1. *The space H is called as the* **reproducing kernel Hilbert space** *(RKHS) if $\forall x \in X$, the evaluation functional $\mathcal{E}_x : H \to \mathbf{K}$ defined as $\mathcal{E}_x(f) := f(x),\ f \in H$ is continuous.*

2. *A function $k : X \times X \to \mathbf{K}$ is called* reproducing kernel *of H if we have:*

   (a) $k(\cdot, x) \in H\ \forall x \in X$, that is $\|k(\cdot, x)\|_H < \infty$, and

   (b) $k(\cdot, \cdot)$ has the reproducing property; that is

$$f(x) = \langle f, k(\cdot, x)\rangle_H\ \forall f \in H \text{ and } x \in X.$$

The norm convergence yields the point-wise convergence inside RKHS. This fact can be readily learned due to the continuity of evaluation functional. This is demonstrated as follows for an arbitrary $f \in H$ and $\{f_n\}_n \in H$ with $\|f - f_n\|_H \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \mathcal{E}_x(f_n) =_{(\text{continuity of } \mathcal{E}_x)} \mathcal{E}_x(f) = f(x).$$

After we recall the definition of RKHS from Definition A.1, for an upcoming theorem we recall following MOORE-ARONSZAJN Theorem.

Theorem A.1 (Aronszajn (1950)). *Let H be an RKHS over an nonempty set X, Then $k : X \times X \to \mathbf{K}$ defined as $k(x, x') := \langle \mathcal{E}_x, \mathcal{E}_{x'}\rangle_H$ for $x, x' \in X$ is the only reproducing kernel of H. Additionally, for some index set $\mathcal{I}$, if we have $\{\mathbf{e}_i\}_{i\in\mathcal{I}}$ as an orthonormal basis then for all $x, x' \in X$, we have*

$$k(x, x') = \sum_{i\in\mathcal{I}} \mathbf{e}_i(x)\overline{\mathbf{e}_i(x')}, \tag{14}$$

with an absolute convergence.

We need to first construct the RKHS of the Laplacian Kernel which we will do by embedding it as an L^2-measure. After that we have determined the RKHS $(H_{\sigma,1,\mathbb{C}^n})$ of the Laplacian Kernel as L^2-measure (Theorem 3.1), we then learn the interplay of the Koopman operators over the RKHS $H_{\sigma,1,\mathbb{C}^n}$. This is performed by determining which bounded Koopman operators (Theorem A.12) is able to act *compactly* over the RKHS $H_{\sigma,1,\mathbb{C}^n}$. Such quantification is paramount because it guides us in executing the Lap-KeDMD algorithm to recover ST modes by employing the finite rank representation of Koopman operators (Interaction-Matrix $\boldsymbol{\mathcal{I}} := [\boldsymbol{\mathcal{I}}]_{i\times j}$ in Step 2 in Algorithm 2). We settle the compactness quantification (Theorem A.19) of the bounded Koopman operators over this RKHS by limiting the essential norm (Theorem A.18) (which measures the norm distance of the Koopman operator to the set of compact operators Shapiro (1987)). Once, we establish the compactness quantification of Koopman operators, we immediately shift our focus on demonstrating its closability over $H_{\sigma,1,\mathbb{C}^n}$.

A.2 ORTHONORMAL BASIS

We will be providing the orthonormal basis (ONB) for the Hilbert space $H_{\sigma,1,\mathbb{C}^n}$ generated by the measure $d\mu_{\sigma,1,\mathbb{C}^n}(\boldsymbol{z})$ embedded into the L^2-measure. Following lemma directs us in that direction.

Lemma A.2. *For $\sigma > 0$, N and $M \in \mathbb{W}$, we have*

$$\langle z^N, z^M\rangle_{\sigma,\mathbb{C}} = \int_{\mathbb{C}} z^N \overline{z^M} e^{-\frac{|z|}{\sigma}} dA(z) = 2\pi\sigma^2\sigma^{N+M}(N + M + 1)!\delta_{NM},$$

where $\langle \cdot, \cdot\rangle_{\sigma,\mathbb{C}}$ is the same inner-product as given in equation 8 but over $\mathbb{C}$.

Proof.

$$\begin{aligned}\langle z^N, z^M\rangle_{\sigma,\mathbb{C}} =& \int_{\mathbb{C}} z^N \overline{z^M} e^{-\frac{|z|}{\sigma}} dA(z) \\ =& \int_0^{2\pi}\int_0^{\infty} r^N r^M e^{i(N-M)\theta} e^{-\frac{r}{\sigma}} r dr d\theta \\ =& 2\pi\delta_{NM}\int_0^{\infty} r^{N+M+1} e^{-\frac{r}{\sigma}} dr \\ =& 2\pi\delta_{NM}\frac{\Gamma(N+M+1+1)}{(1/\sigma)^{N+M+2}} \\ =& 2\pi\delta_{NM}\frac{(N+M+1)!}{(1/\sigma)^{N+M+2}} \\ =& 2\pi\sigma^2\sigma^{N+M}(N+M+1)!\delta_{NM}.\end{aligned}$$

□

Theorem A.3. *For $\sigma > 0$ and $N \in \mathbb{W}$, define $\{\mathbf{e}_N\}_{N\in\mathbb{W}} : \mathbb{C} \to \mathbb{C}$ by*

$$\mathbf{e}_N(z) := \sqrt{\frac{1}{\sigma^{2N}(2N+1)!}}\, z^N. \tag{15}$$

The tensor-product system[2] $(\mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n})_{N_1,\ldots N_n \geq 0}$ *is the ONB of Hilbert space $H_{\sigma,1,\mathbb{C}^n}$.*

Proof. We establish our initial stage of result for single-dimension case to ease our understanding. For this, let us show that $\{\mathbf{e}_N\}_{N\in\mathbb{W}}$ forms an orthonormal system. So, consider $z \in \mathbb{C}$ and let $M, N \in \mathbb{W}$. Then,

$$\begin{aligned}\langle \mathbf{e}_N, \mathbf{e}_M\rangle_{\sigma,\mathbb{C}} =& \int_{\mathbb{C}} \mathbf{e}_N(z)\overline{\mathbf{e}_M(z)} d\mu_{\sigma,\mathbb{C}}(z) \\ =& \frac{1}{2\pi\sigma^2}\int_{\mathbb{C}} \sqrt{\frac{1}{\sigma^{2N}(2N+1)!}} z^N \sqrt{\frac{1}{\sigma^{2M}(2M+1)!}}\overline{z^M} e^{-\frac{|z|}{\sigma}} dA(z) \\ =& \frac{1}{2\pi\sigma^2}\sqrt{\frac{1}{\sigma^{2N}(2N+1)!}}\sqrt{\frac{1}{\sigma^{2M}(2M+1)!}} \cdot 2\pi\sigma^2\sigma^{N+M}(N+M+1)!\delta_{NM}. \\ =& \begin{cases} 1 & \text{if } N = M \\ 0 & \text{otherwise} \end{cases} \qquad (\text{Lemma A.2}).\end{aligned}$$

The above result concludes that $\{\mathbf{e}_n\}_{n\in\mathbb{W}}$ is actually an orthonormal system with respect to normalized Laplacian measure in $\mathbb{C}$. Now, we have to establish that this orthonormal system is also complete. So, for this, pick $f \in H_{\sigma,1,\mathbb{C}}$ with $f(z) = \sum_{l=0}^{\infty} a_l z^l$ and observe that

$$\begin{aligned}\langle f, \mathbf{e}_N\rangle_{\sigma,\mathbb{C}} =& \frac{1}{2\pi\sigma^2}\int_{\mathbb{C}} f(z)\overline{\mathbf{e}_N(z)} d\mu_{\sigma,\mathbb{C}} \\ =& \frac{1}{2\pi\sigma^2}\sqrt{\frac{1}{\sigma^{2N}(2N+1)!}}\int_{\mathbb{C}} f(z)\overline{z^N} e^{-\frac{|z|}{\sigma}} dA(z) \\ =& \frac{1}{2\pi\sigma^2}\sqrt{\frac{1}{\sigma^{2N}(2N+1)!}}\sum_{l=0}^{\infty} a_l \int_{\mathbb{C}} z^l \overline{z^N} e^{-\frac{|z|}{\sigma}} dA(z) \\ =& \frac{1}{2\pi\sigma^2}\sqrt{\frac{1}{\sigma^{2N}(2N+1)!}}\sum_{l=0}^{\infty} a_l \cdot 2\pi\sigma^2\sigma^{l+N}(l+N+1)!\delta_{lN} \\ =& \sqrt{\sigma^{2N}(2N+1)!}\, a_N.\end{aligned}$$

[2] We recall the *tensor product* between two functions, say $f_1,\ f_2 : X \to \mathbb{K}$ given as $f_1 \otimes f_2 : X \times X \to \mathbb{K}$. Then, for all $x, x' \in X$ the tensor product $f_1 \otimes f_2$ is defined as $f_1 \otimes f_2(x, x') := f_1(x)f_2(x')$.

For a given $\sigma > 0$, since the constant $\sqrt{\frac{1}{\sigma^{2N}(2N+1)!}} \neq 0$ for any $N \in \mathbb{W}$, therefore this directly imply that $\langle f, \mathbf{e}_N \rangle_{\sigma,\mathbb{C}} = 0$ if and only if $a_N = 0$, which results into $f \equiv 0$. Hence, $\{\mathbf{e}_N\}_{N\in\mathbb{W}}$ is complete. Now, we establish the same result but in $n-$dimensional case and to this end, we will employ the tensor product notation by considering multi-index notation for $N, M \in \mathbb{W}$ as follows:

$$\langle \mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n}, \mathbf{e}_{M_1} \otimes \cdots \otimes \mathbf{e}_{M_n} \rangle_{\sigma,\mathbb{C}^n} = \prod_{j=1}^{n} \langle \mathbf{e}_{N_j}, \mathbf{e}_{M_j} \rangle_{\sigma,\mathbb{C}}.$$

Hence the orthonormality of $\{\mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n}\}_{N_1,\ldots N_n \in,\mathbb{W}^n}$ is established due to the orthonormality of each individual $\langle \mathbf{e}_{N_j}, \mathbf{e}_{M_j} \rangle_{\sigma,\mathbb{C}}$. We still need to ensure that this $n-$dimensional orthonormal system is complete. We observe

$$\begin{aligned}
\langle f, \mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n} \rangle_{\sigma,1,\mathbb{C}^n} &= \left(\frac{1}{2\pi\sigma^2}\right)^n \int_{\mathbb{C}^n} f(\boldsymbol{z}) \overline{\mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n}(\boldsymbol{z})} d\mu_{\sigma,1,\mathbb{C}^n}(\boldsymbol{z}) \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^n \sum_{l_1,\ldots,l_n}^{\infty} a_{l_1,\ldots,l_n} \mathbb{I}_{l,n},
\end{aligned}$$

where $\mathbb{I}_{l,n} = \int_{\mathbb{C}^n} \boldsymbol{z}^l \left(\mathbf{e}_{N_1} \otimes \cdots \otimes \mathbf{e}_{N_n}(\overline{\boldsymbol{z}})\right) d\mu_{\sigma,1,\mathbb{C}^n}(\boldsymbol{z})$. We further can simplify $\mathbb{I}_{l,n}$ as:

$$\begin{aligned}
\mathbb{I}_{l,n} &= \int_{\mathbb{C}^n} \boldsymbol{z}^l \mathbf{e}_{N_1}(\overline{z_1}) \wedge \cdots \wedge \mathbf{e}_{N_n}(\overline{z_d}) d\mu_{\sigma,\mathbb{C}}(z_1) \wedge \cdots \wedge d\mu_{\sigma,\mathbb{C}}(z_n) \\
&= \prod_{j=1}^{n} \left(\int_{\mathbb{C}} z_j^{l_j} \mathbf{e}_{N_j}(\overline{z_j}) d\mu_{\sigma,\mathbb{C}}(z_j) \right) \\
&= \prod_{j=1}^{n} \left(\int_{\mathbb{C}} z_j^{l_j} \overline{z_j}^{N_j} d\mu_{\sigma,\mathbb{C}}(z_j) \right) \\
&= \prod_{j=1}^{n} \left(2\pi\sigma^2 \sigma^{l_j+N_j} (l_j + N_j + 1)! \right) \delta_{l_j N_j}.
\end{aligned}$$

Finally,

$$\left(\frac{1}{2\pi\sigma^2}\right)^n \sum_{l_1,\ldots,l_n}^{\infty} a_{l_1,\ldots,l_n} \mathbb{I}_{l,n} = \left(\prod_{j=1}^{n} \sqrt{\sigma^{2N_j}(2N_j+1)!} \right) a_{N_1,\ldots,N_n}.$$

Further result for completeness in $n-$dimension follows a routine procedure from single-dimension case as already discussed before. □

A.3 Proof of Theorem 3.1

Proof. We have determined the orthonormal basis given by $\mathbf{e}_N(\boldsymbol{z})$ in Theorem A.3 present in Subsubsection A.2 for the Hilbert space $H_{\sigma,1,\mathbb{C}^n}$. We will use it to construct the reproducing kernel for $H_{\sigma,1,\mathbb{C}^n}$ which eventually makes it the RKHS by recalling Theorem A.1. So,

$$K^\sigma(\boldsymbol{z}, \boldsymbol{w}) = \sum_{N\in\mathbb{W}} \mathbf{e}_N(\boldsymbol{z}) \overline{\mathbf{e}_N(\boldsymbol{w})} = \sum_{N=0}^{\infty} \frac{1}{(2N+1)!} \left(\frac{\boldsymbol{z}\overline{\boldsymbol{w}^\top}}{\sigma^2} \right)^N = \frac{\sinh\left(\sqrt{\boldsymbol{z}\overline{\boldsymbol{w}^\top}/\sigma^2}\right)}{\sqrt{\left(\boldsymbol{z}\overline{\boldsymbol{w}^\top}/\sigma^2\right)}}.$$

Hence, the result is now established. □

A.4 Helper proofs for Theorem A.12

Lemma A.4. *$\forall\, \boldsymbol{z} \in \mathbb{C}^n$, the norm of the reproducing kernel $K_{\boldsymbol{z}}^\sigma$ of RKHS $H_{\sigma,1,\mathbb{C}^n}$ satisfies:*

$$\exp\left(\frac{1}{2}\left[\frac{\|\boldsymbol{z}\|_2}{\sigma}\coth\frac{\|\boldsymbol{z}\|_2}{\sigma} - 1\right]\right) <_{(1)} \|K_{\boldsymbol{z}}^\sigma\|^2 <_{(2)} \exp\left(\frac{\|\boldsymbol{z}\|_2^2}{6\sigma^2}\right). \tag{16}$$

Proof. We will employ the Weierstrass factorization theorem (cf. (Stein & Shakarchi, 2010, Chapter 5)) for $\sinh\zeta/\zeta$ followed by taking the 'log' as demonstrated follows:

$$\frac{\sinh\zeta}{\zeta} = \prod_{j=1}^{\infty}\left(1+\frac{\zeta^2}{j^2\pi^2}\right) \implies \log\frac{\sinh\zeta}{\zeta} = \sum_{j=1}^{\infty}\log\left(1+\frac{\zeta^2}{j^2\pi^2}\right).$$

To establish INEQUALITY (2), we define $\mathfrak{g}_2 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ by $\mathfrak{g}_2(x) = x - \log(1+x)$ for $x \in \mathbb{R}_+ \cup \{0\}$. Then $\mathfrak{g}_2(x) \geq 0$ whenever $x \in \mathbb{R}_+ \cup \{0\}$. Hence, we conclude now that $\mathfrak{g}_2\left(\frac{\zeta^2}{j^2\pi^2}\right) > 0$ whenever $j \geq 1$. Therefore $\log\left(1+\frac{\zeta^2}{j^2\pi^2}\right) < \frac{\zeta^2}{j^2\pi^2}$. Hence, taking the summation of this over $j \in \mathbb{Z}_+$, we have

$$\log\frac{\sinh\zeta}{\zeta} = \sum_{j=1}^{\infty}\log\left(1+\frac{\zeta^2}{j^2\pi^2}\right) < \sum_{j=1}^{\infty}\frac{\zeta^2}{j^2\pi^2} = \frac{\zeta^2}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\zeta^2}{\pi^2}\cdot\frac{\pi^2}{6} = \frac{\zeta^2}{6}.$$

Taking the exponentiation of above yields $\frac{\sinh\zeta}{\zeta} < \exp\left(\frac{\zeta^2}{6}\right)$ and with $\zeta \mapsto \|\boldsymbol{z}\|_{\mathbb{C}^n}/\sigma$, we have finally:

$$\|K_{\boldsymbol{z}}^{\sigma}\|^2 = \left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right)^{-1}\cdot\sinh\left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right) < \exp\left(\frac{\|\boldsymbol{z}\|_2^2}{6\sigma^2}\right). \tag{17}$$

Now, we provide the lower bound for the same. To establish INEQUALITY (1), we define $\mathfrak{g}_1 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ by $\mathfrak{g}_1(x) = \log(1+x) - \frac{x}{1+x}$ for $x \in \mathbb{R}_+ \cup \{0\}$. Then $\mathfrak{g}_1(x) \geq 0$ whenever $x \in \mathbb{R}_+ \cup \{0\}$. Hence we conclude now that $\mathfrak{g}_1\left(\frac{\zeta^2}{j^2\pi^2}\right) > 0$ whenever $j \geq 1$. Therefore, $\frac{\zeta^2}{j^2\pi^2+\zeta^2} < \log\left(1+\frac{\zeta^2}{j^2\pi^2}\right)$. Hence, taking the summation of this over $j \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\log\frac{\sinh\zeta}{\zeta} &= \sum_{j=1}^{\infty}\log\left(1+\frac{\zeta^2}{j^2\pi^2}\right)\\
&> \sum_{j=1}^{\infty}\frac{\zeta^2}{j^2\pi^2+\zeta^2}\\
&= \frac{\zeta^2}{\pi^2}\left[\frac{1}{2}\left[\frac{\pi}{\frac{\zeta}{\pi}}\coth\left(\frac{\zeta}{\pi}\pi\right) - \frac{1}{\frac{\zeta^2}{\pi^2}}\right]\right] \qquad \text{((Stein \& Shakarchi, 2010, Page 128(\#6)))}\\
&= \frac{1}{2}\left[\zeta\coth\zeta - 1\right].
\end{aligned}$$

Exponentiation of above yields $\exp\left(\frac{1}{2}\left[\zeta\coth\zeta - 1\right]\right) < \frac{\sinh\zeta}{\zeta}$ and with $\zeta \mapsto \|\boldsymbol{z}\|_2/\sigma$, we have:

$$\exp\left(\frac{1}{2}\left[\frac{\|\boldsymbol{z}\|_2}{\sigma}\coth\frac{\|\boldsymbol{z}\|_2}{\sigma} - 1\right]\right) < \left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right)^{-1}\cdot\sinh\frac{\|\boldsymbol{z}\|_2}{\sigma} = \|K_{\boldsymbol{z}}^{\sigma}\|^2. \tag{18}$$

□

Lemma A.5. *The adjoint of $\mathcal{K}_\varphi$ denoted by $\mathcal{K}_\varphi^*$ satisfies $\mathcal{K}_\varphi^* K_{\boldsymbol{z}}^{\sigma} = K_{\varphi(\boldsymbol{z})}^{\sigma}$ over the RKHS $H_{\sigma,1,\mathbb{C}^n}$.*

Proof. Let $f \in H_{\sigma,1,\mathbb{C}^n}$ and pick an arbitrary $\boldsymbol{z} \in \mathbb{C}^n$, then

$$\langle f, \mathcal{K}_\varphi^* K_{\boldsymbol{z}}^{\sigma}\rangle = \langle \mathcal{K}_\varphi f, K_{\boldsymbol{z}}^{\sigma}\rangle = \mathcal{K}_\varphi f(\boldsymbol{z}) = f(\varphi(\boldsymbol{z})) = \langle f, K_{\varphi(\boldsymbol{z})}^{\sigma}\rangle.$$

Hence, the desired result is achieved. □

Lemma A.5 makes us realize that the set of reproducing kernel $\{K_{\boldsymbol{z}}^{\sigma} : \boldsymbol{z} \in \mathbb{C}^n\}$ is invariant under the adjoint of $\mathcal{K}_\varphi$ (Cowen, 1983, Chapter 1). Additionally the relation defined in the above lemma provides the unique relationship of $\mathcal{K}_\varphi^* K_{\boldsymbol{z}}^{\sigma}$ via the inner product of the RKHS $H_{\sigma,1,\mathbb{C}^n}$ and hence we can have the Koopman operator $\mathcal{K}_\varphi$ as to be densely defined over the RKHS $H_{\sigma,1,\mathbb{C}^n}$ (Rudin, 1991, Chapter 13, Page 348) and also the adjoint of the Koopman operator is now close in the RKHS $H_{\sigma,1,\mathbb{C}^n}$ (Rudin, 1991, Theorem 13.9).

Lemma A.6. *Let $\Psi : \mathbb{C}^n \to \mathbb{C}$ be holomorphic on a complex domain containing the closed unit ball. If $\Psi(z) = \sum_{|j|=0}^{\infty} a_j z^j$, then*

$$(2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\Psi(\boldsymbol{z})|^2 d\boldsymbol{\vartheta} = \sum_{|j|=0} |a_j|^2 r^{2j}, \quad a_j \in \mathbb{C}.$$

Proof. Recall that $(2\pi)^{-1} \int_{-\pi}^{\pi} e^{i(j-k)\vartheta}\, d\vartheta = \delta_{j,k}$. As $\Psi(\boldsymbol{z}) = \sum_{|j|=0} a_j \boldsymbol{z}^j$, then:

$$\begin{aligned} |\Psi(\boldsymbol{z})|^2 = \Psi(\boldsymbol{z})\overline{\Psi(\boldsymbol{z})} &= \sum_{|j|=0} \sum_{|k|=0} a_j \overline{a_k} z^j \overline{z}^k \\ &= \sum_{|j|=0} \sum_{|k|=0} a_j \overline{a_k} r^{j+k} e^{i(j-k)\theta} \\ &= \sum_{|j|=0} \sum_{|k|=0} a_j \overline{a_k} r_1^{j_1+k_1} \cdots r_n^{j_n+k_n} \left(\prod_{l=1}^{n} e^{i(j_l-k_l)\theta_l} \right). \end{aligned}$$

So,

$$\begin{aligned} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |\Psi(\boldsymbol{z})|^2 d\boldsymbol{\vartheta} &= \sum_{|j|=0} \sum_{|k|=0} a_j \overline{a_k} r_1^{j_1+k_1} \cdots r_n^{j_n+k_n} \overbrace{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi}}^{l-times} \left(\prod_{l=1}^{n} e^{i(j_l-k_l)\vartheta_l} d\vartheta_l \right) \\ &= \sum_{|j|=0} \sum_{|k|=0} a_j \overline{a_k} r_1^{j_1+k_1} \cdots r_n^{j_n+k_n} [(2\pi)^n \delta_{j_1 k_1} \cdots \delta_{j_n k_n}] \\ &= (2\pi)^n \sum_{|j|=0} |a_j|^2 r^{2j}. \end{aligned}$$

Therefore, multiplying by $(2\pi)^{-n}$ in the last equality furnishes the desired proof. $\square$

Proposition A.7 (Jensen's convex inequality). *Let (Ω, Σ, μ) be a probability space, and g a real-valued function that is μ-integrable. If ψ is a convex function then,*

$$\psi \left(\int_{\Omega} g \, d\mu \right) \leq \int_{\Omega} \psi \circ g \, d\mu.$$

Proof. See (Garnett, 2007, Lemma 6.1, Page 33). $\square$

Lemma A.8. *Let $\Xi : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic mapping with $\Xi \equiv (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, where each $\{\xi\}_{i=1,\ldots,n}$ are coordinate functions of Ξ from $\mathbb{C}^n \to \mathbb{C}$ which are holomorphic. As $\|\Xi(\boldsymbol{z})\|_2$ be the Euclidean-norm in $\mathbb{C}^n$ at $\boldsymbol{z} \in \mathbb{C}^n$, then following is satisfied for any $\alpha \geq 1$,*

$$\left(\int_{r\mathbb{B}_n} \|\Xi(\boldsymbol{z})\|_2^2 d\mathbb{P}(\boldsymbol{z}) \right)^{\alpha} \leq \int_{r\mathbb{B}_n} \|\Xi(\boldsymbol{z})\|_2^{2\alpha} d\mathbb{P}(\boldsymbol{z}),$$

where $r\mathbb{B}_n \coloneqq \{\boldsymbol{z} \in \mathbb{C}^n : \|\boldsymbol{z}\|_2 \leq r\}$ for some $r > 0$ against probability measure $d\mu(\boldsymbol{z}) \coloneqq d\mathbb{P}(\boldsymbol{z})$.

Proof. We will use the Jensen's convex inequality Proposition A.7 against the probability measure $d\mathbb{P}(\boldsymbol{z})$ over $r\mathbb{B}_n$. We take $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi_\alpha(x) = x^\alpha$ for some $\alpha \geq 1$, then ψ_α is a convex function (Boyd & Vandenberghe (2004)). Now, $\|\Xi(\boldsymbol{z})\|_2^2 = \sum_{i=1}^n |\xi_i(\boldsymbol{z})|^2$ is a real-valued function integrable with respect to $d\mathbb{P}(\boldsymbol{z})$ over $r\mathbb{B}_n$. Thus, we have

$$\psi_\alpha \left(\int_{r\mathbb{B}_n} \|\Xi(\boldsymbol{z})\|_2^2 d\mathbb{P}(\boldsymbol{z}) \right) = \left(\int_{r\mathbb{B}_n} \|\Xi(\boldsymbol{z})\|_2^2 d\mathbb{P}(\boldsymbol{z}) \right)^{\alpha} \tag{19}$$

by the definition of ψ_α. On the other hand,

$$\int_{r\mathbb{B}_n} \psi_\alpha \circ \|\Xi(\boldsymbol{z})\|_2^2 d\mathbb{P}(\boldsymbol{z}) = \int_{r\mathbb{B}_n} \left(\|\Xi(\boldsymbol{z})\|_2^2 \right)^{\alpha} d\mathbb{P}(\boldsymbol{z}) = \int_{r\mathbb{B}_n} \|\Xi(\boldsymbol{z})\|_2^{2\alpha} d\mathbb{P}(\boldsymbol{z}). \tag{20}$$

Combining the result of equation 19, equation 20 together with the result of Proposition A.7, we achieve the desired result. $\square$

Proposition A.9. *If F is an entire function that satisfy $\sup_{|z|=R}|F(z)| \leq AR^k + B \; \forall \; R > 0$ and for some integer $k \geq 0$ and some constants $A, B > 0$, then F is a polynomial of degree $\leq k$.*

Proof. Follow (Stein & Shakarchi, 2010, Chapter-3, Exercise-15(a), Page 105-106). □

We give a key proposition before we give the main theorem related to the boundedness of the Koopman operators over the RKHS $H_{\sigma,1,\mathbb{C}^n}$.

Proposition A.10. *Let there be a positive finite M such that for a holomorphic function $\varphi : \mathbb{C}^n \to \mathbb{C}^n$. Let $\boldsymbol{z} \in \mathbb{C}^n$, and suppose following holds:*

$$\left[\frac{\sinh\left(\|\varphi(\boldsymbol{z})\|_2/\sigma\right)}{\|\varphi(\boldsymbol{z})\|_2/\sigma}\right]^{\frac{1}{2}} \cdot \left[\frac{\sinh\left(\|\boldsymbol{z}\|_2/\sigma\right)}{\|\boldsymbol{z}\|_2/\sigma}\right]^{-\frac{1}{2}} < M.$$

If $0 < \|\varphi\|_2 \leq \pi\sigma$, then $\varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ with $0 < \|\boldsymbol{A}\|_2 \leq 1$ and $\boldsymbol{b}$ is a complex vector.

Proof. The validity of given inequality still holds even if we square it followed by taking log, so

$$\log\left[\frac{\sinh\left(\|\varphi(\boldsymbol{z})\|_2/\sigma\right)}{\|\varphi(\boldsymbol{z})\|_2/\sigma}\right] < \log(M^2) + \log\left[\frac{\sinh\left(\|\boldsymbol{z}\|_2/\sigma\right)}{\|\boldsymbol{z}\|_2/\sigma}\right].$$

We can now use the results of Lemma A.4 in the above inequality to result into following:

$$\begin{aligned}\frac{1}{2}\left[\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\coth\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right) - 1\right] &< \log(M^2) + \frac{\|\boldsymbol{z}\|_2^2}{6\sigma^2} \\ \implies \frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\coth\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right) &< 2\log(M^2) + 1 + \frac{\|\boldsymbol{z}\|_2^2}{3\sigma^2}. \end{aligned} \tag{21}$$

Now, to further simplify the above inequality, we will simply employ the infinite series expansion of an entire $\coth(\bullet)$ which involves the presence of Bernoulli's number $\{B_j\}_{j\in\mathbb{W}}$; defined as $\coth x = \sum_{j=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1}$. The above equation can be explicitly written as $x\coth x = 1 + \frac{2^2 B_2}{2!}x^2 + \sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}x^{2j}$ under the additional assumption of $0 < |x| < \pi$. Here, for $j = 1$, we have $B_2 = 1/6$, so $x \mapsto \|\varphi(\boldsymbol{z})\|_2/\sigma$ in above yields:

$$\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\coth\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right) = 1 + \frac{1}{3}\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right)^2 + \sum_{j=2}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right)^{2j}. \tag{22}$$

Using the result of equation 22 in equation 21 to have following:

$$\begin{aligned} 1 + \frac{1}{3}\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right)^2 + \sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!}\left(\frac{\|\varphi(\boldsymbol{z})\|_2}{\sigma}\right)^{2j} &< 2\log(M^2) + 1 + \frac{\|\boldsymbol{z}\|_2^2}{3\sigma^2} \\ \|\varphi(\boldsymbol{z})\|_2^2 + (3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}}\|\varphi(\boldsymbol{z})\|_2^{2j} &< 2\log(M^2) + \|\boldsymbol{z}\|_2^2. \end{aligned} \tag{23}$$

Considering $\varphi \equiv (\varphi_1(\boldsymbol{z}), \ldots, \varphi_n(\boldsymbol{z}))^\top \in \mathbb{C}^n$, where each $\{\varphi_i\}_{i=1,\ldots,n}$ is a coordinate function of φ and is a holomorphic mapping from $\mathbb{C}^n \to \mathbb{C}$, then $\|\varphi(\boldsymbol{z})\|_2^2 = \sum_{i=1}^n |\varphi_i(\boldsymbol{z})|^2$. Therefore, with $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in \mathbb{C}^n$ and $\|\boldsymbol{z}\|_2^2 = \sum_{i=1}^n |z_i|^2$, we see that

$$\left\{\sum_{i=1}^n\left[|\varphi_i(\boldsymbol{z})|^2 - |z_i|^2\right]\right\} + \left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}}\|\varphi(\boldsymbol{z})\|_2^{2j}\right\} < 2\log(M^2).$$

Integrating above with respect to $\boldsymbol{\vartheta}$ on $r\mathbb{B}_n$ to have

$$\begin{aligned} &\left\{\int_{r\mathbb{B}_n}\sum_{i=1}^n\left[|\varphi_i(\boldsymbol{z})|^2 - |z_i|^2\right]\frac{d\boldsymbol{\vartheta}}{(2\pi)^n}\right\} \\ &+ \left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}}\int_{r\mathbb{B}_n}\|\varphi(\boldsymbol{z})\|_2^{2j}\frac{d\boldsymbol{\vartheta}}{(2\pi)^n}\right\} < 2\log(M^2). \end{aligned} \tag{24}$$

As here, the infinite summation starts with $j \geq 2 > 1$, therefore by the application of Lemma A.8, we can have following conclusion

$$\begin{aligned}&\left\{\int_{r\mathbb{B}_n}\sum_{i=1}^{n}\left[|\varphi_i(\boldsymbol{z})|^2-|z_i|^2\right]\frac{d\boldsymbol{\vartheta}}{(2\pi)^n}\right\}_{(\spadesuit)}\\&+\left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}(2\pi)^{nj}}\left(\int_{r\mathbb{B}_n}\|\varphi(\boldsymbol{z})\|_2^2 d\boldsymbol{\vartheta}\right)^j\right\}_{(\clubsuit)}\leq \text{LHS of equation 24}\\&<2\log(M^2).\end{aligned}$$

Here we are using $\{\bullet\}_{(\spadesuit)}$ and $\{\bullet\}_{(\clubsuit)}$ to provide smooth understanding of respective manipulations that are happening on respective quantities present inside the curly brackets. Therefore,

$$\begin{aligned}&\left\{\int_{r\mathbb{B}_n}\sum_{i=1}^{n}\left[|\varphi_i(\boldsymbol{z})|^2-|z_i|^2\right]\frac{d\boldsymbol{\vartheta}}{(2\pi)^n}\right\}_{(\spadesuit)}\\&+\left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}(2\pi)^{nj}}\left(\int_{r\mathbb{B}_n}\|\varphi(\boldsymbol{z})\|_2^2 d\boldsymbol{\vartheta}\right)^j\right\}_{(\clubsuit)}<2\log(M^2).\end{aligned}$$

The above inequality can be simplified into following (term by term) in terms of r along with the help of multi-index notation:

$$\begin{aligned}&\left\{\sum_{|j|=0}|a_j^k|^2 r_1^{2j_1}\cdot\ldots\cdot r_n^{2j_n}-\sum_{\ell=1}^{n}r_\ell^2\right\}_{(\spadesuit)}\\&+\left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}(2\pi)^{nj}}\left(\sum_{|j|=0}|a_j^k|^2 r_1^{2j_1}\cdot\ldots\cdot r_n^{2j_n}\right)^j\right\}_{(\clubsuit)}<2\log(M^2).\end{aligned}$$

Note that, here we used a super-script of k to indicate this a decomposition of the k-th component of the function φ. Further, if we let e_i be the multi-index with a 1 in the i-th spot and *zeros* else where, then the above rearranges to:

$$\begin{aligned}&\left\{\overbrace{\sum_{|j|=2}|a_j^k|^2 r_1^{2j_1}\cdot\ldots\cdot r_n^{2j_n}}^{\dagger(r)}+|a_0^k|^2+\sum_{\ell=0}^{n}(|a_{e_\ell}^k|^2-1)r_\ell^2\right\}_{(\spadesuit)}\\&+\left\{(3\sigma^2)\sum_{j=2}^{\infty}\frac{2^{2j}B_{2j}}{(2j)!\sigma^{2j}(2\pi)^{nj}}\left(\underbrace{\sum_{|j|=0}|a_j^k|^2 r_1^{2j_1}\cdot\ldots\cdot r_n^{2j_n}}_{\ddagger(r)}\right)^j\right\}_{(\clubsuit)}<2\log(M^2).\end{aligned}$$

This inequality is true for all $r=(r_1,\ldots,r_n)^\top\in\mathbb{R}_+^n$. Both quantities $\dagger(r)$ and $\ddagger(r)$ grow in the *Big-O* complexity rate, that is $\dagger(r)\propto O(r)$ & $\ddagger(r)\propto O(r^j)$ as $r\to\infty$. Therefore, by the application of Proposition A.9) in this inequality, we conclude that $|a_j^k|=0$ for $j\geq 2$ in $\dagger(r)$. Similarly, Proposition A.9) forces to $\ddagger(r)$ be 0 as well. Hence, with the relabeling of the coordinate function of φ as $\varphi_k(z)=a_{k,1}z_1+\cdots+a_{k,n}z_n+b_k$. Thus, $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$ where, $\boldsymbol{A}=[a_{k,j}]_{k,j=1}^{n,n}$ and $\boldsymbol{b}=(b_1,\ldots,b_n)^\top$. This particular structure is called as an *affine structure*. Now, that we have both $\dagger(r)$ and $\ddagger(r)$ are 0 and $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$, we revisit equation 23 to have

$$\|\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}\|_2^2<2\log(M^2)+\|\boldsymbol{z}\|_2^2\implies\lim_{\|\boldsymbol{z}\|\to\infty}\frac{\|\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}\|_2^2}{\|\boldsymbol{z}\|_2^2}<1.$$

If $\|\boldsymbol{A}\tilde{\boldsymbol{z}}\|_2 > \|\tilde{\boldsymbol{z}}\|_2 = 1$ for some $\tilde{\boldsymbol{z}} \in \mathbb{C}^n$ whose norm is 1. Setting $\boldsymbol{z} = t\tilde{\boldsymbol{z}}$ and $t > 0$ in above yields:

$$\lim_{t\to\infty} \frac{\left\|\boldsymbol{A}\tilde{\boldsymbol{z}} + \frac{1}{t}\boldsymbol{b}\right\|_2}{\|\tilde{\boldsymbol{z}}\|_2} < 1,$$

which is a contradiction. Therefore, $\|\boldsymbol{A}\|_2 \le 1$. □

We give an important inequality for the action of the Koopman operators on the normalized reproducing kernel $\boldsymbol{k}_{\boldsymbol{z}}^{\sigma} := K_{\boldsymbol{z}}^{\sigma}/\|K_{\boldsymbol{z}}^{\sigma}\|$ satisfying $\|\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\| = 1$.This result is captured in the following lemma.

Lemma A.11. *If for some $\boldsymbol{z} \in \mathbb{C}^n$, $\boldsymbol{k}_{\boldsymbol{z}}^{\sigma} \in \mathcal{D}(\mathcal{K}_{\varphi})$, then $\sqrt{\Pi_{\boldsymbol{z}}(\varphi;\sigma)} \le \|\mathcal{K}_{\varphi}\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|$.*

Proof. The proof of the above result involves the application of point-evaluation inequality for the RKHS $H_{\sigma,1,\mathbb{C}^n}$. Further details are given as follows:

$$\|\mathcal{K}_{\varphi}\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|^2\|K_{\boldsymbol{z}}^{\sigma}\|^2 \ge |\mathcal{K}_{\varphi}\boldsymbol{k}_{\varphi(\boldsymbol{z})}^{\sigma}(\boldsymbol{z})|^2 = |\boldsymbol{k}_{\varphi(\boldsymbol{z})}^{\sigma}(\varphi(\boldsymbol{z}))|^2.$$

As $|\boldsymbol{k}_{\varphi(\boldsymbol{z})}^{\sigma}| = |K_{\varphi(\boldsymbol{z})}^{\sigma}/\|K_{\varphi(\boldsymbol{z})}^{\sigma}\||$, hence $|\boldsymbol{k}_{\varphi(\boldsymbol{z})}^{\sigma}(\varphi(\boldsymbol{z}))| = |K_{\varphi(\boldsymbol{z})}^{\sigma}(\varphi(\boldsymbol{z}))/\|K_{\varphi(\boldsymbol{z})}^{\sigma}\|| = |\|K_{\varphi(\boldsymbol{z})}^{\sigma}\|^2/\|K_{\varphi(\boldsymbol{z})}^{\sigma}\|| = \|K_{\varphi(\boldsymbol{z})}^{\sigma}\|$. From this result and above, we have

$$\|\mathcal{K}_{\varphi}\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|^2\|K_{\boldsymbol{z}}^{\sigma}\|^2 \ge \|K_{\varphi(\boldsymbol{z})}^{\sigma}\|^2 = \|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2.$$

Therefore, further dividing above by $\|K_{\boldsymbol{z}}^{\sigma}\|^2 \neq 0$ to have $\|\mathcal{K}_{\varphi}\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|^2 \ge \frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2} = \Pi_{\boldsymbol{z}}(\varphi;\sigma)$. Desired result follows by taking the square-root of above. Hence proved! □

A.5 Boundedness of Koopman operators over the RKHS $H_{\sigma,1,\mathbb{C}^n}$

Theorem A.12. *$\mathcal{K}_{\varphi}$ acts boundedly over $H_{\sigma,1,\mathbb{C}^n} \iff \varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$, where $\|\boldsymbol{A}\|_2 \le 1$ and $\Pi(\varphi;\sigma) < \infty$.*

Proof. $\Longrightarrow$ Suppose that the φ-induced Koopman operator is bounded over RKHS $H_{\sigma,1,\mathbb{C}^n}$, which means that there exists a finite positive M such that $\|\mathcal{K}_{\varphi}\|^2 < M$. As $\|\mathcal{K}_{\varphi}^{*}\|^2 = \|\mathcal{K}_{\varphi}\|^2$ (cf. Hall (2013)) and hence $\|\mathcal{K}_{\varphi}^{*}\|^2 < M < \infty$. Now, observe that

$$\infty > M > \|\mathcal{K}_{\varphi}^{*}\|^2 = \sup_{\boldsymbol{z}\in\mathbb{C}^n} \frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2} \ge \frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2}.$$

The above inequality allow us to have $\frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2} < M$. Thus, employing the result of (our key proposition) Proposition A.10, we have $\varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$ along with $\|\boldsymbol{A}\|_2 \le 1$.

$\Longleftarrow$ Suppose $\varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ with $\|\boldsymbol{A}\|_2 \le 1$. Additionally, suppose that for this φ, $\Pi(\varphi;\sigma) < \infty$ also holds. Recall normalized reproducing kernel $\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}$ at $\boldsymbol{z} \in \mathbb{C}^n$ given as $\boldsymbol{k}_{\boldsymbol{z}}^{\sigma} = K_{\boldsymbol{z}}^{\sigma}/\|K_{\boldsymbol{z}}^{\sigma}\|$. Then,

$$\begin{aligned}
\|\mathcal{K}_{\varphi}^{*}\|^2 = \sup_{\boldsymbol{z}\in\mathbb{C}^n}\left\{\sup_{\|\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|=1} \frac{\|\mathcal{K}_{\varphi}^{*}\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|^2}{\|\boldsymbol{k}_{\boldsymbol{z}}\|^2}\right\} &= \sup_{\boldsymbol{z}\in\mathbb{C}^n}\left\{\sup_{\|\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|=1} \frac{\left\|\mathcal{K}_{\varphi}^{*}\frac{K_{\boldsymbol{z}}^{\sigma}}{\|K_{\boldsymbol{z}}^{\sigma}\|}\right\|^2}{\left\|\frac{K_{\boldsymbol{z}}^{\sigma}}{\|K_{\boldsymbol{z}}^{\sigma}\|}\right\|^2}\right\} \\
&= \sup_{\boldsymbol{z}\in\mathbb{C}^n}\left\{\sup_{\|\boldsymbol{k}_{\boldsymbol{z}}^{\sigma}\|=1}\left[\frac{\|K_{\boldsymbol{z}}^{\sigma}\|^{-2}}{\|K_{\boldsymbol{z}}^{\sigma}\|^{-2}} \cdot \frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2}\right]\right\} \\
&= \sup_{\boldsymbol{z}\in\mathbb{C}^n} \frac{\|\mathcal{K}_{\varphi}^{*}K_{\boldsymbol{z}}^{\sigma}\|^2}{\|K_{\boldsymbol{z}}^{\sigma}\|^2} = \Pi(\varphi;\sigma) < \infty.
\end{aligned}$$

The above chain of inequalities implies that $\|\mathcal{K}_{\varphi}^{*}\|$ is bounded. □

A.6 COMPACTIFICATION OF KOOPMAN OPERATORS OVER THE RKHS $H_{\sigma,1,\mathbb{C}^n}$

Definition A.2. *Define:*

$$\Pi_{\boldsymbol{z}}(\varphi;\sigma) := \|\mathcal{K}_\varphi^* K_{\boldsymbol{z}}^\sigma\|^2 \cdot \|K_{\boldsymbol{z}}^\sigma\|^{-2}, \tag{25}$$
$$\Pi\left(\varphi;\sigma\right) := \sup_{\boldsymbol{z}\in\mathbb{C}^n} \Pi_{\boldsymbol{z}}(\varphi;\sigma). \tag{26}$$

A.6.1 ESSENTIAL NORM AND COMPACTIFICATION

We recall the basic definition for the *essential norm* of a bounded linear operator.

Definition A.3. *For two Banach spaces $\mathbb{X}_1$ and $\mathbb{X}_2$ we denote by $K(\mathbb{X}_1,\mathbb{X}_2)$ the set of all compact operators from $\mathbb{X}_1$ into $\mathbb{X}_2$. The essential norm of a bounded linear operator $\mathrm{A}:\mathbb{X}_1\to\mathbb{X}_2$, denoted as $\|\mathrm{A}\|_{\mathrm{e}}$ is defined as*

$$\|\mathrm{A}\|_{\mathrm{e}} := \inf\left\{\|\mathrm{A}-T\| : T\in K(\mathbb{X}_1,\mathbb{X}_2)\right\}. \tag{27}$$

Recall that a holomorphic function $f:\mathbb{C}^n\to\mathbb{C}$ exhibiting $f(\boldsymbol{z})=\sum_m a_m\boldsymbol{z}^m$ for $\boldsymbol{z}\in\mathbb{C}^n$ where the summation is over all multi-indexes $m=(m_1,\cdots,m_n)$ where each $\{m_i\}$ are positive integer and $\boldsymbol{z}=z_1^{m_1}\cdots z_n^{m_n}$. Follow Zhu (2005) for more details on this. By letting $P_k(\boldsymbol{z})=\sum_{|m|=k} a_m\boldsymbol{z}^m\ \forall\, k\geq 0$ where $|m|=\sum_{i=1}^n m_i$, then the Taylor series of f can be re-written as

$$f(\boldsymbol{z})=\sum_{k=0}^{\infty} P_k(\boldsymbol{z}). \tag{28}$$

The result in equation 28 is called as the *homogeneous polynomial expansion* of holomorphic function f having the degree of k which is uniquely determined by f. Now, for each $m\in\mathbb{Z}_+$, we define the operator $\mathcal{P}_m$ on holomorphic function f as follows:

$$\mathcal{P}_m f\left(\boldsymbol{z}\right)=\sum_{k=m}^{\infty} P_k\left(\boldsymbol{z}\right).$$

Consider the action of the operator $\mathcal{P}_m$ defined above on the reproducing kernel $K_{\boldsymbol{w}}^\sigma$ of the RKHS $H_{\sigma,1,\mathbb{C}^n}$ from equation 10 in Theorem 3.1, then:

$$\mathcal{P}_m K_{\boldsymbol{w}}^\sigma\left(\boldsymbol{z}\right)=\mathcal{P}_m\left(\frac{\sinh\left(\sqrt{\frac{\langle\boldsymbol{z},\boldsymbol{w}\rangle_{\mathbb{C}^n}}{\sigma^2}}\right)}{\sqrt{\frac{\langle\boldsymbol{z},\boldsymbol{w}\rangle_{\mathbb{C}^n}}{\sigma^2}}}\right)=\mathcal{P}_m\left(\sum_{N=0}^{\infty}\frac{\langle\boldsymbol{z},\boldsymbol{w}\rangle_{\mathbb{C}^n}}{(2N+1)!}\right)=\sum_{N=m}^{\infty}\frac{\langle\boldsymbol{z},\boldsymbol{w}\rangle_{\mathbb{C}^n}}{(2N+1)!}.$$

Proposition A.13. *For all $f\in H_{\sigma,1,\mathbb{C}^n}$*

$$|\mathcal{P}_m f(\boldsymbol{z})|\leq\|f\|\sqrt{\sum_{N=m}^{\infty}\frac{\|\boldsymbol{z}\|_2^{2N}}{(2N+1)!}}. \tag{29}$$

Proof. Consider $f\in H_{\sigma,1,\mathbb{C}^n}$, then employ the reproducing property of the reproducing kernel $K_{\boldsymbol{z}}^\sigma$ at $\boldsymbol{z}\in\mathbb{C}^n$ as $\mathcal{P}_m f\left(\boldsymbol{z}\right)=\langle\mathcal{P}_m f, K_{\boldsymbol{z}}^\sigma\rangle$. Then,

$$|\mathcal{P}_m f\left(\boldsymbol{z}\right)|^2=|\langle\mathcal{P}_m f,K_{\boldsymbol{z}}^\sigma\rangle|^2=|\langle f,\mathcal{P}_m^* K_{\boldsymbol{z}}^\sigma\rangle|^2=|\langle f,\mathcal{P}_m K_{\boldsymbol{z}}^\sigma\rangle|^2, \tag{30}$$

where last two step uses the property of $\mathcal{P}_m$ being self-adjoint and idempotent. Now,

$$\begin{aligned}|\langle f,\mathcal{P}_m K_{\boldsymbol{z}}^\sigma\rangle|^2\leq\|f\|^2\|\mathcal{P}_m K_{\boldsymbol{z}}^\sigma\|^2=\|f\|^2\langle\mathcal{P}_m K_{\boldsymbol{z}}^\sigma,\mathcal{P}_m K_{\boldsymbol{z}}^\sigma\rangle &=\langle\mathcal{P}_m^*\mathcal{P}_m K_{\boldsymbol{z}}^\sigma,K_{\boldsymbol{z}}^\sigma\rangle\\ &=\langle\mathcal{P}_m K_{\boldsymbol{z}}^\sigma,K_{\boldsymbol{z}}^\sigma\rangle\\ &=\mathcal{P}_m K_{\boldsymbol{z}}^\sigma\left(\boldsymbol{z}\right)\\ &=\sum_{N=m}^{\infty}\frac{\|\boldsymbol{z}\|_2^{2N}}{(2N+1)!}.\end{aligned} \tag{31}$$

Combining equation 30 and equation 31 followed by taking the square-root proves our result. □

Following details in regards of Hilbert spaces in the light of RKHS $H_{\sigma,1,\mathbb{C}^n}$, are respectfully borrowed from Reed (2012) or (Reed & Simon, April 1980, Chapter VI) or Halmos (2012).

Proposition A.14. *A linear and bounded operator $\mathfrak{B}$ is compact over the RKHS $H_{\sigma,1,\mathbb{C}^n}$ if and only if $\lim_{M\to\infty}\|\mathfrak{B}h_M-\mathfrak{B}h\|=0$ provided that $h_M\to h$ weakly in RKHS $H_{\sigma,1,\mathbb{C}^n}$.*

We use following criteria for *weakly convergence sequence* in the RKHS $H_{\sigma,1,\mathbb{C}^n}$ and as a general approach can be learn from standard references Hai & Rosenfeld (2021), Le (2014) and Le (2017).

Proposition A.15. *A sequence $\{h_M\}_M\in H_{\sigma,1,\mathbb{C}^n}$ converge weakly to 0 in $H_{\sigma,1,\mathbb{C}^n}$ if and only if following conditions are true:*

1. *bounded in the norm topology of the RKHS $H_{\sigma,1,\mathbb{C}^n}$*
2. *uniformly convergent to 0 over the compact subsets of the RKHS $H_{\sigma,1,\mathbb{C}^n}$.*

Proposition A.15 can be used to express the following corollary.

Corollary A.16. *Let $\sigma>0$ and $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$ where $\boldsymbol{A}\not\equiv 0\in\mathbb{C}^{n\times n}$ and $\|\boldsymbol{A}\|_2\le 1$. Consider a sequence of points $\{\boldsymbol{z}_M\}_M\in\mathbb{C}^n$ such that $\|\boldsymbol{z}_M\|_2\to\infty$ as $M\to\infty$. Then, the sequence of normalized reproducing kernels $\left\{\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\right\}_M$ of $H_{\sigma,1,\mathbb{C}^n}$ converges weakly to 0 over $H_{\sigma,1,\mathbb{C}^n}$.*

Lemma A.17. *If $\mathcal{K}_\varphi$ is bounded over the RKHS $H_{\sigma,1,\mathbb{C}^n}$ then the essential norm of $\mathcal{K}_\varphi$ denoted by $\|\mathcal{K}_\varphi\|_{\text{ess}}$ satisfies $\|\mathcal{K}_\varphi\|_{\text{ess}}\le\liminf_{M\to\infty}\|\mathcal{K}_\varphi\mathcal{P}_M\|$, where $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$ with $\|\boldsymbol{A}\|_2\le 1$.*

Proof. As $\mathcal{K}_\varphi:\mathcal{D}(\mathcal{K}_\varphi)\to H_{\sigma,1,\mathbb{C}^n}$ induced by φ is bounded, thus the result of Proposition A.10 holds. Therefore $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$ with $\|\boldsymbol{A}\|_2\le 1$. Let $\mathfrak{C}$ be a compact operator over the RKHS $H_{\sigma,1,\mathbb{C}^n}$. Then, observe following chain of inequalities for some $M\in\mathbb{Z}_+$:

$$\|\mathcal{K}_\varphi-\mathfrak{C}\|=\|\mathcal{K}_\varphi(\mathcal{P}_M+P_M)-\mathfrak{C}\|\le\|\mathcal{K}_\varphi\mathcal{P}_M\|+\|\mathcal{K}_\varphi P_M-\mathfrak{C}\|. \tag{32}$$

Since P_M is finite rank and hence compact. Therefore $\|\mathcal{K}_\varphi P_M-\mathfrak{C}\|=0$ since $\mathfrak{C}$ is also compact over the RKHS $H_{\sigma,1\mathbb{C}^n}$. Now take the $\liminf$ as $M\to\infty$, to have

$$\|\mathcal{K}_\varphi\|_{\text{ess}}\overset{\text{EQUATION 27}}{:=}\liminf_{M\to\infty}\|\mathcal{K}_\varphi-\mathfrak{C}\|\overset{\text{EQUATION 32}}{\le}\liminf_{M\to\infty}\|\mathcal{K}_\varphi\mathcal{P}_M\|.$$

Hence proved! □

Note that, if $\varphi:\mathbb{C}^n\to\mathbb{C}^n$ admits an affine structure with an additional condition that $0\not\equiv\boldsymbol{A}\in\mathbb{C}^{n\times n}$ and also is invertible ($\det(\boldsymbol{A})\ne 0$), then one can define $\varpi(\boldsymbol{u})=\boldsymbol{A}^{-1}\boldsymbol{u}-\boldsymbol{A}^{-1}\boldsymbol{b}$ as the inverse map of φ. In this case, if we define

$$\natural(\varpi(\boldsymbol{u})):=\Pi_{\varpi(\boldsymbol{u})}(\varphi;\sigma)\implies\natural(\varphi)=\Pi_{\boldsymbol{z}}(\varphi;\sigma). \tag{33}$$

$$\implies\natural(\boldsymbol{u})\le\Pi(\varphi;\sigma). \tag{34}$$

Following theorem provides the essential norm estimates for bounded $\mathcal{K}_\varphi$ over $H_{\sigma,1,\mathbb{C}^n}$.

Theorem A.18. *For a bounded $\mathcal{K}_\varphi:\mathcal{D}(\mathcal{K}_\varphi)\to H_{\sigma,1,\mathbb{C}^n}$, the essential norm of $\mathcal{K}_\varphi$ satisfies:*

$$\lim_{\|\boldsymbol{z}\|_2\to\infty}\sqrt{\Pi_{\boldsymbol{z}}(\varphi;\sigma)}\le_{(1)}\|\mathcal{K}_\varphi\|_{\text{ess}}\le_{(2)}\left(\left|\det\left(\boldsymbol{A}^{-1}\right)\right|\right)^{n/2}\lim_{\|\boldsymbol{z}\|_2\to\infty}\sqrt{\Pi_{\boldsymbol{z}}(\varphi;\sigma)}$$

where $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$ with invertible $\boldsymbol{A}$, $0\not\equiv\boldsymbol{A}\in\mathbb{C}^{n\times n}$ and $\|\boldsymbol{A}\|_2<1$.

Proof. Given that $\mathcal{K}_\varphi$ acts boundedly over the RKHS $H_{\sigma,1,\mathbb{C}^n}$, thus φ is of the affine structure, that is $\varphi(\boldsymbol{z})=\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}$, where $\|\boldsymbol{A}\|_2\le 1$. Now, with this φ, we begin now the proof for the INEQUALITY (1). Let $\mathfrak{C}$ be a compact operator over the RKHS $H_{\sigma,1,\mathbb{C}^n}$, then:

$$\|\mathcal{K}_\varphi-\mathfrak{C}\|\ge\limsup_{M\to\infty}\|(\mathcal{K}_\varphi-\mathfrak{C})\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\|\ge\limsup_{M\to\infty}\left[\|\mathcal{K}_\varphi\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\|-\|\mathfrak{C}\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\|\right]. \tag{35}$$

As $\left\{\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\right\}_M$ converges weakly to 0 over $H_{\sigma,1,\mathbb{C}^n}$, therefore $\|\mathfrak{C}\boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\| \to 0$ as $M \to \infty$. Thus,

$$\overbrace{\limsup_{M\to\infty} \sqrt{\Pi_{\boldsymbol{z}_M}(\varphi;\sigma)} \le}^{\text{LEMMA A.11}} \underbrace{\limsup_{M\to\infty} \|\mathcal{K}_\varphi \boldsymbol{k}^{\sigma}_{\varphi(\boldsymbol{z}_M)}\| \le}_{\text{EQUATION 35}} \|\mathcal{K}_\varphi - \mathfrak{C}\|.$$

Therefore, we have established the lower bound for the essential norm of $\mathcal{K}_\varphi$ over $H_{\sigma,1,\mathbb{C}^n}$. Now, we will work on the upper bound of the same. We fix positive r and $M \in \mathbb{Z}_+$. Then, pick an arbitrary $f \in H_{\sigma,1,\mathbb{C}^n}$ and proceed as follows:

$$\|\mathcal{K}_\varphi \mathcal{P}_M f\|^2 = \int_{\mathbb{C}^n} |\mathcal{K}_\varphi \mathcal{P}_M f(\boldsymbol{z})|^2 d\mu_{\sigma,1,\mathbb{C}^n}(\boldsymbol{z}).$$

Note that $\varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$ where $0 \not\equiv \boldsymbol{A} \in \mathbb{C}^{n\times n}$. Recall equation 33 and equation 34 to have:

$$\begin{aligned}
\|\mathcal{K}_\varphi \mathcal{P}_M f\|^2 =& \frac{1}{(2\pi\sigma^2)^n} \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) \left\{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|^n\right\} dV\left(\boldsymbol{u}\right) \\
=& \frac{\left\{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|^n\right\}}{(2\pi\sigma^2)^n} \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
=& \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \\
& \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \left(\boldsymbol{\chi}_{r\mathbb{B}_n} \oplus \boldsymbol{\chi}_{\mathbb{C}^n\setminus r\mathbb{B}_n}\right) \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right),
\end{aligned}$$

where $\boldsymbol{\chi}_{\boxdot}$ is the indicator function for the sub-space $\boxdot \subset \mathbb{C}^n$. Thus,

$$\begin{aligned}
\|\mathcal{K}_\varphi \mathcal{P}_M f\|^2 = \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \cdot \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \boldsymbol{\chi}_{r\mathbb{B}_n} \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
+ \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \boldsymbol{\chi}_{\mathbb{C}^n\setminus r\mathbb{B}_n} \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right).
\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}
\mathcal{I}^{\{M\}}_{r\mathbb{B}_n} &= \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \boldsymbol{\chi}_{r\mathbb{B}_n} \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
&= \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \int_{r\mathbb{B}_n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
&\le \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \Pi\left(\varphi;\sigma\right) \|f\|^2 \left(\sum_{N=M}^{\infty} \frac{r^{2N}}{(2N+1)!}\right) \int_{r\mathbb{B}_n} \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right),
\end{aligned}$$

where in the last step above we used inequality from equation 34 and equation 29. Now, as we allow $M \to \infty$, the quantity $\sum_{N=M}^{\infty} \frac{r^{2N}}{(2N+1)!} \to 0$. Hence $\lim_{M\to\infty} \mathcal{I}^{\{M\}}_{r\mathbb{B}_n} = 0$. After that we have optimized the limit over $r\mathbb{B}_n$, we now revisit equation 36 to optimize the limit for the second part:

$$\begin{aligned}
\mathcal{I}_{(r\mathbb{B}_n)^\complement} &= \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \int_{\mathbb{C}^n} \natural\left(\varpi\left(\boldsymbol{u}\right)\right) \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \boldsymbol{\chi}_{\mathbb{C}^n\setminus r\mathbb{B}_n} \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
&\leq \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \int_{(r\mathbb{B}_n)^\complement} \left\{\sup_{\|\boldsymbol{u}\|_2\geq r} \natural\left(\varpi\left(\boldsymbol{u}\right)\right)\right\} \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
&= \left[\frac{\left|\det\left(\boldsymbol{A}^{-1}\right)\right|}{2\pi\sigma^2}\right]^n \left\{\sup_{\|\boldsymbol{u}\|_2\geq r} \natural\left(\varpi\left(\boldsymbol{u}\right)\right)\right\} \int_{(r\mathbb{B})^\complement} \left|\mathcal{P}_M f\left(\boldsymbol{u}\right)\right|^2 \exp\left(-\frac{\|\boldsymbol{u}\|_2}{\sigma}\right) dV\left(\boldsymbol{u}\right) \\
&= \left(\left|\det\left(\boldsymbol{A}^{-1}\right)\right|\right)^n \left\{\sup_{\|\boldsymbol{u}\|_2\geq r} \natural\left(\varpi\left(\boldsymbol{u}\right)\right)\right\} \|\mathcal{P}_M f\|^2 \\
&\leq \left(\left|\det\left(\boldsymbol{A}^{-1}\right)\right|\right)^n \left\{\sup_{\|\boldsymbol{u}\|_2\geq r} \natural\left(\varpi\left(\boldsymbol{u}\right)\right)\right\} \|f\|^2. \qquad (37)
\end{aligned}$$

Letting $r\to\infty$ and combining the result of equation 37 and result from Lemma A.17, we have

$$\begin{aligned}
\|\mathcal{K}_\varphi\|_{\text{ess}} &\leq \sqrt{\left(\left|\det\left(\boldsymbol{A}^{-1}\right)\right|\right)^n} \|f\| \lim_{r\to\infty} \sqrt{\left\{\sup_{\|\boldsymbol{u}\|_2\geq r} \natural\left(\varpi\left(\boldsymbol{u}\right)\right)\right\}} \\
&= \left(\left|\det\left(\boldsymbol{A}^{-1}\right)\right|\right)^{n/2} \|f\| \lim_{r\to\infty} \sup_{\|\boldsymbol{u}\|_2\geq r} \left\{\sqrt{\natural\left(\varpi\left(\boldsymbol{u}\right)\right)}\right\}.
\end{aligned}$$

Thus, the result prevails. □

Theorem A.19. *A bounded $\mathcal{K}_\varphi$ achieves its compactification over RKHS $H_{\sigma,1,\mathbb{C}^n}$ if and only if $\lim_{\|\boldsymbol{z}\|_2\to 0} \Pi_{\boldsymbol{z}}(\varphi;\sigma) = 0$ where $\varphi(\boldsymbol{z}) = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{b}$ with $0 \not\equiv \boldsymbol{A} \in \mathbb{C}^{n\times n}$ and $\boldsymbol{A}$ is invertible.*

B Review of closable operators in Hilbert space

We recall when we mean an operator T in a Hilbert space $\mathfrak{H}$ to be *closable* or *preclosed* as given in standard references (Rudin, 1991, Chapter 13), (Conway, 2019, Chapter X, Page 304) (Pedersen, 2012, Chapter 5, Page 193), or (Reed, 2012, Chapter VIII, Page 250).

Definition B.1 (Graph of an operator)**.** *For an (unbounded) operator T in Hilbert space $\mathfrak{H}$ with its domain $\mathcal{D}(T)$, we define the* graph of T *in $\mathfrak{H}$ as follows:*

$$\boldsymbol{\Gamma}(T) \coloneqq \left\{(x, Tx) : x \in \mathcal{D}(T)\right\}. \qquad (38)$$

Definition B.2. *Let $T_{\boxdot}$ and T be operators over the Hilbert space $\mathfrak{H}$. Let $\boldsymbol{\Gamma}(T_{\boxdot})$ and $\boldsymbol{\Gamma}(T)$ be the respective graphs of $T_{\boxdot}$ and T as defined in equation 38. If $\boldsymbol{\Gamma}(T) \subset \boldsymbol{\Gamma}(T_{\boxdot})$, then $T_{\boxdot}$ is said to be an extension of T and we write $T \subset T_{\boxdot}$ and equivalently if $T \subset T_{\boxdot}$ if and only if $\mathcal{D}(T) \subset \mathcal{D}(T_{\boxdot})$ and $T_{\boxdot}\Lambda = T\Lambda$ for all $\Lambda \in \mathcal{D}(T)$.*

Definition B.3. *An operator is* ***closable*** *if it has a closed extension.*

Lemma B.1. *The operator T in Hilbert space $\mathfrak{H}$ is closable if and only if for each sequence $\{x_n\}_n \in \mathcal{D}(T)$ converging to 0, the only accumulation point of $\{Tx_n\}_n$ is 0.*

The above lemma (cf. (Pedersen, 2012, Chapter 5, Page 193)) can be interpreted as follows: *a linear operator $T : \mathcal{D}(T) \to \mathfrak{H}$ is closable if and only if for any sequence x_n such that $x_n \to 0$ when $n\to\infty$ and $Tx_n \to y_n$, then $y_n = 0$.*

B.1 Closability Result

Proof. Reproducing kernel $K^\sigma(\cdot,\cdot)$ from equation 10 over $_-\mathfrak{Z}_+$ as defined in equation 6 yields:

$$K^\sigma\left(_-\mathfrak{Z}_+\right) = \frac{\sinh\left(\sqrt{\frac{\langle \boldsymbol{z},-\boldsymbol{z}\rangle_{\mathbb{C}^n}}{\sigma^2}}\right)}{\sqrt{\frac{\langle \boldsymbol{z},-\boldsymbol{z}\rangle_{\mathbb{C}^n}}{\sigma^2}}} = \frac{\sinh\left(\sqrt{-\frac{\langle \boldsymbol{z},\boldsymbol{z}\rangle_{\mathbb{C}^n}}{\sigma^2}}\right)}{\sqrt{-\frac{\langle \boldsymbol{z},\boldsymbol{z}\rangle_{\mathbb{C}^n}}{\sigma^2}}} = \frac{\sinh\left(i\frac{\|\boldsymbol{z}\|_2}{\sigma}\right)}{i\frac{\|\boldsymbol{z}\|_2}{\sigma}} = \frac{\sin\left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right)}{\frac{\|\boldsymbol{z}\|_2}{\sigma}}.$$

We will show that $\|\boldsymbol{z}\|_2 K^{\sigma}\left({}_{-}\mathfrak{Z}_{+}\right) = \sigma \sin\left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right) \in H_{\sigma,1,\mathbb{C}^n}$ and its norm is finite. We have

$$\int_{\mathbb{C}^n} \left|\sigma \sin\left(\frac{\|\boldsymbol{z}\|_2}{\sigma}\right)\right|^2 d\mu_{\sigma,1,\mathbb{C}^n}(\boldsymbol{z}) \leq \sigma^2 \int_{\mathbb{C}^n} d\mu_{\sigma,1,\mathbb{C}^n} = \sigma^2 < \infty.$$

Since, $\sigma < \infty$, therefore, $\|\boldsymbol{z}\|_2 K^{\sigma}({}_{-}\mathfrak{Z}_{+}) \in H_{\sigma,1,\mathbb{C}^n}$ and their norm is bounded by σ. Recall the sequence by $\mathfrak{K}_N$ from the statement of theorem, then

$$\lim_{N\to\infty} \mathfrak{K}_N = \lim_{N\to\infty} \|\boldsymbol{z}_N\|_2 K^{\sigma}({}_{-}\mathfrak{Z}_{+,N}) = \lim_{N\to\infty} \left\{\sigma \sin\left(\frac{\|\boldsymbol{z}_N\|_2}{\sigma}\right)\right\} = 0. \tag{39}$$

Then the corresponding Koopman operator $\mathcal{K}_{\varphi_{\boldsymbol{A}}}$ acting on sequence of function $\mathfrak{K}_N$ yields:

$$\mathcal{K}_{\varphi_{\boldsymbol{A}}} \mathfrak{K}_N = \sigma \sin\left(\frac{\|\boldsymbol{A}\boldsymbol{z}_N\|_2}{\sigma}\right) \implies \lim_{N\to\infty} \mathcal{K}_{\varphi_{\boldsymbol{A}}} \mathfrak{K}_N = \lim_{N\to\infty} \sigma \sin\left(\frac{\|\boldsymbol{A}\boldsymbol{z}_N\|_2}{\sigma}\right) = 0. \tag{40}$$

Note that if $\boldsymbol{b} \not\equiv 0$ in our preceding assumption then we fail to achieve this convergence. Therefore, by the application of Lemma B.1, $\mathcal{K}_{\varphi}$ is closable over the RKHS $H_{\sigma,1,\mathbb{C}^n}$ when $\varphi = \varphi_{\boldsymbol{A}}$.

□

B.2 Failure of closability of Koopman operators over the RKHS of GRBF Kernel

Proof. Recall the domain defined in equation 6 and $K^{2,\sigma}_{\text{EXP}}(\boldsymbol{x}, \boldsymbol{z})$ as the GRBF Kernel and then:

$$K^{2,\sigma}_{\text{EXP}}({}_{-}\mathfrak{Z}_{+}) = \exp\left(-\frac{\|\boldsymbol{z}+\boldsymbol{z}\|_2^2}{\sigma}\right) = \exp\left(-\frac{4\|\boldsymbol{z}\|_2^2}{\sigma}\right).$$

With the subspace given in equation 11, we see that

$$\implies \lim_{N\to 0} \|\boldsymbol{z}_N\|_2 K^{2,\sigma}_{\text{EXP}}({}_{-}\mathfrak{Z}_{+,N}) = \lim_{N\to 0} \left\{\|\boldsymbol{z}_N\|_2 \exp\left(-\frac{4\|\boldsymbol{z}_N\|_2^2}{\sigma}\right)\right\} = 0 \cdot 1 = 0.$$

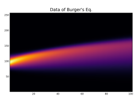
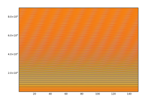
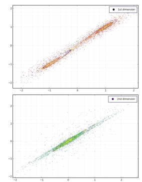
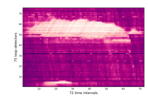
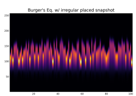
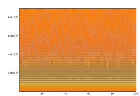
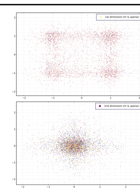
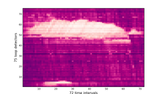
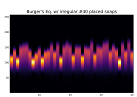
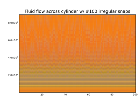
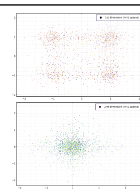
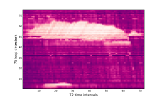
Above sequence converges to 0 but $\|\boldsymbol{z}_N\|_2 K^{2,\sigma}_{\text{EXP}}({}_{-}\mathfrak{Z}_{+,N}) = \|\boldsymbol{z}_N\|_2 \exp\left(-\frac{4\|\boldsymbol{z}\|_2^2}{\sigma}\right) \notin H_{\sigma}$ and can be learned from following:

$$\begin{aligned}\left\|\|\boldsymbol{z}_N\|_2 \exp\left(-\frac{4\|\boldsymbol{z}\|_2^2}{\sigma}\right)\right\|_{\sigma}^2 =& \frac{2^n\sigma^{2n}}{\pi^n}\|\boldsymbol{z}_N\|_2^2 \int_{\mathbb{C}^n} \exp\left(-\frac{4\|\boldsymbol{z}\|_2^2}{\sigma}\right) e^{\sigma^2 \sum_{i=1}^n (z_i - \bar{z}_i)^2} dV(\boldsymbol{z}) \\ \leq& \frac{2^n\sigma^{2n}}{\pi^n}\|\boldsymbol{z}_N\|_2^2 \int_{\mathbb{C}^n} e^{\sigma^2 \sum_{i=1}^n (z_i - \bar{z}_i)^2} dV(\boldsymbol{z}) \not< \infty.\end{aligned}$$

□

C EXPERIMENTAL INFORMATION

Table 5: Data matrix of experiments. Entries of third row corresponds to the third column of Table 6 to generate results from Lap-KeDMD algorithm.

DATA	EXPERIMENT 1	EXPERIMENT 2	EXPERIMENT 3	EXPERIMENT 4
GROUND TRUTH				
IRREGULAR & SPARSE				
PARTIAL RANK				

Details and governing equation of experiments are given as follows:

Table 6: Details of experiments for recovering ST-modes along with their references.

EXPERIMENT	TRUE DATA	PROVIDED DATA	REFERENCES
1. Nonlinear Burger's Equation	256×101	$256 \times \mathbf{40}$	Brunton & Kutz (2022)
2. Fluid flow across cylinder	89351×151	$89351 \times \mathbf{100}$	Brunton & Kutz (2022)
3. Chaotic Duffing's Oscillator	2×50000	$2 \times \mathbf{35000}$	Colbrook (2023)
4. Seattle freeway traffic speed	72×75	$72 \times \mathbf{75}$	Chen et al. (2021b)

Table 7: Koopman eigenvalue distribution over unit circle of all four experiments. For this experiment, real and imaginary parts of eigenvalues delivered by $K^{2,1}_{\text{EXP}}$ are in the order of $O(10^{-q})$ where $q \ggg 1$ due to ill-condition of the Gram Matrix for this data.

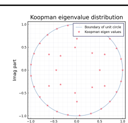
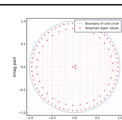
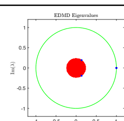
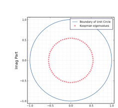
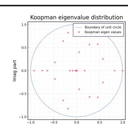
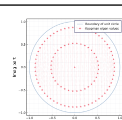
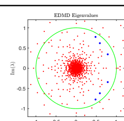
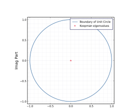
KERNELS	EXPERIMENT 1	EXPERIMENT 2	EXPERIMENT 3	EXPERIMENT 4
$K^{1,1}_{\text{EXP}}$				
$K^{2,1}_{\text{EXP}}$				

Table 8: EXPERIMENT 1: *Nonlinear Burger's Equation*. ST reconstruction through dominant Koopman modes via $K^{1,1}_{\text{EXP}}$ and $K^{2,1}_{\text{EXP}}$ with irregular and sparse 40 snapshots out of actual 100 snapshots.

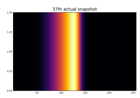
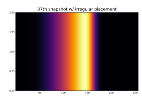
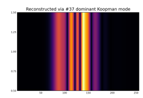
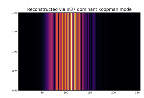
# KOOPMAN MODE	GROUND TRUTH	IRREGULAR & SPARSE	RECON. VIA LAP	RECON. VIA GRBF	KOOPMAN EIGENVALUES (LAP, GRBF)
# 37^{th}					$0.97 - 0.24i$, $0.57 + 0.26i$

Table 9: EXPERIMENT 2: *Fluid flow across cylinder*. ST reconstruction through dominant Koopman modes via $K^{1,1}_{\text{EXP}}$ and $K^{2,1}_{\text{EXP}}$ with irregular and sparse 100 snapshots out of total 151 snapshots.

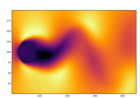
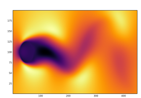
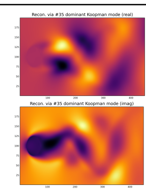
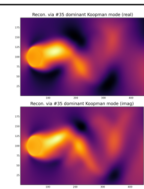
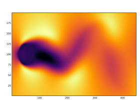
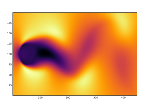
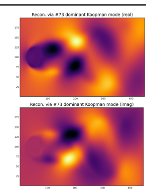
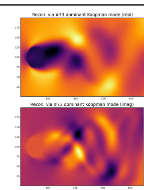
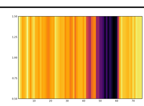
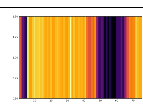
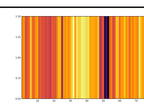
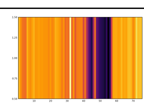
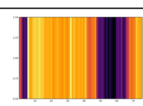
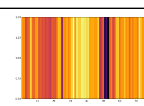
# KOOPMAN MODE	GROUND TRUTH	IRREGULAR & SPARSE	RECON. VIA LAP (REAL/IMAG)	RECON. VIA GRBF (REAL/IMAG)	KOOPMAN EIGENVALUES (LAP, GRBF)
# 35^{th}					$-0.35 - 0.77i$, $-0.3 + 0.82i$
# 73^{th}					$0.58 - 76i$, $0.46 - 0.74i$

Table 10: EXPERIMENT 4: *Seattle Freeway Traffic Speed data*. ST reconstruction through dominant Koopman modes via $K^{1,1}_{\text{EXP}}$ and $K^{2,1}_{\text{EXP}}$.

# KOOPMAN MODE	GROUND TRUTH	RECON. VIA LAP	RECON. VIA GRBF	KOOPMAN EIGENVALUES (LAP, GRBF)
# 53^{th}				$0.38 - 0.39i$, $0 + 0i$
# 59^{th}				$0.47 - 0.28i$, $0 + 0i$

We used the coding platforms of `Julia` and `MATLAB` to investigate our experimental results. To introduce the irregularity and sparsity, we use command `shuffle,dims=1` in `Julia` and `randperm` in `MATLAB`. Experiment 3 was conducted on `MATLAB` and every other experiments were conducted in `Julia`. To build the dataset of $50,000$ trajectory points for experiment 3, we take the advantage of `ode45` inbuilt function in `MATLAB`. If needed, we take the reconstructed results and perform necessary scaling which actually is supported by the boundedness results of Koopman operators over the RKHS $H_{\sigma,1,\mathbb{C}^n}$. In experiment 2, since we had a total spatial values of $449 \times 199 = 89,351$, which is quite at a large scale, we consider only first corresponding 100 spatial $2-$D values for discussion. Doing so not only provide smooth understanding but also helps in clear visualizations and hence saving computational power resources. Same treatment was also given in experiment 3, where we plotted only 50 trajectory values out of $35,000$ trajectory values in both dimensions.