

Incentivising Diffusion while Preserving Differential Privacy (Supplementary Material)

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A IDM

Here, we introduce the first diffusion auction for selling single item, IDM [13]. A key concept of IDM is diffusion critical sequence. Given a profile digraph $G_{\theta'}$, for any buyers $i, j \in V_{\theta'}$, i is θ' -critical to j , denoted by $i \preceq_{\theta'} j$, if all paths from s to j in $G_{\theta'}$ go through i . A *diffusion critical sequence* of i , denoted by C_i , is a sequence of all diffusion critical nodes of i and i itself ordered by θ' -critical relation. That is, $C_i = (x_1, x_2 \dots, x_k, i)$, where $x_1 \preceq_{\theta'} x_2 \preceq_{\theta'} \dots \preceq_{\theta'} x_k \preceq_{\theta'} i$. Based on this concept, IDM works as follows. IDM first locates the buyer m with the highest valuation among all buyers. Then it allocates the item to the buyer w , who has the highest valuation when the buyers after w are not considered. The winner w pays the highest bid without her participation, and each diffusion critical node is rewarded by the increased payment due to her participation.

B PROOF OF LEMMA 4.6

Lemma 4.6. *Given a reported global profile θ' , recursive DPDM REC is $\epsilon_{\max} \Delta \sigma$ -differentially private, where ϵ is the DP parameter of REC.*

Proof. Let θ and θ' be two profiles where a buyer i 's reports i reports v_i in θ and v'_i in θ' such that $v_i \neq v'_i$. Consider the probabilities that $\text{REC}(\theta)$ and $\text{REC}(\theta')$ return a winner w . In a critical diffusion tree T_θ , let d_w denote the depth of w , a_w^ℓ be an ancestor of w with distance ℓ . Also, let $\text{Exp}^\theta(T(a_w^1) - T(w))$ and $\text{Exp}^{\theta'}(T(a_w^1) - T(w))$ denote the value derived from θ and θ' , respectively. Then by Equa-

tion (3), we have

$$\frac{\Pr[\text{REC}(\theta) = o_w]}{\Pr[\text{REC}(\theta') = o_w]} = \frac{\frac{\text{Exp}(w)}{\text{Exp}^\theta(T(a_w^1) - T(w))}}{\frac{\text{Exp}^{\theta'}(w)}{\text{Exp}^{\theta'}(T(a_w^1) - T(w))}} \times \frac{\Pr_{T[a_w^1]}^\theta - \Pr_{a_w^1}^\theta}{\Pr_{T[a_w^1]}^{\theta'} - \Pr_{a_w^1}^{\theta'}}$$

We repeatedly replace $\Pr_{T[a_w^\ell]}^\theta$, $\Pr_{a_w^\ell}^\theta$, $\Pr_{T[a_w^\ell]}^{\theta'}$, $\Pr_{a_w^\ell}^{\theta'}$ by expressions of $a_w^{\ell+1}$ until we get an expression of s . For each distance $0 \leq \ell < d_w$, we denote $\frac{\text{Exp}(T[a_w^\ell])}{\text{Exp}(T(a_w^{\ell+1}))}$ as A_ℓ^θ , $\frac{\text{Exp}(a_w^\ell)}{\text{Exp}(T(a_w^{\ell+1}) \setminus T(a_w^\ell))}$ as B_ℓ^θ . For θ' , we have similar notations as $A_\ell^{\theta'}$ and $B_\ell^{\theta'}$. Then the above ratio can be written as

$$\frac{\Pr[\text{REC}(\theta) = o_w]}{\Pr[\text{REC}(\theta') = o_w]} = \frac{B_0^\theta}{B_0^{\theta'}} \times \prod_{\ell=1}^{d_w-1} \frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}}$$

Next we show for each $0 \leq \ell < d_w$, $\frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}}$ is bounded by $\exp(\epsilon \Delta \sigma)$. To prove it, we first show for each ℓ , $(A_\ell^\theta - A_\ell^{\theta'}) \times (B_\ell^\theta - B_\ell^{\theta'}) \geq 0$ by cases.

- (1) When $i \in T[a_w^\ell]$, we have $A_\ell^\theta - A_\ell^{\theta'} \leq 0$, $B_\ell^\theta - B_\ell^{\theta'} \leq 0$ or $A_\ell^\theta - A_\ell^{\theta'} \geq 0$, $B_\ell^\theta - B_\ell^{\theta'} \geq 0$
- (2) When $i \in T[a_w^{\ell+1}] \setminus T[a_w^\ell]$, then $A_\ell^\theta - A_\ell^{\theta'} \leq 0$, $B_\ell^\theta - B_\ell^{\theta'} \leq 0$ or $A_\ell^\theta - A_\ell^{\theta'} \geq 0$, $B_\ell^\theta - B_\ell^{\theta'} \geq 0$
- (3) When $i \notin T[a_w^{\ell+1}]$, then $A_\ell^\theta - A_\ell^{\theta'} = 0$, $B_\ell^\theta - B_\ell^{\theta'} = 0$.

Without loss of generality, we assume that $A_\ell^{\theta'} = \alpha_1 A_\ell^\theta$, $B_\ell^{\theta'} = \alpha_2 B_\ell^\theta$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$. Plug in these two equations, and we get

$$\frac{A_\ell^\theta - B_\ell^\theta}{A_\ell^{\theta'} - B_\ell^{\theta'}} = \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta}.$$

Then we consider two cases:

- (1) When $\alpha_1 \geq \alpha_2$, we have $\frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta} \leq \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_1 B_\ell^\theta} \leq \frac{1}{\alpha_1}$.

(2) When $\alpha_2 \geq \alpha_1$, we have $\frac{A_\ell^\theta - B_\ell^\theta}{\alpha_1 A_\ell^\theta - \alpha_2 B_\ell^\theta} \leq \frac{A_\ell^\theta - B_\ell^\theta}{\alpha_2 A_\ell^\theta - \alpha_2 B_\ell^\theta} \leq \frac{1}{\alpha_2}$.

After that, we show that both $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ are bounded by $\exp(\epsilon\Delta\sigma)$ as follows. By definition of α_1 , we have $\frac{1}{\alpha_1} = \frac{A_\ell^\theta}{A_\ell^{\theta'}} = \frac{\text{Exp}^\theta(T[a_w^\ell])}{\text{Exp}^{\theta'}(T[a_w^\ell])} \times \frac{\text{Exp}^{\theta'}(T(a_w^{\ell+1}))}{\text{Exp}^\theta(T(a_w^{\ell+1}))}$.

(1) When valuation $v'_i \leq v_i$, the second ratio is at most 1. Then we have

$$\begin{aligned} \frac{1}{\alpha_1} &= \frac{A_\ell^\theta}{A_\ell^{\theta'}} \leq \frac{\text{Exp}^\theta(T[a_w^\ell])}{\text{Exp}^{\theta'}(T[a_w^\ell])} \\ &\leq \frac{\sum_{k \in T[a_w^\ell]} \exp(\epsilon\sigma(\theta, o_k))}{\sum_{k \in T[a_w^\ell]} \exp(\epsilon(\sigma(\theta, o_k) - \Delta\sigma))} \leq \exp(\epsilon\Delta\sigma) \end{aligned}$$

(2) When valuation $v'_i \geq v_i$, the first ratio is at most 1. We have

$$\begin{aligned} \frac{1}{\alpha_1} &= \frac{A_\ell^\theta}{A_\ell^{\theta'}} \leq \frac{\text{Exp}^{\theta'}(T(a_w^{\ell+1}))}{\text{Exp}^\theta(T(a_w^{\ell+1}))} \\ &\leq \frac{\sum_{k \in T(a_w^{\ell+1})} \exp(\epsilon(\sigma(\theta, o_k) + \Delta\sigma))}{\sum_{k \in T(a_w^{\ell+1})} \exp(\epsilon\sigma(\theta, o_k))} \leq \exp(\epsilon\Delta\sigma) \end{aligned}$$

In a similar way, we can show that $\frac{1}{\alpha_2} \leq \exp(\epsilon\Delta\sigma)$.

Therefore we have

$$\begin{aligned} \frac{\Pr[\text{REC}(\theta) = o_w]}{\Pr[\text{REC}(\theta') = o_w]} &\leq \exp(\epsilon\Delta\sigma) \times \prod_{1 \leq \ell < d_w} \exp(\epsilon\Delta\sigma) \\ &\leq \exp(\epsilon d_w \Delta\sigma) \leq \exp(\epsilon d_{\max} \Delta\sigma) \end{aligned}$$

□

C PROOF OF LEMMA 5.2

Lemma 5.2. Given a reported global profile θ' , layered DPDM LAY is $\epsilon\Delta\sigma$ -differential private, where ϵ is the privacy parameter of LAY.

Proof. Given a global profile θ , for each buyer i with (v_i, r_i) , we have

$$\begin{aligned} \mathbf{E}_{\text{LAY}}[u_i(\theta)] &= (v_i - p_i(\theta))\Pr_i(\theta_i) \\ &= \int_0^{v_i} \Pr_i^{\text{LAY}}((x, r_i))dx \geq 0. \end{aligned}$$

Therefore, the lemma holds. □

D PROOF OF LEMMA 5.4

Lemma 5.4. Given a reported global profile θ' , layered DPDM LAY is $\epsilon\Delta\sigma$ -differential private, where ϵ is the privacy parameter of LAY.

Proof. Given two reported global profiles θ and θ' that differ in an arbitrary buyer i 's reported valuation such that i reports v_i in θ and v'_i in θ' , we consider the probabilities that LAY(θ) and LAY(θ') return a winner w .

Without loss of generality, we assume that w is in L_ℓ , then we have

$$\begin{aligned} \frac{\Pr[\text{LAY}(\theta) = o_w]}{\Pr[\text{LAY}(\theta') = o_w]} &= \frac{\Pr_{L_\ell} \times \frac{\text{Exp}^\theta(w)}{\text{Exp}^\theta(L_\ell)}}{\Pr_{L_\ell} \times \frac{\text{Exp}^{\theta'}(w)}{\text{Exp}^{\theta'}(L_\ell)}} \\ &= \frac{\text{Exp}^\theta(w)}{\text{Exp}^{\theta'}(w)} \frac{\text{Exp}^{\theta'}(L_\ell)}{\text{Exp}^\theta(L_\ell)} \end{aligned}$$

When i is not on layer L_ℓ , $\frac{\Pr[\text{LAY}(\theta) = o_w]}{\Pr[\text{LAY}(\theta') = o_w]} = 1 \leq \exp(\epsilon\Delta\sigma)$. Otherwise, when i is on layer L_ℓ , we consider two cases.

(1) $v_i < v'_i$. As $\sigma(\cdot)$ is non-decreasing in v_i , the first ratio is at most 1. Then we have

$$\begin{aligned} \frac{\Pr[\text{LAY}(\theta) = o_w]}{\Pr[\text{LAY}(\theta') = o_w]} &\leq \frac{\text{Exp}^{\theta'}(L_\ell)}{\text{Exp}^\theta(L_\ell)} \\ &\leq \frac{\sum_{j \in L_\ell} \exp(\epsilon(\sigma(\theta, o_j) + \Delta\sigma))}{\sum_{j \in L_\ell} \exp(\epsilon\sigma(\theta, o_j))} \\ &\leq \exp(\epsilon\Delta\sigma) \end{aligned}$$

(2) $v_i > v'_i$. In this case, the second ratio is at most 1. Then we have

$$\begin{aligned} \frac{\Pr[\text{LAY}(\theta) = o_w]}{\Pr[\text{LAY}(\theta') = o_w]} &\leq \frac{\text{Exp}^\theta(w)}{\text{Exp}^{\theta'}(w)} \leq \frac{\exp(\epsilon\sigma(\theta, o_w))}{\exp(\epsilon(\sigma(\theta, o_w) - \Delta\sigma))} \\ &\leq \exp(\epsilon\Delta\sigma) \end{aligned}$$

□

E PROOF OF THEOREM 5.6

Theorem 5.6 Given a global profile θ , layered DPDM LAY has $\mathbf{E}_{\text{LAY}}[sw_{\text{LAY}}(\theta)] \geq \gamma_{d_{\max}} \mathbf{E}_{\text{EMD}}[sw_{\text{EMD}}(\theta)]$.

Proof. Given a global profile θ , the expected social welfare of LAY is

$$\begin{aligned} \mathbf{E}_{\text{LAY}}[sw_{\text{LAY}}(\theta)] &= \sum_{i \in V} (v_i \times \Pr_i^{\text{LAY}}(\theta_i)) \\ &= \sum_{i \in V} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j \in L_{d_i}} \frac{1}{\gamma_{d_i}} \exp(\epsilon, \sigma(\theta, o_j))} \\ &\geq \gamma_{d_{\max}} \sum_{i \in N} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j \in L_{d_i}} \exp(\epsilon, \sigma(\theta, o_j))} \\ &\geq \gamma_{d_{\max}} \sum_{i \in N} v_i \frac{\exp(\epsilon, \sigma(\theta, o_i))}{\sum_{j \in V} \exp(\epsilon, \sigma(\theta, o_j))} \\ &= \gamma_{d_{\max}} \mathbf{E}_{\text{EMD}}[sw_{\text{EMD}}(\theta)] \end{aligned}$$

□