we have the equivalence:*

$$x \in \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q})) \qquad \iff \qquad \forall i \in [n]. \quad x e_i = e_i x. \tag{296}$$

Proof. This is clear as $\mathrm{Cl}(V, \mathfrak{q})$ is generated by $e_{k_1} \cdots e_{k_l}$.

□

Theorem D.47 (The center of the Clifford algebra)**.** *Let $(V, \mathfrak{q})$ be a quadratic space with $\dim V = n < \infty$, $\mathrm{char}\, \mathbb{F} \neq 2$, and let $\mathcal{R} \subseteq V$ be the radical subspace of $(V, \mathfrak{q})$. Then for the center of $\mathrm{Cl}(V, \mathfrak{q})$ we have the following cases:*

1. *If n is odd then:*

$$\mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q})) = \bigwedge^{[0]}(\mathcal{R}) \oplus \mathrm{Cl}^{(n)}(V, \mathfrak{q}). \tag{297}$$

2. *If n is even then:*

$$\mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q})) = \bigwedge^{[0]}(\mathcal{R}). \tag{298}$$

In all cases we have:

$$\bigwedge^{[0]}(\mathcal{R}) \subseteq \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q})). \tag{299}$$

Proof. Let $e_1, \dots, e_n$ be an orthogonal basis for $(V, \mathfrak{q})$. The statement can then equivalently be expressed as:

1. If n is odd or $\mathfrak{q} = 0$ (on all vectors), then:
$$\mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q})) = \mathrm{span}\left\{1, e_A, e_{[n]} \,\middle|\, |A| \text{ even }, \forall i \in A.\, \mathfrak{q}(e_i) = 0\right\} \tag{300}$$
$$= \bigwedge\nolimits^{[0]}(\mathcal{R}) + \mathrm{Cl}^{(n)}(V,\mathfrak{q}). \tag{301}$$

2. If n is even and $\mathfrak{q} \neq 0$ (on some vector), then:
$$\mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q})) = \mathrm{span}\left\{1, e_A \,|\, |A| \text{ even }, \forall i \in A.\, \mathfrak{q}(e_i) = 0\right\} \tag{302}$$
$$= \bigwedge\nolimits^{[0]}(\mathcal{R}). \tag{303}$$

Note that if n is even and $\mathfrak{q} = 0$ then both points would give the same answer, as then $V = \mathcal{R}$ and thus:
$$\mathrm{Cl}^{(n)}(V,\mathfrak{q}) = \bigwedge\nolimits^{(n)}(\mathcal{R}) \subseteq \bigwedge\nolimits^{[0]}(\mathcal{R}). \tag{304}$$

We first only consider basis elements e_A with $A \subseteq [n]$ and check when we have $e_A \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$.

We now consider three cases:

1. $A = \emptyset$. We always have: $e_\emptyset = 1 \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$.
2. $A = [n]$. For every $i \in [n]$ we have:
$$e_{[n]}e_i = (-1)^{n-1} \cdot e_i e_{[n]} = \pm\mathfrak{q}(e_i) \cdot e_{[n]\setminus\{i\}}. \tag{305}$$
So $e_{[n]} \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$ iff either n is odd or $\mathfrak{q} = 0$ (on all e_i).
3. $\emptyset \neq A \subsetneq [n]$. Note that for $i \in A$ we have:
$$e_A e_i = (-1)^{|A|-1} \cdot e_i e_A = \pm\mathfrak{q}(e_i) \cdot e_{A\setminus\{i\}}, \tag{306}$$
while for $j \notin A$ we have:
$$e_A e_j = (-1)^{|A|} \cdot e_j e_A. \tag{307}$$
The latter implies that for $e_A \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$ to hold, $|A|$ necessarily needs to be even. Since in that case $(-1)^{|A|-1} = -1$, we necessarily need by the former case that for every $i \in A$, $\mathfrak{q}(e_i) = 0$.

Now consider a linear combination $x = \sum_{A\subseteq[n]} c_A \cdot e_A \in \mathrm{Cl}(V,\mathfrak{q})$ with $c_A \in \mathbb{F}$. Then abbreviate:
$$\mathcal{Z} := \mathrm{span}\left\{e_A \,|\, A \subseteq [n], e_A \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))\right\} \subseteq \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q})). \tag{308}$$
We need to show that $x \in \mathcal{Z}$. We thus write:
$$x = y + z, \qquad y := \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}}} c_A \cdot e_A, \qquad z := \sum_{\substack{A\subseteq[n]\\ e_A\in\mathcal{Z}}} c_A \cdot e_A \in \mathcal{Z}. \tag{309}$$
So with $x, z \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$ also $y \in \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$ and we are left to show that $y = 0$.

First note that, since $e_\emptyset = 1 \in \mathcal{Z}$, there is no $e_\emptyset$-component in y.

We now have for every $i \in [n]$:
$$0 = ye_i - e_i y \tag{310}$$
$$= \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}}} c_A \cdot (e_A e_i - e_i e_A) \tag{311}$$
$$= \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}\\ i\in A}} c_A \cdot (e_A e_i - e_i e_A) + \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}\\ i\notin A}} c_A \cdot (e_A e_i - e_i e_A) \tag{312}$$
$$= \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}\\ i\in A}} \pm((-1)^{|A|} + 1)\mathfrak{q}(e_i) c_A \cdot e_{A\setminus\{i\}} + \sum_{\substack{A\subseteq[n]\\ e_A\notin\mathcal{Z}\\ i\notin A}} \pm((-1)^{|A|+1} + 1) c_A \cdot e_{A\,\dot\cup\,\{i\}}. \tag{313}$$

Note that for $A, B \subseteq [n]$ with $A \neq B$ we always have:

$$A \setminus \{i\} \neq B \setminus \{i\}, \qquad \text{if } i \in A, i \in B, \tag{314}$$
$$A \setminus \{i\} \neq B \dot\cup \{i\}, \qquad \text{if } i \in A, i \notin B, \tag{315}$$
$$A \dot\cup \{i\} \neq B \setminus \{i\}, \qquad \text{if } i \notin A, i \in B, \tag{316}$$
$$A \dot\cup \{i\} \neq B \dot\cup \{i\}, \qquad \text{if } i \notin A, i \notin B. \tag{317}$$

So the above representation for $ye_i - e_i y$ is already given in basis form. By their linear independence we then get that for every $A \subseteq [n]$ with $e_A \notin \mathcal{Z}$ and every $i \in [n]$:

$$0 = ((-1)^{|A|} + 1)\mathfrak{q}(e_i)c_A, \qquad \text{for } i \in A, \tag{318}$$
$$0 = ((-1)^{|A|+1} + 1)c_A, \qquad \text{for } i \notin A. \tag{319}$$

First consider the case $e_{[n]} \notin \mathcal{Z}$. By the previous result we then know that n is even and $\mathfrak{q} \neq 0$. So there exists e_i with $\mathfrak{q}(e_i) \neq 0$. So the above condition for $i \in [n]$ then reads:

$$0 = 2\mathfrak{q}(e_i)c_{[n]}, \qquad \text{for } i \in [n], \tag{320}$$

which implies $c_{[n]} = 0$ as $\mathfrak{q}(e_i) \neq 0$. So y does not have a $e_{[n]}$-component.

Similarly, for $A \subseteq [n]$ with $A \neq [n]$ and $e_A \notin \mathcal{Z}$ and $|A|$ odd there exists $i \notin A$ and the above condition reads:

$$0 = 2c_A, \qquad \text{for } i \notin A, \tag{321}$$

which implies $c_A = 0$. So y does not have any e_A-components with odd $|A|$.

Now let $A \subseteq [n]$ with $A \neq [n]$ and $e_A \notin \mathcal{Z}$ and $|A|$ even. Then by our previous analysis we know that there exists $i \in A$ with $\mathfrak{q}(e_i) \neq 0$. Otherwise $e_A \in \mathcal{Z}$. So the above condition reads:

$$0 = 2\mathfrak{q}(e_i)c_A, \qquad \text{for } i \in A, \tag{322}$$

which implies $c_A = 0$ as $\mathfrak{q}(e_i) \neq 0$. This shows that y does not have any e_A-component with even $|A|$.

Overall, this shows that $y = 0$ and thus the claim. □

D.9 The Twisted Center of the Clifford Algebra

Notation D.48. *Let $e_1, \dots, e_n$ be an orthogonal basis of $(V, \mathfrak{q})$. For $A \subseteq [n] := \{1, \dots, n\}$ let $e_A := \prod_{i \in A}^{<} e_i$ be the product in $\mathrm{Cl}(V, \mathfrak{q})$ in increasing index order, $e_\emptyset := 1$. Then $(e_A)_{A \subseteq [n]}$ forms a basis for $\mathrm{Cl}(V, \mathfrak{q})$.*

Definition D.49 (The twisted center of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra)**.** *We define the* twisted center *of a $\mathbb{Z}/2\mathbb{Z}$-graded algebra $\mathcal{A}$ as the following subset:*

$$\mathfrak{K}(\mathcal{A}) := \left\{ y \in \mathcal{A} \,\middle|\, \forall x \in \mathcal{A}.\; yx^{[0]} + (y^{[0]} - y^{[1]})x^{[1]} = xy \right\}. \tag{323}$$

Theorem D.50. *We have the following identification of the twisted center with the radical subalgebra of the Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$ and the set:*

$$\mathfrak{K}(\mathrm{Cl}(V, \mathfrak{q})) = \bigwedge(\mathcal{R}) = \{ y \in \mathrm{Cl}(V, \mathfrak{q}) \,|\, \forall v \in V.\; \alpha(y)v = vy \}. \tag{324}$$

Proof. Let $y \in \bigwedge(\mathcal{R})$ then by Lemma D.42 we get:

$$yx^{[0]} + \alpha(y)x^{[1]} = x^{[0]}y + x^{[1]}y = xy. \tag{325}$$

This shows that:

$$y \in \mathfrak{K}(\mathrm{Cl}(V, \mathfrak{q})), \tag{326}$$

and thus:

$$\bigwedge(\mathcal{R}) \subseteq \mathfrak{K}(\mathrm{Cl}(V, \mathfrak{q})). \tag{327}$$

Note that the following inclusion is clear as $V \subseteq \mathrm{Cl}(V, \mathfrak{q})$:

$$\mathfrak{K}(\mathrm{Cl}(V, \mathfrak{q})) \subseteq \{y \in \mathrm{Cl}(V, \mathfrak{q}) \,|\, \forall v \in V.\, \alpha(y)v = vy\}\,. \tag{328}$$

For the final inclusion, let $y = \sum_{B \subseteq [n]} c_B \cdot e_B \in \mathrm{Cl}(V, \mathfrak{q})$ such that for all $v \in V$ we have $\alpha(y)v = vy$. Then for all orthogonal basis vectors e_i we get the requirement:

$$e_i y = \alpha(y) e_i, \tag{329}$$

which always holds if $\mathfrak{q}(e_i) = 0$, and is only a condition for $\mathfrak{q}(e_i) \neq 0$. For such e_i we get:

$$\begin{aligned}
& e_i \Bigg(\sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \notin B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \notin B}} c_B \cdot e_B \Bigg) && (330) \\
&= e_i y && (331) \\
&= \alpha(y) e_i && (332) \\
&= \alpha \Bigg(\sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \notin B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \notin B}} c_B \cdot e_B \Bigg) e_i && (333) \\
&= \Bigg(\sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \notin B}} c_B \cdot e_B - \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \in B}} c_B \cdot e_B - \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \notin B}} c_B \cdot e_B \Bigg) e_i && (334) \\
&= e_i \Bigg(- \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ even} \\ i \notin B}} c_B \cdot e_B - \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \in B}} c_B \cdot e_B + \sum_{\substack{B \subseteq [n] \\ |B| \text{ odd} \\ i \notin B}} c_B \cdot e_B \Bigg). && (335)
\end{aligned}$$

Since e_i with $\mathfrak{q}(e_i) \neq 0$ is invertible, we can cancel e_i on both sides and make use of the linear independence of $(e_B)_{B \subseteq [n]}$ to get that:

$$c_B = 0 \qquad \text{if} \qquad i \in B. \tag{336}$$

Since for given B this can be concluded from every e_i with $\mathfrak{q}(e_i) \neq 0$ we can only have $c_B \neq 0$ if $\mathfrak{q}(e_j) = 0$ for all $j \in B$. Note that elements e_j with $\mathfrak{q}(e_j) = 0$ that are part of an orthogonal basis satisfy $e_j \in \mathcal{R}$. This shows that:

$$y = \sum_{\substack{B \subseteq [n] \\ \forall j \in B.\, \mathfrak{q}(e_j) = 0}} c_B \cdot e_B \in \mathrm{span}\,\{e_A \,|\, A \subseteq [n], \forall i \in A.\, e_i \in \mathcal{R}\} = \bigwedge(\mathcal{R}). \tag{337}$$

This shows the remaining inclusion:

$$\{y \in \mathrm{Cl}(V, \mathfrak{q}) \,|\, \forall v \in V.\, \alpha(y)v = vy\} \subseteq \bigwedge(\mathcal{R}). \tag{338}$$

This shows the equality of all three sets. □

E The Clifford Group and its Clifford Algebra Representations

We saw that Cartan-Dieudonné (Theorem C.13) generates the orthogonal group of a (non-degenerate) quadratic space by composing reflections. Considering this, we seek in the following a group representation that acts on the entire Clifford algebra, but reduces to a reflection when restricted to V. Further, we ensure that the action is an algebra homomorphism and will therefore respect the geometric product.

E.1 Adjusting the Twisted Conjugation

Recall the notation $\mathrm{Cl}^\times(V,\mathfrak{q}) := \{x \in \mathrm{Cl}(V,\mathfrak{q}) \,|\, \exists y \in \mathrm{Cl}(V,\mathfrak{q}).\, xy = yx = 1\}$.

Motivation E.1 (Generalizing reflection operations)**.** *For $v, w \in V$ with $\mathfrak{q}(w) \neq 0$ the reflection of v onto the hyperplane that is normal to w is given by the following formula, which we then simplify:*

$$r_w(v) = v - 2\frac{\mathfrak{b}(w,v)}{\mathfrak{b}(w,w)}w \tag{339}$$

$$= wwv/\mathfrak{q}(w) - 2\frac{\mathfrak{b}(w,v)}{\mathfrak{q}(w)}w \tag{340}$$

$$= -w(-wv + 2\mathfrak{b}(w,v))/\mathfrak{q}(w) \tag{341}$$

$$= -wvw/\mathfrak{q}(w) \tag{342}$$

$$= -wvw^{-1}. \tag{343}$$

So we have $r_w(v) = -wvw^{-1}$ for $v, w \in V$ with $\mathfrak{q}(w) \neq 0$. We would like to generalize this to an operation ρ for w from a subgroup of $\Gamma \subseteq \mathrm{Cl}^\times(V,\mathfrak{q})$ (as large as possible) onto all elements $x \in \mathrm{Cl}(V,\mathfrak{q})$. More explicitely, we want for all $w_1, w_2 \in \Gamma$ and $x_1, x_2 \in \mathrm{Cl}(V,\mathfrak{q})$, $v, w \in V$, $\mathfrak{q}(w) \neq 0$:

$$\rho(w)(v) = -wvw^{-1} = r_w(v), \tag{344}$$

$$(\rho(w_2) \circ \rho(w_1))(x_1) = \rho(w_2 w_1)(x_1), \tag{345}$$

$$\rho(w_1)(x_1 + x_2) = \rho(w_1)(x_1) + \rho(w_1)(x_2), \tag{346}$$

$$\rho(w_1)(x_1 x_2) = \rho(w_1)(x_1)\rho(w_1)(x_2). \tag{347}$$

The second condition makes sure that $\mathrm{Cl}(V,\mathfrak{q})$ will be a group representation of Γ, i.e. we get a group homomorphism:

$$\rho : \Gamma \to \mathrm{Aut}(\mathrm{Cl}(V,\mathfrak{q})), \tag{348}$$

where $\mathrm{Aut}(\mathrm{Cl}(V,\mathfrak{q}))$ denotes the set of automorphisms $\mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V,\mathfrak{q})$.

In the literature, the following versions of reflection operations were studied:

$$\rho_0(w) : x \mapsto wxw^{-1}, \qquad \rho_1(w) : x \mapsto \alpha(w)xw^{-1}, \qquad \rho_2(w) : x \mapsto wx\alpha(w)^{-1}. \tag{349}$$

Since $\rho_0(w)$ is missing the minus sign it only generalizes compositions of reflections for elements $w = w_1 \cdots w_k$, $w_l \in V$, $\mathfrak{q}(v_l) \neq 0$, of even parity k. The map $\rho_1(w)$, on the other hand, takes the minus sign into account and generalizes also to such elements $w = w_1 \cdots w_k$ of odd party $k = \mathrm{prt}(w)$ as then: $\alpha(w) = (-1)^{\mathrm{prt}(w)}w$. However, in contrast to $\rho_0(w)$, which is an algebra homomorphism for all $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$, the map $\rho_1(w)$ is not multiplicative in x, as can be seen with $v_1, v_2 \in V$ and $w \in V$ with $\mathfrak{q}(w) \neq 0$:

$$\rho_1(w)(v_1 v_2) = \alpha(w)v_1 v_2 w^{-1} \tag{350}$$

$$= (-1)^{\mathrm{prt}(w)} wv_1 w^{-1} (-1)^{\mathrm{prt}(w)} (-1)^{\mathrm{prt}(w)} wv_2 w^{-1} \tag{351}$$

$$= (-1)^{\mathrm{prt}(w)} \rho_1(w)(v_1)\rho_1(w)(v_2) \tag{352}$$

$$\neq \rho_1(w)(v_1)\rho_1(w)(v_2). \tag{353}$$

The lack of multiplicativity means that reflection and taking geometric product does not commute.

To fix this, it makes sense to first restrict $\rho_1(w)$ to V, where it coincides with r_w and also still is multiplicative in w, and then study under which conditions on w it extends to an algebra homomorphism $\mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V,\mathfrak{q})$.

More formally, by the universal property of the Clifford algebra, the obstruction for:

$$\rho(w) : V \to \mathrm{Cl}(V,\mathfrak{q}), \quad \rho(w)(v) := \alpha(w)vw^{-1} = w\eta(w)vw^{-1}, \quad \eta(w) := w^{-1}\alpha(w), \tag{354}$$

with general invertible $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$, to extend to an $\mathbb{F}$-algebra homomorphism:

$$\rho(w) : \mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V,\mathfrak{q}), \tag{355}$$

is the following:

$$\forall v \in V. \qquad \mathfrak{q}(v) \overset{!}{=} (\rho(w)(v))^2 \tag{356}$$
$$= \rho(w)(v)\rho(w)(v) \tag{357}$$
$$= (w\eta(w)vw^{-1})(w\eta(w)vw^{-1}) \tag{358}$$
$$= w\eta(w)v\eta(w)vw^{-1}, \tag{359}$$

which reduces to:

$$\forall v \in V. \qquad \mathfrak{q}(v) \overset{!}{=} \eta(w)v\eta(w)v. \tag{360}$$

The latter is, for instance, satisfied if $\eta(w)$ commutes with every $v \in V$ and $1 = \eta(w)^2$. In particular, the above requirement is satisfied for all $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ with $\eta(w) \in \{\pm 1\}$, which is equivalent to $\alpha(w) = \pm w$, which means that w is a homogeneous element of $\mathrm{Cl}(V, \mathfrak{q})$ in the parity grading. This discussion motivates the following definitions and analysis.

Notation E.2 (The coboundary of α)**.** *The* coboundary *η of α is defined on $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ as follows:*

$$\eta : \mathrm{Cl}^\times(V, \mathfrak{q}) \to \mathrm{Cl}^\times(V, \mathfrak{q}), \qquad \eta(w) := w^{-1}\alpha(w). \tag{361}$$

Remark E.3.

1. *η is a* crossed group homomorphism *(aka 1-cocycle), i.e. for $w_1, w_2 \in \mathrm{Cl}^\times(V, \mathfrak{q})$ we have:*

$$\eta(w_1 w_2) = \eta(w_1)^{w_2}\eta(w_2), \qquad \textit{with} \qquad \eta(w_1)^{w_2} := w_2^{-1}\eta(w_1)w_2. \tag{362}$$

2. *For $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ we have:*

$$\alpha(\eta(w)) = \alpha(w)^{-1}w = \eta(w)^{-1}. \tag{363}$$

3. *For $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ we have that w is an homogeneous element in $\mathrm{Cl}(V, \mathfrak{q})$, in the sense of parity, if and only if $\eta(w) \in \{\pm 1\}$.*

Definition E.4 (The group of homogeneous invertible elements)**.** *With the introduced notation we can define the group of all invertible elements of $\mathrm{Cl}(V, \mathfrak{q})$ that are also homogeneous (in the sense of parity) as:*

$$\mathrm{Cl}^{[\times]}(V, \mathfrak{q}) := \left(\mathrm{Cl}^\times(V, \mathfrak{q}) \cap \mathrm{Cl}^{[0]}(V, \mathfrak{q})\right) \cup \left(\mathrm{Cl}^\times(V, \mathfrak{q}) \cap \mathrm{Cl}^{[1]}(V, \mathfrak{q})\right) \tag{364}$$
$$= \left\{ w \in \mathrm{Cl}^\times(V, \mathfrak{q}) \,\middle|\, \eta(w) \in \{\pm 1\} \right\}. \tag{365}$$

Notation E.5 (The main involution - revisited)**.** *We now make the following abbreviations:*

$$\alpha^0 := \mathrm{id} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha^0(x) := x^{[0]} + x^{[1]} = x, \tag{366}$$
$$\alpha^1 := \alpha : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha^1(x) := x^{[0]} - x^{[1]}. \tag{367}$$

For $w \in \mathrm{Cl}(V, \mathfrak{q})$ we then have:

$$\alpha^{\mathrm{prt}(w)} : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha^{\mathrm{prt}(w)}(x) = x^{[0]} + (-1)^{\mathrm{prt}(w)}x^{[1]}. \tag{368}$$

Note that $\alpha^{\mathrm{prt}(w)}$ is an $\mathbb{F}$-algebra involution of $\mathrm{Cl}(V, \mathfrak{q})$ that preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$.

*We also need the following slight variation α^w, which in many, but not all cases, coincides with $\alpha^{\mathrm{prt}(w)}$. For $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ we define the w-*twisted map*:*

$$\alpha^w : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \alpha^w(x) := x^{[0]} + \eta(w)x^{[1]}. \tag{369}$$

Remark E.6. *Note that for $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ we have that $\eta(w) = (-1)^{\mathrm{prt}(w)}$ and thus $\alpha^w = \alpha^{\mathrm{prt}(w)}$, in which case α^w is an $\mathbb{F}$-algebra involution of $\mathrm{Cl}(V, \mathfrak{q})$ that preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$.*

Definition E.7 (Adjusted twisted conjugation)**.** *For $w \in \mathrm{Cl}^\times(V, \mathfrak{q})$ we define the* twisted conjugation*:*

$$\rho(w) : \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \rho(w)(x) := wx^{[0]}w^{-1} + \alpha(w)x^{[1]}w^{-1}. \tag{370}$$
$$= w\left(x^{[0]} + \eta(w)x^{[1]}\right)w^{-1} \tag{371}$$
$$= w\alpha^w(x)w^{-1}. \tag{372}$$

We now want to re-investigate the action of ρ on $\mathrm{Cl}(V, \mathfrak{q})$.

Lemma E.8. *For every $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ the map:*

$$\rho(w):\ \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad x \mapsto \rho(w)(x) = w\left(x^{[0]} + \eta(w) x^{[1]}\right) w^{-1}, \tag{373}$$

is an $\mathbb{F}$-algebra automorphism that preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$. Its inverse it given by $\rho(w^{-1})$.

Proof. First note that α^w agrees with the involution $\alpha^{\mathrm{prt}(w)}$ for $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$. Since $\alpha^{\mathrm{prt}(w)}$ is an $\mathbb{F}$-algebra automorphism that preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$ so is α^w for $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$.

Furthermore, the conjugation $\rho_0(w):\ x \mapsto wxw^{-1}$ is an $\mathbb{F}$-algebra automorphism, which preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$ if $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$. To see this, note that $0 = \mathrm{prt}(1) = \mathrm{prt}(ww^{-1}) = \mathrm{prt}(w) + \mathrm{prt}(w^{-1})$. As such, $\mathrm{prt}(w) = \mathrm{prt}(w^{-1})$. Then, $\mathrm{prt}(wxw^{-1}) = \mathrm{prt}(w) + \mathrm{prt}(x) + \mathrm{prt}(w^{-1}) = 2\,\mathrm{prt}(w) + \mathrm{prt}(x) = \mathrm{prt}(x)$. Here, we use the fact that the Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$-graded.

So, their composition $\rho(w) = \rho_0(w) \circ \alpha^w$ is also an $\mathbb{F}$-algebra automorphism that preserves the parity grading $\mathrm{Cl}(V, \mathfrak{q})$ for $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$. □

As a direct corollary we get:

Corollary E.9. *Let $F(T_{0,1}, \dots, T_{1,\ell}) \in \mathbb{F}[T_{0,1}, \dots, T_{1,\ell}]$ be a polynomial in 2ℓ variables with coefficients in $\mathbb{F}$. Let $x_1, \dots, x_\ell \in \mathrm{Cl}(V, \mathfrak{q})$ be ℓ elements of the Clifford algebra and $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ be a homogeneous invertible element of $\mathrm{Cl}(V, \mathfrak{q})$. Then we have the following equivariance property:*

$$\rho(w)\left(F(x_1^{[0]}, \dots, x_l^{[i]}, \dots, x_\ell^{[1]})\right) = F(\rho(w)(x_1)^{[0]}, \dots, \rho(w)(x_l)^{[i]}, \dots, \rho(w)(x_\ell)^{[1]}). \tag{374}$$

Proof. This directly follows from Lemma E.8. □

Futhermore, we get the following result:

Theorem E.10. *The map:*

$$\rho:\ \mathrm{Cl}^{[\times]}(V, \mathfrak{q}) \to \mathrm{Aut}_{\mathbf{Alg},\mathrm{prt}}\left(\mathrm{Cl}(V, \mathfrak{q})\right), \qquad w \mapsto \rho(w), \tag{375}$$

is a well-defined group homomorphism from the group of all homogeneous invertible elements of $\mathrm{Cl}(V, \mathfrak{q})$ to the group of $\mathbb{F}$-algebra automorphisms of $\mathrm{Cl}(V, \mathfrak{q})$ that preserve the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$. In particular, $\mathrm{Cl}(V, \mathfrak{q})$, $\mathrm{Cl}^{[0]}(V, \mathfrak{q})$, $\mathrm{Cl}^{[1]}(V, \mathfrak{q})$ are group representations of $\mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ via ρ.

Proof. By the previous Lemma E.8 we already know that ρ is a well-defined map. We only need to check if it is a group homomorphism. Let $w_1, w_2 \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ and $x \in \mathrm{Cl}(V, \mathfrak{q})$, then we get:

$$\left(\rho(w_2) \circ \rho(w_1)\right)(x) = \rho(w_2)(\rho(w_1)(x)) \tag{376}$$

$$= \rho(w_2)\left(w_1 \alpha^{\mathrm{prt}(w_1)}(x) w_1^{-1}\right) \tag{377}$$

$$= w_2 \alpha^{\mathrm{prt}(w_2)}\left(w_1 \alpha^{\mathrm{prt}(w_1)}(x) w_1^{-1}\right) w_2^{-1} \tag{378}$$

$$= w_2 \alpha^{\mathrm{prt}(w_2)}(w_1) \alpha^{\mathrm{prt}(w_2)}(\alpha^{\mathrm{prt}(w_1)}(x)) \alpha^{\mathrm{prt}(w_2)}(w_1)^{-1} w_2^{-1} \tag{379}$$

$$= w_2 (-1)^{\mathrm{prt}(w_2)\,\mathrm{prt}(w_1)} w_1 \alpha^{\mathrm{prt}(w_2)+\mathrm{prt}(w_1)}(x) (-1)^{\mathrm{prt}(w_2)\,\mathrm{prt}(w_1)} w_1^{-1} w_2^{-1} \tag{380}$$

$$= w_2 w_1 \alpha^{\mathrm{prt}(w_2)+\mathrm{prt}(w_1)}(x) w_1^{-1} w_2^{-1} \tag{381}$$

$$= (w_2 w_1) \alpha^{\mathrm{prt}(w_2 w_1)}(x) (w_2 w_1)^{-1} \tag{382}$$

$$= \rho(w_2 w_1)(x), \tag{383}$$

where we used the multiplicativity of $\alpha^{\mathrm{prt}(w)}(x)$:

$$
\begin{aligned}
\alpha^w(x)\alpha^w(y) &= \left((-1)^{\mathrm{prt}(w)}x^{[1]} + x[0]\right)\left((-1)^{\mathrm{prt}(w)}y^{[1]} + y^{[0]}\right) && (384)\\
&= (-1)^{\mathrm{prt}(w)}\left(x^{[0]}y^{[1]} + x^{[1]}y^{[0]}\right) + x^{[1]}y^{[1]} + x^{[0]}y^{[0]} && (385)\\
&= \alpha^w\left((xy)^{[1]}\right) + (xy)^{[0]} && (386)\\
&= \alpha^w(xy). && (387)
\end{aligned}
$$

This implies:

$$\rho(w_2) \circ \rho(w_1) = \rho(w_2 w_1), \tag{388}$$

which shows the claim. □

Finally, we want to re-check that our newly defined ρ, despite its different appearance, still has the proper interpretation of a reflection.

Remark E.11. *Let $w, v \in V$ with $\mathfrak{q}(w) \neq 0$. Then $\rho(w)(v)$ is the reflection of v w.r.t. the hyperplane that is normal to w:*

$$\rho(w)(v) = w\alpha^w(v)w^{-1} = -wvw^{-1} = r_w(v). \tag{389}$$

Remark E.12. *The presented results in this subsection can be slightly generalized as follows. If $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$ such that $\eta(w) \in \bigwedge(\mathcal{R})$ then by Theorem D.50 we get for all $v \in V$:*

$$v\eta(w) = \alpha(\eta(w))v = \eta(w)^{-1}v. \tag{390}$$

This implies that for all $v \in V$:

$$\eta(w)v\eta(w)v = \eta(w)\eta(w)^{-1}vv = \mathfrak{q}(v), \tag{391}$$

and thus for all $v \in V$:

$$(\alpha(w)vw^{-1})(\alpha(w)vw^{-1}) = \mathfrak{q}(v). \tag{392}$$

By the universal property of the Clifford algebra the map:

$$\rho(w) : V \to \mathrm{Cl}(V,\mathfrak{q}), \qquad v \mapsto \alpha(w)vw^{-1}, \tag{393}$$

uniquely extends to an $\mathbb{F}$-algebra homomorphism:

$$\rho(w) : \mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V,\mathfrak{q}), \tag{394}$$

with:

$$
\begin{aligned}
x &= c_0 + \sum_{i\in I} c_i \cdot v_{i,1}\cdots v_{i,k_i} && (395)\\
&\mapsto c_0 + \sum_{i\in I} c_i \cdot \alpha(w)v_{i,1}w^{-1}\cdots\alpha(w)v_{i,k_i}w^{-1} && (396)\\
&= w\left(c_0 + \sum_{i\in I} c_i \cdot \eta(w)v_{i,1}\cdots\eta(w)v_{i,k_i}\right)w^{-1} && (397)\\
&= w\left(c_0 + \sum_{\substack{i\in I\\ k_i \text{ even}}} c_i \cdot v_{i,1}\cdots v_{i,k_i} + \eta(w)\cdot\sum_{\substack{i\in I\\ k_i \text{ odd}}} c_i \cdot v_{i,1}\cdots v_{i,k_i}\right)w^{-1} && (398)\\
&= w\left(x^{[0]} + \eta(w)\cdot x^{[1]}\right)w^{-1}. && (399)
\end{aligned}
$$

To further ensure that the elements $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$ with the above property form a group we might need to further restrict to require that $\eta(w) \in \bigwedge^{[\times]}(\mathcal{R})$. At least in this case we get for $w_1, w_2 \in \mathrm{Cl}^\times(V,\mathfrak{q})$ with $\eta(w_1), \eta(w_2) \in \bigwedge^{[\times]}(\mathcal{R}) \subseteq \mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$, see Remark D.43, that:

$$\eta(w_2w_1) = w_1^{-1}\eta(w_2)w_1\eta(w_1) = w_1^{-1}w_1\eta(w_2)\eta(w_1) = \eta(w_2)\eta(w_1) \in \bigwedge^{[\times]}(\mathcal{R}). \tag{400}$$

So the following set:

$$C := \left\{ w \in \mathrm{Cl}^\times(V,\mathfrak{q}) \,\middle|\, \eta(w) \in \bigwedge^{[\times]}(\mathcal{R}) \right\} \tag{401}$$

is a subgroup of $\mathrm{Cl}^\times(V,\mathfrak{q})$ where every $w \in C$ defines an algebra homomorphism:

$$\rho(w) : \mathrm{Cl}(V,\mathfrak{q}) \to \mathrm{Cl}(V,\mathfrak{q}), \qquad \rho(w)(x) = w\left(x^{[0]} + \eta(w)x^{[1]}\right)w^{-1}. \tag{402}$$

E.2 The Clifford Group

Motivation E.13. *We have seen in the last section that if we choose homogeneous invertible elements $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ then the action $\rho(w)$, the (adjusted) twisted conjugation, is an algebra automorphism of $\mathrm{Cl}(V, \mathfrak{q})$ that also preserves the parity grading of $\mathrm{Cl}(V, \mathfrak{q})$, in particular, $\rho(w)$ is linear and multiplicative.*

We now want to investigate under which conditions on w the algebra automorphism $\rho(w)$ also preserves the multivector grading:

$$\mathrm{Cl}(V, \mathfrak{q}) = \mathrm{Cl}^{(0)}(V, \mathfrak{q}) \oplus \mathrm{Cl}^{(1)}(V, \mathfrak{q}) \oplus \cdots \oplus \mathrm{Cl}^{(m)}(V, \mathfrak{q}) \oplus \cdots \oplus \mathrm{Cl}^{(n)}(V, \mathfrak{q}). \tag{403}$$

If this was the case then each component $\mathrm{Cl}^{(m)}(V, \mathfrak{q})$ would give rise to a corresponding group representation.

To preserve the multivector grading we at least need that for $v \in V = \mathrm{Cl}^{(1)}(V, \mathfrak{q})$ we have that also $\rho(w)(v) \in \mathrm{Cl}^{(1)}(V, \mathfrak{q}) = V$. We will see that for $w \in \mathrm{Cl}^{\times}(V, \mathfrak{q})$ such that $\rho(w)$ is an algebra homomorphism, i.e. for w homogeneous and invertible, and such that $\rho(w)(v) \in V$ for all $v \in V$, we already get the preservation of the whole multivector grading of $\mathrm{Cl}(V, \mathfrak{q})$.

Again, let $(V, \mathfrak{q})$ be a quadratic space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and $\dim V = n < \infty$ and $e_1, \dots, e_n$ and orthogonal basis of $(V, \mathfrak{q})$.

Remark E.14. *In the following, we elaborate on the term* Clifford group *in constrast to previous literature. First, the* unconstrained *Clifford group is also referred to as the* Lipschitz group *or* Clifford-Lipschitz *group in honor of its creator Rudolf Lipschitz [LS09]. Throughout its inventions, several generalizations and versions have been proposed, often varying in details, leading to slightly non-equivalent definitions. For instance, some authors utilize conjugation as an action, others apply the the twisted conjugation, while yet another group employs the twisting on the other side of the conjugation operation. Furthermore, some authors require that the elements are homogeneous in the parity grading, where others do not. We settle for a definition that requires homogeneous invertible elements of the Clifford algebra that act via our* adjusted *twisted conjugation such that elements from the vector space V land also in V. The reason for our definition is that we want the adjusted twisted conjugation to act on the whole Clifford algebra $\mathrm{Cl}(V, \mathfrak{q})$, not just on the vector space V. Furthermore, we want it to respect, besides the vector space structure of $\mathrm{Cl}(V, \mathfrak{q})$, also the product structure, leading to algebra homomorphisms. In addition, we also want that the action to respect the extended bilinear form $\mathfrak{b}$, the orthogonal structure, and the multivector grading. These properties might not (all) be ensured in other definitions with different nuances.*

Also note that the name Clifford group might be confused with different groups with the same name in other literature, e.g., with the group of unitary matrices that normalize the Pauli group or with the finite group inside the Clifford algebra that is generated by an orthogonal basis via the geometric product.

Definition E.15.

1. *We denote the* unconstrained Clifford group *of $\mathrm{Cl}(V, \mathfrak{q})$ as follows:*

$$\tilde{\Gamma}(V, \mathfrak{q}) := \left\{ w \in \mathrm{Cl}^{\times}(V, \mathfrak{q}) \,\middle|\, \forall v \in V.\, \rho(w)(v) \in V \right\}. \tag{404}$$

2. *We denote the* Clifford group *of $\mathrm{Cl}(V, \mathfrak{q})$ as follows:*

$$\Gamma(V, \mathfrak{q}) := \mathrm{Cl}^{[\times]}(V, \mathfrak{q}) \cap \tilde{\Gamma}(V, \mathfrak{q}) \tag{405}$$

$$= \left\{ w \in \mathrm{Cl}^{\times}(V, \mathfrak{q}) \,\middle|\, \eta(w) \in \{\pm 1\} \wedge \forall v \in V.\, \rho(w)(v) \in V \right\}. \tag{406}$$

3. *We define the* special Clifford group *as follows:*

$$\Gamma^{[0]}(V, \mathfrak{q}) := \tilde{\Gamma}(V, \mathfrak{q}) \cap \mathrm{Cl}^{[0]}(V, \mathfrak{q}) = \Gamma(V, \mathfrak{q}) \cap \mathrm{Cl}^{[0]}(V, \mathfrak{q}). \tag{407}$$

Theorem E.16. *For $w \in \Gamma(V, \mathfrak{q})$ and $x \in \mathrm{Cl}(V, \mathfrak{q})$ we have for all $m = 0, \dots, n$:*

$$\rho(w)(x^{(m)}) = \rho(w)(x)^{(m)}. \tag{408}$$

In particular, for $x \in \mathrm{Cl}^{(m)}(V, \mathfrak{q})$ we also have $\rho(w)(x) \in \mathrm{Cl}^{(m)}(V, \mathfrak{q})$.

Proof. We first claim that for $w \in \Gamma(V, \mathfrak{q})$ the set of elements:

$$b_1 := \rho(w)(e_1), \dots, b_n := \rho(w)(e_n), \tag{409}$$

forms an orthogonal basis of $(V, \mathfrak{q})$. Indeed, since $\rho(w)(e_t) \in V$, by definition of $\Gamma(V, \mathfrak{q})$, the orthogonality relation, $i \neq j$:

$$0 = 2\mathfrak{b}(e_i, e_j) = e_i e_j + e_j e_i, \tag{410}$$

transforms under $\rho(w)$ to:

$$\begin{aligned} 0 = \rho(w)(0) &= \rho(w)\left(e_i e_j + e_j e_i\right) && (411)\\ &= \rho(w)(e_i)\rho(w)(e_j) + \rho(w)(e_j)\rho(w)(e_i) && (412)\\ &= b_i b_j + b_j b_i && (413)\\ &= 2\mathfrak{b}(b_i, b_j). && (414) \end{aligned}$$

This shows that $b_1, \dots, b_n$ is an orthogonal system in V. Using $\rho(w^{-1})$ we also see that the system is linear independent and thus an orthognal basis of V. By the basis-independence of the multivector grading $\mathrm{Cl}^{(m)}(V, \mathfrak{q})$, see Theorem D.27, we then get for:

$$x = \sum_{i_1 < \dots < i_m} c_{i_1, \dots, i_m} \cdot e_{i_1} \cdots e_{i_m} \in \mathrm{Cl}^{(m)}(V, \mathfrak{q}), \tag{415}$$

the relation:

$$\rho(w)(x) = \sum_{i_1 < \dots < i_m} c_{i_1, \dots, i_m} \cdot b_{i_1} \cdots b_{i_m} \in \mathrm{Cl}^{(m)}(V, \mathfrak{q}). \tag{416}$$

This shows the claim. □

Corollary E.17. *The map:*

$$\rho : \Gamma(V, \mathfrak{q}) \to \mathrm{Aut}_{\mathbf{Alg}, \mathrm{grd}}\left(\mathrm{Cl}(V, \mathfrak{q})\right), \qquad w \mapsto \rho(w), \tag{417}$$

is a well-defined group homomorphism from the Clifford group to the group of $\mathbb{F}$*-algebra automorphisms of* $\mathrm{Cl}(V, \mathfrak{q})$ *that preserve the multivector grading of* $\mathrm{Cl}(V, \mathfrak{q})$*. In particular,* $\mathrm{Cl}(V, \mathfrak{q})$ *and* $\mathrm{Cl}^{(m)}(V, \mathfrak{q})$ *for* $m = 0, \dots, n$*, are group representations of* $\Gamma(V, \mathfrak{q})$ *via* ρ*.*

Corollary E.18. *Let* $F(T_1, \dots, T_\ell) \in \mathbb{F}[T_1, \dots, T_\ell]$ *be a polynomial in* ℓ *variables with coefficients in* $\mathbb{F}$ *and let* $k \in \{0, \dots, n\}$*. Further, consider* ℓ *elements* $x_1, \dots, x_\ell \in \mathrm{Cl}(V, \mathfrak{q})$*. Then for every* $w \in \Gamma(V, \mathfrak{q})$ *we get the equivariance property:*

$$\rho(w)\left(F(x_1, , \dots, x_\ell)^{(k)}\right) = F(\rho(w)(x_1), \dots, \rho(w)(x_\ell))^{(k)}, \tag{418}$$

where the superscript (k) *indicates the projection onto the multivector grade-k-part of the whole expression.*

Example E.19. *Let* $w \in V$ *with* $\mathfrak{q}(w) \neq 0$*, then* $w \in \Gamma(V, \mathfrak{q})$*.*

Proof. It is clear that w is homogeneous in the parity grading, as $\eta(w) = -1$. For $v \in V$ we get:

$$\rho(w)(v) = r_w(v) = v - 2\frac{\mathfrak{b}(v, w)}{\mathfrak{q}(w)} w \in V. \tag{419}$$

This shows $w \in \Gamma(V, \mathfrak{q})$. □

Example E.20. *Let* $e, f \in V$ *with* $\mathfrak{b}(v, f) = 0$ *for all* $v \in V$*, and put:* $\gamma := 1 + ef \in \mathrm{Cl}^{[0]}(V, \mathfrak{q})$*. Then* $\gamma \in \Gamma(V, \mathfrak{q})$*.*

Proof. First, note that for all $v \in V$ we get:

$$fv = -vf + 2\mathfrak{b}(f, v) = -vf. \tag{420}$$

Next, we see that:

$$\gamma^{-1} = 1 - ef. \tag{421}$$

Indeed, we get:

$$(1+ef)(1-ef) = 1 + ef - ef - efef \tag{422}$$
$$= 1 + effe \tag{423}$$
$$= 1 + \underbrace{\mathfrak{q}(f)}_{=0}\mathfrak{q}(e) \tag{424}$$
$$= 1. \tag{425}$$

Now consider $v \in V$. We then have:

$$\rho(\gamma)(v) = \alpha(\gamma) v \gamma^{-1} \tag{426}$$
$$= (1+ef)v(1-ef) \tag{427}$$
$$= (1+ef)(v - vef) \tag{428}$$
$$= v + efv - vef - efvef \tag{429}$$
$$= v - evf - vef - eveff \tag{430}$$
$$= v - (ev + ve)f - \underbrace{\mathfrak{q}(f)}_{=0} eve \tag{431}$$
$$= v - 2\mathfrak{b}(e,v)f \tag{432}$$
$$\in V. \tag{433}$$

This shows $\gamma \in \Gamma(V,\mathfrak{q})$. □

Example E.21. *Let $f_1,\dots,f_r \in V$ be a basis of the radical subspace $\mathcal{R}$ of $(V,\mathfrak{q})$. In particular, we have $\mathfrak{b}(v,f_j) = 0$ for all $v \in V$. Then we put:*

$$g := 1 + h, \quad \textit{with} \quad h \in \operatorname{span}\{f_{k_1}\cdots f_{k_s} \,|\, s \geq 2 \text{ \underline{even}},\, 1 \leq k_1 < k_2 < \dots k_s \leq r\} \subseteq \mathrm{Cl}(V,\mathfrak{q}). \tag{434}$$

Then we claim that $g \in \Gamma(V,\mathfrak{q})$ and $\rho(g) = \mathrm{id}_{\mathrm{Cl}(V,\mathfrak{q})}$.

Proof. Since we restrict to even products it is clear that $g \in \mathrm{Cl}^{[0]}(V,\mathfrak{q})$. Furthermore, note that, since h lives in the radical subalgebra, there exists a number $k \geq 1$ such that:

$$h^{k+1} = 0. \tag{435}$$

Then we get that:

$$g^{-1} = 1 - h + h^2 + \cdots + (-1)^k h^k. \tag{436}$$

Indeed, we get:

$$(1+h)\left(\sum_{l=0}^{k}(-1)^l h^l\right) = \sum_{l=0}^{k}(-1)^l h^l + \sum_{l=0}^{k}(-1)^l h^{l+1} \tag{437}$$
$$= 1 + \sum_{l=1}^{k}(-1)^l h^l - \sum_{l=1}^{k}(-1)^l h^l + (-1)^k \underbrace{h^{k+1}}_{=0} \tag{438}$$
$$= 1. \tag{439}$$

Furthermore, g lies in the center $\mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))$ of $\mathrm{Cl}(V,\mathfrak{q})$ by Theorem D.47. Then for $v \in V$ we get:

$$\rho(g)(v) = \alpha(g) v g^{-1} \tag{440}$$
$$= g v g^{-1} \tag{441}$$
$$= v g g^{-1} \tag{442}$$
$$= v \tag{443}$$
$$\in V. \tag{444}$$

So, $g \in \Gamma(V,\mathfrak{q})$ and acts as the identity on V, and thus on $\mathrm{Cl}(V,\mathfrak{q})$, via ρ. □

E.3 The Structure of the Clifford Group

We have identified the Clifford group and its action on the algebra. In particular, our adjusted twisted conjugation preserves the parity and multivector grading, and reduces to a reflection when restricted to V. We now want to further investigate how the Clifford and the Clifford groups act via the twisted conjugation ρ on V and $\mathrm{Cl}(V,\mathfrak{q})$. We again denote by $\mathcal{R} \subseteq V$ the radical subspace of V w.r.t. $\mathfrak{q}$. We follow and extend the analysis of [Cru80, Cru90, DKL10].

We first investigate the kernel of the twisted action.

Corollary E.22 (The kernel of the twisted conjugation)**.** *1. We have the following identity for the twisted conjugation:*

$$\ker\left(\rho|_{\mathrm{Cl}^\times(V,\mathfrak{q})}\right) := \left\{w \in \mathrm{Cl}^\times(V,\mathfrak{q}) \,\middle|\, \rho(w) = \mathrm{id}_{\mathrm{Cl}(V,\mathfrak{q})}\right\} \stackrel{!}{=} \bigwedge^\times(\mathcal{R}). \tag{445}$$

2. For the twisted conjugation restricted to the unconstrained Clifford, group and to V the kernel is given by:

$$\ker\left(\rho :\, \tilde{\Gamma}(V,\mathfrak{q}) \to \mathrm{GL}(V)\right) := \left\{w \in \tilde{\Gamma}(V,\mathfrak{q}) \,\middle|\, \rho(w)|_V = \mathrm{id}_V\right\} \stackrel{!}{=} \bigwedge^\times(\mathcal{R}). \tag{446}$$

3. For the twisted conjugation restricted to the Clifford group, the kernel is given by:

$$\ker\left(\rho :\, \Gamma(V,\mathfrak{q}) \to \mathrm{Aut}_{\mathbf{Alg}}(\mathrm{Cl}(V,\mathfrak{q}))\right) := \left\{w \in \Gamma(V,\mathfrak{q}) \,\middle|\, \rho(w) = \mathrm{id}_{\mathrm{Cl}(V,\mathfrak{q})}\right\} \tag{447}$$

$$\stackrel{!}{=} \bigwedge^{[\times]}(\mathcal{R}). \tag{448}$$

Proof. This follows directly from the characterizing of the twisted center of $\mathrm{Cl}(V,\mathfrak{q})$ by Theorem D.50. Note that we have for $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$:

$$\rho(w) = \mathrm{id}_{\mathrm{Cl}(V,\mathfrak{q})} \iff \forall x \in \mathrm{Cl}(V,\mathfrak{q}).\ \rho(w)(x) = x \tag{449}$$

$$\iff \forall x \in \mathrm{Cl}(V,\mathfrak{q}).\ wx^{[0]}w^{-1} + \alpha(w)x^{[1]}w^{-1} = x \tag{450}$$

$$\iff \forall x \in \mathrm{Cl}(V,\mathfrak{q}).\ wx^{[0]} + \alpha(w)x^{[1]} = xw \tag{451}$$

$$\iff w \in \mathfrak{K}(\mathrm{Cl}(V,\mathfrak{q})) = \bigwedge(\mathcal{R}). \tag{452}$$

From this follows that:

$$w \in \mathrm{Cl}^\times(V,\mathfrak{q}) \cap \bigwedge(\mathcal{R}) = \bigwedge^\times(\mathcal{R}). \tag{453}$$

The other points follow similarly with Theorem D.50.

For the last point also note:

$$\Gamma(V,\mathfrak{q}) = \tilde{\Gamma}(V,\mathfrak{q}) \cap \mathrm{Cl}^{[\times]}(V,\mathfrak{q}), \qquad \bigwedge^\times(\mathcal{R}) \cap \mathrm{Cl}^{[\times]}(V,\mathfrak{q}) = \bigwedge^{[\times]}(\mathcal{R}). \tag{454}$$

This shows the claims. □

Lemma E.23. *1. For every $w \in \mathrm{Cl}^\times(V,\mathfrak{q})$ we have:*

$$\rho(w)|_{\mathcal{R}} = \mathrm{id}_{\mathcal{R}}. \tag{455}$$

2. For every $w \in \mathrm{Cl}^{[\times]}(V,\mathfrak{q})$ we have:

$$\rho(w)|_{\bigwedge(\mathcal{R})} = \mathrm{id}_{\bigwedge(\mathcal{R})}. \tag{456}$$

3. For every $g \in \bigwedge^\times(\mathcal{R})$ we have:

$$\rho(g)|_V = \mathrm{id}_V. \tag{457}$$

4. For every $g \in \bigwedge^{[\times]}(\mathcal{R})$ we have:

$$\rho(g)|_{\mathrm{Cl}(V,\mathfrak{q})} = \mathrm{id}_{\mathrm{Cl}(V,\mathfrak{q})}. \tag{458}$$

Proof. This directly follows from the twisted commutation relationship, see Lemma D.42. For $w \in \mathrm{Cl}^{\times}(V, \mathfrak{q})$ and $f \in \mathcal{R} \subseteq V$ we have:

$$\rho(w)(f) = \alpha(w) f w^{-1} = f w w^{-1} = f. \tag{459}$$

For $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$ the map $\rho(w)$ is an algebra automorphism of $\mathrm{Cl}(V, \mathfrak{q})$ and satisfies $\rho(w)(f) = f$ for all $f \in \mathcal{R}$. So for every $y \in \bigwedge(\mathcal{R})$ we have:

$$\rho(w)(y) = \rho(w)\left(\sum_{i \in I} c_i \cdot f_{k_i} \cdots f_{l_i}\right) \tag{460}$$

$$= \sum_{i \in I} c_i \cdot \rho(w)(f_{k_i}) \cdots \rho(w)(f_{l_i}) \tag{461}$$

$$= \sum_{i \in I} c_i \cdot f_{k_i} \cdots f_{l_i} \tag{462}$$

$$= y. \tag{463}$$

For $g \in \bigwedge^{\times}(\mathcal{R})$ and $v \in V$ we get:

$$\rho(g)(v) = \alpha(g) v g^{-1} = v g g^{-1} = v. \tag{464}$$

For $g \in \bigwedge^{[\times]}(\mathcal{R})$ the map $\rho(g)$ is an algebra automorphism of $\mathrm{Cl}(V, \mathfrak{q})$ and satisfies $\rho(g)(v) = v$ for all $v \in V$. As above we see that for $x \in \mathrm{Cl}(V, q)$ we get:

$$\rho(g)(x) = \rho(g)\left(\sum_{i \in I} c_i \cdot v_{k_i} \cdots v_{l_i}\right) \tag{465}$$

$$= \sum_{i \in I} c_i \cdot \rho(g)(v_{k_i}) \cdots \rho(g)(v_{l_i}) \tag{466}$$

$$= \sum_{i \in I} c_i \cdot v_{k_i} \cdots v_{l_i} \tag{467}$$

$$= x. \tag{468}$$

This shows all the claims. ☐

From Lemma E.23 we see that $\rho(w)|_{\bigwedge(\mathcal{R})} = \mathrm{id}_{\bigwedge(\mathcal{R})}$ for $w \in \mathrm{Cl}^{[\times]}(V, \mathfrak{q})$. Together with Corollary E.18 we arrive at a slightly more general version that allows one to parameterize polynomials not just with coefficients from $\mathbb{F}$, but also with elements from $\bigwedge(\mathcal{R})$, and still get the equivariance w.r.t. the Clifford group $\Gamma(V, \mathfrak{q})$:

Corollary E.24. *Let $F(T_1, \dots, T_{\ell+s}) \in \mathbb{F}[T_1, \dots, T_{\ell+s}]$ be a polynomial in $\ell + s$ variables with coefficients in $\mathbb{F}$ and let $k \in \{0, \dots, n\}$. Further, consider ℓ elements $x_1, \dots, x_\ell \in \mathrm{Cl}(V, \mathfrak{q})$ and s elements $y_1, \dots, y_s \in \bigwedge(\mathcal{R})$. Then for every $w \in \Gamma(V, \mathfrak{q})$ we get the equivariance property:*

$$\rho(w)\left(F(x_1, , \dots, x_\ell, y_1, \dots, y_s)^{(k)}\right) = F(\rho(w)(x_1), \dots, \rho(w)(x_\ell), y_1, \dots, y_s)^{(k)}, \tag{469}$$

where the superscript (k) indicates the projection onto the multivector grade-k-part of the whole expression.

We now investigate the image/range of the twisted conjugation.

Theorem E.25 (The range of the twisted conjugation)**.** *The image/range of the Clifford group $\Gamma(V, \mathfrak{q})$ under the twisted conjugation restricted to V coincides with all orthogonal automorphisms of $(V, \mathfrak{q})$ that restrict to the identity $\mathrm{id}_{\mathcal{R}}$ of the radical subspace $\mathcal{R} \subseteq V$ of $(V, \mathfrak{q})$:*

$$\mathrm{ran}\,(\rho :\, \Gamma(V, \mathfrak{q}) \to \mathrm{GL}(V)) = \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}). \tag{470}$$

Again, recall that the kernel is given by:

$$\ker\,(\rho :\, \Gamma(V, \mathfrak{q}) \to \mathrm{GL}(V)) = \bigwedge^{[\times]}(\mathcal{R}). \tag{471}$$

Proof. We first show that the range of ρ when restricted to V will consistute an orthogonal automorphism of $(V,\mathfrak{q})$. For this let $e_1,\dots,e_n$ be an orthogonal basis of $(V,\mathfrak{q})$ and $\mathcal{R}$ the radical subspace of $(V,\mathfrak{q})$, $r=\dim\mathcal{R}$. W.l.o.g. we can assume that $e_1,\dots,e_m,e_{m+1},\dots,e_{m+r}$ with:

$$\mathcal{R}=\operatorname{span}\{e_{m+1},\dots,e_{m+r}\},\qquad E:=\operatorname{span}\{e_1,\dots,e_m\}. \tag{472}$$

For $w\in\Gamma(V,\mathfrak{q})$ we now apply $\rho(w)\in\mathrm{GL}(V)$ to the basis elements e_i. Note that by definition of $\Gamma(V,\mathfrak{q})$ we have $\rho(w)(e_i)\in V$. With this we get:

$$2\mathfrak{b}(\rho(w)(e_i),\rho(w)(e_j))=\rho(w)(e_i)\rho(w)(e_j)+\rho(w)(e_j)\rho(w)(e_i) \tag{473}$$

$$=\rho(w)\left(e_ie_j+e_je_i\right) \tag{474}$$

$$=\rho(w)\left(2\mathfrak{b}(e_i,e_j)\right) \tag{475}$$

$$=2\mathfrak{b}(e_i,e_j). \tag{476}$$

This shows for $v=\sum_{i=1}^n a_i\cdot e_i$:

$$\mathfrak{q}(\rho(w)(v))=\mathfrak{q}(\sum_{i=1}^n a_i\cdot\rho(w)(e_i)) \tag{477}$$

$$=\sum_{i=1}^n a_i^2\cdot\mathfrak{q}(\rho(w)(e_i)) \tag{478}$$

$$=\sum_{i=1}^n a_i^2\cdot\mathfrak{q}(e_i) \tag{479}$$

$$=\mathfrak{q}(\sum_{i=1}^n a_i\cdot e_i) \tag{480}$$

$$=\mathfrak{q}(v). \tag{481}$$

Since $\rho(w)$ is also a linear automorphism of V with inverse $\rho(w^{-1})$ we see that:

$$\rho(w)|_V\in\mathrm{O}(V,\mathfrak{q}). \tag{482}$$

By Lemma E.23 we also see that:

$$\rho(w)|_{\mathcal{R}}=\mathrm{id}_{\mathcal{R}}. \tag{483}$$

Together this shows that:

$$\rho(w)|_V\in\mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}). \tag{484}$$

This shows the inclusion:

$$\operatorname{ran}\left(\rho:\ \Gamma(V,\mathfrak{q})\to\mathrm{GL}(V)\right)\subseteq\mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}). \tag{485}$$

Recall the definition of the set of radical preserving orthogonal automorphisms:

$$\mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}):=\{\Phi\in\mathrm{O}(V,\mathfrak{q})\,|\,\Phi|_{\mathcal{R}}=\mathrm{id}_{\mathcal{R}}\}\cong\begin{pmatrix}\mathrm{O}(E,\mathfrak{q}|_E) & 0_{m\times r}\\ \mathrm{M}(r,m) & \mathrm{id}_{\mathcal{R}}\end{pmatrix} \tag{486}$$

$$\cong\mathrm{O}(E,\mathfrak{q}|_E)\ltimes\mathrm{M}(r,m). \tag{487}$$

So an element $\Phi\in\mathrm{O}_{\mathcal{R}}(V,\mathfrak{q})$ can equivalently be written as:

$$\Phi=\begin{pmatrix}O & 0\\ M & I\end{pmatrix}=\begin{pmatrix}O & 0\\ 0 & I\end{pmatrix}\circ\begin{pmatrix}I & 0\\ M & I\end{pmatrix} \tag{488}$$

$$=\begin{pmatrix}O_1 & 0\\ 0 & I\end{pmatrix}\circ\cdots\circ\begin{pmatrix}O_k & 0\\ 0 & I\end{pmatrix}\circ\begin{pmatrix}I & 0\\ M_{1,1} & I\end{pmatrix}\circ\cdots\circ\begin{pmatrix}I & 0\\ M_{m,r} & I\end{pmatrix}, \tag{489}$$

where $O=O_1\cdots O_k$ is a product of $k\le m$ reflection matrices O_l by the Theorem of Cartan-Dieudonné C.13, and $M=\sum_{i=1}^m\sum_{j=1}^r M_{i,j}$ where the matrix $M_{i,j}=c_{i,j}\cdot I_{i,j}$ only has the entry $c_{i,j}\in\mathbb{F}$ at (i,j) (and 0 otherwise). Now let $w_l\in V$ be the normal vector of the reflection O_l with $\mathfrak{q}(w_l)\neq 0$ for $l=1,\dots,k$, and for $i=1,\dots,m$ and $j=1,\dots,r$ put:

$$\gamma_{i,j}:=1+c_{i,j}\cdot e_ie_{m+j}\in\mathrm{Cl}(V,\mathfrak{q}), \tag{490}$$

and further:

$$w := w_1 \cdots w_k \cdot \gamma_{1,1} \cdots \gamma_{m,r} \in \mathrm{Cl}(V, \mathfrak{q}). \tag{491}$$

Note that by Examples E.19 and E.20 we have:

$$\{w \in V \mid \mathfrak{q}(w) \neq 0\} \subseteq \Gamma(V, \mathfrak{q}), \tag{492}$$

$$\left\{ \gamma_i := 1 + e_i \sum_{j=1}^{r} c_{i,j} e_{m+j} \,\middle|\, c_{i,j} \in \mathbb{F}, i = 1, \dots, m, \; j = 1, \dots, r \right\} \subseteq \Gamma(V, \mathfrak{q}), \tag{493}$$

which implies that $w \in \Gamma(V, \mathfrak{q})$. With this we get:

$$\rho(w) = \rho(w_1) \circ \cdots \circ \rho(w_k) \circ \rho(\gamma_{1,1}) \circ \cdots \circ \rho(\gamma_{m,r}) \tag{494}$$

$$= \begin{pmatrix} O_1 & 0 \\ 0 & I \end{pmatrix} \circ \cdots \circ \begin{pmatrix} O_k & 0 \\ 0 & I \end{pmatrix} \circ \begin{pmatrix} I & 0 \\ M_{1,1} & I \end{pmatrix} \circ \cdots \circ \begin{pmatrix} I & 0 \\ M_{m,r} & I \end{pmatrix} \tag{495}$$

$$= \Phi. \tag{496}$$

This thus shows the surjectivity of the map:

$$\rho : \; \Gamma(V, \mathfrak{q}) \to \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}). \tag{497}$$

This shows the claim. □

We can summarize our finding in the following statement.

Corollary E.26. *We have the short exact sequence:*

$$1 \longrightarrow \bigwedge^{[\times]}(\mathcal{R}) \xrightarrow{\text{incl}} \Gamma(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}) \longrightarrow 1. \tag{498}$$

From the above Theorem we can now also derive the structure of the elements of the Clifford group.

Corollary E.27 (Elements of the Clifford group)**.** *Let $(V, \mathfrak{q})$ be a finite dimensional quadratic vector space of dimension $n := \dim V < \infty$ over a fields $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. Let $\mathcal{R} \subseteq V$ be the radical vector subspace of $(V, \mathfrak{q})$ with dimension $r := \dim \mathcal{R}$. Put $m := n - r \geq 0$. Let $e_1, \dots, e_n$ be an orthogonal basis of $(V, \mathfrak{q})$ ordered in such a way that $e_{m+1}, \dots e_{m+r} = e_n$ are the basis vectors inside $\mathcal{R}$, while $e_1, \dots, e_m$ are spanning a non-degenerate orthogonal subspace to $\mathcal{R}$ inside $(V, \mathfrak{q})$.*

Then every element of the Clifford group $w \in \Gamma(V, \mathfrak{q})$ is of the form:

$$w = c \cdot v_1 \cdots v_k \cdot \gamma_1 \cdots \gamma_m \cdot g, \tag{499}$$

with $c \in \mathbb{F}^\times$, $k \in \mathbb{N}_0$, $v_l \in V$ with $\mathfrak{q}(v_l) \neq 0$ for $l = 1, \dots, k$,

$$\gamma_i = 1 + e_i \sum_{j=1}^{r} c_{i,j} e_{m+j}, \tag{500}$$

with $c_{i,j} \in \mathbb{F}$ for $i = 1, \dots, m$, $j = 1, \dots, r$, and some $g \in \bigwedge^{[*]}(\mathcal{R})$.

E.4 Orthogonal Representations of the Clifford Group

Lemma E.28. *Let $e_1, \dots, e_n$ be an orthogonal basis of $(V, \mathfrak{q})$ and $w \in \Gamma(V, \mathfrak{q})$. If we put for $j \in [n]$:*

$$b_j := \rho(w)(e_j), \tag{501}$$

and for $A \subseteq [n]$:

$$b_A := \prod_{i \in A}^{<} b_i = \rho(w)(e_A), \tag{502}$$

then $b_1, \dots, b_n$ is an orthogonal basis for $(V, \mathfrak{q})$ and both $(e_A)_{A \in [n]}$ and $(b_A)_{A \in [n]}$ are orthogonal bases for $(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}})$.

Proof. First note that $b_1, \dots, b_n$ is a basis of V. Indeed, the releation:

$$0 = \sum_{i=1}^{n} a_i \cdot b_i, \tag{503}$$

with $a_i \in \mathbb{F}$ implies:

$$\begin{aligned} 0 &= \rho(w)^{-1}(0) && (504)\\ &= \rho(w)^{-1}\left(\sum_{i=1}^{n} a_i \cdot b_i\right) && (505)\\ &= \sum_{i=1}^{n} a_i \cdot \rho(w)^{-1}(b_i) && (506)\\ &= \sum_{i=1}^{n} a_i \cdot e_i. && (507) \end{aligned}$$

Since $e_1, \dots, e_n$ is linear independent we get $a_i = 0$ for all $i \in [n]$. So also $b_1, \dots, b_n$ is linear independent and thus constitute a basis of V.

To show that $b_1, \dots, b_n$ is an orthogonal basis of $(V, \mathfrak{q})$ let $i \neq j$ and then consider the following:

$$\begin{aligned} 2 \cdot \mathfrak{b}(b_i, b_j) &= b_i b_j + b_j b_i && (508)\\ &= \rho(w)(e_i)\rho(w)(e_j) + \rho(w)(e_j)\rho(w)(e_i) && (509)\\ &= \rho(w)\left(e_i e_j + e_j e_i\right) && (510)\\ &= \rho(w)(2 \cdot \underbrace{\mathfrak{b}(e_i, e_j)}_{=0}) && (511)\\ &= 0. && (512) \end{aligned}$$

This shows that $b_1, \dots, b_n$ is an orthogonal basis of $(V, \mathfrak{q})$.

Theorem D.26 then shows that both $(e_A)_{A \in [n]}$ and $(b_A)_{A \in [n]}$ are orthogonal bases for $(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}})$. □

Theorem E.29. *Let $w \in \Gamma(V, \mathfrak{q})$ and $x \in \mathrm{Cl}(V, \mathfrak{q})$ then we get:*

$$\bar{\mathfrak{q}}(\rho(w)(x)) = \bar{\mathfrak{q}}(x). \tag{513}$$

In other words, $\rho(w) \in \mathrm{O}(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}})$. Furthermore, we have:

$$\rho(w)|_{\bigwedge(\mathcal{R})} = \mathrm{id}_{\bigwedge(\mathcal{R})}. \tag{514}$$

In other words, $\rho(w) \in \mathrm{O}_{\bigwedge(\mathcal{R})}(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}})$.

Proof. Let $e_1, \dots, e_n$ be an orthogonal basis for $(V, \mathfrak{q})$ and $b_j := \rho(w)(e_j)$ for $j \in [n]$. Then by Lemma E.28 we know that $b_1, \dots, b_n$ is an orthogonal basis of $(V, \mathfrak{q})$ and both $(e_A)_{A \subseteq [n]}$ and $(b_A)_{A \subseteq [n]}$ are orthogonal basis for $(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}})$. Now let $x \in \mathrm{Cl}(V, \mathfrak{q})$ and write it as:

$$x = \sum_{A \subseteq [n]} x_A \cdot e_A, \qquad \rho(w)(x) = \sum_{A \subseteq [n]} x_A \cdot \rho(w)(e_A) = \sum_{A \subseteq [n]} x_A \cdot b_A. \tag{515}$$

Then we get:

$$\bar{\mathfrak{q}}(\rho(w)(x)) = \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \mathfrak{q}(b_i) \tag{516}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \mathfrak{q}(\rho(w)(e_i)) \tag{517}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \rho(w)(e_i)\rho(w)(e_i) \tag{518}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \rho(w)(e_i^2) \tag{519}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \rho(w)(\mathfrak{q}(e_i)\cdot 1) \tag{520}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \mathfrak{q}(e_i)\cdot \rho(w)(1) \tag{521}$$

$$= \sum_{A\subseteq[n]} x_A \cdot \prod_{i\in A} \mathfrak{q}(e_i) \tag{522}$$

$$= \bar{\mathfrak{q}}(x). \tag{523}$$

This shows the claim. The remaining point follows from Lemma E.23. □

Similarly, and more detailed, we also get the following, using Theorem D.27, Corollary E.17 and Lemma E.23:

Corollary E.30. *For every $w \in \Gamma(V,\mathfrak{q})$ and $m = 0,\dots,n$ we have:*

$$\rho(w)|_{\mathrm{Cl}^{(m)}(V,\mathfrak{q})} \in \mathrm{O}_{\bigwedge^{(m)}(\mathcal{R})}(\mathrm{Cl}^{(m)}(V,\mathfrak{q}),\bar{\mathfrak{q}}). \tag{524}$$

In words, $\rho(w)$, when restricted to the m-th homogeneous multivector component $\mathrm{Cl}^{(m)}(V,\mathfrak{q})$ of $\mathrm{Cl}(V,\mathfrak{q})$ acts as an orthogonal automorphism of $\mathrm{Cl}^{(m)}(V,\mathfrak{q})$ w.r.t. $\bar{\mathfrak{q}}$. Furthermore, it acts as the identity when further restricted to the m-th homogeneous multivector component of the radical subalgebra: $\bigwedge^{(m)}(\mathcal{R}) \subseteq \mathrm{Cl}^{(m)}(V,\mathfrak{q})$.

Remark E.31. *For $m \in [n]$ we use $\rho^{(m)}$ to denote the group homomorphism ρ when restricted to act on the subvector space $\mathrm{Cl}^{(m)}(V,\mathfrak{q})$:*

$$\rho^{(m)} : \Gamma(V,\mathfrak{q}) \to \mathrm{O}_{\bigwedge^{(m)}(\mathcal{R})}(\mathrm{Cl}^{(m)}(V,\mathfrak{q}),\bar{\mathfrak{q}}), \qquad \rho^{(m)}(w) := \rho(w)|_{\mathrm{Cl}^{(m)}(V,\mathfrak{q})}. \tag{525}$$

By Theorem D.50 or Corollary E.22 we see that $\bigwedge^{[\times]}(\mathcal{R})$ always lies inside the kernel of $\rho^{(m)}$:

$$\bigwedge^{[\times]}(\mathcal{R}) \subseteq \ker \rho^{(m)} \subseteq \Gamma(V,\mathfrak{q}). \tag{526}$$

So we get a well-defined group homorphism on the quotient:

$$\bar{\rho}^{(m)} : \Gamma(V,\mathfrak{q})/\bigwedge^{[\times]}(\mathcal{R}) \to \mathrm{O}_{\bigwedge^{(m)}(\mathcal{R})}(\mathrm{Cl}^{(m)}(V,\mathfrak{q}),\bar{\mathfrak{q}}), \quad \bar{\rho}^{(m)}([w]) := \rho(w)|_{\mathrm{Cl}^{(m)}(V,\mathfrak{q})}. \tag{527}$$

Furthermore, by Theorem E.25 we have the isomorphism:

$$\bar{\rho}^{(1)} : \Gamma(V,\mathfrak{q})/\bigwedge^{[\times]}(\mathcal{R}) \cong \mathrm{O}_{\bigwedge^{(1)}(\mathcal{R})}(\mathrm{Cl}^{(1)}(V,\mathfrak{q}),\bar{\mathfrak{q}}) = \mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}), \quad \bar{\rho}^{(1)}([w]) = \rho(w)|_V. \tag{528}$$

Consequently, for all $m \in [n]$ we get the composition of group homorphisms:

$$\tilde{\rho}^{(m)} : \mathrm{O}_{\mathcal{R}}(V,\mathfrak{q}) \overset{(\bar{\rho}^{(1)})^{-1}}{\cong} \Gamma(V,\mathfrak{q})/\bigwedge^{[\times]}(\mathcal{R}) \overset{\bar{\rho}^{(m)}}{\to} \mathrm{O}_{\bigwedge^{(m)}(\mathcal{R})}(\mathrm{Cl}^{(m)}(V,\mathfrak{q}),\bar{\mathfrak{q}}), \tag{529}$$

$$\tilde{\rho}^{(m)}(\Phi) = \rho(w)|_{\mathrm{Cl}^{(m)}(V,\mathfrak{q})}, \quad \text{for any } w \in \Gamma(V,\mathfrak{q}) \text{ with } \rho^{(1)}(w) = \Phi. \tag{530}$$

Similarly, we get an (injective) group homomorphism:

$$\tilde{\rho}: \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}) \to \mathrm{O}_{\bigwedge(\mathcal{R})}(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}}), \tag{531}$$

$$\tilde{\rho}(\Phi) := \rho(w), \quad \textit{for any } w \in \Gamma(V, \mathfrak{q}) \textit{ with } \rho^{(1)}(w) = \Phi. \tag{532}$$

So, the group $\mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$ acts on $\mathrm{Cl}(V, \mathfrak{q})$ and all subvector spaces $\mathrm{Cl}^{(m)}(V, \mathfrak{q})$, $m = 0, \dots, n$, in the same way as $\Gamma(V, \mathfrak{q})$ does via ρ, when using the surjective map $\rho^{(1)}$ to lift elements $\Phi \in \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$ to elements $w \in \Gamma(V, \mathfrak{q})$ with $\rho^{(1)}(w) = \Phi$.

Specifically, let $x \in \mathrm{Cl}(V, \mathfrak{q})$ be of the form $x = \sum_{i \in I} c_i \cdot v_{i,1} \cdots v_{i,k_i}$ with $v_{i,j} \in V$, $c_i \in \mathbb{F}$ and $\Phi \in \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$ is given by $\Phi = \bar{\rho}^{(1)}([w])$ with $w \in \Gamma(V, \mathfrak{q})$, then we have:

$$\rho(w)(x) = \sum_{i \in I} c_i \cdot \rho(w)(v_{i,1}) \cdots \rho(w)(v_{i,k_i}) = \sum_{i \in I} c_i \cdot \Phi(v_{i,1}) \cdots \Phi(v_{i,k_i}). \tag{533}$$

This means that the action $\tilde{\rho}$ of $\mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$ transforms a multivector x by transforming all its vector components $v_{i,j}$ by Φ, acting through the usual orthogonal transformation on vectors.

As such, we have the following equivariance property with respect to $\mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$.

Corollary E.32. *Let $F(T_1, \dots, T_{\ell+s}) \in \mathbb{F}[T_1, \dots, T_{\ell+s}]$ be a polynomial in $\ell + s$ variables with coefficients in $\mathbb{F}$ and let $k \in \{0, \dots, n\}$. Further, consider ℓ elements $x_1, \dots, x_\ell \in \mathrm{Cl}(V, \mathfrak{q})$ and s elements $y_1, \dots, y_s \in \bigwedge(\mathcal{R})$. Then for every $\Phi \in \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})$ we get the equivariance property:*

$$\tilde{\rho}(\Phi)\left(F(x_1, , \dots, x_\ell, y_1, \dots, y_s)^{(k)}\right) = F(\tilde{\rho}(\Phi)(x_1), \dots, \tilde{\rho}(\Phi)(x_\ell), y_1, \dots, y_s)^{(k)}, \tag{534}$$

where the superscript (k) indicates the projection onto the multivector grade-k-part of the whole expression.

E.5 The Spinor Norm and the Clifford Norm

In this subsection we shortly introduce the three slightly different versions of a norm that appear in the literature: the Spinor norm, the Clifford norm and the extended quadratic form. We are interested under which conditions do they have multiplicative behaviour.

Definition E.33 (The Spinor norm and the Clifford norm)**.** *We define the* Spinor norm *and the* Clifford norm *of* $\mathrm{Cl}(V, \mathfrak{q})$ *as the maps:*

$$\mathrm{SN}: \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \mathrm{SN}(x) := \beta(x)x, \tag{535}$$

$$\mathrm{CN}: \mathrm{Cl}(V, \mathfrak{q}) \to \mathrm{Cl}(V, \mathfrak{q}), \qquad \mathrm{CN}(x) := \gamma(x)x. \tag{536}$$

Also recall the extended quadratic form*:*

$$\bar{\mathfrak{q}}: \mathrm{Cl}(V, \mathfrak{q}) \to \mathbb{F}, \qquad \bar{\mathfrak{q}}(x) = \zeta(\beta(x)x) = \zeta(\mathrm{SN}(x)). \tag{537}$$

As a first preliminary Lemma we need to study when the projection onto the zero-component is multiplicative:

Lemma E.34. *Let $x \in \bigwedge(\mathcal{R})$ and $y \in \mathrm{Cl}(V, \mathfrak{q})$ then we have:*

$$\zeta(xy) = \zeta(x)\,\zeta(y). \tag{538}$$

As a result, the projection onto the zero component induces an $\mathbb{F}$-algebra homomorphism:

$$\zeta: \bigwedge(\mathcal{R}) \to \mathbb{F}, \qquad y \mapsto \zeta(y). \tag{539}$$

Proof. We now use the notations from D.40 and Lemma D.41. We distinguish two cases: $x \in \bigwedge^{(\geq 1)}(\mathcal{R})$ and:

$$x \in \bigwedge(\mathcal{R}) \setminus \bigwedge^{(\geq 1)}(\mathcal{R}) = \bigwedge^{\times}(\mathcal{R}) = \mathbb{F}^{\times} + \bigwedge^{(\geq 1)}(\mathcal{R}. \tag{540}$$

In the first case, we have: $\zeta(xy) = 0 = \zeta(x)\,\zeta(y)$, as multiplying with $x \in \bigwedge^{(\geq 1)}(\mathcal{R})$ can only increase the grade of occurring terms or make them vanish.

In the second case, we can write $x = a + f$ with $a \in \mathbb{F}^{\times}$ and $f \in \bigwedge^{(\geq 1)}(\mathcal{R})$. Clearly, $\zeta(x) = a$. We then get by linearity and the first case:

$$\zeta(xy) = \zeta(ay + fy) = a\,\zeta(y) + \zeta(fy) = \zeta(x)\,\zeta(y) + 0. \tag{541}$$

This shows the claim. ☐

Lemma E.35. *Let* $x_1, x_2 \in \mathrm{Cl}(V, \mathfrak{q})$.

1. *If* $\mathrm{SN}(x_1) \in \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$ *then we have:*
$$\mathrm{SN}(x_1 x_2) = \mathrm{SN}(x_1)\,\mathrm{SN}(x_2). \tag{542}$$
2. *If* $\mathrm{CN}(x_1) \in \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$ *then we have:*
$$\mathrm{CN}(x_1 x_2) = \mathrm{CN}(x_1)\,\mathrm{CN}(x_2). \tag{543}$$
3. *If* $\mathrm{SN}(x_1) \in \bigwedge^{[0]}(\mathcal{R})$ *or* $\mathfrak{q} = 0$ *then we have:*
$$\bar{\mathfrak{q}}(x_1 x_2) = \bar{\mathfrak{q}}(x_1)\bar{\mathfrak{q}}(x_2). \tag{544}$$

Proof. $\mathrm{SN}(x_1) \in \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$ implies:
$$\begin{aligned}
\mathrm{SN}(x_1 x_2) &= \beta(x_1 x_2) x_1 x_2 &(545)\\
&= \beta(x_2)\beta(x_1) x_1 x_2 &(546)\\
&= \beta(x_2)\,\mathrm{SN}(x_1) x_2 &(547)\\
&\overset{\mathrm{SN}(x_1)\in\mathfrak{Z}(\mathrm{Cl}(V,\mathfrak{q}))}{=} \mathrm{SN}(x_1)\beta(x_2)x_2 &(548)\\
&= \mathrm{SN}(x_1)\,\mathrm{SN}(x_2). &(549)
\end{aligned}$$
Similarly for CN.

Together with Lemma E.34 and $\mathrm{SN}(x_1) \in \bigwedge^{[0]}(\mathcal{R}) \subseteq \bigwedge(\mathcal{R}) \cap \mathfrak{Z}(\mathrm{Cl}(V, \mathfrak{q}))$ we get:
$$\begin{aligned}
\bar{\mathfrak{q}}(x_1 x_2) &= \zeta\left(\mathrm{SN}(x_1 x_2)\right) &(550)\\
&= \zeta\left(\mathrm{SN}(x_1)\,\mathrm{SN}(x_2)\right) &(551)\\
&= \zeta(\mathrm{SN}(x_1))\,\zeta(\mathrm{SN}(x_2)) &(552)\\
&= \bar{\mathfrak{q}}(x_1)\bar{\mathfrak{q}}(x_2). &(553)
\end{aligned}$$
This shows the claim. □

Lemma E.36. *Consider the following subset of* $\mathrm{Cl}(V, \mathfrak{q})$*:*
$$\Gamma^{[-]}(V, \mathfrak{q}) := \left\{ x \in \mathrm{Cl}^{[0]}(V, \mathfrak{q}) \cup \mathrm{Cl}^{[1]}(V, \mathfrak{q}) \,\middle|\, \forall v \in V\ \exists v' \in V.\ \alpha(x)v = v'x \right\}. \tag{554}$$
Then $\Gamma^{[-]}(V, \mathfrak{q})$ *is closed under multiplication and for every* $x \in \Gamma^{[-]}(V, \mathfrak{q})$ *we have:*
$$\mathrm{SN}(x) \in \bigwedge^{[0]}(\mathcal{R}), \qquad \mathrm{CN}(x) \in \bigwedge^{[0]}(\mathcal{R}). \tag{555}$$

Proof. For $x, y \in \Gamma^{[-]}(V, \mathfrak{q})$ we also have that xy is homogeneous. Furthermore, we get for $v \in V$:
$$\begin{aligned}
\alpha(xy)v &= \alpha(x)\alpha(y)v &(556)\\
&= \alpha(x)v'y &(557)\\
&= \tilde{v}xy, &(558)
\end{aligned}$$
for some $v', \tilde{v} \in V$. So, $xy \in \Gamma^{[-]}(V, \mathfrak{q})$ and $\Gamma^{[-]}(V, \mathfrak{q})$ is closed under multiplication.

With the above conditions on x we get for every $v \in V$:
$$\begin{aligned}
\alpha(\mathrm{SN}(x))v &= \alpha(\beta(x))\alpha(x)v &(559)\\
&= \alpha(\beta(x))v'x &(560)\\
&= \alpha(\beta(x))(-\alpha(\beta(v')))x &(561)\\
&= -\alpha(\beta(x)\beta(v'))x &(562)\\
&= -\alpha(\beta(v'x))x &(563)\\
&= -\alpha(\beta(\alpha(x)v))x &(564)\\
&= -\beta(\alpha(\alpha(x)v))x &(565)\\
&= -\beta(x\alpha(v))x &(566)\\
&= \beta(xv)x &(567)\\
&= \beta(v)\beta(x)x &(568)\\
&= v\,\mathrm{SN}(x). &(569)
\end{aligned}$$

This implies by Theorem D.50 that:

$$\mathrm{SN}(x) \in \bigwedge(\mathcal{R}). \tag{570}$$

Since for homogeneous $x \in \mathrm{Cl}(V, \mathfrak{q})$ we have: $\mathrm{prt}(\beta(x)) = \mathrm{prt}(x)$ and thus: $\mathrm{SN}(x) = \beta(x)x \in \mathrm{Cl}^{[0]}(V, \mathfrak{q})$. This implies:

$$\mathrm{SN}(x) \in \bigwedge(\mathcal{R}) \cap \mathrm{Cl}^{[0]}(V, \mathfrak{q}) = \bigwedge^{[0]}(\mathcal{R}). \tag{571}$$

This shows the claim. □

Theorem E.37 (Multiplicativity of the three different norms)**.** *Both, the Spinor norm and the Clifford norm, when restricted to the Clifford group, are well-defined group homomorphisms:*

$$\mathrm{SN} : \Gamma(V, \mathfrak{q}) \to \bigwedge^{[\times]}(\mathcal{R}), \qquad w \mapsto \mathrm{SN}(w) = \beta(w)w, \tag{572}$$

$$\mathrm{CN} : \Gamma(V, \mathfrak{q}) \to \bigwedge^{[\times]}(\mathcal{R}), \qquad w \mapsto \mathrm{CN}(w) = \gamma(w)w. \tag{573}$$

Furthermore, the extended quadratic form $\bar{\mathfrak{q}}$ of $\mathrm{Cl}(V, \mathfrak{q})$ restricted to the Clifford group is a well-defined group homomorphism:

$$\bar{\mathfrak{q}} : \Gamma(V, \mathfrak{q}) \to \mathbb{F}^{\times}, \qquad w \mapsto \bar{\mathfrak{q}}(w) = \zeta(\beta(w)w). \tag{574}$$

Proof. This directly follows from Lemma E.35 and Lemma E.36. Note that $\Gamma(V, \mathfrak{q}) \subseteq \Gamma^{[-]}(V, \mathfrak{q})$. □

Example E.38.

1. *For $a \in \mathbb{F}$ we get:*

$$\mathrm{SN}(a) = \beta(a)a = a^2. \tag{575}$$

This also shows: $\bar{\mathfrak{q}}(a) = a^2\bar{\mathfrak{q}}(1) = a^2$.

2. *For $w \in V$ we have:*

$$\mathrm{SN}(w) = \beta(w)w = w^2 = \mathfrak{q}(w). \tag{576}$$

This also shows: $\bar{\mathfrak{q}}(w) = \mathfrak{q}(w)$.

3. *For $\gamma = 1 + ef$ with $e \in V$ and $f \in \mathcal{R}$ we have:*

$$\begin{aligned} \mathrm{SN}(\gamma) &= (1 + fe)(1 + ef) &(577)\\ &= 1 + fe + ef + feef &(578)\\ &= 1 + 2\mathfrak{b}(e, f) + \mathfrak{q}(f) \cdot \mathfrak{q}(e) &(579)\\ &= 1. &(580) \end{aligned}$$

This also shows: $\bar{\mathfrak{q}}(\gamma) = 1$.

4. *For $g = 1 + h \in \bigwedge^{*}(\mathcal{R})$ we get:*

$$\begin{aligned} \mathrm{SN}(g) &= (1 + \beta(h))(1 + h) &(581)\\ &= 1 + \beta(h) + h + \beta(h)h, &(582) \end{aligned}$$

and thus: $\bar{\mathfrak{q}}(g) = 1$.

E.6 The Pin Group and the Spin Group

We have investigated the Clifford group and its action on the algebra through the twisted conjugation. In many fields of study, the Clifford group is further restricted to the Pin or Spin group. There can arise a few issues regarding the exact definition of these groups, especially when also considering fields other than the reals. We elaborate on these concerns here and leave the general definition for future discussion.

Motivation E.39 (The problem of generalizing the definition of the Spin group)**.** *For a positive definite quadratic form* $\mathfrak{q}$ *on the real vector space* $V = \mathbb{R}^n$ *with* $n \geq 3$ *the* Spin group $\mathrm{Spin}(n)$ *is defined via the kernel of the Spinor norm (=extended quadratic form on* $\mathrm{Cl}(V, \mathfrak{q})$*) restricted to the special Clifford group* $\Gamma^{[0]}(V, \mathfrak{q})$*:*

$$\mathrm{Spin}(n) := \ker\left(\bar{\mathfrak{q}} : \Gamma^{[0]}(V, \mathfrak{q}) \to \mathbb{R}^\times\right) = \left\{ w \in \Gamma^{[0]}(V, \mathfrak{q}) \,\middle|\, \bar{\mathfrak{q}}(w) = 1 \right\} = \bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(1). \quad (583)$$

$\mathrm{Spin}(n)$ *is thus a normal subgroup of the special Clifford group* $\Gamma^{[0]}(V, \mathfrak{q})$*, and, as it turns out, a double cover of the* special orthogonal group $\mathrm{SO}(n)$ *via the twisted conjugation* ρ*. The latter can be summarized by the short exact sequence:*

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \mathrm{Spin}(n) \xrightarrow{\rho} \mathrm{SO}(n) \longrightarrow 1. \quad (584)$$

We intend to generalize this in several directions: 1. from Spin to Pin group, 2. from $\mathbb{R}^n$ *to vector spaces* V *over general fields* $\mathbb{F}$ *with* $\mathrm{char}(\mathbb{F}) \neq 2$*, 3. from non-degenerate to degenerate quadratic forms* $\mathfrak{q}$*, 4. from positive (semi-)definite to non-definite quadratic forms* $\mathfrak{q}$*. This comes with several challenges and ambiguities.*

If we want to generalize the above to define the Pin group we would allow for elements not just of (pure) even parity $w \in \Gamma^{[0]}(V, \mathfrak{q})$*. Here the question arises if one should generalize to the unconstrained Clifford group* $\tilde{\Gamma}(V, \mathfrak{q})$ *or the (homogeneous) Clifford group* $\Gamma(V, \mathfrak{q})$*. As discussed before, to ensure that the (adjusted) twisted conjugation* ρ *is a well-defined algebra automorphism of* $\mathrm{Cl}(V, \mathfrak{q})$ *the parity homogeneity assumptions is crucial. Furthermore, the elements of* $\tilde{\Gamma}(V, \mathfrak{q})$ *that lead to non-trivial orthogonal automorphisms, e.g.* $v \in V$ *with* $\mathfrak{q}(v) \neq 0$ *and* $\gamma = 1 + ef$ *with* $e \in V$*,* $f \in \mathcal{R}$*, are already homogeneous. So it is arguably safe and reasonable to restrict to the Clifford group* $\Gamma(V, \mathfrak{q})$ *and define the Pin group* $\mathrm{Pin}(V, \mathfrak{q})$ *as some subquotient of* $\Gamma(V, \mathfrak{q})$*.*

In the non-definite (but still non-degenerate, real) case $\mathbb{R}^{(p,q)}$*,* $p, q \geq 1$*, the special orthogonal group* $\mathrm{SO}(p, q)$ *contains combinations of reflections* $r_0 \circ r_1$ *where the corresponding normal vectors* $v_0, v_1 \in V$ *satisfy* $\mathfrak{q}(v_0) = 1$ *and* $\mathfrak{q}(v_1) = -1$*. Their product* $v_0 v_1$ *would lie in the special Clifford group* $\Gamma^{[0]}(V, \mathfrak{q})$*. However, their spinor norm would be different from* 1*:*

$$\bar{\mathfrak{q}}(v_0 v_1) = \mathfrak{q}(v_0)\mathfrak{q}(v_1) = -1 \neq 1. \quad (585)$$

Here now the question arises if we would like to preserve the former definition of $\mathrm{Spin}(p, q)$ *as* $\bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(1)$ *and exclude* $v_0 v_1$ *from* $\mathrm{Spin}(p, q)$*, or, if we adjust the definition of* $\mathrm{Spin}(p, q)$ *and include* $v_0 v_1$*. The former definition has the effect that* $\mathrm{Spin}(p, q)$ *does, in general, not map surjectively onto* $\mathrm{SO}(p, q)$ *anymore, and we would only get a short exact sequence:*

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \mathrm{Spin}(p, q) \xrightarrow{\rho} \mathrm{SO}(p, q) \xrightarrow{\bar{\bar{\mathfrak{q}}}} \underbrace{\mathbb{R}^\times / (\mathbb{R}^\times)^2}_{\cong \{\pm 1\}}. \quad (586)$$

The alternative would be to define $\mathrm{Spin}(p, q)$ *as* $\bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(\pm 1)$*. This would allow for* $v_0 v_1 \in \mathrm{Spin}(p, q)$ *and lead to the short exact sequence:*

$$1 \longrightarrow \underbrace{\mu_4(\mathbb{R})}_{= \{\pm 1\}} \xrightarrow{\text{incl}} \mathrm{Spin}(p, q) \xrightarrow{\rho} \mathrm{SO}(p, q) \longrightarrow 1, \quad (587)$$

which exactly recovers the former behaviour for $\mathrm{Spin}(n)$*, and, which makes* $\mathrm{Spin}(p, q)$ *a double cover of* $\mathrm{SO}(p, q)$*.*

However, for other fields $\mathbb{F}$*,* $\mathrm{char}(\mathbb{F}) \neq 2$*, and non-degenerate* $(V, \mathfrak{q})$*,* $\dim V \geq 3$*, one would get, with the last definition* $\bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(\pm 1)$*, the exact sequence:*

$$1 \longrightarrow \mu_4(\mathbb{F}) \xrightarrow{\text{incl}} \mathrm{Spin}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}(V, \mathfrak{q}) \xrightarrow{\bar{\bar{\mathfrak{q}}}} \mathbb{F}^\times / (\mathbb{F}^\times)^2, \quad (588)$$

and one would need to live with the fact that: a) $\mu_4(\mathbb{F}) := \left\{x \in \mathbb{F}^\times \,\middle|\, x^4 = 1\right\}$ could contain a number of elements k different from 2, rendering ρ a $(k:1)$-map, in contrast to a $(2:1)$-map, and, again, b) the non-surjectivity of ρ. The latter comes from the fact that a combination of reflections $r_0 \circ r_1$ where the corresponding normal vectors $v_0, v_1 \in V$ with $\mathfrak{q}(v_0), \mathfrak{q}(v_1) \neq 0$ cannot, in general, be normalized such that the $\mathfrak{q}(v_i)$'s lie in $\{\pm 1\}$ by multiplying/dividing the v_i's with some scalars $c_i \in \mathbb{F}^\times$. Note that we get:

$$\mathfrak{q}(v_i/c_i) = \mathfrak{q}(v_i)/c_i^2. \tag{589}$$

This shows that we can only normalize the v_i's such that the $\mathfrak{q}(v_i)$'s lie inside a fixed system of representatives $S \subseteq \mathbb{F}^\times$ of $\mathbb{F}^\times/(\mathbb{F}^\times)^2$, which is thus of size:

$$\#S = \#\left(\mathbb{F}^\times/(\mathbb{F}^\times)^2\right), \tag{590}$$

which can be different from $2 = \#\{\pm 1\}$.

So, in this general setting, the first definition of $\mathrm{Spin}(V, \mathfrak{q})$ as $\bar{\mathfrak{q}}|^{-1}_{\Gamma^{[0]}(V,\mathfrak{q})}(1)$ would at least correct the map ρ to be a $(2:1)$-map. We would get the following short exact seequence:

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \mathrm{Spin}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}(V, \mathfrak{q}) \xrightarrow{\bar{\bar{\mathfrak{q}}}} \mathbb{F}^\times/(\mathbb{F}^\times)^2. \tag{591}$$

The normalization argument around Equation 589 for general fields $\mathbb{F}$ now would also give us a third option: we could, instead of restricting elements to have a fixed value $\mathfrak{q}(v_i) \in S$, which depends on the choice of S, identify elements $w_1, w_2 \in \Gamma^{[0]}(V, \mathfrak{q})$ if they differ by a scalar $c \in \mathbb{F}^\times$:

$$w_1 = c \cdot w_2. \tag{592}$$

Then for their spinor norms (modulo $(\mathbb{F}^\times)^2$) we would get:

$$\bar{\mathfrak{q}}(w_1) = c^2 \cdot \bar{\mathfrak{q}}(w_2), \qquad [\bar{\mathfrak{q}}(w_1)] = [\bar{\mathfrak{q}}(w_2)] \in \mathbb{F}^\times/(\mathbb{F}^\times)^2. \tag{593}$$

So, the spinor norms of w_1 and w_2 would be represented by the same representative $s \in S$, as desired. However, this definition would just identify $\mathrm{Spin}(V, \mathfrak{q})$ with $\mathrm{SO}(V, \mathfrak{q})$ via ρ:

$$\mathrm{Spin}(V, \mathfrak{q}) = \Gamma^{[0]}(V, \mathfrak{q})/\mathbb{F}^\times \overset{\rho}{\cong} \mathrm{SO}(V, \mathfrak{q}), \tag{594}$$

and nothing new would emerge from this. Note that the latter isomorphisms always holds for non-degenerate $(V, \mathfrak{q})$ with $\dim V \geq 3$, $\mathrm{char}(\mathbb{F}) \neq 2$, and can be expressed as the exact sequence:

$$1 \longrightarrow \mathbb{F}^\times \xrightarrow{\text{incl}} \Gamma^{[0]}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}(V, \mathfrak{q}) \longrightarrow 1. \tag{595}$$

A fourth option would be to mod out the scalar squares $(\mathbb{F}^\times)^2$ instead of $\mathbb{F}^\times$ and use $\Gamma^{[0]}(V, \mathfrak{q})/(\mathbb{F}^\times)^2$ as the definition of $\mathrm{Spin}(V, \mathfrak{q})$. This would lead to the exact sequence:

$$1 \longrightarrow \mathbb{F}^\times/(\mathbb{F}^\times)^2 \xrightarrow{\text{incl}} \mathrm{Spin}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}(V, \mathfrak{q}) \longrightarrow 1, \tag{596}$$

which would again coincide with the real case of $\mathrm{Spin}(n)$ as then $\mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \{\pm 1\}$. However, in the general case, again, ρ is here a $(k:1)$-map instead of a $(2:1)$-map with $k := \#\left(\mathbb{F}^\times/(\mathbb{F}^\times)^2\right)$. Furthermore, the spinor norm $\bar{\mathfrak{q}}$ on $\mathrm{Spin}(V, \mathfrak{q})$ would not be well-defined anymore, in its current form, as different representatives of elements $[w_1] = [w_2] \in \Gamma^{[0]}(V, \mathfrak{q})/(\mathbb{F}^\times)^2$ would differ by a scalar square: $w_1 = c^2 \cdot w_2$. Their spinor norms would thus differ by by a forth scalar power:

$$\bar{\mathfrak{q}}(w_1) = \bar{\mathfrak{q}}(c^2 \cdot w_2) = c^4 \cdot \bar{\mathfrak{q}}(w_2). \tag{597}$$

So, the spinor norm on $\Gamma^{[0]}(V, \mathfrak{q})/(\mathbb{F}^\times)^2$ would only be well defined modulo $(\mathbb{F}^\times)^4 \subseteq (\mathbb{F}^\times)^2$:

$$[\bar{\mathfrak{q}}] : \Gamma^{[0]}(V, \mathfrak{q})/(\mathbb{F}^\times)^2 \to \mathbb{F}^\times/(\mathbb{F}^\times)^4, \qquad [w] \mapsto [\bar{\mathfrak{q}}(w)]. \tag{598}$$

Things become even more complicated in the degenerate case. At least we always have a short exact sequence for $m \geq 3$:

$$1 \longrightarrow \bigwedge^{[\times]}(\mathcal{R}) \xrightarrow{\text{incl}} \Gamma^{[0]}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}_{\mathcal{R}}(V, \mathfrak{q}) \longrightarrow 1, \tag{599}$$

where $\mathrm{SO}_{\mathcal{R}}(V,\mathfrak{q})$ indicates the set of those special orthogonal automorphisms Φ of $(V,\mathfrak{q})$ with $\Phi|_{\mathcal{R}} = \mathrm{id}_{\mathcal{R}}$, where $\mathcal{R}$ is the radical subspace of $(V,\mathfrak{q})$ with $r := \dim \mathcal{R}$, $n := \dim V$, $m := n - r$. Recall that we have:

$$\bigwedge^{[\times]}(\mathcal{R}) = \mathbb{F}^\times \cdot \bigwedge^{[*]}(\mathcal{R}), \tag{600}$$

$$\bigwedge^{[*]}(\mathcal{R}) = 1 + \mathrm{span}\left\{f_1 \cdots f_k \,|\, k \geq 2 \text{ even}, f_l \in \mathcal{R}, l = 1, \dots, k\right\}. \tag{601}$$

Note that for $g \in \bigwedge^{[*]}(\mathcal{R})$ we have $\rho(g)|_V = \mathrm{id}_V$ and $\bar{\mathfrak{q}}(g) = 1$. So the elements from $\bigwedge^{[*]}(\mathcal{R})$ are only blowing up the kernels of ρ and $\bar{\mathfrak{q}}$ and can be considered redundant for our analysis. So one can argue that one can mod out $\bigwedge^{[*]}(\mathcal{R})$ in the above groups. We thus get a short exact sequence:

$$1 \longrightarrow \mathbb{F}^\times \longrightarrow \underbrace{\Gamma^{[0]}(V,\mathfrak{q})/\bigwedge^{[*]}(\mathcal{R})}_{=:\tilde{\Gamma}^{[0]}(V,\mathfrak{q})} \xrightarrow{\rho} \mathrm{SO}_{\mathcal{R}}(V,\mathfrak{q}) \longrightarrow 1, \tag{602}$$

which now looks similar to the non-degenerate case. We can now consider the same 4 options for the definition of the Spin group as before:

$$\bar{\mathfrak{q}}|^{-1}_{\tilde{\Gamma}^{[0]}(V,\mathfrak{q})}(1), \qquad \bar{\mathfrak{q}}|^{-1}_{\tilde{\Gamma}^{[0]}(V,\mathfrak{q})}(\pm 1), \qquad \tilde{\Gamma}^{[0]}(V,\mathfrak{q})/\mathbb{F}^\times, \qquad \tilde{\Gamma}^{[0]}(V,\mathfrak{q})/(\mathbb{F}^\times)^2. \tag{603}$$

As before, the third option can easily be discarded. If we want to preserve generality, we have the option to either preserve $\bar{\mathfrak{q}}$ and the $(2:1)$-property of ρ and pick the first option, or, preserve the surjectivity of ρ and take the fourth option. If we are only interested in the $\mathbb{R}$-case, then the second option preserves all properties. Note that in the $\mathbb{R}$-case the groups of the second and forth option are isomorphic as groups anyways:

$$\bar{\mathfrak{q}}|^{-1}_{\tilde{\Gamma}^{[0]}(V,\mathfrak{q})}(\pm 1) \cong \tilde{\Gamma}^{[0]}(V,\mathfrak{q})/(\mathbb{R}^\times)^2. \tag{604}$$

The reason is that we always have that: $\bar{\mathfrak{q}}(\mathbb{F}^\times) = (\mathbb{F}^\times)^2$, and, in the $\mathbb{R}$-case, the left group already contains the relevant elements to also map surjectively onto $\mathrm{SO}_{\mathcal{R}}(V,\mathfrak{q})$ via ρ.

One could further discuss if one wanted to replace the extended quadratic form $\bar{\mathfrak{q}}$, which is given by $\bar{\mathfrak{q}}(x) = \zeta(\beta(x)x) \in \mathbb{F}^\times$, by the other possible definition of the spinor norm SN, which is only given by $\mathrm{SN}(x) = \beta(x)x \in \bigwedge^{[\times]}(\mathcal{R})$. However, for all relevant elements of $\Gamma^{[0]}(V,\mathfrak{q})$ both definitions agree, but the description of the set of the rather irrelevant elements, which satisfy $\mathrm{SN}(x) = 1$ and $\rho(x)|_V = \mathrm{id}_V$, becomes more complicated than the set $\bigwedge^{[\times]}(\mathcal{R})$. As we mod those irrelevant elements out anyways and we prefer to have our "norm map" to map to the scalars $\mathbb{F}^\times$ instead of to $\bigwedge^{[\times]}(\mathcal{R})$, it is safe and reasonable to work with the extended quadratic form $\bar{\mathfrak{q}}$ in all definitions.

In this paper we are mostly interested in working with the orthogonal groups and thus are interested in preserving the surjectivity of ρ. Since, for computational reasons, we usually restrict ourselves to the $\mathbb{R}$-case, it is easier to work with (a restricted set of) elements of a group than equivalence classes, and, the spinor norm/extended quadratic form has computational meaning, we side with the second definition in this paper, but only state it for the $\mathbb{R}$-case below. We leave the general definition for future discussion.

Definition E.40 (The real Pin group and the real Spin group)**.** *Let V be a finite dimensional $\mathbb{R}$-vector space V, $\dim V = n < \infty$, and $\mathfrak{q}$ a (possibly degenerate) quadratic form on V. We define the (real)* Pin group *and (real)* Spin group*, resp., of $(V,\mathfrak{q})$ as the following subquotients of the Clifford group $\Gamma(V,\mathfrak{q})$ and its even parity part $\Gamma^{[0]}(V,\mathfrak{q})$, resp.:*

$$\mathrm{Pin}(V,\mathfrak{q}) := \{x \in \Gamma(V,\mathfrak{q}) \,|\, \bar{\mathfrak{q}}(x) \in \{\pm 1\}\} \,/ \bigwedge^{[*]}(\mathcal{R}), \tag{605}$$

$$\mathrm{Spin}(V,\mathfrak{q}) := \left\{x \in \Gamma^{[0]}(V,\mathfrak{q}) \,\middle|\, \bar{\mathfrak{q}}(x) \in \{\pm 1\}\right\} / \bigwedge^{[*]}(\mathcal{R}). \tag{606}$$

If $(V,\mathfrak{q}) = \mathbb{R}^{(p,q,r)}$ is the standard quadratic $\mathbb{R}$-vector space with signature (p,q,r) then we denote:

$$\mathrm{Pin}(p,q,r) := \mathrm{Pin}(\mathbb{R}^{(p,q,r)}), \tag{607}$$

$$\mathrm{Spin}(p,q,r) := \mathrm{Spin}(\mathbb{R}^{(p,q,r)}). \tag{608}$$

Corollary E.41. *Let $(V, \mathfrak{q})$ be a finite dimensional quadratic vector space over $\mathbb{R}$. Then the twisted conjugation induces a well-defined and surjective group homomorphism onto the group of radical preserving orthogonal automorphisms of $(V, \mathfrak{q})$:*

$$\rho: \operatorname{Pin}(V, \mathfrak{q}) \to \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}), \tag{609}$$

with kernel:

$$\ker\left(\rho: \operatorname{Pin}(V, \mathfrak{q}) \to \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q})\right) = \{\pm 1\}. \tag{610}$$

Correspondingly, for the $\operatorname{Spin}(V, \mathfrak{q})$ *group. In short, we have short exact sequences:*

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \operatorname{Pin}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{O}_{\mathcal{R}}(V, \mathfrak{q}) \longrightarrow 1, \tag{611}$$

$$1 \longrightarrow \{\pm 1\} \xrightarrow{\text{incl}} \operatorname{Spin}(V, \mathfrak{q}) \xrightarrow{\rho} \mathrm{SO}_{\mathcal{R}}(V, \mathfrak{q}) \longrightarrow 1. \tag{612}$$

The examples E.19, E.20, E.21, E.38 allow us to describe the elements of the Pin and Spin group as follows.

Corollary E.42 (The elements of the real Pin group and the real Spin group)**.** *Let $(V, \mathfrak{q})$ be a finite dimensional quadratic vector space over $\mathbb{R}$ with signature (p, q, r). We get the following description of the elements of the Pin group:*

$$\operatorname{Pin}(V, \mathfrak{q}) = \left\{ \pm v_1 \cdots v_k \cdot \gamma_1 \cdots \gamma_{p+q} \cdot g \,\middle|\, k \in \mathbb{N}_0, \mathfrak{q}(v_l) \in \{\pm 1\}, g \in \bigwedge^{[*]}(\mathcal{R}) \right\} / \bigwedge^{[*]}(\mathcal{R}), \tag{613}$$

where $\gamma_i = 1 + e_i \sum_{j=1}^{r} c_{i,j} e_{p+q+j}$, $c_{i,j} \in \mathbb{R}$, for $i = 1, \ldots, p+q$ and $j = 1, \ldots, r$, and, $v_l \in V$ with $\mathfrak{q}(v_l) \in \{\pm 1\}$ with $l = 1, \ldots, l$ and $k \in \mathbb{N}_0$. Note that $\{e_{p+q+j} \,|\, j \in [r]\}$ is meant to span the radical subspace $\mathcal{R}$ of $(V, \mathfrak{q})$.

We similarly can describe the Spin group as follows:

$$\operatorname{Spin}(V, \mathfrak{q}) = \left\{ \pm v_1 \cdots v_k \cdot \gamma_1 \cdots \gamma_{p+q} \cdot g \,\middle|\, k \in 2\mathbb{N}_0, \mathfrak{q}(v_l) \in \{\pm 1\}, g \in \bigwedge^{[*]}(\mathcal{R}) \right\} / \bigwedge^{[*]}(\mathcal{R}), \tag{614}$$

with the same conditions as above, but where k needs to be an even number.

Corollary E.43. *Note that we also get well-defined group representations:*

$$\rho: \operatorname{Pin}(V, \mathfrak{q}) \to \operatorname{Aut}_{\mathbf{Alg},\mathrm{grd}}(\mathrm{Cl}(V, \mathfrak{q})) \cap \mathrm{O}_{\bigwedge(\mathcal{R})}(\mathrm{Cl}(V, \mathfrak{q}), \bar{\mathfrak{q}}), \tag{615}$$

with kernel $\ker \rho = \{\pm 1\}$.

In particular, $\mathrm{Cl}(V, \mathfrak{q})$ *and* $\mathrm{Cl}^{(m)}(V, \mathfrak{q})$ *for $m = 0, \ldots, n$, are orthogonal group representations of* $\operatorname{Pin}(V, \mathfrak{q})$ *via ρ:*

$$\rho: \operatorname{Pin}(V, \mathfrak{q}) \to \mathrm{O}_{\bigwedge^{(m)}(\mathcal{R})}(\mathrm{Cl}^{(m)}(V, \mathfrak{q}), \bar{\mathfrak{q}}). \tag{616}$$

Also, if $F(T_1, \ldots, T_{\ell+s}) \in \mathbb{R}[T_1, \ldots, T_{\ell+s}]$ is a polynomial in $\ell + s$ variables with coefficients in $\mathbb{R}$ and $x_1, \ldots, x_\ell \in \mathrm{Cl}(V, \mathfrak{q})$, $y_1, \ldots, y_s \in \bigwedge(\mathcal{R})$, and $k \in \{0, \ldots, n\}$. Then for every $w \in \operatorname{Pin}(V, \mathfrak{q})$ we have the equivariance property:

$$\rho(w)\left(F(x_1, \ldots, x_\ell, y_1, \ldots, y_s)^{(k)}\right) = F(\rho(w)(x_1), \ldots, \rho(w)(x_\ell), y_1, \ldots, y_s)^{(k)}. \tag{617}$$