Mixture of Normalizing Flows for European Option Pricing (Supplementary Material)

1 CONSTRAINTS TRANSLATION

Given our NF model for option pricing,

$$f_\theta(K;z) = e^{-r\tau}F_\tau \int_{-\infty}^{+\infty} [\mathbb{I}(z)(\frac{e^x}{\mathrm{SG}(\mu_\theta)} - \frac{K}{F_\tau})]_+ q_\theta(x)\mathrm{d}x \tag{1}$$

where $\mu_\theta = \int_{-\infty}^{+\infty} e^x q_\theta(x)\mathrm{d}x$. We are going to prove that, as long as q_θ is a valid density function, i.e.,

$$q_\theta \geq 0 \qquad \int_{-\infty}^{+\infty} q_\theta(x)\mathrm{d}(x) = 1 \tag{2}$$

The constraints (C1) – C(4) for call option pricing and the constraints (P1) – P(4) for put option pricing will be met.

For the ease of notation, we drop some subscripts and write the pricing functions for call and put option separately.

$$C(K) = e^{-r\tau}F \int_{\log(\frac{K\mu}{F})}^{+\infty} (\frac{e^x}{\mu} - \frac{K}{F})q(x)\mathrm{d}x \tag{C}$$

$$P(K) = e^{-r\tau}F \int_{-\infty}^{\log(\frac{K\mu}{F})} (\frac{K}{F} - \frac{e^x}{\mu})q(x)\mathrm{d}x \tag{P}$$

The following constraints must be met for a non-arbitrage pricing model.

$$\frac{\partial C(K)}{\partial K} \leq 0 \tag{C1}$$

$$\frac{\partial^2 C(K)}{\partial K^2} \geq 0 \tag{C2}$$

$$C(\infty) = 0 \tag{C3}$$

$$\max(0, e^{-r\tau}(F-K)) \leq C(K) \leq e^{-r\tau}F \tag{C4}$$

$$\frac{\partial P(K)}{\partial K} \geq 0 \tag{P1}$$

$$\frac{\partial^2 P(K)}{\partial K^2} \geq 0 \tag{P2}$$

$$P(0) = 0 \tag{P3}$$

$$\max(0, e^{-r\tau}(K-F)) \leq P(K) \leq e^{-r\tau}K \tag{P4}$$

For (C1),

$$\frac{\partial C(K)}{\partial K} = -e^{-r\tau}\int_{\log(\frac{K\mu}{F})}^{+\infty} q(x)\mathrm{d}x \le 0 \tag{3}$$

For (C2)

$$\frac{\partial^2 C(K)}{\partial K^2} = \frac{e^{-r\tau}}{K} q(\log(\frac{K\mu}{F})) \ge 0 \tag{4}$$

For (C3),

$$C(\infty) = e^{-r\tau}F\int_{+\infty}^{+\infty}(\frac{e^x}{\mu} - \frac{K}{F})q(x)\mathrm{d}x = 0 \tag{5}$$

For (C4), the upper bound is achieved when $K = 0$ (C1), and

$$C(0) = e^{-r\tau}F\int_{-\infty}^{+\infty}\frac{e^x}{\mu}q(x)\mathrm{d}x = e^{-r\tau}F \tag{6}$$

Note that we have $C'(0) = \frac{\partial C(K)}{\partial K}|_{K=0} = -e^{-r\tau}$, and $C'(\cdot)$ is a non-decreasing function (C2), which means $C'(K) \ge C'(0)$. Then,

$$\begin{aligned} C(K) &= C(0) + \int_0^K C'(y)\mathrm{d}y && (7)\\ &\ge C(0) + \int_0^K C'(0)\mathrm{d}y && (8)\\ &= C(0) + KC'(0) && (9)\\ &= e^{-r\tau}F - Ke^{-r\tau} && (10)\\ &= e^{-r\tau}(F-K) && (11) \end{aligned}$$

Besides, $C(K) \ge C(\infty) = 0$, thus the tighter lower bound should be $\max(0, e^{-r\tau}(F-K))$.

For (P1),

$$\frac{\partial P(K)}{\partial K} = e^{-r\tau}\int_{-\infty}^{\log(\frac{K\mu}{F})} q(x)\mathrm{d}x \ge 0 \tag{12}$$

For (P2),

$$\frac{\partial^2 P(K)}{\partial K^2} = \frac{e^{-r\tau}}{K} q(\log(\frac{K\mu}{F})) \ge 0 \tag{13}$$

For (P3),

$$P(0) = e^{-r\tau}F\int_{-\infty}^{-\infty}(\frac{K}{F} - \frac{e^x}{\mu})q(x)\mathrm{d}x = 0 \tag{14}$$

For (P4), we can follow the similar procedure as we prove (C4). However, since C(4) has already been proved, we can simply apply put-call parity,

$$P(K) = C(K) - e^{-r\tau}F + e^{-r\tau}K \tag{15}$$

Thus we have

$$\max(0, e^{-r\tau}(K-F)) \le P(K) \le e^{-r\tau}K \tag{16}$$