Identification of the Generalized Condorcet Winner in Multi-dueling Bandits

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Abstract

The reliable identification of the "best" arm while keeping the sample complexity as low as possible is a common task in the field of multi-armed bandits. In the multi-dueling variant of multi-armed bandits, where feedback is provided in the form of a winning arm among a set of k chosen ones, a reasonable notion of best arm is the generalized Condorcet winner (GCW). The latter is an arm that has the greatest probability of being the winner in each subset containing it. In this paper, we derive lower bounds on the sample complexity for the task of identifying the GCW under various assumptions. As a by-product, our lower bound results provide new insights for the special case of dueling bandits ($k = 2$). We propose the Dvoretzky–Kiefer–Wolfowitz tournament (DKWT) algorithm, which we prove to be nearly optimal. In a numerical study, we show that DKWT empirically outperforms current state-of-the-art algorithms, even in the special case of dueling bandits or under a Plackett-Luce assumption on the feedback mechanism.

1 Introduction

The standard multi-armed bandit (MAB) problem describes a sequential decision scenario, in which one of finitely many choice alternatives must be selected in each time step, resulting in the observation of a numerical reward of stochastic nature. One important and extensively studied variant of the MAB setting is the dueling bandits problem, where a duel consisting of two arms is chosen in each time step and one of the duelling arms is observed as the winner [4]. Recently, the multi-dueling bandits setting has been introduced [7, 40, 31] as a generalization with multiple practically relevant applications, such as algorithm configuration [13] or online retrieval evaluation [36]. Instead of pairs of arms, in this generalization a set consisting of $k \geq 2$ arms can be chosen in each time step. These arms compete against each other and determine a single winner, which is observed as feedback by the learner. The outcomes of the (multi-)duels in the (multi-)dueling bandit scenario are typically assumed to be of time-stationary stochastic nature in the sense that whenever arms $a_1, \ldots, a_k$ compete against each other, then a_i wins with some underlying (unknown) ground-truth probability $\mathbf{P}(a_i|\{a_1, \ldots, a_k\})$.

One often targeted learning task in the context of multi-armed bandits and its variants is the problem of identifying the best among all arms. While for standard MABs, the canonical definition of the

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"best arm" is the arm with highest expected reward, the picture is less clear for its variants. In the realm of dueling bandits, any arm that is likely to win (i.e., with probability $> 1/2$) in each duel against another arm is called the *Condorcet winner* (CW). This notion dates back to the 18th century [8] and also appears in the social choice literature [16, 17], where the data is typically assumed to be available in the form of a list containing total rankings over all alternatives from different voters. In practice, the Condorcet winner does not necessarily exist due to the presence of preferential cycles in the probabilistic model in the sense that a_i is likely to win against a_j, a_j against a_k, and a_k against a_i. For the theoretical analysis of the best-arm-identification problem, this issue is overcome in the literature either by the consideration of alternative optimality concepts such as *Borda winner* or *Copeland winner*, which are guaranteed to exist, or by simply assuming the existence of the CW.

In this paper, we focus on finding a generalized variant of the CW in the multi-dueling bandits setting under the assumption that it exists. There have been several suggestions for generalizations of the CW in social choice. For example, a weighted variant is introduced in [30], where the weights control the relevance given to the ranking positions of the alternatives, while in [25] the notion of a k-winner is defined as an alternative that (in some appropriate sense) outperforms all other arms among any k alternatives. In contrast to our work, these papers focus on offline learning tasks and suppose full rankings over all alternatives to be given. In this paper, we adapt the notion of *generalized Condorcet winner* (GCW) as in [1], i.e., a GCW is an arm a_i that outperforms each arm a_j in every query set S containing both a_i and a_j, in the sense that $\mathbf{P}(a_i|S) \geq \mathbf{P}(a_j|S)$.

Regarding the dueling bandits setting as the multi-dueling setting where the allowed multi-duels S are exactly those with $|S| = 2$, the GCW is indeed a generalization of the Condorcet winner. We analyze the sample complexity of (probabilistic) algorithms that are able to identify the GCW with high probability under the assumption of mere existence as well as more restrictive assumptions. We provide upper and lower bounds for this task, which depend on the desired confidence, the total number m of alternatives, the size k of allowed query sets as well as the underlying unknown preference probabilities $\mathbf{P}(a_i|S)$.

We start in Section 2 with a brief literature overview on the multi-dueling bandits scenario. Section 3 introduces the basic formalism and a precise definition of the considered GCW identification problem. It also gives a rough, simplified overview of the sample complexity bounds obtained in this paper. In Section 4, we discuss the special case $m = k$, in which the GCW identification problem essentially boils down to the task of finding the mode of a categorical distribution. We provide solutions to this problem and prove their sample complexity to be optimal up to logarithmic factors in the worst-case sense. Section 5 focuses on lower bounds for the general case $m \geq k$, and in Section 6, we discuss several upper bounds. In Section 7, we empirically compare the algorithms discussed before, prior to concluding in Section 8. For the sake of convenience, detailed proofs of all theoretical results presented in the paper are deferred to the supplemental material.

2 Related Work

Initially, the multi-dueling bandit problem was studied intensively in the case of pairs as actions of the learner, which is also better known as *dueling bandits* [42]. The extension to the scenario considered in this paper, where more general sets as pairs of arms are selectable as an action, has been the focus of recent work. Part of these works model the feedback process by essentially tracing it back to the dueling bandit case [7, 40, 31]. The majority of papers, however, assume latent utility values for the arms and model the feedback process using a random utility model (RUM) [3] based on these utility values. Thanks to the latent utility values, an ordering of the arms is obtained quite naturally, which in turn makes it easy to define an objective such as the optimal arm or the top-k arms. Under these assumptions, the PB-MAB problem was investigated with respect to various performance metrics such as the regret [32, 5, 1] or the sampling complexity in an (ϵ, δ)-PAC setting [33, 34, 35].

In [1] a different approach is taken by generalizing the concept for the naturally optimal arm in the dueling bandit case, namely the Condorcet winner (CW), under the term generalized Condorcet winner (GCW). The optimal arm defined in this way coincides with the optimal arm if latent utility values for the arms and a RUM for the feedback process are assumed. While in [1] the problem for finding this GCW is investigated in a regret minimization scenario, we are interested in the minimum sampling complexity. In light of this, the work by [35] is the most related to ours, although the authors assume a PL model (a special case of a RUM).

If we restrict the learner's actions to pairs of arms in our more general setting, i.e., the dueling bandits case, the GCW and the CW coincide. This special case of our problem setting has been dealt with by [26], [20] and [29].

Finally, it remains to mention that there are a number of similar problem scenarios, namely the Stochastic click model (SCM) [43], the dynamic assortment problem (DAS) [9] and the best-of-k-bandits [39]. However, all these scenarios take into account other specific aspects in the modelling such as the order of the arms in the action subset (SCM), known revenues associated with the arms (DAS) or a so-called "no-choice option" (all three). Accordingly, these problem scenarios are fundamentally different from our learning scenario (see also Sec. 6.6 in [4] for a more detailed discussion). The same is true for combinatorial bandits [10], which also allow subsets of arms as actions, but differ fundamentally in the nature of feedback (quantitative vs. qualitative feedback).

3 The GCW Identification Problem

For adequately stating our results, we introduce in the following some basic terminology and notations used throghout this paper. For the sake of convenience, Table 1 summarizes the most frequently used notations.

3.1 The Notion of a GCW

If not explicitly stated otherwise, we suppose throughout the paper the total number of arms m, the query set size $k \in \{2, \dots, m\}$, a desired confidence $1-\gamma \in (0,1)$ and a complexity parameter $h \in (0,1)$ to be arbitrary but fixed. We write $[m] \coloneqq \{1, \dots, m\}$ and $[m]_k \coloneqq \{S \subseteq [m] \,|\, |S| = k\}$.

Parameter spaces of categorical distributions. For any subset of size k, i.e., $S \in [m]_k$, define $\Delta_S \coloneqq \{\mathbf{p} = (p_j)_{j \in S} \in [0,1]^{|S|} \,|\, \sum_{j \in S} p_j = 1\}$ as the set of all possible parameters for a categorical random variable $X \sim \mathrm{Cat}((p_j)_{j \in S})$, i.e., $\mathbb{P}(X = j) = p_j$ for any $j \in S$. For $\mathbf{p} \in \Delta_S$, we write $\mathrm{mode}(\mathbf{p}) \coloneqq \arg\max_{j \in S} p_j$ and in case $|\mathrm{mode}(\mathbf{p})| = 1$ we denote by $\mathrm{mode}(\mathbf{p})$ — with a slight abuse of notation — also the unique element in $\mathrm{mode}(\mathbf{p})$. Let us define for $h \in (0,1]$ the sets

$$\Delta_S^h \coloneqq \left\{\mathbf{p} \in \Delta_S \,|\, \exists i \in S \text{ s.t. } p_i \geq \max_{j \in S \setminus \{i\}} p_j + h\right\},$$

and with this $\Delta_S^0 \coloneqq \bigcup_{h \in (0,1)} \Delta_S^h$. These sets are nested in the sense that $\Delta_S^h \subseteq \Delta_S^{h'} \Leftrightarrow h \geq h'$. If $\mathbf{p} \in \Delta_S$ is fixed, the value $h(\mathbf{p}) \coloneqq \max\{h \in [0,1] \,|\, \mathbf{p} \in \Delta_S^h\}$ is well-defined and we have $\mathbf{p} \in \Delta_S^h$ iff $h \leq h(\mathbf{p})$. Obviously, the equivalence $|\mathrm{mode}(\mathbf{p})| = 1 \Leftrightarrow \mathbf{p} \in \Delta_S^0$ holds for all $\mathbf{p} \in \Delta_S$.

Probability models on $[m]_k$**.** A family $\mathbf{P} = \{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k}$ of parameters $\mathbf{P}(\cdot \,|\, S) \in \Delta_S$, $S \in [m]_k$, is called a **probability model (short: PM) on** $[m]_k$. We write PM_k^m for the set of all probability models on $[m]_k$ and define the following subsets of PM_k^m :

$$PM_k^m(\Delta^0) \coloneqq \{\mathbf{P} = \{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, \forall\, S \in [m]_k \,:\, \mathbf{P}(\cdot \,|\, S) \in \Delta_S^0\},$$
$$PM_k^m(\Delta^h) \coloneqq \{\mathbf{P} = \{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, \forall S \in [m]_k \,:\, \mathbf{P}(\cdot \,|\, S) \in \Delta_S^h\},$$
$$PM_k^m(\mathrm{PL}) \coloneqq \{\{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, \exists \boldsymbol{\theta} \in (0,\infty)^m \,\forall\, S \in [m]_k \,:\, \mathbf{P}(i \,|\, S) = \theta_i / (\sum_{j \in S} \theta_j)\}.$$

Note that $PM_k^m(\mathrm{PL})$ denotes the set of all probability models $\mathbf{P}$ consistent with a Plackett-Luce (PL) model [27, 23]. Let $h(\mathbf{P}) \coloneqq \max_{h \in [0,1]}\{\mathbf{P} \in PM_k^m(\Delta^h)\} = \min_{S \in [m]_k} h(\mathbf{P}(\cdot \,|\, S))$, then it is easy to see that $\mathbf{P} \in PM_k^m(\Delta^h)$ iff $h \leq h(\mathbf{P})$.

An element $i \in [m]$ is called a **generalized Condorcet Winner** (short: GCW) of $\mathbf{P}$ if

$$\forall S \in [m]_k \text{ with } i \in S, \forall j \in S : \mathbf{P}(i \,|\, S) - \mathbf{P}(j \,|\, S) \geq 0$$

and we write $\mathrm{GCW}(\mathbf{P})$ for the set of all GCWs of $\mathbf{P}$. With this, we define the following subsets of PM_k^m related to the concept of the GCW:

$$PM_k^m(\exists\mathrm{GCW}) \coloneqq \{\mathbf{P} = \{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, \mathrm{GCW}(\mathbf{P}) \neq \emptyset\},$$
$$PM_k^m(\exists\mathrm{GCW}^*) \coloneqq \{\mathbf{P} = \{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, |\mathrm{GCW}(\mathbf{P})| = 1\},$$
$$PM_k^m(\exists h\mathrm{GCW}) \coloneqq \{\{\mathbf{P}(\cdot \,|\, S)\}_{S \in [m]_k} \,\big|\, \exists i : \forall S \in [m]_k, j \in S \setminus \{i\} : \mathbf{P}(i \,|\, S) - \mathbf{P}(j \,|\, S) \geq h\}.$$

Table 1: A list of frequently used notation

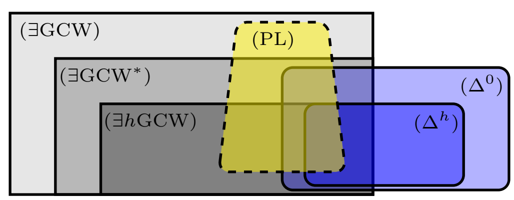
m	the total number of arms
k	the query set size
γ	the desired error rate bound
$[m]$	the set $\{1,\dots,m\}$
$[m]_k$	the set of all subsets of $[m]$ of size k
$\mathbf{1}_A$	indicator function, which is 1 if A is a true statement and 0 otherwise; also denoted by $\mathbf{1}_{\{A\}}$
S	an element from $[m]_k$
PM_k^m	set of all parameters $\{\mathbf{P}(\cdot\|S)\}_{S\in[m]_k} \subseteq [0,1]^{\binom{m}{k}}$ with $\sum_{j\in S}\mathbf{P}(j\|S)=1\ \forall S\in[m]_k$
$PM_k^m(\mathrm{X})$	the set of all $\mathbf{P}$, which fulfill the condition(s) X
$PM_k^m(\mathrm{X}\wedge Y)$	the set $PM_k^m(\mathrm{X})\cap PM_k^m(\mathrm{Y})$
$\mathbf{P}$	an element from PM_k^m
$\mathrm{GCW}(\mathbf{P})$	set of all GCWs of $\mathbf{P}$; if $\|\mathrm{GCW}(\mathbf{P})\|=1$ it denotes the only element in $\mathrm{GCW}(\mathbf{P})$
	Δ^h, Δ^0, PL, $\exists$GCW, $\exists h$GCW and $\exists$GCW*, cf. Section 3.1
Δ_S	set of all $\mathbf{w}=(w_i)_{i\in S}\in[0,1]^{\|S\|}$ with $\sum_{i\in[m]} w_i=1$; here, S is a finite set
Δ_S^h	set of all $\mathbf{w}\in\Delta_S$, for which $i\in S$ exists with $\forall j\in S\setminus\{i\}: w_i\geq w_j+h$
Δ_k,Δ_k^h	$\Delta_{[k]}$ resp. $\Delta_{[k]}^h$
$\mathbf{p}$	an element from Δ_k or an element from Δ_S for some $S\in[m]_k$
$\mathrm{mode}(\mathbf{p})$	$\arg\max_{i\in[k]} p_i$ for $\mathbf{p}=(p_1,\dots,p_k)$; the term $\mathrm{mode}(\mathbf{P}(\cdot\|S))$ is defined accordingly
$h(\mathbf{p})$	$\max\{h\in[0,1]\,\|\,\mathbf{p}\in\Delta_k^h\}$ for $\mathbf{p}\in\Delta_k$
$h(\mathbf{P})$	$\max\{h\in[0,1]\,\|\,\mathbf{P}\in PM_k^m(\Delta^h)\}$
$\mathcal{A}$	an algorithm
$\mathbf{D}(\mathcal{A})$	the return value of $\mathcal{A}$
$T^{\mathcal{A}}$	the sample complexity of $\mathcal{A}$, i.e., the number of samples observed by $\mathcal{A}$ before termination
$\mathcal{A}(x_1,\dots,x_l)$	An algorithm $\mathcal{A}$ called with the parameters $x_1,\dots,x_l$
$\mathcal{P}_k^{m,\gamma}(\mathrm{X})$	Problem of finding for any $\mathbf{P}\in PM_k^m(\mathrm{X})$ with error prob. $\leq\gamma$ the GCW, cf. Def. 3.1

Clearly, it holds that $PM_k^m(\exists\mathrm{GCW}^*) = \bigcup_{h>0} PM_k^m(\exists h\mathrm{GCW})$ and every probability model $\mathbf{P}\in PM_k^m(\exists\mathrm{GCW})$ has at least one GCW, while for a probability model $\mathbf{P}\in PM_k^m(\exists\mathrm{GCW}^*)$ the GCW is unique.

The figure on the right illustrates the relationships between the introduced subsets of PM_k^m. For the sake of convenience, we write simply (X) instead of $PM_k^m(\mathrm{X})$, where

$$\mathrm{X}\in\{\exists\mathrm{GCW},\exists\mathrm{GCW}^*,\exists h\mathrm{GCW},\Delta^0,\Delta^h,\mathrm{PL}\}.$$

This convention will be used several times in the course of the paper.



3.2 Problem Formulation

We are interested in algorithms $\mathcal{A}$ able to find the GCW of some $\mathbf{P}=\{\mathbf{P}(\cdot\,|\,S)\}_{S\in[m]_k}\in PM_k^m$, which is unknown and only observable via sampling from $\mathbf{P}$. More precisely, we suppose that at each time step $t\in\mathbb{N}$, such an algorithm $\mathcal{A}$ is allowed to choose one **query set** $S_t\in[m]_k$, for which it then observes a sample $S_t\ni X_t\sim\mathrm{Cat}(\mathbf{P}(\cdot\,|\,S_t))$. At some time, $\mathcal{A}$ may decide to make no more queries and output a prediction $\mathbf{D}(\mathcal{A})\in[m]$ for the GCW. We write $T^{\mathcal{A}}\in\mathbb{N}\cup\{\infty\}$ for the sample complexity of $\mathcal{A}$, i.e., the total number of queries made by $\mathcal{A}$ before termination. Note that both $\mathbf{D}(\mathcal{A})$ and $T^{\mathcal{A}}$ are random variables w.r.t. the sigma-algebra generated by the stochastic feedback mechanism. We write $\mathbb{P}_{\mathbf{P}}$ for the probability measure corresponding to the stochastic feedback mechanism if the unknown ground-truth PM is given by $\mathbf{P}$.

Definition 3.1 (The GCW Identification Problem)**.** *Let* (X) *be any of the assumptions from above with* $PM_k^m(\mathrm{X})\subseteq PM_k^m(\exists\mathrm{GCW})$, *and let* $\gamma\in(0,1)$ *be fixed. An algorithm* $\mathcal{A}$ ***solves the problem*** $\mathcal{P}_k^{m,\gamma}(\mathrm{X})$ *if* $\mathbb{P}_{\mathbf{P}}\big(\mathbf{D}(\mathcal{A})\in\mathrm{GCW}(\mathbf{P})\big)\geq 1-\gamma$ *holds for any* $\mathbf{P}\in PM_k^m(\mathrm{X})$.

3.3 Overview of Results

In this paper, we provide several upper and lower sample complexity bounds for solutions to the GCW identification problem $\mathcal{P}_k^{m,\gamma}(\mathrm{X})$ under different assumptions (X). We start in Section 4 with the discussion of the special case $m = k$, in which $\mathcal{P}_k^{k,\gamma}(\mathrm{X})$ can simply be thought of as finding the mode of a categorical distribution on $[k]$. In Sections 5 and 6, we discuss lower and upper bounds for the general case $m \geq k$, respectively.

Table 2 summarizes the obtained worst-case sample complexity bounds[1] of solutions to $\mathcal{P}_k^{m,\gamma}(\mathrm{X})$, where the worst-case is meant w.r.t. $PM_k^m(\mathrm{X} \wedge \mathrm{Y})$, for different choices of X and Y, and the Bachmann-Landau notations $\Omega(\cdot)$ and $\mathcal{O}(\cdot)$ are to be understood w.r.t. m, k, h^{-1} and γ^{-1}. In addition, we also provide instance-wise bounds in Theorems 5.2 and E.1.

Table 2: Sample complexity bounds of solutions to $\mathcal{P}_k^{m,\gamma}(\mathrm{X})$

(X)	(Y)	Type	Asymptotic bounds	References
(PL)	$(\exists h\mathrm{GCW})$	in exp.	$\Omega(\frac{m}{h^2 k}(\frac{1}{k} + h) \ln \frac{1}{\gamma})$	Thm. 5.1
$(\Delta^h \wedge \exists\mathrm{GCW})$	(Δ^h)	in exp.	$\Omega(\frac{m}{h^2 k} \ln \frac{1}{\gamma})$	Thm. 5.2
$(\mathrm{PL} \wedge \exists\mathrm{GCW}^*)$	$(\exists h\mathrm{GCW})$	w.h.p.	$\mathcal{O}(\frac{m}{h^2 k}(\frac{1}{k} + h) \ln(\frac{k}{\gamma} \ln \frac{1}{h}))$	Thm. 6.1
$(\exists\mathrm{GCW} \wedge \Delta^0)$	(Δ^h)	w.h.p.	$\mathcal{O}(\frac{m}{h^2 k} \ln(\frac{m}{k})(\ln\ln \frac{1}{h} + \ln \frac{1}{\gamma}))$	Thm. 6.2
$(\exists h\mathrm{GCW} \wedge \Delta^0)$	$(\exists h\mathrm{GCW})$	a.s.	$\mathcal{O}(\frac{m}{h^2 k} \ln(\frac{m}{k\gamma}))$	Thm. E.2

Due to $PM_k^m(\Delta^h \wedge \exists\mathrm{GCW}) \subsetneq PM_k^m(\exists h\mathrm{GCW}) \subsetneq PM_k^m(\exists\mathrm{GCW})$, Thm. 5.2 implies in particular that any solution $\mathcal{A}$ to $\mathcal{P}_k^{m,\gamma}(\exists\mathrm{GCW})$ fulfills $\sup_{\mathbf{P} \in PM_k^m(\exists h\mathrm{GCW})} \mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] \in \Omega(\frac{m}{kh^2} \ln(\gamma^{-1}))$. As Thm. 6.1 and Thm. 6.2 indicate that the bounds in Thm. 5.1 and Thm. 5.2 are asymptotically sharp up to logarithmic factors, the GCW identification problem seems to be easier under the PL assumption by a factor $1/k + h$.

4 The Single Bandit Case $m = k$

In this section, we address the problem $\mathcal{P}_k^{m,\gamma}(\mathrm{X})$ for the special case $k = m$. For sake of convenience, we abbreviate $\Delta_k := \Delta_{[k]}$ and similarly $\Delta_k^h := \Delta_{[k]}^h$ for any $h \in [0,1]$. Due to $[k]_k = \{[k]\}$, any probability model $\mathbf{P} \in PM_k^k$ is completely characterized by $\mathbf{P}(\cdot|[k])$ and the GCW of $\mathbf{P}$ is simply $\mathrm{mode}(\mathbf{P}(\cdot|[k]))$. Since the latter one always exists, we have $PM_k^k \subseteq PM_k^k(\exists\mathrm{GCW})$. Note that $\mathcal{P}_k^{k,\gamma}(\Delta^0) = \mathcal{P}_k^{k,\gamma}(\exists\mathrm{GCW}^*)$ as well as $\mathcal{P}_k^{k,\gamma}(\Delta^h) = \mathcal{P}_k^{k,\gamma}(\exists h\mathrm{GCW})$ are fulfilled trivially – i.e., we do not have to distinguish between the assumptions Δ^0 and $\exists\mathrm{GCW}^*$ resp. Δ^h and $\exists h\mathrm{GCW}$ throughout this section. For the sake of convenience we will identify $\mathbf{P} = \{\mathbf{P}(\cdot|[k])\} \in PM_k^k$ with $\mathbf{p} := (p_1, \ldots, p_k) := (\mathbf{P}(1|[k]), \ldots, \mathbf{P}(k|[k])) \in \Delta_k$. Due to $h(\mathbf{P}) = h(\mathbf{p})$, the set $PM_k^k(\Delta^h)$ is identified with Δ_k^h this way for any $h \in [0,1]$, and thus an algorithm $\mathcal{A}$ solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$ for $h \in [0,1]$ iff it fulfills $\mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) = \mathrm{mode}(\mathbf{p})) \geq 1 - \gamma$ for any $\mathbf{p} \in \Delta_k^h$.

4.1 Lower Bounds

Based on Wald's identity, the optimality of the sequential probability ratio test and a result by [14] we are able to prove the following two results, each of which are proven in Section C. In the appendix, we state with Prop. C.1 a more explicit but technical version of Prop. 4.1.

Proposition 4.1. *For any $\gamma_0 \in (0, 1/2)$ and $h_0 \in (0,1)$ there exists a constant $c(h_0, \gamma_0) > 0$ with the following property: Whenever $h \in (0, h_0)$, $\gamma \in (0, \gamma_0)$ and $\mathcal{A}$ is a solution to $\mathcal{P}_k^{k,\gamma}(\Delta^h)$, then*

$$\forall \mathbf{p} \in \Delta_k^h : \mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq 2c(h_0, \gamma_0)(h(\mathbf{p}))^{-2} \ln(\gamma^{-1}) \left({}^{1}\!/\!_{k} + h\right).$$

In particular, $\sup_{\mathbf{p} \in \Delta_k^h} \mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq 4c(h_0, \gamma_0) h^{-2} \ln(\gamma^{-1})$.

Proposition 4.2. *Let $\gamma \in (0, 1/2)$ be fixed and suppose $\mathcal{A}$ is a solution to $\mathcal{P}_k^{k,\gamma}(\Delta^0)$. Let $\mathbf{p} \in \Delta_k^0$ be arbitrary, $i := \mathrm{mode}(\mathbf{p})$ and $j := \arg\max_{j \in [k] \setminus \{i\}} p_j$. Then, the family $\{\mathbf{p}(h)\}_{h \in (0, p_i - p_j)} \subseteq \Delta_k^0$ defined via $(\mathbf{p}(h))_i := {}^{(p_i + p_j + h)}\!/\!_{2}$, $(\mathbf{p}(h))_j := {}^{(p_i + p_j - h)}\!/\!_{2}$ and $(\mathbf{p}(h))_l := p_l$ for $l \in [k] \setminus \{i, j\}$ fulfills $\mathbf{p}(h) \in \Delta_k^h$ as well as*

$$\limsup_{h \to 0} {}^{\mathbb{E}_{\mathbf{p}(h)}[T^{\mathcal{A}}]}\!/\!_{(h^{-2} \ln\ln h^{-1})} \geq (1 - 2\gamma)(p_i + p_j) > 0.$$

[1]Here, "in exp." means *in expectation*, "w.h.p." means *with prob.* $\geq 1 - \gamma$ and "a.s." *with prob.* 1.

[37] have recently proven a result similar to Proposition 4.1. In contrast to theirs, our bound provides as additional information also the asymptotical behavior as $k \to \infty$. Moreover, our proof is based on the optimality of the Sequential Probability Ratio Test [41, 38] instead of a measure-changing argument [21].

4.2 Upper Bounds and Further Prerequisites

To construct a solution $\mathcal{A}$ to $\mathcal{P}_k^{k,\gamma}(\Delta^h)$, we have to decide in a sequential manner at each time t, whether we want to make a further query $S_t \in [k]_k$ resulting in a sample X_t or to output an answer $D(\mathcal{A}) \in [k]$. As $[k]_k = \{[k]\}$, we can only choose $S_t = [k]$ in each time step t, upon which we observe as feedback $X_t \sim \mathrm{Cat}(\mathbf{p})$, i.e., $\mathbb{P}_{\mathbf{p}}(X_t = i) = p_i$ for any $i \in [k]$. Having observed $X_1, \ldots, X_t$, a straightforward idea for a prediction $\mathbf{D}(\mathcal{A})$ would be to use the mode of the empirical distribution $\hat{\mathbf{p}}^t := (\hat{p}_1^t, \ldots, \hat{p}_k^t)$ given by $\hat{p}_i^t := \frac{1}{t}\sum_{t' \leq t} \mathbf{1}_{\{X_t = i\}}$. As the Dvoretzky-Kiefer-Wolfowitz (DKW) inequality assures us that

$$\mathbb{P}_{\mathbf{p}}\left(\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > \varepsilon\right) \leq 4e^{-t\varepsilon^2/2} \tag{1}$$

holds for any $\varepsilon > 0$ (Lem. D.1), we can infer that $\hat{\mathbf{p}}^t$ is close to $\mathbf{p}$ with high confidence for large values of t. Hence, if t is large enough, predicting the mode of $\hat{\mathbf{p}}^t$ would be the correct prediction for $\mathrm{mode}(\mathbf{p})$ with high probability. In the following we show which choice of t is sufficient to assure a confidence $\geq 1 - \gamma$.

Let us first consider the case $\mathbf{p} \in \Delta_k^h$. It can be shown that

$$(\exists i : \tilde{p}_i - \max_{j \neq i} \tilde{p}_j \geq \varepsilon \text{ and } p_i \neq \max_j p_j) \;\Rightarrow\; ||\tilde{\mathbf{p}} - \mathbf{p}||_\infty \geq (h + \varepsilon)/2$$

holds for any $h \in [0, 1], \varepsilon \in (-h, 1], \mathbf{p} \in \Delta_k^h$ and $\tilde{\mathbf{p}} \in \Delta_k$ (Lemma D.2). This result is optimal in the sense that the term $(h + \varepsilon)/2$ therein cannot be improved (Remark D.3). Choosing $\varepsilon = 0$ and $\tilde{\mathbf{p}} = \hat{\mathbf{p}}^t$ shows us that $||\hat{\mathbf{p}}^t - \mathbf{p}||_\infty > h/2$ is necessary for $\mathrm{mode}(\hat{\mathbf{p}}^t) \neq \mathrm{mode}(\mathbf{p})$. Combining this with (1) based on the DKW inequality, we could simply query $S_t = [k]$ for $T = \lceil 8\ln(4/\gamma)h^{-2}\rceil$ many times and return the mode of $\hat{\mathbf{p}}^T$ as the decision. This (non-sequential) strategy solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$ and terminates after exactly $\lceil 8\ln(4/\gamma)h^{-2}\rceil$ time steps (Proposition D.4). Note that according to Prop. 4.1, this strategy is asymptotically optimal.

Next, we intend to solve the more challenging problem $\mathcal{P}_k^{k,\gamma}(\Delta^0)$. Note that any solution to $\mathcal{P}_k^{k,\gamma}(\Delta^0)$ is also a solution to $\mathcal{P}_k^{k,\gamma}(\Delta^h)$ for any $h > 0$, whence Prop. 4.1 shows that $\mathcal{P}_k^{k,\gamma}(\Delta^0)$ cannot be solved by any non-sequential algorithm, i.e., one which decides a priori the number of samples it observes. To construct a solution, we make use of Alg. 1, which also tackles the problem of finding the mode of $\mathbf{p}$ in a non-sequential manner but is allowed to return UNSURE as an indicator that it is not confident enough for its prediction. In other words, the algorithm is allowed to abstain from making a decision. Since

$$\forall i : \tilde{p}_i \leq \max_{j \neq i} \tilde{p}_j + h \quad \Rightarrow \quad ||\mathbf{p} - \tilde{\mathbf{p}}||_\infty \geq h.$$

holds for any $h > 0$, $\mathbf{p} \in \Delta_k^{3h}$ and $\tilde{\mathbf{p}} \in \Delta_k$ (Lem. D.5), Alg. 1 can be shown to return with probability at least $1 - \gamma$ the correct mode in case $\mathbf{p} \in \Delta_k^{3h}$ is fulfilled. The constraint $\mathbf{p} \in \Delta_k^{3h}$ in the statement above is sharp in the sense that we show in Lem. D.6 for any $h \in (0, 1/8)$ that

$$\inf\left\{s > 0 \,\middle|\, \forall \mathbf{p} \in \Delta_k^{sh}\, \forall \tilde{\mathbf{p}} \in \Delta_k : (\forall i : \tilde{p}_i \leq \max_{j \neq i} \tilde{p}_j + h \;\Rightarrow\; ||\mathbf{p} - \tilde{\mathbf{p}}||_\infty \geq h)\right\} = 3.$$

Algorithm 1 DKW mode-identification with abstention

Input: $\gamma \in (0, 1)$, $h \in (0, 1)$, access to iid samples $X_t \sim \mathrm{Cat}(\mathbf{p})$

1: $T \leftarrow \lceil 8\ln(4/\gamma)/h^2\rceil$
2: Observe samples $X_1, \ldots, X_T$
3: Calculate $\hat{\mathbf{p}}^T = (\hat{p}_1^T, \ldots, \hat{p}_k^T)$ as $\hat{p}_i^T := \frac{1}{T}\sum_{t=1}^T \mathbf{1}_{\{X_t = i\}}$, $i \in [k]$
4: Choose $i^* \in \mathrm{mode}(\hat{\mathbf{p}}^T)$
5: **if** $\hat{p}_{i^*}^T > \max_{j \neq i^*} \hat{p}_j^T + h$ **then return** i^*
6: **else return** UNSURE

Lemma 4.3. *$\mathcal{A} \coloneqq$ Alg. 1 initialized with parameters $\gamma, h \in (0,1)$ fulfills $T^{\mathcal{A}} = \lceil 8\ln(4/\gamma)/h^2 \rceil$,*

$$\forall \mathbf{p} \in \Delta_k \quad : \mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) \in [k] \text{ and } p_{\mathbf{D}(\mathcal{A})} < \max_{j\in[k]} p_j) \leq \gamma, \tag{2}$$

$$\forall \mathbf{p} \in \Delta_k^0 \quad : \mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) \in \{\mathrm{mode}(\mathbf{p}), \mathrm{UNSURE}\}) \geq 1-\gamma, \tag{3}$$

$$\forall \mathbf{p} \in \Delta_k^{3h} : \mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) = \mathrm{mode}(\mathbf{p})) \geq 1-\gamma. \tag{4}$$

Lemma 4.3 (proven in Section D) reveals that Alg. 1 has a low failure rate (2) by appropriate choice of γ, while in turn by an appropriate choice of h, namely $h \leq \frac{1}{3}h(\mathbf{p})$, the correct decision will be returned (4) with high probability. However, there are two problems arising: Alg. 1 can also abstain from making a decision (3) and more importantly, the value of $h(\mathbf{p})$ is unknown. As a remedy, we could run Alg. 1 successively with appropriately decreasing choices for γ and h until a (real) decision is returned. This approach is followed by Alg. 2 and the following proposition shows that it is indeed a solution to $\mathcal{P}_k^{k,\gamma}(\Delta^0)$; its proof is an adaptation of Lem. 11 in [28] and given in Section D.

Algorithm 2 DKW mode-identification – Solution to $\mathcal{P}_k^{k,\gamma}(\Delta^0)$

Input: $\gamma \in (0,1)$, sample access to $\mathrm{Cat}(\mathbf{p})$
Initialization: $\widetilde{\mathcal{A}} \coloneqq$ Alg. 1, $s \leftarrow 1$, $\forall r \in \mathbb{N} : \gamma_r \coloneqq \frac{6\gamma}{\pi^2 r^2}, h_r \coloneqq 2^{-r-1}$

```
feedback ← UNSURE
while feedback is UNSURE do
    feedback ← Ã(γ_s, h_s, sample access to Cat(p))
    s ← s + 1
return feedback
```

Proposition 4.4. *$\mathcal{A} \coloneqq$ Alg. 2 initialized with the parameter $\gamma \in (0,1)$ solves $\mathcal{P}_k^{k,\gamma}(\Delta^0)$ s.t.*

$$\mathbb{P}_{\mathbf{p}}(T^{\mathcal{A}} < \infty) = 1 \quad \textit{and} \quad \mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}) = \mathrm{mode}(\mathbf{p}) \textit{ and } T^{\mathcal{A}} \leq t_0(\gamma, h(\mathbf{p}))\right) \geq 1-\gamma$$

for any $\mathbf{p} \in \Delta_k^0$, where $t_0(\gamma, h)$ is mon. decr. w.r.t. h with $t_0(\gamma, h) \in \mathcal{O}\left(h^{-2}\left(\ln\ln h^{-1} + \ln\gamma^{-1}\right)\right)$.

The sample complexity of $\mathcal{A}$ in Proposition 4.4 improves upon the existing alternative solution for $\mathcal{P}_k^{k,\gamma}(\Delta^0)$ in Theorem 2 in [37] with respect to two essential aspects: First, its sample complexity bound is constant instead of increasing in k and second, the dependence on the hardness parameter $h(\mathbf{p})$ is $h(\mathbf{p})^{-2}\ln\ln h(\mathbf{p})^{-1}$ instead of $h(\mathbf{p})^{-2}\ln h(\mathbf{p})^{-1}$.

5 Lower Bounds on the General GCW Identification Problem

In this section we provide lower sample complexity bounds for solutions to the GCW identification problem for arbitrary $2 \leq k \leq m$. The following theorem is based on a result by [35], which we state as Thm. B.1 in the appendix.

Theorem 5.1. *Any solution $\mathcal{A}$ to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL})$ fulfills*

$$\sup\nolimits_{\mathbf{P}\in PM_k^m(\mathrm{PL}\wedge\exists h\mathrm{GCW})} \mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] \in \Omega\left(m(1/k+h)\ln(1/\gamma)/(kh^2)\right). \tag{5}$$

One of the key ingredients for proving Thm. B.1 and thus for Thm. 5.1 is a change-of-measure argument by [21]. By means of the latter technique, we are also able to show the following instance-based as well as worst-case lower bounds for any solution to $\mathcal{P}_k^{m,\gamma}(\Delta^h \wedge \exists\mathrm{GCW})$, which is proven in Section F.

Theorem 5.2. *Suppose $\mathcal{A}$ solves $\mathcal{P}_k^{m,\gamma}(\Delta^h \wedge \exists\mathrm{GCW})$ and let $\mathbf{P} \in PM_k^m(\Delta^h \wedge \exists\mathrm{GCW})$ be arbitrary with $\min_{S\in[m]_k}\min_{j\in S}\mathbf{P}(j|S) > 0$. For $S \in [m]_k$ write $m_S \coloneqq \mathrm{mode}(\mathbf{P}(\cdot|S))$ and for any $l \in S \setminus \{m_S\}$ define $\mathbf{P}^{[l]}(\cdot|S) \in \Delta_S$ via*

$$\mathbf{P}^{[l]}(l|S) \coloneqq \mathbf{P}(m_S|S), \quad \mathbf{P}^{[l]}(m_S|S) \coloneqq \mathbf{P}(l|S), \quad \forall j \in S \setminus \{l, m_S\} : \mathbf{P}^{[l]}(j|S) \coloneqq \mathbf{P}(j|S).$$

Then,

$$\mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] \geq \ln((2.4\gamma)^{-1})/(k-1) \sum\nolimits_{l\in[m]\setminus\{\mathrm{GCW}(\mathbf{P})\}} \min\nolimits_{S\in[m]_k : l\in S\setminus\{m_S\}} 1/\mathrm{KL}(\mathbf{P}(\cdot|S), \mathbf{P}^{[l]}(\cdot|S)),$$

where $\mathrm{KL}(\mathbf{P}(\cdot|S), \mathbf{P}^{[l]}(\cdot|S))$ denotes the Kullback-Leibler divergence between two categorical distributions $X \sim \mathrm{Cat}(\mathbf{P}(\cdot|S))$ and $Y \sim \mathrm{Cat}(\mathbf{P}^{[l]}(\cdot|S))$. Moreover, we have

$$\sup\nolimits_{\mathbf{P}\in PM_k^m(\Delta^h\wedge\exists\mathrm{GCW})} \mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] \geq m(1-h^2)\ln((2.4\gamma)^{-1})/(4kh^2).$$

In the special case of dueling bandits ($k = 2$), the instance-dependent lower bound is a novel result,[2] it actually leads to a slightly larger worst-case lower bound than the worst-case bound in Theorem 5.2 for the dueling bandit case (Cor. F.4). For $k \geq 3$, the worst-case bound in Theorem 5.2 is not a consequence of the instance-wise version (Rem. F.3), instead it requires a more involved proof than the latter. For $m = k$, the instance-wise lower bound underlying Prop. 4.1 is apparently larger than that of Thm. 5.2 (Rem. F.5). The reason is that the proof for the instance-wise bound in Theorem 5.2 is tailored to the problem class $PM_k^m(\Delta^h \wedge \exists \mathrm{GCW})$ and consequently has to deal with combinatorial issues arising in case $k < m$.

6 Upper Bounds on the General GCW Identification Problem

In [35] the PAC-WRAPPER algorithm is introduced, which is an algorithm able to identify the GCW under the Plackett-Luce assumption with (up to logarithmic terms) optimal instance-wise sample complexity, see Section B. By translating the sample complexity result of PAC-WRAPPER into our setting, we obtain the following result (see Section B for its proof), which is also by Thm. 5.1 suggested to be optimal up to logarithmic factors.

Theorem 6.1. *There exists a solution $\mathcal{A}$ to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL} \wedge \exists \mathrm{GCW}^*)$ s.t.*

$$\inf_{\mathbf{P} \in PM_k^m(\mathrm{PL} \wedge \exists h \mathrm{GCW})} \mathbb{P}_{\mathbf{P}} \left(\mathbf{D}(\mathcal{A}) \in \mathrm{GCW}(\mathbf{P}) \text{ and } T^{\mathcal{A}} \leq t'(m,h,k,\gamma) \right) \geq 1 - \gamma$$

holds with $t'(h,m,k,\gamma) \in \mathcal{O}\left(m(1/k+h) \ln(k/\gamma \ln(1/h)) / (kh^2) \right)$.

Algorithm 3 DVORETZKY–KIEFER–WOLFOWITZ TOURNAMENT – Solution to $\mathcal{P}_k^{m,\gamma}(\exists \mathrm{GCW} \wedge \Delta^0)$

Input: $k, m \in \mathbb{N}$, $\gamma \in (0,1)$, sample access to $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S \in [m]_k}$

Initialization: $\widetilde{\mathcal{A}} \coloneqq$ Alg. 2, choose $S_1 \in [m]_k$ arbitrary, $F_1 \leftarrow [m]$, $\gamma' \leftarrow \frac{\gamma}{\lceil m/(k-1) \rceil}$, $s \leftarrow 1$

▷ S_s : candidates in round s, F_s : remaining elements in round s, i_s : output of $\widetilde{\mathcal{A}}$ in round s

while $s \leq \lceil \frac{m}{k-1} \rceil - 1$ **do**
  $i_s \leftarrow \widetilde{\mathcal{A}}(\gamma'$, sample access to $\mathbf{P}(\cdot|S_s))$
  $F_{s+1} \leftarrow F_s \setminus S_s$
  Write $F_{s+1} = \{j_1, \ldots, j_{|F_{s+1}|}\}$.
  if $|F_{s+1}| < k$ **then**
    Fix distinct $j_{|F_{s+1}|+1}, \ldots, j_{k-1} \in [m] \setminus F_{s+1}$.
  $S_{s+1} \leftarrow \{i_s, j_1, \ldots, j_{k-1}\}$
  $s \leftarrow s + 1$
$i_s \leftarrow \widetilde{\mathcal{A}}(\gamma'$, sample access to $\mathbf{P}(\cdot|S_s))$
return i_s

Next, we consider the problem class $\mathcal{P}_k^{m,\gamma}(\exists \mathrm{GCW} \wedge \Delta^0)$, for which we propose the DVORETZKY–KIEFER–WOLFOWITZ TOURNAMENT (DKWT) algorithm (see Alg. 3). DKWT is a simple round-based procedure eliminating in each round those arms from a candidate set of possible GCWs that have been discarded by Alg. 2 with high confidence as being the GCW. In the following theorem we derive theoretical guarantees for DKWT, while a more sophisticated sample complexity bound is provided in Thm. E.1.

Theorem 6.2. $\mathcal{A} \coloneqq \mathrm{DKWT}$ *initialized with the parameter $\gamma \in (0,1)$ solves $\mathcal{P}_k^{m,\gamma}(\exists \mathrm{GCW} \wedge \Delta^0)$ s.t.*

$$\mathbb{P}_{\mathbf{P}} \left(\mathbf{D}(\mathcal{A}) \in \mathrm{GCW}(\mathbf{P}) \text{ and } T^{\mathcal{A}} \leq T'(h(\mathbf{P}),m,k,\gamma) \right) \geq 1 - \gamma$$

holds for all $\mathbf{P} \in PM_k^m(\exists \mathrm{GCW} \wedge \Delta^0)$ with $T'(h,m,k,\gamma) \in \mathcal{O}\left(\frac{m}{kh^2} \ln\left(\frac{m}{k}\right) \left(\ln\ln\frac{1}{h} + \ln\frac{1}{\gamma} \right) \right)$.

The result stated in Table 2 for $(\mathrm{X}) = (\exists \mathrm{GCW} \wedge \Delta^0)$ and $(\mathrm{Y}) = (\Delta^h)$ follows from this by noting that $h(\mathbf{P}) \geq h$ holds for any $\mathbf{P} \in PM_k^m(\exists h \mathrm{GCW} \wedge \Delta^h)$. Regarding Prop. 4.2, the additional factor

[2] So far, existing lower sample complexity bounds for solutions to $\mathcal{P}_2^{m,\gamma}(\Delta^h \wedge \exists \mathrm{GCW})$ are either restricted to worst-case scenarios [6] or to the special case where $\mathbf{P}$ belongs to a Thurstone model [29] or a Plackett-Luce model [35].

Table 3: Comparison of DKWT with PAC-WRAPPER (PW) on $\boldsymbol{\theta} = (1, 0.8, 0.6, 0.4, 0.2)$

	$T^{\mathcal{A}}$		Accuracy	
k	DKWT	PW	DKWT	PW
3	**44293** (3695.6)	1631668498 (1453661392.0)	1.0	1.0
4	**32427** (2516.2)	263543687 (127401593.7)	1.0	1.0

$\ln \ln h^{-1}$ in the upper bounds from Thm. 6.1 and Thm. 6.2 appears indispensable. Since $PM_k^m(\text{PL} \wedge \exists\text{GCW}^*) \not\subseteq PM_k^m(\exists\text{GCW} \wedge \Delta^0)$ and $PM_k^m(\text{PL} \wedge \exists\text{GCW}^*) \not\supseteq PM_k^m(\exists\text{GCW} \wedge \Delta^0)$ hold, a solution to $\mathcal{P}_k^{m,\gamma}(\text{PL} \wedge \exists\text{GCW}^*)$ is in general not comparable with a solution to $\mathcal{P}_k^{m,\gamma}(\exists\text{GCW} \wedge \Delta^0)$, i.e., neither Thm. 6.1 nor Thm. 6.2 implies the other one.

Replacing $\exists\text{GCW} \wedge \Delta^0$ with the more restrictive assumption $\exists h\text{GCW} \wedge \Delta^0$ (as an assumption on $\mathbf{P}$) makes the GCW identification task much easier. This is similar to the case of $\mathcal{P}_k^{k,\gamma}(\Delta^h)$ and $\mathcal{P}_k^{k,\gamma}(\Delta^0)$ discussed in Section 4.2. For $\mathcal{P}_k^{m,\gamma}(\exists h\text{GCW} \wedge \Delta^0)$ we can modify Alg. 3 in order to incorporate the knowledge of h as follows: choose in round s a query set $S_s \subseteq F_s$ (filled up with $|F_s| - k$ further elements from $[m] \setminus F_s$ if $|F_s| < k$) and execute Alg. 1 with parameters $\frac{h}{3}$, $\frac{\gamma}{\lceil m/(k-1) \rceil}$ and sample access to $\mathbf{P}(\cdot|S_s)$. In case Alg. 1 returns as decision an element $i \in S_s$, we let $F_{s+1} = F_s \setminus (S_s \setminus \{i\})$, and otherwise $F_{s+1} = F_s$. Then we proceed with the next round $s+1$. We repeat this procedure until $|F_s| = 1$, and return the unique element in F_s as the prediction for the GCW. In Sec. E we provide detailed a pseudocode for this algorithm (Alg. 5) and show that it indeed solves $\mathcal{P}_k^{m,\gamma}(\exists h\text{GCW} \wedge \Delta^0)$ with the guarantee that it terminates almost surely for any $\mathbf{P} \in PM_k^m(\exists h\text{GCW} \wedge \Delta^0)$ before some time $t'(m,k,h,\gamma) \in \mathcal{O}\left(m \ln(m/(k\gamma)) / (kh^2)\right)$ (see Thm. E.2). A look at Thm. 5.2 reveals that this solution to $\mathcal{P}_k^m(\exists h\text{GCW} \wedge \Delta^0)$ is asymptotically optimal up to logarithmic factors in a worst-case sense w.r.t. $PM_k^m(\exists h\text{GCW} \wedge \Delta^0)$.

7 Empirical Evaluation

In the following, we present experimental results on the performance of our GCW identification solution.[3] We restrict ourselves in the main paper to DKWT, which is our solution of the most general problem $\mathcal{P}_k^{m,\gamma}(\exists h\text{GCW} \wedge \Delta^0)$. Throughout all experiments, if not specified differently in the pseudocode, every choice of an element within a specific set made by DKWT is performed uniformly at random. All experiments were conducted on a machine with an Intel® Core™ i7-4700MQ Processor, executing all experiments (including those in the supplemental material) with only one CPU core in use took less than 72 hours.

At first, we compare DKWT with PAC-WRAPPER (PW), which is the solution to $\mathcal{P}_k^{m,\gamma}(\text{PL})$ in [35] underlying Thm. 6.1 and so far the only solution in the literature to the best of our knowledge for identifying the GCW in multi-dueling bandits with an error probability at most γ. Table 3 shows the results of both algorithms when started on an instance $\mathbf{P} \in PM_k^5(\text{PL})$ with underlying PL-parameter $\boldsymbol{\theta} = (1, 0.8, 0.6, 0.4, 0.2)$ and $\gamma = 0.05$, for different values of k. The observed termination time $T^{\mathcal{A}}$, the corresponding standard error (in brackets) and the accuracy are averaged over 10 repetitions.

Both algorithms achieve the desired accuracy $\geq 95\%$ in every case, but DKWT requires far less samples than PW to find the GCW. Further experiments in the appendix (cf. Section G) demonstrate the superiority of DKWT over PW also for other values of m, $\boldsymbol{\theta}$ and γ, including the problem instances considered in [35]. Note that the observed extremely large sample complexity of PW appears to be consistent with the experimental results in [35] and is supposedly caused by multiple runs of a costly procedure PAC-BEST-ITEM, which does not exploit the DKW inequality but is rather based on applications of Chernoff's bound.

In the case $k = 2$, the GCW identification problem coincides with the Condorcet winner (CW) identification problem in dueling bandits. Thus, we can compare DKWT to state-of-the art solutions for finding the CW if it exists: SELECT [26], SEEBS[4] [29] and EXPLORE-THEN-VERIFY (EtV) [20]. Formally, SELECT requires $h \in (0,1)$ as a parameter as it solves $\mathcal{P}_2^{m,\gamma}(\exists\text{GCW} \wedge \Delta^h)$,

[3] Our implementation is provided at `https://github.com/bjoernhad/GCWidentification`.

[4] We include SEEBS even though it techniqually requires $\mathbf{P}$ to fulfill *strong stochastic transitivity* and the *stochastic triangle inequality*, cf. Sec. G.2.

Table 4: Comparison of DKWT, SEEBS and EXPLORE-THEN-VERIFY (EtV)

m	h	$T^{\mathcal{A}}$		
		DKWT	SEEBS	EtV
5	0.20	**6010** (293.2)	7305 (432.1)	8601 (589.2)
5	0.15	**8874** (460.0)	13393 (904.5)	11899 (986.9)
5	0.10	**15769** (1457.1)	19802 (1543.2)	260171 (210678.1)
5	0.05	**31454** (4127.4)	36855 (3533.2)	156534 (115903.1)
10	0.20	**14334** (492.8)	16956 (617.9)	26115 (969.2)
10	0.15	**18563** (734.5)	27527 (1126.7)	32548 (2514.6)
10	0.10	**33040** (1625.1)	47330 (2138.2)	68858 (11304.5)
10	0.05	**78660** (6517.2)	83877 (5842.6)	220098 (92484.9)

while DKWT, SEEBS and EtV solve the more challenging problem $\mathcal{P}_2^{m,\gamma}(\exists \mathrm{GCW} \wedge \Delta^0)$. As a consequence, we compare here only the latter three algorithms on probability models $\mathbf{P}$ sampled uniformly at random from $PM_k^m(\exists \mathrm{GCW} \wedge \Delta^h)$ for various values of h *without* providing these algorithms with the explicit value of h. We also compare SELECT with the three considered algorithms in Section G. Without great surprise, it turns out that SELECT has a much smaller sample complexity due to its advantage of knowing the explicit value of h.

Table 4 reports the observed sample complexities, together with the standard errors in brackets, obtained for $\gamma = 0.05$ and different choices of m and h, the numbers are averaged over 100 repetitions. Every algorithm achieves an accuracy of 1 in each case. DKWT clearly outperforms SEEBS and EtV in any case, which is consistent with similar results for larger values of m in the appendix. Overall, these results show that DKWT is also well suited for the dueling bandit case.

We complement our empirical study in Section G.3 with a comparison of DKWT with Alg. 5 showing that the latter outperforms the former in the case where $h(\mathbf{P})$ is small and $\mathbf{P} \in PM_k^{m,\gamma}(\exists h'\mathrm{GCW} \wedge \Delta^0)$ for some a priori known $h' > h(\mathbf{P})$.

8 Conclusion

We investigated the sample complexity required for identifying the generalized Condorcet winner (GCW) in multi-dueling bandits within a fixed confidence setting. We provided lower bound results, which as a special case yield a novel instance-wise lower sample complexity bound for identifying the Condorcet winner in the realm of dueling bandits. We introduced DVORETZKY-KIEFER-WOLFOWITZ TOURNAMENT (DKWT), an algorithmic solution to the GCW identification task with asymptotically nearly optimal worst-case sample complexity. In our experiments, DKWT outperformed competing state-of-the art algorithms, even in the special case of dueling bandits. Last but not least, we pointed out that and to which extent incorporating a Plackett-Luce assumption on the feedback mechanism makes the GCW identification problem asymptotically easier w.r.t. the worst-case required sample complexity.

There are several directions in which this work could be extended. First, one could investigate the GCW identification problem in the so-called *probably approximately correct* (PAC) setting and search not for the GCW but instead for an arm that outperforms any other arm only with some margin $\varepsilon > 0$. Secondly, one may generalize this problem to the identification of the GCW *without* assuming its existence, where one has to check on-the-fly whether a GCW exists, i.e., a testification (*test* and *identify*) problem as in [19] in the dueling bandit case for the Condorcet Winner. Moreover, one may also extend our problem to the case where query sets *up to size* k are allowed at each time step. This variant has already been discussed in a regret minimization scenario [32, 1] or a PAC setting [33].

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References

[1] A. Agarwal, N. Johnson, and S. Agarwal. Choice bandits. In *Proceedings of Advances in Neural Information Processing Systems (NeurIPS)*, pages 18399–18410, 2020.

[2] H. Bauer and R. B. Burckel. *Probability Theory*. De Gruyter Studies in Mathematics. De Gruyter, 1996.

[3] M. Ben-Akiva and M. Bierlaire. Discrete choice methods and their applications to short term travel decisions. In *Handbook of transportation science*, pages 5–33. Springer, 1999.

[4] V. Bengs, R. Busa-Fekete, A. El Mesaoudi-Paul, and E. Hüllermeier. Preference-based online learning with dueling bandits: A survey. *Journal of Machine Learning Research*, 22(7):1–108, 2021.

[5] V. Bengs and E. Hüllermeier. Preselection bandits. In *Proceedings of International Conference on Machine Learning (ICML)*, pages 778–787, 2020.

[6] M. Braverman and E. Mossel. Noisy sorting without resampling. In *Proceedings of Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 268–276, 2008.

[7] B. Brost, Y. Seldin, I. J. Cox, and C. Lioma. Multi-dueling bandits and their application to online ranker evaluation. In *Proceedings of ACM International Conference on Information and Knowledge Management (CIKM)*, pages 2161–2166, 2016.

[8] M. J. A. N. Caritat. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. De l'Imprimerie royale, 1785.

[9] F. Caro and J. Gallien. Dynamic assortment with demand learning for seasonal consumer goods. *Management Science*, 53(2):276–292, 2007.

[10] N. Cesa-Bianchi and G. Lugosi. Combinatorial bandits. *Journal of Computer and System Sciences*, 78(5):1404–1422, 2012.

[11] X. Chen and Y. Wang. A note on a tight lower bound for capacitated MNL-bandit assortment selection models. *Operations Research Letters*, 46(5):534–537, 2018.

[12] A. Dvoretzky, J. Kiefer, and J. Wolfowitz. Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *The Annals of Mathematical Statistics*, 27(3):642 – 669, 1956.

[13] A. El Mesaoudi-Paul, D. Weiß, V. Bengs, E. Hüllermeier, and K. Tierney. Pool-based realtime algorithm configuration: A preselection bandit approach. In *Proceedings of International Conference on Learning and Intelligent Optimization (LION)*, pages 216–232, 2020.

[14] R. H. Farrell. Asymptotic behavior of expected sample size in certain one sided tests. *The Annals of Mathematical Statistics*, 35(1):36–72, 1964.

[15] T. S. Ferguson. *Mathematical Statistics: A Decision Theoretic Approach*. Probability and mathematical statistics. Academic Press, 1967.

[16] P. C. Fishburn. Paradoxes of voting. *American Political Science Review*, 68(2):537–546, 1974.

[17] P. C. Fishburn and W. V. Gehrlein. Borda's rule, positional voting, and Condorcet's simple majority principle. *Public Choice*, 28:79–88, 1976.

[18] A. Garivier and E. Kaufmann. Optimal best arm identification with fixed confidence. In *Proceedings of Annual Conference on Learning Theory (COLT)*, pages 998–1027, 2016.

[19] B. Haddenhorst, V. Bengs, J. Brandt, and E. Hüllermeier. Testification of condorcet winners in dueling bandits. In *Proceedings of Conference on Uncertainty in Artificial Intelligence (UAI)*, 2021.

[20] Z. Karnin. Verification based solution for structured MAB problems. In *Proceedings of Advances in Neural Information Processing Systems (NIPS)*, pages 145–153, 2016.

[21] E. Kaufmann, O. Cappé, and A. Garivier. On the complexity of best-arm identification in multi-armed bandit models. *The Journal of Machine Learning Research*, 17(1):1–42, 2016.

[22] M. R. Kosorok. *Introduction to Empirical Processes and Semiparametric Inference*. Springer, New York, USA, 2008.

[23] R. D. Luce. *Individual Choice Behavior: A Theoretical Analysis*. Wiley, 1959.

[24] P. Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *The Annals of Probability*, 18(3):1269 – 1283, 1990.

[25] A. Meyers, M. E. Orrison, J. Townsend, S. Wolff, and A. Wu. Generalized Condorcet winners. *Social Choice and Welfare*, 43(1):11–27, 2014.

[26] S. Mohajer, C. Suh, and A. Elmahdy. Active learning for top-k rank aggregation from noisy comparisons. In *Proceedings of International Conference on Machine Learning (ICML)*, pages 2488–2497, 2017.

[27] R. L. Plackett. The analysis of permutations. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 24(1):193–202, 1975.

[28] W. Ren, J. Liu, and N. B. Shroff. On sample complexity upper and lower bounds for exact ranking from noisy comparisons. In *Proceedings of Advances in Neural Information Processing Systems (NeurIPS)*, pages 10014–10024, 2019.

[29] W. Ren, J. Liu, and N. B. Shroff. The sample complexity of best-k items selection from pairwise comparisons. In *Proceedings of the International Conference on Machine Learning (ICML)*, pages 8051–8072, 2020.

[30] D. G. Saari. Symmetry extensions of "neutrality" I: Advantage to the Condorcet loser. *Social Choice and Welfare*, 9(4):307–336, 1992.

[31] A. Saha and A. Gopalan. Battle of bandits. In *Proceedings of Conference on Uncertainty in Artificial Intelligence (UAI)*, 2018.

[32] A. Saha and A. Gopalan. Combinatorial bandits with relative feedback. In *Advances in Neural Information Processing Systems (NeurIPS)*, pages 983–993, 2019.

[33] A. Saha and A. Gopalan. PAC Battling Bandits in the Plackett-Luce Model. In *Proceedings of the International Conference on Algorithmic Learning Theory (ALT)*, pages 700–737, 2019.

[34] A. Saha and A. Gopalan. Best-item learning in random utility models with subset choices. In *Proceedings of International Conference on Artificial Intelligence and Statistics (AISTATS)*, pages 4281–4291, 2020.

[35] A. Saha and A. Gopalan. From PAC to instance-optimal sample complexity in the Plackett-Luce model. In *Proceedings of International Conference on Machine Learning (ICML)*, pages 8367–8376, 2020.

[36] A. Schuth, H. Oosterhuis, S. Whiteson, and M. de Rijke. Multileave gradient descent for fast online learning to rank. In *Proceedings of ACM International Conference on Web Search and Data Mining (WSDM)*, page 457–466, 2016.

[37] D. Shah, T. Choudhury, N. Karamchandani, and A. Gopalan. Sequential mode estimation with oracle queries. In *Proceedings of AAAI Conference on Artificial Intelligence (AAAI)*, volume 34, pages 5644–5651, 2020.

[38] D. Siegmund. *Sequential Analysis: Tests and Confidence Intervals*. Springer Series in Statistics. Springer, 1985.

[39] M. Simchowitz, K. Jamieson, and B. Recht. Best-of-k bandits. In *Proceedings of Annual Conference on Learning Theory (COLT)*, pages 1440–1489, 2016.

[40] Y. Sui, V. Zhuang, J. W. Burdick, and Y. Yue. Multi-dueling bandits with dependent arms. In *Proceedings of Conference on Uncertainty in Artificial Intelligence (UAI)*, 2017.

[41] A. Wald and J. Wolfowitz. Optimum character of the sequential probability ratio test. *The Annals of Mathematical Statistics*, 19(3):326–339, 1948.

[42] Y. Yue and T. Joachims. Beat the mean bandit. In *Proceedings of International Conference on Machine Learning (ICML)*, pages 241–248, 2011.

[43] M. Zoghi, T. Tunys, M. Ghavamzadeh, B. Kveton, C. Szepesvari, and Z. Wen. Online learning to rank in stochastic click models. In *Proceedings of International Conference on Machine Learning (ICML)*, pages 4199–4208. PMLR, 2017.

A Relationships between the Probability Models

Lemma A.1. *For any $k, m \in \mathbb{N}$ and $h \in (0,1)$ we have the implications*

$$\begin{aligned} PM_k^m(\exists h\mathrm{GCW}) \subsetneq PM_k^m(\exists \mathrm{GCW}^*) \subsetneq PM_k^m(\exists \mathrm{GCW}), \\ PM_k^m(\Delta^h) \subsetneq PM_k^m(\Delta^0), \\ PM_k^m(\mathrm{PL}) \subsetneq PM_k^m(\exists \mathrm{GCW}), \\ PM_k^m(\Delta^h) \cap PM_k^m(\exists \mathrm{GCW}) \subsetneq PM_k^m(\exists h\mathrm{GCW}), \\ PM_k^m(\Delta^0) \cap PM_k^m(\exists \mathrm{GCW}) \subsetneq PM_k^m(\exists \mathrm{GCW}^*). \end{aligned}$$

Proof. This is a direct consequence of the definitions. □

B GCW Identification under the Plackett-Luce Assumption

In this section, we prove the lower and upper bounds of solutions to the GCW identification problem under the Plackett-Luce assumption stated in Theorems 5.1 and 6.1. For $\boldsymbol{\theta} \in (0,\infty)^m$ we denote by $\mathbf{P}(\boldsymbol{\theta}) \in PM_k^m(\mathrm{PL})$ the corresponding PM, which is consistent with the Plackett-Luce model with parameter $\boldsymbol{\theta}$ on S_m, i.e., $\mathbf{P}(\boldsymbol{\theta}) = \{\mathbf{P}(\boldsymbol{\theta})(\cdot|S)\}_{S\in[m]_k}$ is defined via

$$\mathbf{P}(\boldsymbol{\theta})(i|S) := \frac{\theta_i}{\sum_{a\in S}\theta_a} \quad \text{for any } S \in [m]_k \text{ and } i \in S.$$

As $\mathbf{P}(x\boldsymbol{\theta}) = \mathbf{P}(\boldsymbol{\theta})$ holds for any $x > 0$ and $\boldsymbol{\theta} \in (0,\infty)^m$, we may restrict ourselves w.l.o.g. to those $\mathbf{P}(\boldsymbol{\theta})$ with $\max_{i\in[m]}\theta_i = 1$.

In [35], the following lower resp. upper sample complexity bounds for solutions to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL})$ resp. $\mathcal{P}_k^{m,\gamma}(\mathrm{PL} \wedge \exists\mathrm{GCW}^*)$ depending on the ground-truth Plackett-Luce parameter have been proven.

Theorem B.1. *Any solution $\mathcal{A}$ to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL})$ fulfills*

$$\mathbb{E}_{\mathbf{P}(\boldsymbol{\theta})}\left[T^{\mathcal{A}}\right] \in \Omega\left(\max\left(\sum\nolimits_{j=2}^m \frac{\theta_j}{(1-\theta_j)^2}\ln\frac{1}{\gamma}, \frac{m}{k}\ln\frac{1}{\gamma}\right)\right)$$

for any $\boldsymbol{\theta} \in (0,1]^m$ with $1 = \theta_1 > \max_{j\geq 2}\theta_j$.

Proof. Confer Theorem 7 in [35]. □

Theorem B.2. *There is a solution $\mathcal{A}$ to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL} \wedge \exists\mathrm{GCW}^*)$, which fulfills for any $\boldsymbol{\theta} \in (0,1]^m$ with $1 = \theta_1 > \max_{j\geq 2}\theta_j$ the estimate*

$$\mathbb{P}_{\mathbf{P}(\boldsymbol{\theta})}\left(\mathbf{D}(\mathcal{A}) \in \mathrm{GCW}(\mathbf{P}) \text{ and } T^{\mathcal{A}} \leq t'(\boldsymbol{\theta},k,\gamma)\right) \geq 1-\gamma$$

with

$$t'(\boldsymbol{\theta},k,\gamma) \in \mathcal{O}\left(\frac{\Theta_{[k]}}{k}\sum\nolimits_{j=2}^m \frac{1}{(1-\theta_j)^2}\ln\left(\frac{k}{\gamma}\ln\left(\frac{1}{1-\theta_j}\right)\right)\right)$$

and $\Theta_{[k]} := \max_{S\in[m]_k}\sum_{a\in S}\theta_a$.

Proof. Confer Theorem 3 in [35] and note that $\min_{j\geq 2}(1-\theta_j)^{-2} \geq 1$ holds for any $\boldsymbol{\theta} \in (0,1]^m$ with $1 = \theta_1 > \max_{j\geq 2}\theta_j$. □

To translate the preceding results into our setting, we need a better understanding of the set $PM_k^m(\mathrm{PL} \wedge \exists h\mathrm{GCW})$. This is achieved by means of the following observation. For the sake of completeness, we also provide a characterization of $PM_k^m(\mathrm{PL} \wedge \Delta^h)$.

Lemma B.3. *For $\boldsymbol{\theta} \in (0,\infty)^m$ with $\theta_1 \geq \cdots \geq \theta_m$ we have*

$$\begin{aligned} \mathbf{P}(\boldsymbol{\theta}) \in PM_k^m(\exists h\mathrm{GCW}) &\Leftrightarrow \forall j \in \{2,\ldots,k\} : h(\theta_1 + \cdots + \theta_k) + \theta_j - \theta_1 \leq 0 \\ &\Leftrightarrow h(\theta_1 + \cdots + \theta_k) + \theta_2 - \theta_1 \leq 0 \end{aligned}$$

and

$$\mathbf{P}(\boldsymbol{\theta}) \in PM_k^m(\Delta^h) \Leftrightarrow \forall i \in [m-k] : h(\theta_i + \cdots + \theta_{i+k-1}) + \theta_{i+1} - \theta_i \leq 0.$$

Proof. This follows directly from the definitions. □

From this, we obtain the following result, which is not explicitly needed anywhere but rather stated for the sake of completeness.

Corollary B.4. *For any $h \in (0,1)$ and $m, k \in \mathbb{N}$ with $k \leq m$ we have $PM_k^m(\mathrm{PL} \wedge \exists h\mathrm{GCW}) \supseteq PM_k^m(\mathrm{PL} \wedge \Delta^h) \neq \emptyset$.*

Proof. Note that $PM_k^m(\mathrm{PL} \wedge \exists h\mathrm{GCW}) \supseteq PM_k^m(\mathrm{PL} \wedge \Delta^h)$ is a direct consequence from the definitions. To see $PM_k^m(\mathrm{PL} \wedge \Delta^h) \neq \emptyset$ we fix $x > 1$ with $h + \frac{h}{x} \leq 1$ and define $\boldsymbol{\theta} \in (0,1]^m$ via $\theta_j := \frac{h^j}{(kx)^j}$ for any $j \in [m]$. Then,

$$\begin{aligned} &h(\theta_i + \cdots + \theta_{i+k-1}) + \theta_{i+1} - \theta_i \\ &= \frac{h^{i+1}}{(kx)^i} + \left(\frac{h^{i+2}}{(kx)^{i+1}} + \cdots + \frac{h^{i+k}}{(kx)^{i+k-1}} + \frac{h^{i+1}}{(kx)^{i+1}}\right) - \frac{h^i}{(kx)^i} \\ &\leq \frac{h^{i+1}}{(kx)^i} + \frac{kh^{i+1}}{(kx)^{i+1}} - \frac{h^i}{(kx)^i} = \frac{h^i}{(kx)^i}\left(h + \frac{h}{x} - 1\right) \leq 0 \end{aligned}$$

holds for any $i \in [m-k]$ and thus $\mathbf{P}(\boldsymbol{\theta}) \in PM_k^m(\mathrm{PL} \wedge \Delta^h)$ follows from Lemma B.3. □

Proof of Theorem 5.1. Define $\boldsymbol{\theta} \in (0,1]^m$ via $\theta_1 := 1$ and $\theta_j := \frac{1-h}{h(k-1)+1}$ for $2 \leq j \leq m$. Then,

$$\begin{aligned} h\sum\nolimits_{j=1}^k \theta_j + \theta_2 - \theta_1 &= h\left(1 + \frac{(k-1)(1-h)}{h(k-1)+1}\right) + \frac{1-h-h(k-1)-1}{h(k-1)+1} \\ &= \frac{h(h(k-1)+1+(k-1)(1-h)) - hk}{h(k-1)+1} = 0 \end{aligned}$$

shows with regard to Lemma B.3 that $\mathbf{P}(\boldsymbol{\theta}) \in PM_k^m(\exists h\mathrm{GCW})$ is fulfilled. Moreover, for $j \in \{2, \ldots, m\}$ we have $1 - \theta_j = \frac{hk}{h(k-1)+1}$ and thus

$$\frac{\theta_j}{(1-\theta_j)^2} = \frac{(h(k-1)+1)(1-h)}{h^2k^2} = \frac{hk(1-h) + (1-h)^2}{h^2k^2},$$

which is in $\Theta(\frac{1}{hk} + \frac{1}{h^2k^2}) = \Theta\left(\frac{1}{kh^2}\left(\frac{1}{k} + h\right)\right)$, since $1 - h \in \Theta(1)$ as $h \searrow 0$. In particular,

$$\sum_{j=2}^m \frac{\theta_j}{(1-\theta_j)^2} \in \Theta\left(\frac{m}{kh^2}\left(\frac{1}{k} + h\right)\right)$$

and thus the statement follows from Theorem B.1. □

Proof of Theorem 6.1. Suppose $\gamma \in (0,1)$, $h \in (0,1)$ and $m, k \in \mathbb{N}_{\geq 2}$ with $k \leq m$ to be arbitrary but fixed for the moment and let $\mathcal{A}$ be the solution to $\mathcal{P}_k^{m,\gamma}(\mathrm{PL} \wedge \exists \mathrm{GCW}^*)$ from Theorem B.2. For $l \in \{2, \ldots, k\}$ define $g_l : [0,1]^m \to \mathbb{R}$ via $g_l(\boldsymbol{\theta}) := h(1 + \theta_2 + \cdots + \theta_k) + \theta_l - 1$ and denote by $\mathfrak{B}$ the set

$$\{\boldsymbol{\theta} \in (0,1]^m \,|\, 1 = \theta_1 > \theta_2 \geq \cdots \geq \theta_m \text{ and } \forall l \in \{2, \ldots, k\} : g_l(\boldsymbol{\theta}) \leq 0\}.$$

According to Lemma B.3, any $\mathbf{P} \in PM_k^m(\mathrm{PL})$ with $\mathrm{GCW}(\mathbf{P}) = 1$ fulfills $\mathbf{P} \in PM_k^m(\exists h\mathrm{GCW})$ iff $\mathbf{P} = \mathbf{P}(\boldsymbol{\theta})$ for some $\boldsymbol{\theta} \in \mathfrak{B}$. Consequently, it is with regard to Theorem B.2 sufficient to show that

$$\frac{\Theta_{[k]}}{k}\sum\nolimits_{j=2}^m \frac{1}{(1-\theta_j)^2}\ln\left(\frac{k}{\gamma}\ln\left(\frac{1}{1-\theta_j}\right)\right) \leq \frac{6m}{kh^2}\left(\frac{1}{k} + h\right)\ln\left(\frac{k}{\gamma}\ln(h^{-1})\right) \tag{6}$$

holds for any $\boldsymbol{\theta} \in \mathfrak{B}$. We prove this in several steps.

Claim 1: For any $\boldsymbol{\theta} \in \mathfrak{B}$ we have

$$\sum\nolimits_{j=2}^k \frac{1 + \theta_2 + \cdots + \theta_k}{(1-\theta_j)^2} \leq \frac{3(1+hk)}{h^2}. \tag{7}$$

Proof of Claim 1: Let $\mathfrak{B}'$ be the set of all $\boldsymbol{\theta} = (1, \theta_2, \dots, \theta_k)$ with $1 \geq \theta_2 \geq \dots \geq \theta_k \geq 0$ and $g_l(\boldsymbol{\theta}) \leq 0$ for all $l \in \{2, \dots, k\}$. As $(1, \theta_2, \dots, \theta_k) \in \mathfrak{B}'$ holds for any $(1, \theta_2, \dots, \theta_m) \in \mathfrak{B}$, it is sufficient to show that (7) holds for any $\boldsymbol{\theta} = (1, \theta_2, \dots, \theta_k) \in \mathfrak{B}'$.

Claim 1a: For any $\boldsymbol{\theta} \in \mathfrak{B}'$ and $l \in \{2, \dots, k\}$ we have $\theta_l \leq 1 - h$.
Proof: For $\boldsymbol{\theta} = (1, \theta_2, \dots, \theta_k) \in \mathfrak{B}'$ and $l \in \{2, \dots, k\}$ we have

$$0 \geq g_l(\boldsymbol{\theta}) = h(1 + \theta_2 + \dots + \theta_k) + \theta_l - 1 \geq h + \theta_l - 1,$$

and thus $\theta_l \leq 1 - h$. ♣

According to Claim 1a, $\mathfrak{B}'$ is a compact subset of $\{1\} \times [0, 1-h]^{k-1}$. Consequently, the continuous function $f : \mathfrak{B}' \to \mathbb{R}$, $f(\boldsymbol{\theta}) := \sum_{j=2}^{k} \frac{1+\theta_2+\dots+\theta_k}{(1-\theta_j)^2}$ is well-defined and takes its maximum on $\mathfrak{B}'$ in a point $\boldsymbol{\theta}^* \in \mathfrak{B}'$.

Claim 1b: There is some $j \in \{2, \dots, k\}$ s.t. $g_2(\boldsymbol{\theta}^*) = \dots = g_j(\boldsymbol{\theta}^*) = 0$ and $\theta^*_{j+2} = \dots = \theta^*_k = 0$.
Proof: To show indirectly the existence of some $j \in \{2, \dots, k\}$ with $g_j(\boldsymbol{\theta}^*) = 0$ assume on the contrary that $g_l(\boldsymbol{\theta}^*) < 0$ for any $l \in \{2, \dots, k\}$. Then, if $\varepsilon > 0$ is small enough, $\boldsymbol{\theta}_\varepsilon := (1, \theta^*_2 + \varepsilon, \theta^*_3, \dots, \theta^*_k)$ is an element of $\mathfrak{B}'$. Since

$$\frac{\partial f}{\partial \theta_2}(\boldsymbol{\theta}) = \frac{2\theta_2(1 + \theta_2 + \dots + \theta_k)}{(1-\theta_2)^3} + \sum\nolimits_{l=2}^{k} \frac{1}{(1-\theta_l)^2} > 0$$

holds for any $\boldsymbol{\theta}$ in the interior of $\mathfrak{B}'$, we would obtain $f(\boldsymbol{\theta}_\varepsilon) > f(\boldsymbol{\theta}^*)$ in contradiction to the optimality of $\boldsymbol{\theta}^*$. Hence, there has to be a $j \in \{2, \dots, k\}$ with $g_j(\boldsymbol{\theta}^*) = 0$. In case $j \geq 3$, we may infer from $g_{j-1}(\boldsymbol{\theta}^*) - g_j(\boldsymbol{\theta}^*) = \theta^*_{j-1} - \theta^*_j \geq 0$ inductively $0 = g_{j-1}(\boldsymbol{\theta}^*) = \dots = g_2(\boldsymbol{\theta}^*)$.
It remains to prove $\theta^*_{j+2} = \dots = \theta^*_k = 0$. Assume this was not the case, i.e., $j \leq k-2$ and $j' := \max\{l \in \{2, \dots, k\} \mid \theta^*_l > 0\} \geq j+2$. By definition of j we have $g_j(\boldsymbol{\theta}^*) < 0$. Consequently,

$$\boldsymbol{\theta}'_\varepsilon := (1, \theta^*_2, \dots, \theta^*_j, \theta^*_{j+1} + \varepsilon, \theta^*_{j+2}, \dots, \theta^*_{j'} - \varepsilon, 0, \dots, 0)$$

is for small values of $\varepsilon \geq 0$ an element of $\mathfrak{B}'$. Using $\sum_{l=2}^{k} (\boldsymbol{\theta}'_\varepsilon)_k = \sum_{l=2}^{k} \theta^*_l$ we see that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(\boldsymbol{\theta}'_\varepsilon) = \frac{2}{(1 - \theta^*_{j+1} - \varepsilon)^3} - \frac{2}{(1 - \theta^*_{j'} + \varepsilon)^3},$$

which is due to $\theta^*_{j+1} \geq \theta^*_{j'}$ positive for small values of $\varepsilon > 0$. In particular, $f(\boldsymbol{\theta}'_\varepsilon) > f(\boldsymbol{\theta}'_0) = f(\boldsymbol{\theta}^*)$ holds for small $\varepsilon > 0$, which contradicts the optimality of $\boldsymbol{\theta}^*$. This completes the proof of Claim 1b. ♣

According to Claim 1b we may fix some $j \in \{2, \dots, k\}$ with $g_2(\boldsymbol{\theta}^*) = \dots = g_j(\boldsymbol{\theta}^*) = 0$ and $\theta^*_{j+2} = \dots = \theta^*_k = 0$. Since $g_l(\boldsymbol{\theta}^*) - g_{l'}(\boldsymbol{\theta}^*) = \theta^*_l - \theta^*_{l'} = 0$ holds for any $l, l' \in \{2, \dots, k\}$, we have $\theta^*_2 = \dots = \theta^*_j$. From $0 \geq g_2(\boldsymbol{\theta}^*) \geq h(1 + (j-1)\theta^*_2) + \theta^*_2 - 1$ we infer

$$\theta^*_2 = \dots = \theta^*_j \leq \frac{1-h}{1+(j-1)h} = 1 - \frac{hj}{1+h(j-1)}.$$

Together with $\theta^*_j \geq \theta^*_{j+1} \geq 0 = \theta^*_{j+2} = \dots = \theta^*_k$ we obtain

$$\begin{aligned}
\frac{1 + \theta^*_2 + \dots + \theta^*_k}{(1-\theta^*_2)^2} &\leq \frac{1 + j\theta^*_2}{(1-\theta^*_2)^2} \leq \frac{(1+h(j-1))^2}{h^2 j^2} \left(1 + \frac{j(1-h)}{1+h(j-1)}\right) \\
&= \frac{(1+h(j-1))(1-h+j)}{h^2 j^2} \leq 2\left(\frac{1}{h^2 j} + \frac{h(j-1)}{h^2 j}\right) \\
&\leq \frac{2}{h^2}\left(\frac{1}{j} + h\right),
\end{aligned}$$

where we have used that $1-h+j \leq 2j$ holds trivially. Combining this with the fact that $g_2(\boldsymbol{\theta}^*) \leq 0$ implies $(1+\theta_2^* + \cdots + \theta_k^*) \leq \frac{1-\theta_2^*}{h} \leq \frac{1}{h}$ yields

$$\begin{aligned} f(\boldsymbol{\theta}^*) &= \sum\nolimits_{l=2}^{k} \frac{1+\theta_2^* + \cdots + \theta_k^*}{(1-\theta_l^*)^2} \\ &\leq (1+\theta_2^* + \cdots + \theta_k^*) \left(\sum\nolimits_{l=2}^{j+1} \frac{1}{(1-\theta_2^*)^2} + \sum\nolimits_{l=j+2}^{k} 1 \right) \\ &\leq \frac{2j}{h^2}\left(\frac{1}{j} + h\right) + \frac{k-j-1}{h} \leq \frac{3(1+hk)}{h^2}. \end{aligned}$$

Since $\boldsymbol{\theta}^*$ was a maximum point of f in $\mathfrak{B}'$, Claim 1 follows. ■

Claim 2: For any $\boldsymbol{\theta} \in \mathfrak{B}$ we have $\sum_{j=2}^{m} \frac{1}{(1-\theta_j)^2} \leq \frac{m-1}{k-1} \sum_{j=2}^{k} \frac{1}{(1-\theta_j)^2}$.
Proof of Claim 2: Using $1 \geq \theta_2 \geq \cdots \geq \theta_m$, this follows directly from comparing the $(m-1)(k-1)$ summands in $(k-1)\sum_{j=2}^{m} \frac{1}{(1-\theta_j)^2} = \sum_{j=2}^{m} \frac{1}{(1-\theta_j)^2} + \cdots + \sum_{j=2}^{m} \frac{1}{(1-\theta_j)^2}$ with those in $(m-1)\sum_{j=2}^{k} \frac{1}{(1-\theta_j)^2}$. ■

Claim 3: Inequality (6) holds for any $\boldsymbol{\theta} \in \mathfrak{B}$.
Proof of Claim 3: Let $\boldsymbol{\theta} \in \mathfrak{B}$ be fixed and note that $\Theta_{[k]} = 1 + \theta_2 + \cdots + \theta_k$ holds. From $1 \geq \theta_2 \geq \cdots \geq \theta_m \geq 0$ we get $\Theta_{[k]} \in [1,k]$. Together with $\frac{1-\theta_2}{\Theta_{[k]}} = \frac{h\Theta_{[k]} - g_2(\boldsymbol{\theta})}{\Theta_{[k]}} \geq h$ this shows $1-\theta_j \geq 1-\theta_2 \geq h$ and in particular $\ln(1/(1-\theta_j)) \leq \ln(h^{-1})$ for each $j \in \{2,\ldots,m\}$. In combination with Claims 1 and 2 this allows us to conclude

$$\begin{aligned} &\frac{\Theta_{[k]}}{k} \sum\nolimits_{j=2}^{m} \frac{1}{(1-\theta_j)^2} \ln\left(\frac{k}{\gamma} \ln\left(\frac{1}{1-\theta_j}\right)\right) \\ &\leq \frac{1}{k} \ln\left(\frac{k}{\gamma}\ln(h^{-1})\right) \sum\nolimits_{j=2}^{m} \frac{1+\theta_2+\cdots+\theta_k}{(1-\theta_j)^2} \\ &\leq \frac{m-1}{k(k-1)} \ln\left(\frac{k}{\gamma}\ln(h^{-1})\right) \sum\nolimits_{j=2}^{k} \frac{1+\theta_2+\cdots+\theta_k}{(1-\theta_j)^2} \\ &\leq \frac{3(m-1)(1+hk)}{k(k-1)h^2} \ln\left(\frac{k}{\gamma}\ln(h^{-1})\right) \\ &\leq \frac{6m}{kh^2}\left(\frac{1}{k}+h\right) \ln\left(\frac{k}{\gamma}\ln(h^{-1})\right), \end{aligned}$$

where we have used that $\frac{m-1}{k-1} \leq \frac{2m}{k}$ holds due to $k \geq 2$. This completes the proof of Claim 3 and of the theorem.

□

C Proofs for Section 4.1

Proposition C.1 (Detailed version of Proposition 4.1)**.** *Let $0 < \gamma < \gamma_0 < 1/2$ and $0 < h < h_0 < 1$ be fixed. Suppose $\mathcal{A}$ solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$, let $\mathbf{p} \in \Delta_k^h$ be arbitrary and write $i := \mathrm{mode}(\mathbf{p})$. Then,*

$$\mathbb{E}_{\mathbf{p}}\left[T^{\mathcal{A}}\right] \geq \frac{f\left(\frac{p_i - p_j}{2(p_i+p_j)}, \gamma\right)}{p_i + p_j}$$

holds for all $j \in [k] \setminus \{i\}$ with $f(z,\gamma) := \frac{1-2\gamma}{2z}\left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln((1/2+z)/(1/2-z))} \right\rceil$, which fulfills $\forall z \in (0, h_0/2)$: $f(z,\gamma) \geq c(h_0,\gamma_0) z^{-2}\ln(\gamma^{-1})$ for some appropriate constant $c(h_0,\gamma_0) > 0$ that does not depend on γ or h. In particular, we obtain the worst-case bound

$$\sup\nolimits_{\mathbf{p}\in\Delta_k^h} \mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq 4c(h_0,\gamma_0)h^{-2}\ln(\gamma^{-1}) \tag{8}$$

and the instance-wise bound

$$\forall \mathbf{p} \in \Delta_k^h : \mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq 2c(h_0, \gamma_0)(h(\mathbf{p}))^{-2} \ln(\gamma^{-1}) \left(\frac{1}{k} + h\right). \tag{9}$$

We prepare the proof of Proposition C.1 with sample complexity lower bounds of solutions to $\mathcal{P}_2^{2,\gamma}(\Delta^h)$. For the sake of convenience, we write p for $(p, 1-p) \in \Delta_2$. Note that solving $\mathcal{P}_2^{2,\gamma}(\Delta^h)$ resp. $\mathcal{P}_2^{2,\gamma}(\Delta^0)$ reduces to deciding with error probability $\leq \gamma$

$$\mathbf{H}_0 : p > 1/2 \quad \text{vs.} \quad \mathbf{H}_1 : p < 1/2 \tag{10}$$

based on iid samples $X_1, X_2, \dots \sim \mathrm{Ber}(p)$ for any $p \in [0,1]$ with $|p - 1/2| \geq h$ resp. $|p - 1/2| > 0$.

Lemma C.2. *Let $0 < \gamma < \gamma_0 < 1/2$ and $0 < h < h_0 < 1/2$ and suppose $\mathcal{A}$ is able to decide* (10) *with confidence $\geq 1 - \gamma$ for any $p \in \{1/2 \pm h\}$, i.e.,*

$$\mathbb{P}_{1/2+h}(\mathbf{D}(\mathcal{A}) = 0) \geq 1 - \gamma \quad \textit{and} \quad \mathbb{P}_{1/2-h}(\mathbf{D}(\mathcal{A}) = 1) \geq 1 - \gamma.$$

There exists a constant $c(h_0, \gamma_0) > 0$, which does not depend on γ or h s.t.

$$\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}}] = \frac{1-2\gamma}{2h} \left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln\left({(1/2+h)}/{(1/2-h)}\right)} \right\rceil \geq c(h_0, \gamma_0) h^{-2} \ln(\gamma^{-1}).$$

Proof. Let $\mathcal{A}'$ be the corresponding *Sequential Probability Ratio Test* (cf. [41]) for (10), i.e. it samples $X_1, X_2, \dots$ until the first time n, where $\frac{1}{n}\sum_{k=1}^n X_k \notin [1/2 \pm C_{h,\gamma}(n)]$ with $C_{h,\gamma}(n) := \frac{1}{2n}\left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln({(1/2+h)}/{(1/2-h)})} \right\rceil$ and decides for 0 in case $\frac{1}{n}\sum_{k=1}^n X_k > 1/2 + C_{h,\gamma}(n)$ and for 1 in case $\frac{1}{n}\sum_{k=1}^n X_k < 1/2 - C_{h,\gamma}(n)$. On p.10–15 in [38] it is shown that $\mathcal{A}'$ fulfills

$$\mathbb{P}_{1/2+h}(\mathbf{D}(\mathcal{A}') = 0) \geq 1 - \gamma \quad \text{and} \quad \mathbb{P}_{1/2-h}(\mathbf{D}(\mathcal{A}') = 1) \geq 1 - \gamma,$$

as well as

$$\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}'}] = \frac{1-2\gamma}{2h} \left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln\left({(1/2+h)}/{(1/2-h)}\right)} \right\rceil =: g(h, \gamma).$$

According to pages 19–22 in [38] or [15, Theorem 2, p. 365] or the original proof from [41], $\mathcal{A}'$ is a test $\mathcal{A}''$ with error $\leq \gamma$ (on any instance $p \in \{1/2 \pm h\}$) for (10), for which $\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}''}]$ is minimal. In particular, we have

$$\mathbb{E}_{1/2\pm h}[T^{\mathcal{A}}] \geq \mathbb{E}_{1/2\pm h}[T^{\mathcal{A}'}] \geq g(h, \gamma).$$

Since $w : (0,1) \to \mathbb{R}, \gamma \mapsto \frac{\ln((1-\gamma)/\gamma)\cdot(1-2\gamma)}{\ln(1/\gamma)}$ fulfills $w(1/2) = 0$ and

$$w'(\gamma) = \frac{(1-2\gamma)\ln(\gamma^{-1}) - (\gamma - 1)\ln(\gamma^{-1} - 1)(2\gamma + 2\gamma\ln(\gamma^{-1}) - 1))}{(\gamma - 1)\gamma \ln^2(\gamma^{-1})} < 0$$

for every $\gamma \in (0, 1/2)$, there exists some $c'(\gamma_0) > 0$ with $\ln((1-\gamma)/\gamma)(1-2\gamma) \geq c'(\gamma_0)\ln(1/\gamma)$ for each $\gamma \in (0, \gamma_0)$. Moreover, as $\ln(1 + x) < x$ for $x > -1$, we obtain for $h \in (0, h_0)$ the inequality

$$\ln\left(\frac{1/2 + h}{1/2 - h}\right) = \ln\left(1 + \frac{4h}{1-2h}\right) < \frac{4h}{1-2h} < \frac{4h}{1-2h_0}.$$

Combining these estimates, we get $g(h, \gamma) \geq c(h_0, \gamma_0) h^{-2} \ln(\gamma^{-1})$ with $c(h_0, \gamma_0) := \frac{c'(\gamma_0)(1-2h_0)}{8}$. □

Before proving Proposition C.1, we state two further auxiliary lemmata. The first one is a simplified version of *Walds identity* (cf. e.g. Thm. 17.7 in [2]), which we shortly prove for the sake of convenience. The second lemma is only required for the instance-wise bound in Proposition C.1.

Lemma C.3. *Let $k \in \mathbb{N}$ and $(p_1, \dots, p_k) \in \Delta_k$ be fixed. Suppose $\{X_t\}_{t\in\mathbb{N}}$ to be an iid family of random variables $X_t \sim \mathrm{Cat}(p_1, \dots, p_k)$ on some joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\in\mathbb{N}} \subseteq \mathcal{F}$ to be a filtration, such that $\{X_t\}_t$ is $\{\mathcal{F}_t\}_t$-adapted and $\forall t : X_t \perp\!\!\!\perp \mathcal{F}_{t-1}$, e.g. $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$. If τ is an $\{\mathcal{F}_t\}_t$-stopping time, then the random variables*

$$T_i(\tau) := \sum\nolimits_{t\leq\tau} \mathbf{1}_{\{X_t = i\}}, \quad i \in [k],$$

fulfill $\mathbb{E}[T_i(\tau)] = p_i\mathbb{E}[\tau]$ *for each* $i \in [k]$. *In particular, we obtain*

$$\mathbb{E}[\tau] = \frac{\sum_{i\in I}\mathbb{E}[T_i(\tau)]}{\sum_{i\in I} p_i}$$

for any $I \subseteq [k]$ *with* $\sum_{i\in I} p_i > 0$.

Proof. Since $\{t \le \tau\} = \{t > \tau\}^c = \{\tau \le t-1\}^c \in \mathcal{F}_{t-1}$ holds for any $t \in \mathbb{N}$ and $X_t \perp\!\!\!\perp \mathcal{F}_{t-1}$, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{X_t=i\}}\mathbf{1}_{\{t\le\tau\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{X_t=i\}}\mathbf{1}_{\{t\le\tau\}}\middle|\mathcal{F}_{t-1}\right]\right] \\ &= \mathbb{E}[\mathbf{1}_{\{t\le\tau\}}\mathbb{E}[\mathbf{1}_{\{X_t=i\}}|\mathcal{F}_{t-1}]] = p_i\mathbb{E}[\mathbf{1}_{\{t\le\tau\}}].\end{aligned}$$

Via an application of the monotone convergence theorem we infer

$$\begin{aligned}\mathbb{E}\left[T_i(\tau)\right] &= \lim_{T\to\infty}\mathbb{E}[T_i(\tau\wedge T)] \\ &= \lim_{T\to\infty}\sum_{t\le T}\mathbb{E}\left[\mathbf{1}_{\{X_t=i\}}\mathbf{1}_{\{t\le\tau\}}\right] \\ &= p_i\lim_{T\to\infty}\sum_{t\le T}\mathbb{E}\left[\mathbf{1}_{\{t\le\tau\}}\right] \\ &= p_i\lim_{T\to\infty}\mathbb{E}[\tau\wedge T] = p_i\mathbb{E}[\tau].\end{aligned}$$

and thus in particular $\sum_{i\in I}\mathbb{E}[T_i(\tau)] = \mathbb{E}[\tau]\sum_{i\in I} p_i$. □

Lemma C.4. *Suppose* $\mathbf{p} \in \Delta_k^h \setminus \Delta_k^{\tilde{h}}$ *for some* $0 < h < \tilde{h} < 1$ *and let* $i := \mathrm{mode}(\mathbf{p})$ *and* $j \in \arg\max_{l\in[k]\setminus\{i\}} p_l$. *Then, we have* $p_i + p_j \ge \frac{2+(k-2)h}{k}$ *and* $p_i - p_j < \tilde{h}$.

Proof. From $\mathbf{p} \in \Delta_k^h$ and $\mathrm{mode}(\mathbf{p}) = i$ we infer that $p_l \le p_i - h$ holds for each $l \in [k]\setminus\{i\}$. Thus,

$$1 = \sum_{l\in[k]} p_l \le p_i + \sum_{l\ne i}(p_i - h) = kp_i - (k-1)h$$

shows us that $p_i = \frac{1+(k-1)h}{k} + \varepsilon$ for some $\varepsilon \ge 0$. Due to $\sum_{l\ne i} p_l = 1 - p_i$ and $p_j = \max_{l\in[k]\setminus\{i\}} p_l$, we have

$$p_j \ge \frac{1-p_i}{k-1} = \frac{1 - \frac{1+(k-1)h}{k} - \varepsilon}{k-1} = \frac{1+h}{k} - \frac{\varepsilon}{k-1}.$$

Consequently,

$$\begin{aligned}p_i + p_j &\ge \frac{1+(k-1)h}{k} + \varepsilon + \frac{1+h}{k} - \frac{\varepsilon}{k-1} \\ &= \frac{2+(k-2)h}{k} + \frac{(k-2)\varepsilon}{k-1} \\ &\ge \frac{2+(k-2)h}{k}.\end{aligned}$$

Moreover, $\mathbf{p} \notin \Delta_k^{\tilde{h}}$ assures the existence of some $j' \in [m]\setminus\{i\}$ with $p_i < p_{j'} + \tilde{h}$. Since the choice of j guarantees $p_{j'} + \tilde{h} \le p_j + \tilde{h}$, this implies $p_i - p_j < \tilde{h}$. □

Proof of Proposition C.1. We may suppose w.l.o.g. $i = 1$ and fix $j = 2$. Let us define $a := \frac{p_1}{p_1+p_2}$ and suppose we have a coin $C \sim \mathrm{Ber}(p)$ with $p \in \{a, 1-a\}$. By simulating $\mathcal{A}$, we will construct an algorithm $\mathcal{A}'$ for testing

$$\mathbf{H}_0' : p = a \qquad \mathbf{H}_1' : p = 1-a$$

in the following way: Whenever $\mathcal{A}$ makes a query at time t, we generate an independent sample $U_t \sim \mathcal{U}([0,1])$. Then, we return the feedback $X_t = i' \in \{3,\dots,k\}$ iff $U_t \in (\sum_{j'\le i'-1} p_{j'}, \sum_{j'\le i'} p_{j'}]$ and in case $U_t \in [0, p_1+p_2]$ we generate an independent sample $C_t \sim \mathrm{Ber}(p)$ from our coin C and return

$$X_t = \begin{cases}1, & \text{if } C_t = 1,\\ 2, & \text{if } C_t = 0.\end{cases}$$

As soon as $\mathcal{A}$ terminates, we terminate and return $\mathbf{D}(\mathcal{A}') = 0$ if $\mathbf{D}(\mathcal{A}) = 1$ and $\mathbf{D}(\mathcal{A}') = 1$ otherwise. By our construction, we have $\mathbb{P}_p(X_t = i) = p_i$ for each $i \in \{3, \ldots, k\}$

$$\mathbb{P}_a(X_t = 1) = (p_1 + p_2)\mathbb{P}(C_t = 1) = p_1, \;\; \mathbb{P}_a(X_t = 2) = (p_1 + p_2)\mathbb{P}(C_t = 0) = p_2$$

and similarly $\mathbb{P}_{1-a}(X_t = 1) = p_2$ and $\mathbb{P}_{1-a}(X_t = 2) = p_1$. Thus if $p = a$, $\mathcal{A}$ behaves as started on $\mathbf{p}$ and if $p = 1 - a$, $\mathcal{A}$ behaves as started on $\mathbf{p}' \coloneqq (p_2, p_1, p_3, \ldots, p_k) \in \Delta_k^h$. Since $\mathcal{A}$ solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$, we obtain

$$\mathbb{P}_a(\mathbf{D}(\mathcal{A}') = 0) = \mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}) = 1\right) \geq 1 - \gamma$$

and (due to $2 = \arg\max_{j' \in [k]} p'_j$)

$$\mathbb{P}_{1-a}(\mathbf{D}(\mathcal{A}') = 1) = \mathbb{P}_{\mathbf{p}'}(\mathbf{D}(\mathcal{A}) \neq 1) \geq \mathbb{P}_{\mathbf{p}'}(\mathbf{D}(\mathcal{A}) = 2) \geq 1 - \gamma,$$

i.e., $\mathcal{A}'$ is able to decide $\mathbf{H}'_0$ versus $\mathbf{H}'_1$ with error probability $\leq \gamma$. From Lemma C.2 we infer that it has to throw the coin C (in both cases $p \in \{a, 1-a\}$) in expectation at least $f(a - 1/2, \gamma)$ times for this. Regarding that C is thrown in our construction iff we return as feedback an element from $\{1, 2\}$, we get that

$$\mathbb{E}_{\mathbf{p}}[T_1(T^{\mathcal{A}}) + T_2(T^{\mathcal{A}})] \geq f(a - 1/2, \gamma) \quad \text{where } T_i(T^{\mathcal{A}}) \coloneqq \sum\nolimits_{t \leq T^{\mathcal{A}}} \mathbf{1}_{\{X_t = i\}}.$$

An application of Lemma C.3 yields

$$\mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq \frac{f(a - 1/2, \gamma)}{p_1 + p_2} = \frac{f\left(\frac{p_1 - p_2}{2(p_1 + p_2)}, \gamma\right)}{p_1 + p_2},$$

which completes the proof of the first statement.

The worst-case bound (8) then follows from the just proven bound via

$$\sup\nolimits_{\mathbf{p} \in \Delta_k^h} \mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq \mathbb{E}_{\left(\frac{1+h}{2}, \frac{1-h}{2}, 0, \ldots, 0\right)}[T^{\mathcal{A}}] \geq f(h/2, \gamma) \geq 4c(h_0, \gamma_0) h^{-2} \ln(\gamma^{-1})$$

for some $c(h_0, \gamma_0) > 0$, that is assured to exist by Lemma C.2. To prove (9) suppose at first $\tilde{h} \in (h, 1)$ and $\mathbf{p} \in \Delta_k^h \setminus \Delta_k^{\tilde{h}}$ to be fixed and write $i \coloneqq \mathrm{mode}(\mathbf{p})$. Lemma C.4 reveals that there exists some $j \in [k] \setminus \{i\}$ with $p_i + p_j \geq \frac{2 + (k-2)h}{k}$ and $p_i - p_j < \tilde{h}$. Consequently, the above proven bound and the estimate $f(h, \gamma) \geq c(h_0, \gamma_0) h^{-2} \ln(\gamma^{-1})$ yield

$$\begin{aligned}
\mathbb{E}_{\mathbf{p}}[T^{\mathcal{A}}] \geq \frac{f\left(\frac{p_i - p_j}{2(p_i + p_j)}, \gamma\right)}{p_i + p_j} &\geq 4c(h_0, \gamma_0) \frac{p_i + p_j}{(p_i - p_j)^2} \ln(\gamma^{-1}) \\
&\geq 4c(h_0, \gamma_0) \tilde{h}^{-2} \ln(\gamma^{-1}) \frac{2 + (k-2)h}{k} \\
&\geq 2c(h_0, \gamma_0) \tilde{h}^{-2} \ln(\gamma^{-1}) \left(\frac{1}{k} + h\right).
\end{aligned}$$

Since $\mathbf{p} \in \Delta_k^{h(\mathbf{p})} \setminus \left(\bigcup_{\tilde{h} > h(\mathbf{p})} \Delta_k^{\tilde{h}}\right) = \bigcap_{\tilde{h} > h(\mathbf{p})} (\Delta_k^{h(\mathbf{p})} \setminus \Delta_k^{\tilde{h}})$ for any $\mathbf{p} \in \Delta_k^h$, (9) can be inferred from this by taking the limit $\tilde{h} \searrow h(\mathbf{p})$. □

From Lemma C.2 we can infer that any solution $\mathcal{A}$ to $\mathcal{P}_2^{2,\gamma}(\Delta^0)$ fulfills $\lim_{h \to 0} \mathbb{E}_{1/2 \pm h}[T^{\mathcal{A}}] \in \Omega(h^{-2})$ as $h \to 0$. The following lemma improves upon this bound and is the key ingredient for the proof of Proposition 4.2.

Lemma C.5. *Let $\gamma \in (0, 1/2)$ be fixed and suppose $\mathcal{A}$ to be an algorithm, which terminates a.s. for any $p \neq 1/2$ and is able to decide* (10) *for any $p \neq 1/2$ with confidence $\geq 1 - \gamma$, i.e.,*

$$\forall p > 1/2 : \mathbb{P}_p(\mathbf{D}(\mathcal{A}) = 0) \geq 1 - \gamma \quad \textit{and} \quad \forall p < 1/2 : \mathbb{P}_p(\mathbf{D}(\mathcal{A}) = 1) \geq 1 - \gamma.$$

Then,

$$\limsup_{h \to 0} \frac{\mathbb{E}_{1/2 \pm h}\left[T^{\mathcal{A}}\right]}{h^{-2} \ln \ln h^{-1}} \geq \frac{1}{2} \mathbb{P}_{1/2}(T^{\mathcal{A}} = \infty) \geq \frac{1}{2}(1 - 2\gamma) > 0.$$

Proof. This is stated in Theorem 1 in [14]. To verify this, note that $|\ln|\ln|h|||^{-1} = (\ln\ln h^{-1})^{-1}$ holds for $h < \frac{1}{e}$ and also confer the remark directly after Theorem 1 therein. $\square$

Proof of Proposition 4.2. We suppose w.l.o.g. $(i,j) = (1,2)$ throughout the proof. For $h \in (0, p_1 - p_2)$ we have $(\mathbf{p}(h))_1 > (\mathbf{p}(h))_2 > (\mathbf{p}(h))_l$ for every $l \in \{3,\ldots,k\}$ and together with $|(\mathbf{p}(h))_1 - (\mathbf{p}(h))_2| = h$ this shows $\mathbf{p}(h) \in \Delta_k^h$. Suppose we have a coin $C \sim \mathrm{Ber}(p)$ for $p \neq 1/2$. By simulating $\mathcal{A}$ as in the proof of Proposition 4.1 we obtain an algorithm $\mathcal{A}'$ for testing $\mathbf{H}_0 : p > 1/2$ versus $\mathbf{H}_1 : p < 1/2$, which has (due to the theoretical guarantees of $\mathcal{A}$) an error probability $\leq \gamma$ for every $p \neq 1/2$. Consequently, Lemma C.5 guarantees the existence of a sequence $\{h'_l\}_{l\in\mathbb{N}} \subseteq (0, e^{-4})$ with

$$\forall l \in \mathbb{N} : \frac{\mathbb{E}_{1/2\pm h'_l}[T^{\mathcal{A}'}]}{{h'_l}^{-2}\ln\ln {h'_l}^{-1}} \geq \frac{1-2\gamma}{2} - \varepsilon > 0$$

for some arbitrarily small but fixed $\varepsilon \in (0, \frac{1-2\gamma}{2})$. If we choose $h_l := 2(p_1+p_2)h'_l$, then the corresponding bias of the coin C in the reduction (cf. the proof of Proposition 4.1) is exactly

$$\frac{(\mathbf{p}(h_l))_1}{(\mathbf{p}(h_l))_1 + (\mathbf{p}(h_l))_2} = \frac{\frac{p_1+p_2}{2} + \frac{h_l}{2}}{p_1+p_2} = \frac{1}{2} + \frac{h_l}{2(p_1+p_2)} = \frac{1}{2} + h'_l$$

Hence, if $\mathcal{A}'$ is started on $1/2 + h'_l$, its internal method $\mathcal{A}$ works as if started on $\mathbf{p}(h_l)$. From $h_l \leq e^{-4}$ we obtain $4 = (1/2)^{-2} \leq \ln(h_l^{-1})$ and thus $-2\ln(1/2) \leq \ln\ln(h_l^{-1})$, i.e., $\ln(1/2) \geq -1/2\ln\ln(h_l^{-1}) \geq -1/2\ln(h_l^{-1})$. Consequently,

$$\begin{aligned}\ln\ln {h'_l}^{-1} = \ln\ln\left(\frac{h_l^{-1}}{2(p_1+p_2)}\right) &\geq \ln\left(\ln(1/2) + \ln(h_l^{-1})\right)\right) \geq \ln\left(\frac{1}{2}\ln(h_l^{-1})\right) \\ &= \ln(1/2) + \ln\ln(h_l^{-1}) \geq \frac{1}{2}\ln\ln(h_l^{-1})\end{aligned}$$

holds, and we obtain similarly as in the proof of Proposition 4.1

$$\begin{aligned}&\mathbb{E}_{\mathbf{p}(h_l)}[T_1(T^{\mathcal{A}}) + T_2(T^{\mathcal{A}})] \geq \mathbb{E}_{1/2+h'_l}[T^{\mathcal{A}'}] \geq \left(\frac{1}{2}(1-2\gamma) - \varepsilon\right){h'_l}^{-2}\ln\ln {h'_l}^{-1} \\ &\geq 2(p_1+p_2)^2\left(\frac{1}{2}(1-2\gamma) - \varepsilon\right)h_l^{-2}\ln\ln h_l^{-1}\end{aligned}$$

Regarding that his holds for arbitrarily small $\varepsilon > 0$, Lemma C.3 shows[5] that

$$\frac{\mathbb{E}_{\mathbf{p}(h_l)}[T^{\mathcal{A}}]}{h_l^{-2}\ln\ln h_l^{-1}} \geq (1-2\gamma)(p_1+p_2)$$

holds for every $l \in \mathbb{N}$, which completes the proof. $\square$

D Proofs of Section 4.2

Our upper bounds for both the cases $m = k$ and $m \geq k$ rely on the Kiefer-Dvoretzky-Wolfowitz inequality, which we state in the following for convenience only for categorical random variables.

Lemma D.1. *Suppose $X_1, X_2, \ldots$ to be iid random variables $X_n \sim \mathrm{Cat}(\mathbf{p})$ for some $\mathbf{p} \in \Delta_k$. For $t \in \mathbb{N}$ let $\hat{\mathbf{p}}^t$ be the corresponding empirical distribution after the t observations $X_1, \ldots, X_t$, i.e., $\hat{p}_i^t = \frac{1}{t}\sum_{s=1}^t \mathbf{1}_{\{X_s = i\}}$ for all $i \in [k]$. Then, we have for any $\varepsilon > 0$ and $t \in \mathbb{N}$ the estimate*

$$\mathbb{P}\left(\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > \varepsilon\right) \leq 4e^{-t\varepsilon^2/2}.$$

Proof. Confer [12, 24] as well as Theorem 11.6 in [22]. Moreover, note that the cumulative distribution functions F resp. $\hat{F}^t$ of $X_1 \sim \mathrm{Cat}(\mathbf{p})$ resp. $\hat{\mathbf{p}}^t$ fulfill $p_j = F(j) - F(j-1)$ and $\hat{p}_j^t = \hat{F}^t(j) - \hat{F}^t(j-1)$ and thus

$$|\hat{p}_j^t - p_j| \leq |\hat{F}^t(j) - F(j)| + |\hat{F}^t(j-1) - F(j-1)|.$$

for each $j \in [k]$. $\square$

[5] Note here that $(\mathbf{p}(h))_1 + (\mathbf{p}(h))_2 = p_1 + p_2$.

Lemma D.2. *For $h \in [0,1], \varepsilon \in (-h,1], \mathbf{p} \in \Delta_k^h$ and $\tilde{\mathbf{p}} \in \Delta_k$ we have*

$$(\exists i : \tilde{p}_i - \max_{j \neq i} \tilde{p}_j \geq \varepsilon \text{ and } p_i \neq \max_j p_j) \quad \Rightarrow \quad ||\tilde{\mathbf{p}} - \mathbf{p}||_\infty \geq (h+\varepsilon)/2.$$

Proof. Suppose there is some $i \in [k]$ s.t. $\tilde{p}_i - \max_{j \neq i} \tilde{p}_j \geq \varepsilon$ and $p_i \neq \max_j p_j$ hold. Then, there exists some $j \in [k] \setminus \{i\}$ with

$$p_j \geq p_i + h \quad \text{and} \quad \tilde{p}_i \geq \tilde{p}_j + \varepsilon$$

and we conclude

$$2\,||\tilde{\mathbf{p}} - \mathbf{p}||_\infty \geq |p_j - \tilde{p}_j| + |\tilde{p}_i - p_i| \geq (p_j - p_i) + (\tilde{p}_i - \tilde{p}_j) \geq h + \varepsilon.$$

□

Remark D.3. *The bounds from Lemma D.2 are sharp: Consider e.g. $\mathbf{p} \in \Delta_k^h$ and $\tilde{\mathbf{p}} \in \Delta_k$ defined via*

$$p_i = \begin{cases} 1/2 - h/2, & \text{if } i = 1, \\ 1/2 + h/2, & \text{if } i = 2, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{p}_i = \begin{cases} 1/2 + \varepsilon/2, & \text{if } i = 1, \\ 1/2 - \varepsilon/2, & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have $\tilde{p}_1 - \max_{j \neq 1} \tilde{p}_j = \varepsilon$ and $p_1 \neq \max_{j \in [k]} p_j$ and at the same time $||\mathbf{p} - \tilde{\mathbf{p}}||_\infty = \frac{h+\varepsilon}{2}$.

For sake of convenience, we give a pseudo-code for the straightforward strategy described in Section 4.2 for solving $\mathcal{P}_k^{k,\gamma}(\Delta^h)$.

Algorithm 4 DKW mode identification – (non-sequential) solution to $\mathcal{P}_k^{k,\gamma}(\Delta^h)$

Input: $\gamma \in (0,1)$, $h \in (0,1)$, $k \in \mathbb{N}$, access to iid samples $X_t \sim \mathrm{Cat}(\mathbf{p})$

1: Let $T \leftarrow \lceil 8 \ln(4/\gamma) h^{-2} \rceil$

2: Observe $X_1, \ldots, X_T \sim \mathrm{Cat}(\mathbf{p})$

3: **return** $\mathrm{mode}(\hat{\mathbf{p}}^T) = \arg\max_{i \in [k]} \sum_{t=1}^T \mathbf{1}_{\{X_t = i\}}$

As a direct consequence of Lemma D.1 and Lemma D.2 we obtain the following result.

Proposition D.4. *For any $k \in \mathbb{N}$, $h \in (0,1)$ and $\gamma \in (0,1)$, Algorithm 4 called with parameters γ, h, k solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$ and terminates after exactly $\lceil 8 \ln(4/\gamma) h^{-2} \rceil$ time steps.*

Lemma D.5. *Let $h > 0$, $\mathbf{p} \in \Delta_k^{3h}$ and $\tilde{\mathbf{p}} \in \Delta_k$ be fixed. Then,*

$$\forall i : \tilde{p}_i \leq \max_{j \neq i} \tilde{p}_j + h \quad \Rightarrow \quad ||\mathbf{p} - \tilde{\mathbf{p}}||_\infty \geq h.$$

Proof. To prove the contraposition, we suppose $||\mathbf{p} - \tilde{\mathbf{p}}||_\infty < h$ to be fulfilled. Let $i := \mathrm{mode}(\mathbf{p}) \in [k]$ and fix some arbitrary $j \in [k] \setminus \{i\}$. Since $\mathbf{p} \in \Delta_k^{3h}$ assures $p_i \geq p_j + 3h$, we obtain

$$\begin{aligned} \tilde{p}_i - \tilde{p}_j &= p_i + (\tilde{p}_i - p_i) + (p_j - \tilde{p}_j) - p_j \geq p_i - p_j - 2\,||\mathbf{p} - \tilde{\mathbf{p}}||_\infty \\ &> p_i - p_j - 2h \geq h. \end{aligned}$$

As j was arbitrary, we conclude that $\tilde{p}_i > \max_{j \neq i} \tilde{p}_j + h$, which completes the proof. □

Lemma D.6. *For any $h \in (0,1/8)$, $\varepsilon \in (0,1/3)$ and $k \in \mathbb{N}_{\geq 3}$ there exist $\mathbf{p} \in \Delta_k^{(3-\varepsilon)h}$ and $\tilde{\mathbf{p}} \in \Delta_k$ such that*

$$\forall i \in [k] : \tilde{p}_i \leq \max_{j \neq i} \tilde{p}_j + h \quad \text{and} \quad ||\mathbf{p} - \tilde{\mathbf{p}}||_\infty < h.$$

Proof. Suppose $h \in (0,1/8)$, $\varepsilon \in (0,1/3)$ and $k \in \mathbb{N}_{\geq 3}$ to be fixed. Now, define $\mathbf{p} \in \Delta_k$ and $\tilde{\mathbf{p}} \in \Delta_k$ via

$$p_j := \begin{cases} \frac{1}{2} + h, & \text{if } j = 1, \\ \frac{1}{2} - (2-\varepsilon)h, & \text{if } j = 2, \\ \frac{(1-\varepsilon)h}{k-2}, & \text{if } j \geq 3, \end{cases}$$

and

$$\tilde{p}_j := \begin{cases} p_1 - (1-\frac{\varepsilon}{4})h = \frac{1}{2} + \frac{\varepsilon h}{4}, & \text{if } j = 1, \\ p_2 + (1-\frac{\varepsilon}{4})h = \frac{1}{2} + (\frac{3\varepsilon}{4} - 1)h, & \text{if } j = 2, \\ \frac{(1-\varepsilon)h}{k-2}, & \text{if } j \geq 3. \end{cases}$$

From $h < 1/8$ we infer $1/2 - (2-\varepsilon)h > 1/2 - 2h > 1/4$ and thus

$$\forall j \geq 3: \ \frac{(k-2)p_j}{p_2} = \frac{(1-\varepsilon)h}{1/2 - (2-\varepsilon)h} < 4(1-\varepsilon)h < 4h < 1/2 < k-2.$$

This shows $p_1 - (3-\varepsilon)h = p_2 > \max_{j\geq 3} p_j$ and consequently $\mathbf{p} \in \Delta_k^{(3-\varepsilon)h}$. Since $\tilde{p}_j = p_j$ is fulfilled for each $j \geq 3$, we have $\tilde{p}_1 > \tilde{p}_2 > p_2 > \max_{j\geq 3} \tilde{p}_j$, and together with

$$\tilde{p}_1 - \tilde{p}_2 = \frac{\varepsilon h}{4} - \frac{3\varepsilon h}{4} + h = \left(1 - \frac{\varepsilon}{2}\right) h < h$$

we see that $\tilde{p}_i \leq \max_{j\neq i} \tilde{p}_j + h$ holds for each $i \in [m]$. Finally $||\mathbf{p} - \tilde{\mathbf{p}}||_\infty < h$ follows from $|p_1 - \tilde{p}_1| = (1-\frac{\varepsilon}{4})h = |p_2 - \tilde{p}_2|$ as well as $p_j = \tilde{p}_j$ for all $j \geq 3$. ☐

Proof of Lemma 4.3. Let $\mathbf{p} \in \Delta_k$ be fixed, and note that Algorithm 1 terminates after exactly $\lceil 8\ln(4/\gamma)h^{-2}\rceil$ time steps. Lemma D.2 and Lemma D.1 let us directly infer

$$\begin{aligned} &\mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}) \in [k] \text{ and } p_{\mathbf{D}(\mathcal{A})} < \max\nolimits_{j\in[k]} p_j\right) \\ &\quad = \mathbb{P}\left(\exists i \in [k] : \hat{p}_i^t - \max\nolimits_{j\neq i} \hat{p}_j^t > h \text{ and } p_i \neq \max\nolimits_{j\in[k]} p_j\right) \\ &\quad \leq \mathbb{P}\left(\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > h/2\right) \leq \gamma. \end{aligned} \tag{11}$$

Next, suppose $\mathbf{p} \in \Delta_k^0$ and let $i' := \mathrm{mode}(\mathbf{p}) \in [k]$. Again, Lemma D.2 yields

$$\begin{aligned} \{\mathbf{D}(\mathcal{A}) \in [k] \setminus \{i'\}\} &= \left\{\exists i \neq i' : \hat{p}_i^t - \max\nolimits_{j\neq i} \hat{p}_j^t > h \text{ and } p_{i'} > \max\nolimits_{j\neq i'} p_j\right\} \\ &\subseteq \left\{\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > h/2\right\}, \end{aligned} \tag{12}$$

and thus

$$\mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) \notin \{i', \text{UNSURE}\}) \leq \mathbb{P}_{\mathbf{p}}\left(\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > h/2\right) \leq \gamma$$

follows from Lemma D.1 and the choice of t. Now, let us suppose $\mathbf{p} \in \Delta_k^{3h}$. A look at Lemma D.5 reveals

$$\{\mathbf{D}(\mathcal{A}) = \text{UNSURE}\} = \left\{\forall i \in [k] : \hat{p}_i^t \leq \max\nolimits_{j\neq i} \hat{p}_j^t + h\right\} \subseteq \left\{\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > h\right\},$$

and combining this with (12) yields

$$\begin{aligned} \mathbb{P}_{\mathbf{p}}(\mathbf{D}(\mathcal{A}) \neq \mathrm{mode}(\mathbf{p})) &= \mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}) \in [k] \setminus \{i'\} \text{ or } \mathbf{D}(\mathcal{A}) = \text{UNSURE}\right) \\ &\leq \mathbb{P}_{\mathbf{p}}\left(\left|\left|\hat{\mathbf{p}}^t - \mathbf{p}\right|\right|_\infty > h/2\right) \leq \gamma, \end{aligned}$$

where the last estimate is again due to Lemma D.1. ☐

We proceed with the proof of Proposition 4.4.

Proof of Proposition 4.4. Let $\mathbf{p} \in \Delta_k^0$ be fixed and abbreviate $h := h(\mathbf{p})$. Moreover, denote by $\mathbf{D}(\mathcal{A}_s)$ the output of the instance of Algorithm 1 with parameters γ_s, h_s that is called in iteration s of the while loop of $\mathcal{A}$ (Algorithm 2). Let us define for each $s \in \mathbb{N}$ the set

$$\begin{aligned} \mathcal{E}_1^s &:= \{h_s > h/3 \text{ and } \mathbf{D}(\mathcal{A}_s) \in \{\text{UNSURE}, \mathrm{mode}(\mathbf{p})\}\}, \\ \mathcal{E}_2^s &:= \{h_s \leq h/3 \text{ and } \mathbf{D}(\mathcal{A}_s) = \mathrm{mode}(\mathbf{p})\} \end{aligned}$$

and

$$\mathcal{E} := \bigcup\nolimits_{s\in\mathbb{N}} \left(\mathcal{E}_1^s \cup \mathcal{E}_2^s\right)^c.$$

From the equivalence $h' \le \frac{1}{3}h(\mathbf{p}) \Leftrightarrow \mathbf{p} \in \Delta_k^{3h'}$ and Lemma 4.3 we infer

$$\mathbb{P}_{\mathbf{p}}\left((\mathcal{E}_1^s \cup \mathcal{E}_2^s)^c\right) = \left.\begin{cases} \mathbb{P}_{\mathbf{p}}\left((\mathcal{E}_1^s)^c\right), & \text{if } h_s > h/3 \\ \mathbb{P}_{\mathbf{p}}\left((\mathcal{E}_2^s)^c\right), & \text{if } h_s \le h/3 \end{cases}\right\} \le \gamma_s$$

and therefore

$$\mathbb{P}_{\mathbf{p}}(\mathcal{E}) \le \sum\nolimits_{s\in\mathbb{N}} \gamma_s = \sum\nolimits_{s\in\mathbb{N}} \frac{6\gamma}{\pi^2 s^2} = \gamma. \tag{13}$$

Now, let $s_0 \coloneqq s_0(h) \in \mathbb{N}$ be such that $h_{s_0} \le h/3 < h_{s_0-1}$ and note that

$$\begin{aligned}\mathcal{E}^c &\subseteq \mathcal{E}_2^{s_0} \subseteq \{\mathbf{D}(\mathcal{A}_{s_0}) \neq \text{UNSURE}\} \\ &\subseteq \{\mathcal{A} \text{ terminates at latest after the } s_0\text{-th iteration of the while loop}\}. \end{aligned} \tag{14}$$

In particular, $\mathcal{A}$ terminates almost surely on $\mathcal{E}^c$. Regarding the construction[6] of $\mathcal{A}$ we also have

$$\begin{aligned}\mathcal{E}^c = \bigcap\nolimits_{s\in\mathbb{N}} (\mathcal{E}_1^s \cup \mathcal{E}_2^s) &\subseteq \bigcap\nolimits_{s\in\mathbb{N}} \{\mathbf{D}(\mathcal{A}_s) \in \{\text{UNSURE}, \text{mode}(\mathbf{p})\}\} \\ &\subseteq \{\mathbf{D}(\mathcal{A}) = \text{mode}(\mathbf{p})\}. \end{aligned} \tag{15}$$

Since $\mathcal{A}$ makes in its s-th iteration of the while loop (according to Algorithm 1) exactly $\lceil 8\ln(4/\gamma_s)h_s^{-2}\rceil$ queries, combining (13), (14) and (15) yields

$$\mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}) = \text{mode}(\mathbf{p}) \text{ and } T^{\mathcal{A}} \le t_0(h,\gamma)\right) \ge \mathbb{P}_{\mathbf{p}}\left(\mathcal{E}^c\right) \ge 1-\gamma,$$

with $t_0(h,\gamma) \coloneqq \sum_{s\le s_0(h)} \lceil 8\ln(4/\gamma_s)h_s^{-2}\rceil$. As the choice of $s_0 = s_0(h)$ guarantees $\frac{h}{3} < h_{s_0-1} = 2^{-s_0}$ and thus $s_0 < \log_2(3h^{-1})$, we obtain with regard to the choices of $h_s = 2^{-s-1}$ and $\gamma_s = \frac{6\gamma}{\pi^2 s^2}$ that

$$\begin{aligned} t_0(h,\gamma) \le 2^7 \sum\nolimits_{s=1}^{s_0(h)} 2^{2s-1} \ln\left(\frac{2\pi^2 s^2}{3\gamma}\right) &\in \mathcal{O}\left(\sum\nolimits_{s=1}^{s_0(h)} 2^{2s-1} \ln\left(\frac{s_0(h)}{\gamma}\right)\right) \\ &\subseteq \mathcal{O}\left(4^{s_0(h)} \ln\left(\frac{s_0(h)}{\gamma}\right)\right) \\ &\subseteq \mathcal{O}\left(4^{\log_2(3/h)} \ln\left(\log_2(3h^{-1})\gamma^{-1}\right)\right) \\ &\subseteq \mathcal{O}\left(h^{-2}\left(\ln\ln h^{-1} + \ln\gamma^{-1}\right)\right) \end{aligned}$$

as $\min\{h,\gamma\} \to 0$. It remains to show that $T^{\mathcal{A}}$ is almost surely finite w.r.t. $\mathbb{P}_{\mathbf{p}}$. For an arbitrary integer $s \ge \log_2(3/h)$ we have $h_s \le h/3$ and thus

$$\begin{aligned}\mathbb{P}_{\mathbf{p}}\left(T^{\mathcal{A}} = \infty\right) &\le \mathbb{P}_{\mathbf{p}}\left(\forall s' \in \mathbb{N} \text{ with } h_{s'} \le h/3 : \mathbf{D}(\mathcal{A}_{s'}) = \text{UNSURE}\right) \\ &\le \mathbb{P}_{\mathbf{p}}\left(\mathbf{D}(\mathcal{A}_s) = \text{UNSURE}\right) \le \mathbb{P}_{\mathbf{p}}((\mathcal{E}_2^s)^c) \le \gamma_s, \end{aligned}$$

which directly implies $\mathbb{P}_{\mathbf{p}}(T^{\mathcal{A}} = \infty) \le \lim_{s\to\infty} \gamma_s = 0$. □

E Remaining Proofs for Section 6

We prove the following more detailed version of Theorem 6.2.

Theorem E.1. *Let $\mathcal{A}$ be Algorithm 3 called with the parameters $k, m \in \mathbb{N}$ with $k \le m$ and $\gamma \in (0,1)$. Then, $\mathcal{A}$ solves $\mathcal{P}_k^{m,\gamma}(\exists\text{GCW} \wedge \Delta^0)$ and fulfills for any $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S\in[m]_k} \in PM_k^m(\exists\text{GCW}\wedge\Delta^0)$*

$$\mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}) = \text{GCW}(\mathbf{P}) \text{ and } T^{\mathcal{A}} \le t'(\mathbf{P}, m, k, \gamma)\right) \ge 1-\gamma,$$

where $t'(\mathbf{P}, m, k, \gamma)$ is given as

$$\max\left\{\sum\nolimits_{s\le s'} t_0(h(\mathbf{P}(\cdot|B_s)),\gamma') \ : \ B_1, B_2, \ldots, B_{s'} \in [m]_k \text{ s.t. } \bigcup\nolimits_{s\le s'} B_s = [m]\right\} \tag{16}$$

with $s' \coloneqq \lceil \frac{m}{k-1} \rceil$, $\gamma' \coloneqq \frac{\gamma}{s'}$ and $t_0(h,\gamma)$ defined as in Proposition 4.4, i.e., $t_0(h,\gamma) = \sum_{s\le s_0(h)} \lceil 8\ln(4/\gamma_s)h_s^{-2}\rceil$ with $s_0(h) = \lceil \log_2(3/h)\rceil - 1$.

[6] Note here that $\mathbf{D}(\mathcal{A}) \in [m]$ holds, i.e., $\mathcal{A}$ cannot terminate with UNSURE as output.

Proof. Suppose $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S\in[m]_k} \in PM_k^m(\exists\text{GCW} \wedge \Delta^0)$ to be fixed and abbreviate $i := \text{GCW}(\mathbf{P})$. Recall the internal values s, S_s and F_s of Algorithm 3. If $\mathcal{A}$ terminates, then the value of s is $s' := \lceil \frac{m}{k-1} \rceil$. Let us write $\widetilde{\mathcal{A}}_s$ for the instance of Algorithm 2, which is called with parameters m, γ' and sample access to $\mathbf{P}(\cdot|S_s)$ in Step 2 (or 9), i.e., we have $i_s = \mathbf{D}(\widetilde{\mathcal{A}}_s) \in S_s$ for each $s \le s'$. For $s \ge 2$, S_s and F_s depend on the outcome of $\widetilde{\mathcal{A}}_{s-1}$ and are thus random variables.

Claim 1: On the event $\{T^{\mathcal{A}} < \infty\}$ we have

(i) $F_{s'} = \emptyset$ and $\bigcup_{s\le s'} S_s = [m]$, i.e., $\sum_{s\le s'} t_0(\gamma', h(\mathbf{P}(\cdot|S_s))) \le t'(\mathbf{P}, m, k, \gamma)$ holds a.s.,

(ii) $\{\mathbf{D}(\mathcal{A}) \neq i\} \subseteq \bigcup_{s\le s'}\{\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|S_s))\}$.

Proof of Claim 1: Suppose $T^{\mathcal{A}} < \infty$. Clearly, $|F_s|$ is monotonically decreasing in s. Whenever $|F_s| \ge k$, then $|S_s \cap F_s| \ge k-1$ and thus $|F_{s+1}| \le |F_s| - (k-1)$ are fulfilled. Hence, $|F_s| \le m - s(k-1)$ holds for any $s \le s'-1$. In particular, we have $|F_{s'-1}| \le k-1$, which implies $F_{s'} = \emptyset$.
From $[m] = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{s'} = \emptyset$ and $\forall s \le s' : F_{s+1} = F_s \setminus S_s$ we infer $\bigcup_{s\le s'} S_s = [m]$, which proves (i). Regarding that the implications

$$j \in S_s \setminus S_{s'} \;\Rightarrow\; \exists l \in \{0,\ldots,s'-s\} : j \in S_{s+l-1} \setminus S_{s+l}$$

and

$$j \in S_s \setminus S_{s+1} \;\Rightarrow\; j \neq i_s$$

are trivially fulfilled for all $j \in [m]$ and $s \in \{0,\ldots,s'-1\}$, we obtain

$$\begin{aligned}\{i \notin S_{s'}\} &\subseteq \{\exists s < s' : i \in S_s \text{ and } i \notin S_{s+1}\}\\ &\subseteq \{\exists s < s' : i \in S_s \text{ and } i_s \neq i\}.\end{aligned}$$

Due to $\{i \in S_s \text{ and } i_s \neq i\} \subseteq \{\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|S_s))\}$, this implies

$$\begin{aligned}\{\mathbf{D}(\mathcal{A}) \neq i\} &= \{i \in S_{s'} \text{ and } i \neq i_{s'}\} \cup \{i \notin S_{s'}\}\\ &\subseteq \bigcup_{s\le s'} \{i \in S_s \text{ and } i \neq i_s\}\\ &\subseteq \bigcup_{s\le s'} \{\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|S_s))\}.\end{aligned}$$

■

Claim 2: We have the estimate

$$\mathbb{P}_{\mathbf{P}}\left(\exists s \le s' : \mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|S_s))) \text{ or } T^{\widetilde{\mathcal{A}}_s} > t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right) \le \gamma.$$

Proof of Claim 2: For $s \le s'$ let

$$E_s := \left\{\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|S_s))) \text{ or } T^{\widetilde{\mathcal{A}}_s} > t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right\}$$

denote the set, where $\mathcal{A}$ fails at round s in the sense that $\widetilde{\mathcal{A}}_s$ either makes an error in finding $\text{mode}(\mathbf{P}(\cdot|S_s))$ or queries "too many" samples for this. For $B \in [m]_k$ and $s \le s'-1$ with $\mathbb{P}_{\mathbf{P}}(\{S_s = B\} \cap \bigcap_{\tilde{s}\le s-1} E^c_{\tilde{s}}) > 0$ we have with regard to Proposition 4.4

$$\begin{aligned}&\mathbb{P}_{\mathbf{P}}\left(E_s \,\middle|\, \{S_s = B\} \cap \bigcap_{\tilde{s}\le s-1} E^c_{\tilde{s}}\right)\\ &= \mathbb{P}_{\mathbf{P}(\cdot|B)}\left(\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \text{mode}(\mathbf{P}(\cdot|B)) \text{ or } T^{\widetilde{\mathcal{A}}_s} > t_0(\gamma', h(\mathbf{P}(\cdot|B)))\right) \le \gamma',\end{aligned}$$

where we have used that both $\bigcap_{\tilde{s}\leq s-1} E^c_{\tilde{s}}$ and the choice $\{S_s = B\}$ are independent of the samples observed by $\widetilde{\mathcal{A}}_s$. We conclude

$$\begin{aligned}
&\mathbb{P}_{\mathbf{P}}\left(\bigcup\nolimits_{s\leq s'} E_s\right) = \mathbb{P}_{\mathbf{P}}\left(\bigcup\nolimits_{s\leq s'} E_s \setminus \left(\bigcup\nolimits_{\tilde{s}\leq s-1} E_{\tilde{s}}\right)\right)\\
&\leq \sum\nolimits_{s\leq s'} \sum\nolimits_{B\in[m]_k} \mathbb{P}_{\mathbf{P}}\left(E_s \cap \{S_s = B\} \cap \bigcap\nolimits_{\tilde{s}\leq s-1} E^c_{\tilde{s}}\right)\\
&= \sum\nolimits_{s\leq s'} \left[\sum\nolimits_{B} \mathbb{P}_{\mathbf{P}}\left(E_s \middle| \{S_s = B\} \cap \bigcap\nolimits_{\tilde{s}\leq s-1} E^c_{\tilde{s}}\right) \mathbb{P}_{\mathbf{P}}\left(\{S_s = B\} \cap \bigcap\nolimits_{\tilde{s}\leq s-1} E^c_{\tilde{s}}\right)\right]\\
&\leq \sum\nolimits_{s\leq s'} \gamma' \leq \gamma,
\end{aligned}$$

where we have written $\sum_B$ for the sum over all $B \in [m]_k$ with $\mathbb{P}_{\mathbf{P}}\left(\{S_s = B\} \cap \bigcap_{\tilde{s}\leq s-1} E^c_{\tilde{s}}\right) > 0$. ■

Now, let us define for $s \leq s'$ the events

$$\mathcal{R}_s := \left\{T^{\widetilde{\mathcal{A}}_s} \leq t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right\}$$

and $\mathcal{R} := \bigcap_{s\leq s'} \mathcal{R}_s$. Due to $T^{\mathcal{A}} = \sum_{s\leq s'} T^{\widetilde{\mathcal{A}}_s}$ we have

$$\mathcal{R} \subseteq \left\{T^{\mathcal{A}} \leq \sum\nolimits_{s\leq s'} t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right\} \subseteq \left\{T^{\mathcal{A}} < \infty\right\}.$$

The equality $\mathcal{R}^c = \bigcup_{s\leq s'} \mathcal{R}^c_s$ together with Part (ii) of Claim 1 and Claim 2 let us infer

$$\begin{aligned}
&\mathbb{P}_{\mathbf{P}}\left(\{\mathbf{D}(\mathcal{A}) \neq i\} \cup \mathcal{R}^c\right) = \mathbb{P}_{\mathbf{P}}\left((\{\mathbf{D}(\mathcal{A}) \neq i\} \cap \mathcal{R}) \cup \mathcal{R}^c\right)\\
&\leq \mathbb{P}_{\mathbf{P}}\left(\bigcup\nolimits_{s\leq s'} \left\{\mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \mathrm{mode}(\mathbf{P}(\cdot|S_s))\right\} \cup \mathcal{R}^c_s\right)\\
&= \mathbb{P}_{\mathbf{P}}\left(\exists s \leq s' : \mathbf{D}(\widetilde{\mathcal{A}}_s) \neq \mathrm{mode}(\mathbf{P}(\cdot|S_s)) \text{ or } T^{\widetilde{\mathcal{A}}_s} > t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right)\\
&\leq \gamma
\end{aligned}$$

and we can thus conclude with the help of Part (i) of Claim 1 that

$$\begin{aligned}
&\mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}) = i \text{ and } T^{\mathcal{A}} \leq t'(\mathbf{P}, m, k, \gamma)\right)\\
&\geq \mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}) = i \text{ and } T^{\mathcal{A}} \leq \sum\nolimits_{s\leq s'} t_0(\gamma', h(\mathbf{P}(\cdot|S_s)))\right)\\
&\geq \mathbb{P}_{\mathbf{P}}\left(\{\mathbf{D}(\mathcal{A}) = i\} \cap \mathcal{R}\right)\\
&\geq 1 - \gamma.
\end{aligned}$$

□

Proof of Theorem 6.2. According to Theorem E.1, $\mathcal{A}$ solves $\mathcal{P}^{m,\gamma}_k(\exists\mathrm{GCW} \wedge \Delta^0)$. Let $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S\in[m]_k} \in PM^m_k(\exists\mathrm{GCW} \wedge \Delta^0)$ be arbitrary. Theorem E.1 ensures that

$$\mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}) \in \mathrm{GCW}(\mathbf{P}) \text{ and } T^{\mathcal{A}} \leq t'(\mathbf{P}, m, k, \gamma)\right) \geq 1 - \gamma$$

holds with $t'(\mathbf{P}, m, k, \gamma)$ as in (16). By definition of $h(\mathbf{P})$ we have $h(\mathbf{P}(\cdot|S)) \geq h(\mathbf{P})$ for any $S \in [m]_k$, whence monotonicity of $t_0(h, \gamma)$ from Proposition 4.4 w.r.t. h shows us that $t_0(h(\mathbf{P}(\cdot|S)), \gamma) \geq t_0(h(\mathbf{P}), \gamma)$ for any $S \in [m]_k$. Thus, a look at (16) reveals that

$$t'(\mathbf{P}, m, k, \gamma) \leq T'(h(\mathbf{P}), m, k, \gamma)$$

with $T'(h, m, k, \gamma) := \left\lceil \frac{m}{k-1} \right\rceil t_0\left(h, \frac{\gamma}{\lceil m/(k-1)\rceil}\right)$, which is according to Proposition 4.4 in $\mathcal{O}\left(\frac{m}{kh^2} \ln\left(\frac{m}{k}\right)\left(\ln\ln h^{-1} + \ln\gamma^{-1}\right)\right)$. □

The following algorithm is a solution to $\mathcal{P}^{m,\gamma}_k(\exists h\mathrm{GCW} \wedge \Delta^0)$.

Theorem E.2. *Let $\mathcal{A}$ be Algorithm 5 called with parameters $m, k \in \mathbb{N}$ with $k \leq m$ and $\gamma, h \in (0, 1)$. Then, $\mathcal{A}$ solves $\mathcal{P}^{m,\gamma}_k(\exists h\mathrm{GCW} \wedge \Delta^0)$ and terminates a.s. for any $\mathbf{P} \in PM^m_k(\exists h\mathrm{GCW} \wedge \Delta^0)$ before some time $t'(m, k, h, \gamma) \in \mathcal{O}\left(\frac{m}{kh^2} \ln\left(\frac{m}{k\gamma}\right)\right)$.*

Algorithm 5 Solution to $\mathcal{P}_k^{m,\gamma}(\exists h\text{GCW} \wedge \Delta^0)$

Input: $k, m \in \mathbb{N}$, $\gamma \in (0,1)$, $h \in (0,1)$, sample access to $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S \in [m]_k}$,
Initialization: $\widetilde{\mathcal{A}} :=$ Alg. 1, $i_0 \leftarrow$ UNSURE, $h' \leftarrow \frac{h}{3}$, $\gamma' \leftarrow \frac{\gamma}{\lceil m/(k-1) \rceil}$, let $S_1 \in [m]_k$ arbitrary, $F_1 \leftarrow [m]$, $s \leftarrow 1$

$\triangleright$ S_s : candidate set in round s, F_s : remaining elements in round s
$\triangleright$ $i_s \in S_s \cup \{\text{UNSURE}\}$: output of $\widetilde{\mathcal{A}}$ in round s

while $|F_s| > 0$ **do**
  $i_s \leftarrow \widetilde{\mathcal{A}}(h', \gamma'$, sample access to $\mathbf{P}(\cdot|S_s))$
  $F_{s+1} \leftarrow F_s \setminus S_s$
  Write $F_{s+1} = \{j_1, \ldots, j_{|F_{s+1}|}\}$.
  if $|F_{s+1}| < k$ **then**
    Fix distinct $j_{|F_{s+1}|+1}, \ldots, j_k \in [m] \setminus (F_{s+1} \cup \{i_s\})$.
  if $i_s \in [m]$ **then** $S_{s+1} \leftarrow \{i_s, j_1, \ldots, j_{k-1}\}$
  else $S_{s+1} \leftarrow \{j_1, \ldots, j_k\}$
  $s \leftarrow s + 1$
$i_s \leftarrow \widetilde{\mathcal{A}}(h', \gamma'$, sample access to $\mathbf{P}(\cdot|S_s))$
if $i_s \in [m]$ **then return** i_s
else return 1

Proof of Theorem E.2. Let us define the random variable $s^{\mathcal{A}} := \min\{s \in \mathbb{N} \,|\, F_s = \emptyset\} \in \mathbb{N} \cup \{\infty\}$ and suppose $\mathbf{P} \in PM_k^m$ to be arbitrary but fixed for the moment.
Claim 1: We have $s^{\mathcal{A}} \leq s' := \lceil \frac{m}{k-1} \rceil$ a.s. w.r.t. $\mathbb{P}_{\mathbf{P}}$.
Proof of Claim 1: Assume on the contrary that $s^{\mathcal{A}} > s'$. Note that $|F_s|$ is monotonically decreasing in s. Whenever $|F_s| \geq k$, then $|S_s \cap F_s| \geq k-1$ and thus $|F_{s+1}| \leq |F_s| - (k-1)$ are fulfilled. Hence, $|F_s| \leq m - s(k-1)$ holds for any $s \leq s'-1$. In particular, we have $|F_{s'-1}| \leq k-1$, which implies $F_{s'} = \emptyset$, contradicting the assumption $s^{\mathcal{A}} > s'$. This proves that $s^{\mathcal{A}} \leq s'$ is fulfilled a.s. ■

Using that $\mathcal{A}$ makes exactly $s^{\mathcal{A}}$ calls of $\tilde{\mathcal{A}}$ (i.e., Algorithm 1) with parameters h', γ' and each such call is executed with a sample complexity of exactly $\lceil 8\ln(4/\gamma')/{h'}^2 \rceil$, the total sample complexity of $\mathcal{A}$ is at most

$$s' \lceil 8\ln(4/\gamma')/{h'}^2 \rceil = \left\lceil \frac{m}{k-1} \right\rceil \left\lceil \frac{72}{h^2} \ln\left(\frac{4\lceil m/(k-1) \rceil}{\gamma}\right) \right\rceil,$$

which is in $\mathcal{O}\left(\frac{m}{kh^2} \ln\left(\frac{m}{k\gamma}\right)\right)$ as $\max\{m, k, h^{-1}, \gamma^{-1}\} \to \infty$. It remains to prove correctness of $\mathcal{A}$. Write $\mathcal{A}'$ for Algorithm 6 called with the same parameters as $\mathcal{A}$.
Claim 2: For any $\mathbf{P} \in PM_k^m$, we have

$$\mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}) \neq \text{GCW}(\mathbf{P})\right) = \mathbb{P}_{\mathbf{P}}\left(\mathbf{D}(\mathcal{A}') \neq \text{GCW}(\mathbf{P})\right).$$

Proof of Claim 2: This follows directly from the fact that for any $S \in [m]_k$, different calls of $\tilde{\mathcal{A}}$ on $\mathbf{P}(\cdot|S)$ are by assumption executed on different samples of $\mathbf{P}(\cdot|S)$ and thus independent of each other. ■

This result shows that it is sufficient to prove correctness of $\mathcal{A}'$. In the following, we denote by s, i_s F_s and S_s the internal statistics of $\mathcal{A}'$ and write $\widetilde{\mathcal{A}}_s$ for that instance of $\widetilde{\mathcal{A}}$, which is executed in $\mathcal{A}'$ to determine i_s. Let $\mathbf{P} \in PM_k^m(\exists h\text{GCW} \wedge \Delta^0)$ be fixed and define $i := \text{GCW}(\mathbf{P})$.

Claim 3: For all $s \leq s'$ we have

$$\mathbb{P}_{\mathbf{P}}\left(i \in S_s \text{ and } i_s \neq i\right) \leq \gamma'.$$

Proof of Claim 3: Suppose $B \in [m]_k$ with $i \in [m]$ and $\mathbb{P}_{\mathbf{P}}(S_s = B) > 0$ to be arbitrary but fixed for the moment. By assumption on $\mathbf{P}$ we have $\mathbf{P}(\cdot|B) \in \Delta_k^{3h'}$ and since $\widetilde{\mathcal{A}}_s$ is Algorithm 1 executed with parameters h', γ' and sample access to $\mathbf{P}(\cdot|S_s)$ only, Lemma 4.3 assures

$$\begin{aligned} &\mathbb{P}_{\mathbf{P}}\left(i \in S_s \text{ and } i_s \neq i | S_s = B\right) \\ &= \mathbb{P}_{\mathbf{P}(\cdot|B)}(\text{Alg. 1 started with } h', \gamma' \text{ does not output } \text{mode}(\mathbf{P}(\cdot|B))) \leq \gamma'. \end{aligned}$$

Claim 3 thus follows via summation over all such B. ■

On the event $\{T^{\mathcal{A}'} < \infty\}$, we infer from $[m] = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{s^{\mathcal{A}}} = \cdots = F_{s'} = \emptyset$ and $\forall s \leq s' : F_{s+1} = F_s \setminus S_s$ similarly as in the proof of Theorem E.1

$$\{\mathbf{D}(\mathcal{A}') \neq i\} \subseteq \bigcup\nolimits_{s \leq s'} \{i \in S_s \text{ and } i_s \neq i\}.$$

As $T^{\mathcal{A}'} < \infty$ holds a.s. w.r.t. $\mathbb{P}_{\mathbf{P}}$, combining this with Claim 3 directly yields

$$\mathbb{P}_{\mathbf{P}}(\mathbf{D}(\mathcal{A}') \neq i) \leq \sum\nolimits_{s \leq s'} \gamma' = \gamma,$$

which completes the proof. □

Algorithm 6 Modification of Algorithm 5 for the proof of Theorem E.2

Input: $k, m \in \mathbb{N}$, $\gamma \in (0,1)$, $h \in (0,1)$, sample access to $\mathbf{P} = \{\mathbf{P}(\cdot|S)\}_{S \in [m]_k}$,
Initialization: $\widetilde{\mathcal{A}} :=$ Algorithm 1, $i_0 \leftarrow$ UNSURE, $h' \leftarrow \frac{h}{3}$, $\gamma' \leftarrow \frac{\gamma}{\lceil m/(k-1) \rceil}$
$S_1 \leftarrow [k]$, $F_1 \leftarrow [m]$, $s \leftarrow 1$

```
Execute steps 1–8 of Algorithm 5.
let s' ← ⌈m/(k−1)⌉
while s < s' do
    i_s ← Ã(h', γ', sample access to P(·|S_s))
    F_{s+1} ← F_s, S_{s+1} ← S_s
    s ← s + 1
i_s ← Ã(h', γ', sample access to P(·|S_s))
return i_s
```

F Proof of Theorem 5.2

Before proving Theorem 5.2, we require some preparation. For $S \in [m]_k$ and $\mathbf{p}, \mathbf{q} \in \Delta_S$ let us write $\mathrm{KL}(\mathbf{p}, \mathbf{q})$ for the *Kullback-Leibler divergence* of random variables $X \sim \mathrm{Cat}(\mathbf{p})$ and $Y \sim \mathrm{Cat}(\mathbf{q})$, i.e.,

$$\mathrm{KL}(\mathbf{p}, \mathbf{q}) = \begin{cases} \sum_{x \in S: p_x > 0} p_x \ln\left(\frac{p_x}{q_x}\right), & \text{if } \forall y \in S : q_y = 0 \Rightarrow p_y = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

For the sake of convenience, we write in the binary case $k = 2$ simply $\mathrm{kl}(x, y) := \mathrm{KL}((x, 1-x), (y, 1-y))$ for any $x, y \in [0,1]$.

Lemma F.1. *(i) For any $S \in [m]_k$ and $\mathbf{p}, \mathbf{q} \in \Delta_S$ we have*

$$\mathrm{KL}(\mathbf{p}, \mathbf{q}) \leq \sum\nolimits_{x \in S} \frac{(p_x - q_x)^2}{q_x}.$$

(ii) The inequality $\mathrm{kl}(\gamma, 1-\gamma) \geq \ln((2.4\gamma)^{-1})$ holds for any $\gamma \in (0,1)$.

Proof. The statement from (i) is Lemma 3 in [11] and for (ii) cf. Equation (3) in [21]. □

Given an algorithm $\mathcal{A}$, which tackles the problem $\mathcal{P}_k^{m,\gamma}(\Delta^h)$, let us write $S_t^{\mathcal{A}}$ for the query (element of $[m]_k$) made at time step t. Moreover, define $T_S^{\mathcal{A}}$ to be the number of times $\mathcal{A}$ makes the query $S \in [m]_k$ before termination, i.e., $T_S^{\mathcal{A}} = \sum_{t=1}^{T^{\mathcal{A}}} \mathbf{1}_{\{S_s^{\mathcal{A}} = S\}}$ and $T^{\mathcal{A}} = \sum_{S \in [m]_k} T_S^{\mathcal{A}}$ are fulfilled. Let $i_t^{\mathcal{A}} \in S_t^{\mathcal{A}}$ be the feedback observed by $\mathcal{A}$ at time step t, after having queried $S_t^{\mathcal{A}}$, and write $\mathcal{F}_t^{\mathcal{A}} := \sigma(S_1^{\mathcal{A}}, i_t^{\mathcal{A}}, \ldots, S_t^{\mathcal{A}}, i_t^{\mathcal{A}})$ for the sigma algebra generated by the behaviour and observed feedback of $\mathcal{A}$ until time t, and as usual $\mathcal{F}_{T^{\mathcal{A}}} := \mathcal{F}_{T^{\mathcal{A}}}^{\mathcal{A}} = \sigma\left(\bigcup_{t \leq T^{\mathcal{A}}} \mathcal{F}_t^{\mathcal{A}}\right)$.

Since $\mathcal{A}$ may be thought of as a multi-armed bandit with $\binom{m}{k}$ arms (one for each $S \in [m]_k$) and "rewards" $i_t^{\mathcal{A}} \in S_t^{\mathcal{A}}$, we may translate Lemma 1 from [21] to our setting in the following way:

Lemma F.2. *Let* $\mathbf{P}, \mathbf{P}' \in PM_k^m(\Delta^h \wedge \exists \mathrm{GCW})$ *with*[7] $\mathbf{P}(j|S), \mathbf{P}'(j|S) > 0$ *for any* $S \in [m]_k$ *and* $j \in S$. *If an algorithm* $\mathcal{A}$ *tackles* $\mathcal{P}_k^{m,\gamma}(\Delta^h \wedge \exists \mathrm{GCW})$ *and fulfills* $\mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}], \mathbb{E}_{\mathbf{P}'}[T^{\mathcal{A}}] < \infty$*, then*

$$\sum\nolimits_{S \in [m]_k} \mathbb{E}_{\mathbf{P}}\left[T_S^{\mathcal{A}}\right] \mathrm{KL}(\mathbf{P}(\cdot|S), \mathbf{P}'(\cdot|S)) \geq \sup\nolimits_{\mathcal{E} \in \mathcal{F}_{T^{\mathcal{A}}}} \mathrm{kl}\left(\mathbb{P}_{\mathbf{P}}(\mathcal{E}), \mathbb{P}_{\mathbf{P}'}(\mathcal{E})\right)$$

We are now ready to prove Theorem 5.2. The proof idea is similar to the one followed in the proof of Theorem 7 in [35].

Proof of Theorem 5.2. We prove the instance-wise and asymptotic lower bound separately.

Part 1: Proof of the instance-wise bound

After relabeling the items in $[m]$, we may suppose w.l.o.g. $\mathrm{GCW}(\mathbf{P}) = 1$ throughout the proof. Write for convenience $\mathbf{P}^{[1]} := \mathbf{P}$, recall that $m_S = \mathrm{mode}(\mathbf{P}^{[1]}(\cdot|S))$ for any $S \in [m]_k$ and define $\mathbf{P}^{[l]} \in PM_k^m(\Delta^h)$ for each $l \in \{2, \ldots, m\}$ via

$$\begin{aligned} &\mathbf{P}^{[l]}(l|S) := \mathbf{P}^{[1]}(m_S|S), \quad \mathbf{P}^{[l]}(m_S|S) := \mathbf{P}^{[1]}(l|S), \\ &\mathbf{P}^{[l]}(j|S) := \mathbf{P}^{[1]}(j|S) \text{ for all } j \in S \setminus \{l, m_S\} \end{aligned} \tag{17}$$

for any $S \in [m]_k$ with $l \in S$ and

$$\mathbf{P}^{[l]}(j|S) := \mathbf{P}^{[1]}(j|S) \text{ for all } j \in S$$

for any $S \in [m]_k$ with $l \notin S$. Abbreviating $\mathbf{P}_S^{[r]} := \mathbf{P}^{[r]}(\cdot|S)$ we directly obtain $\mathrm{KL}\left(\mathbf{P}_S^{[1]}, \mathbf{P}_S^{[l]}\right) = 0$ whenever $S \notin [m]_k^{(l)} := \{S \in [m]_k \,|\, l \in S \text{ and } l \neq m_S\}$. Define

$$\Sigma(l) := \sum\nolimits_{S \in [m]_k^{(l)}} \mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right]$$

for each $l \in \{2, \ldots, m\}$. Now, suppose l to be fixed for the moment and note that $\mathrm{GCW}(\mathbf{P}^{[l]}) = l$ holds by construction of $\mathbf{P}^{[l]}$. As $\mathcal{A}$ solves $\mathcal{P}_k^{m,\gamma}(\Delta^h)$, the event $\mathcal{E} := \{\mathbf{D}(\mathcal{A}) = 1\} \in \mathcal{F}_{T^{\mathcal{A}}}$ fulfills $\mathbb{P}_{\mathbf{P}^{[1]}}(\mathcal{E}) \geq 1 - \gamma$ and $\mathbb{P}_{\mathbf{P}^{[l]}}(\mathcal{E}) \leq \gamma$. Consequently, by applying part (ii) of Lemma F.1 and Lemma F.2, we obtain

$$\begin{aligned} \ln\left((2.4\gamma)^{-1}\right) &\leq \mathrm{kl}\left(\mathbb{P}_{\mathbf{P}^{[1]}}(\mathcal{E}), \mathbb{P}_{\mathbf{P}^{[l]}}(\mathcal{E})\right) \\ &\leq \sum\nolimits_{S \in [m]_k} \mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right] \mathrm{KL}\left(\mathbf{P}_S^{[1]}, \mathbf{P}_S^{[l]}\right) \\ &= \sum\nolimits_{S \in [m]_k^{(l)}} \mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right] \mathrm{KL}\left(\mathbf{P}_S^{[1]}, \mathbf{P}_S^{[l]}\right), \end{aligned}$$

that is,

$$\Sigma(l) \geq \ln\left((2.4\gamma)^{-1}\right) \min\nolimits_{S \in [m]_k^{(l)}} \frac{1}{\mathrm{KL}\left(\mathbf{P}_S^{[1]}, \mathbf{P}_S^{[l]}\right)}. \tag{18}$$

For any $S = \{i_1, \ldots, i_k\} \in [m]_k$ with $i_1 := m_S$ the term $\mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right]$ appears exactly $k-1$ times as summand in

$$\Sigma(2) + \cdots + \Sigma(m) = \sum\nolimits_{l=2}^{m} \sum\nolimits_{S \in [m]_k : m_S \neq l \in S} \mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right],$$

namely as one summand in $\Sigma(i_2), \ldots, \Sigma(i_k)$ each. Hence, (18) lets us infer

$$\begin{aligned} (k-1)\mathbb{E}_{\mathbf{P}^{[1]}}\left[T^{\mathcal{A}}\right] &= \sum\nolimits_{S \in [m]_k} (k-1)\mathbb{E}_{\mathbf{P}^{[1]}}\left[T_S^{\mathcal{A}}\right] \\ &\geq \Sigma(2) + \cdots + \Sigma(m) \\ &\geq \ln\left((2.4\gamma)^{-1}\right) \sum\nolimits_{l=2}^{m} \min\nolimits_{S \in [m]_k^{(l)}} \frac{1}{\mathrm{KL}\left(\mathbf{P}_S^{[1]}, \mathbf{P}_S^{[l]}\right)}. \end{aligned}$$

[7] We put these conditions on $\mathbf{P}$ and $\mathbf{P}'$ in order to guarantee mutually absolutely continuity of the "rewards" $i_S \sim \mathrm{Cat}(\mathbf{P}(\cdot|S))$ resp. $i'_S \sim \mathrm{Cat}(\mathbf{P}'(\cdot|S))$, $S \in [m]_k$, which is formally required in Lemma 1 in [21].

This completes our proof of the instance-wise bound. ■

Part 2: Proof of the worst-case bound
Since the statement is trivial for $h = 1$, we may assume w.l.o.g. $h \in (0,1)$ in the following. Let us abbreviate $\Delta_{[m]_k} := \{\mathbf{w} = (w_S)_{S\in[m]_k} \in [0,1]^{[m]_k} \mid \sum_{S\in[m]_k} w_S = 1\}$. For $S \in [m]_k$, write $S = \{S_{(1)}, \dots, S_{(k)}\}$ with $S_{(1)} < \dots < S_{(k)}$. Suppose $\varepsilon \in (0, 1/2)$ to be arbitrary but fixed for the moment and define $\mathbf{P}^{[1,\varepsilon]} \in PM_k^m(\exists\text{GCW} \wedge \Delta^h)$ via

$$\mathbf{P}^{[1,\varepsilon]}(S_{(1)}|S) := \frac{1+h+2\varepsilon}{2}, \quad \mathbf{P}^{[1]}(S_{(2)}|S) := \frac{1-h}{2}$$

and

$$\forall j \in \{3,\dots,k\} : \mathbf{P}^{[1]}(S_{(j)}|S) := \frac{\varepsilon}{k-2}.$$

for any $S \in [m]_k$. For $l \in \{2,\dots,m\}$ let $\mathbf{P}^{[l,\varepsilon]}$ be as $\mathbf{P}^{[1,\varepsilon]}$ with $[m]$ being relabeled via the l-shift $\nu_l : [m] \to [m]$ given by

$$1 \mapsto l, \quad 2 \mapsto l+1, \quad \dots \quad m-l-1 \mapsto m, \quad m-l \mapsto 1, \quad \dots \quad m \mapsto l-1,$$

i.e., $\mathbf{P}^{[l,\varepsilon]}(\nu_l(i_r)|\{\nu_l(i_1),\dots,\nu_l(i_k)\}) = \mathbf{P}^{[1,\varepsilon]}(i_r|\{i_1,\dots,i_k\})$ for any $\{i_1,\dots,i_k\} \in [m]_k$ and $r \in [k]$. Then, $\mathbf{P}^{[l]} \in PM_k^m(\exists\text{GCW} \wedge \Delta^h)$ and $\text{GCW}(\mathbf{P}^{[l]}) = l$ hold for any $l \in [m]$. Write

$$\mathfrak{P}^*(\varepsilon) := \{\mathbf{P}^{[1,\varepsilon]}, \mathbf{P}^{[2,\varepsilon]}, \dots \mathbf{P}^{[m,\varepsilon]}\}$$

and define

$$\mathfrak{P}_*(\neg l) := \left\{\mathbf{P} \in PM_k^m(\exists\text{GCW} \wedge \Delta^h) \mid \text{GCW}(\mathbf{P}) \neq l \text{ and } \forall S \in [m]_k : \min_{j\in S} \mathbf{P}(j|S) > 0\right\}.$$

For any $\mathbf{P}, \mathbf{P}' \in PM_k^m(\exists\text{GCW} \wedge \Delta^h)$ fulfilling $\min_{S\in[m]_k} \min_{j\in S} \mathbf{P}(j|S) > 0$ as well as $\min_{S\in[m]_k} \min_{j\in S} \mathbf{P}'(j|S) > 0$ and $\text{GCW}(\mathbf{P}) \neq \text{GCW}(\mathbf{P}')$ Lemma F.2 guarantees similarly as above

$$\ln((2.4\gamma)^{-1}) \le \sum_{S\in[m]_k} \mathbb{E}_{\mathbf{P}}\left[T_S^{\mathcal{A}}\right] \text{KL}(\mathbf{P}_S, \mathbf{P}'_S),$$

where we have written $\mathbf{P}_S$ resp. $\mathbf{P}'_S$ for $\mathbf{P}(\cdot|S)$ resp. $\mathbf{P}'(\cdot|S)$. Regarding arbitrariness of $\mathbf{P}$ and $\mathbf{P}'$ therein and using that $\mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] > 0$ and $\left(\mathbb{E}_{\mathbf{P}}[T_S^{\mathcal{A}}]/\mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}]\right)_{S\in[m]_k} \in \Delta_{[m]_k}$ hold trivially for any $\mathbf{P} \in PM_k^m$, we may follow an idea from [18] (cf. the proof of Theorem 1 therein) and estimate

$$\begin{aligned}
\ln((2.4\gamma)^{-1}) &\le \min_{\mathbf{P}\in\mathfrak{P}^*(\varepsilon)} \inf_{\mathbf{P}'\in\mathfrak{P}_*(\neg\text{GCW}(\mathbf{P}))} \sum_{S\in[m]_k} \mathbb{E}_{\mathbf{P}}\left[T_S^{\mathcal{A}}\right] \text{KL}(\mathbf{P}_S, \mathbf{P}'_S) \\
&\le \min_{\mathbf{P}\in\mathfrak{P}^*(\varepsilon)} \mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] \inf_{\mathbf{P}'\in\mathfrak{P}_*(\neg\text{GCW}(\mathbf{P}))} \sum_{S\in[m]_k} \frac{\mathbb{E}_{\mathbf{P}}\left[T_S^{\mathcal{A}}\right]}{\mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}]} \text{KL}(\mathbf{P}_S, \mathbf{P}'_S) \\
&\le \sup_{\mathbf{w}\in\Delta_{[m]_k}} \min_{\mathbf{P}\in\mathfrak{P}^*(\varepsilon)} \mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] \inf_{\mathbf{P}'\in\mathfrak{P}_*(\neg\text{GCW}(\mathbf{P}))} \sum_{S\in[m]_k} w_S \text{KL}(\mathbf{P}_S, \mathbf{P}'_S). \qquad (19)
\end{aligned}$$

Suppose $\mathbf{w} \in \Delta_{[m]_k}$ to be arbitrary but fixed for the moment. The identity

$$k = k\sum_{S\in[m]_k} w_S = \sum_{l\in[m]} \sum_{S\in[m]_k : l\in S} w_S$$

assures the existence of some $l = l(\mathbf{w}) \in [m]$ with $\sum_{S\in[m]_k : l\in S} w_S \le \frac{k}{m}$. Abbreviate $\mathbf{P} := \mathbf{P}^{[l,\varepsilon]}$. After relabeling $[m]$ via ν_l^{-1}, we may assume w.l.o.g. $l = 1$ in the following, i.e. $\mathbf{P} = \mathbf{P}^{[1,\varepsilon]} \in \mathfrak{P}^*(\varepsilon)$. Define $\mathbf{P}' \in PM_k^m$ via

$$\mathbf{P}'(2|S) := \frac{1+h+2\varepsilon}{2}, \quad \mathbf{P}'(\min S \setminus \{2\} \mid S) := \frac{1-h}{2} \quad \text{and} \quad \mathbf{P}'(j|S) := \frac{\varepsilon}{k-2}$$

for any $j \in S \setminus \{2, \min(S \setminus \{2\})\}$, if $2 \in S$, and

$$\mathbf{P}'(j|S) := \mathbf{P}(j|S)$$

for any $j \in S$, if $2 \notin S$. From $\mathbf{P}' \in PM_k^m(\exists \mathrm{GCW} \wedge \Delta^h)$ and $\mathrm{GCW}(\mathbf{P}') = 2 \neq 1 = \mathrm{GCW}(\mathbf{P})$ we infer $\mathbf{P}' \in \mathfrak{P}_*(\neg\mathrm{GCW}(\mathbf{P}))$. In case $\{1,2\} \not\subseteq S$, we have $\mathbf{P}(j|S) = \mathbf{P}'(j|S)$ for any $j \in S$ and thus $\mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) = 0$. In the remaining case $\{1,2\} \subseteq S$ Lemma F.1 allows us to estimate

$$\begin{aligned}
&\mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) \\
&= \mathrm{KL}\left(\left(\frac{1+h+2\varepsilon}{2}, \frac{1-h}{2}, \frac{\varepsilon}{k-2} \dots, \frac{\varepsilon}{k-2}\right), \left(\frac{1-h}{2}, \frac{1+h+2\varepsilon}{2}, \frac{\varepsilon}{k-2} \dots, \frac{\varepsilon}{k-2}\right)\right) \\
&\leq (h+\varepsilon)^2 \left(\frac{2}{1-h} + \frac{2}{1+h+\varepsilon}\right) = \frac{(4+2\varepsilon)(h+\varepsilon)^2}{(1-h)(1+h+\varepsilon)}.
\end{aligned}$$

Regarding the choice of $l = 1$ we infer

$$\begin{aligned}
\sum\nolimits_{S \in [m]_k} w_S \mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) &= \sum\nolimits_{S \in [m]_k : \{1,2\} \subseteq S} w_S \mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) \\
&\leq \frac{(4+2\varepsilon)(h+\varepsilon)^2}{(1-h)(1+h+\varepsilon)} \sum\nolimits_{S \in [m]_k : 1 \in S} w_S \leq \frac{k(4+2\varepsilon)(h+\varepsilon)^2}{m(1-h)(1+h+\varepsilon)}
\end{aligned}$$

and thus clearly

$$\mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] \sum\nolimits_{S \in [m]_k} w_S \mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) \leq \frac{k(4+2\varepsilon)(h+\varepsilon)^2}{m(1-h)(1+h+\varepsilon)} \mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}].$$

Since $\mathbf{w}$ was arbitrary and $\mathbf{P} = \mathbf{P}^{[l(\mathbf{w}),\varepsilon]}$, combining this with (19) yields

$$\begin{aligned}
&\ln((2.4\gamma)^{-1}) \\
&\leq \sup\nolimits_{\mathbf{w} \in \Delta_{[m]_k}} \min\nolimits_{\mathbf{P} \in \mathfrak{P}^*(\varepsilon)} \mathbb{E}_{\mathbf{P}}[T^{\mathcal{A}}] \inf\nolimits_{\mathbf{P}' \in \mathfrak{P}_*(\neg\mathrm{GCW}(\mathbf{P}))} \sum\nolimits_{S \in [m]_k} w_S \mathrm{KL}(\mathbf{P}_S, \mathbf{P}'_S) \\
&\leq \sup\nolimits_{\mathbf{w} \in \Delta_{[m]_k}} \mathbb{E}_{\mathbf{P}^{[l(\mathbf{w}),\varepsilon]}}[T^{\mathcal{A}}] \inf\nolimits_{\mathbf{P}' \in \mathfrak{P}_*(\neg\mathrm{GCW}(\mathbf{P}^{[l(\mathbf{w}),\varepsilon]}))} \sum\nolimits_{S \in [m]_k} w_S \mathrm{KL}\left(\mathbf{P}_S^{[l(\mathbf{w}),\varepsilon]}, \mathbf{P}'_S\right) \\
&\leq \frac{k(4+2\varepsilon)(h+\varepsilon)^2}{m(1-h)(1+h+\varepsilon)} \sup\nolimits_{\mathbf{w} \in \Delta_{[m]_k}} \mathbb{E}_{\mathbf{P}^{[l(\mathbf{w}),\varepsilon]}}[T^{\mathcal{A}}] \\
&\leq \frac{k(4+2\varepsilon)(h+\varepsilon)^2}{m(1-h)(1+h+\varepsilon)} \max\nolimits_{l \in [m]} \mathbb{E}_{\mathbf{P}^{[l,\varepsilon]}}[T^{\mathcal{A}}].
\end{aligned}$$

As $\varepsilon \in (0, 1/2)$ was arbitrary, we finally conclude

$$\begin{aligned}
\sup\nolimits_{\mathbf{P} \in PM_k^m(\exists\mathrm{GCW} \wedge \Delta^h)} \mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] &\geq \sup\nolimits_{\varepsilon \in (0,1/2)} \max\nolimits_{l \in [m]} \mathbb{E}_{\mathbf{P}^{[l,\varepsilon]}}\left[T^{\mathcal{A}}\right] \\
&\geq \sup\nolimits_{\varepsilon \in (0,1/2)} \frac{m(1-h)(1+h+\varepsilon)}{k(4+2\varepsilon)(h+\varepsilon)^2} \ln((2.4\gamma)^{-1}) \\
&\geq \frac{m(1-h^2)}{4kh^2} \ln((2.4\gamma)^{-1}).
\end{aligned}$$

□

Remark F.3. *The instance-wise bound in Theorem 5.2 appears to be maximal on an instance* $\mathbf{P} \in PM_k^m$ *defined via*

$$\mathbf{P}(m_S|S) := \frac{1-h+hk}{k} \quad \textit{and} \quad \mathbf{P}(j|S) := \frac{1-h}{k} \textit{ for each } j \in S \setminus \{m_S\}.$$

with $m_S := \min S$ *for each* $S \in [m]_k$*. Note that* $\mathbf{P}(m_S|S) = \mathbf{P}(j|S) + h$ *is fulfilled for each* $S \in [m]_k$*,* $j \in S \setminus \{m_S\}$*. Regarding the definition of* m_S *we thus have* $\mathbf{P} \in PM_k^m(\Delta^h)$ *with* $\mathrm{GCW}(\mathbf{P}) = 1$*. With* $\mathbf{P}^{[l]}(\cdot|S)$ *defined as in Theorem 5.2 we can estimate for each* $l \in \{2, \dots, m\}$

and $S \in [m]_k$ with $l \in S \setminus \{m_S\}$ via Lemma F.1

$$\begin{aligned}
\mathrm{KL}\left(\mathbf{P}(\cdot|S), \mathbf{P}^{[l]}(\cdot|S)\right) &\leq \sum_{j \in S} \frac{(\mathbf{P}(j|S) - \mathbf{P}^{[l]}(j|S))^2}{\mathbf{P}(j|S)} \\
&= \frac{(\mathbf{P}(m_S|S) - \mathbf{P}^{[l]}(m_S|S))^2}{\mathbf{P}^{[l]}(m_S|S)} + \frac{(\mathbf{P}(l|S) - \mathbf{P}^{[l]}(l|S))^2}{\mathbf{P}^{[l]}(l|S)} \\
&= \frac{(\mathbf{P}(m_S|S) - \mathbf{P}(l|S))^2}{\mathbf{P}(l|S)} + \frac{(\mathbf{P}(l|S) - \mathbf{P}(m_S|S))^2}{\mathbf{P}(m_S|S)} \\
&= \frac{(\mathbf{P}(m_S|S) - \mathbf{P}(l|S))^2(\mathbf{P}(m_S|S) + \mathbf{P}(l|S))}{\mathbf{P}(m_S|S)\mathbf{P}(l|S)} \\
&= \frac{h^2 k(1-h+hk+1-h)}{(1-h+hk)(1-h)} \leq \frac{2kh^2}{1-h},
\end{aligned}$$

where we have used $hk \geq 0$ in the last step. Consequently, the instance-wise bound from Theorem 5.2 yields

$$\mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] \geq \frac{(m-1)(1-h)\ln((2.4\gamma)^{-1})}{2h^2 k(k-1)} \in \Omega\left(\frac{m}{k^2 h^2} \ln \frac{1}{\gamma}\right),$$

which is by a factor $1/k$ asymptotically smaller than the worst-case bound stated in Theorem 5.2.

In the case of dueling bandits ($k = 2$), the instance-dependent bound from Theorem 5.2 reduces to

$$\begin{aligned}
\mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] &\geq \ln((2.4\gamma)^{-1}) \sum_{l \in [m]\setminus\{i\}} \frac{1}{\mathrm{KL}(\mathbf{P}(\cdot|\{i,l\}), \mathbf{P}^{[l]}(\cdot|\{i,l\}))} \\
&= \ln((2.4\gamma)^{-1}) \sum_{l \in [m]\setminus\{i\}} \frac{1}{\mathrm{kl}(\mathbf{P}(i|\{i,l\}), \mathbf{P}(l|\{i,l\}))}
\end{aligned}$$

for any $\mathbf{P} \in PM_2^m(\exists\mathrm{GCW} \wedge \Delta^h)$ with $\mathrm{GCW}(\mathbf{P}) = i$ and any solution $\mathcal{A}$ to $\mathcal{P}_2^{m,\gamma}(\exists\mathrm{GCW} \wedge \Delta^h)$. By means of this, we obtain the following worst-case sample lower bound, which is by a factor $\frac{2(m-1)}{m}$ larger than the one stated in Theorem 5.2.

Corollary F.4. *If $\mathcal{A}$ solves $\mathcal{P}_2^{m,\gamma}(\exists\mathrm{GCW} \wedge \Delta^h)$, then*

$$\sup_{\mathbf{P} \in PM_2^m(\exists\mathrm{GCW} \wedge \Delta^h)} \mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] \geq \frac{(m-1)(1-h^2)}{4h^2} \ln((2.4\gamma)^{-1}).$$

Proof. Define $\mathbf{P} \in PM_2^m(\exists\mathrm{GCW} \wedge \Delta^h)$ via $\mathbf{P}(i|\{i,j\}) := \frac{1+h}{2}$ for any $1 \leq i < j \leq m$. Theorem 5.2 and Lemma F.1 allow us to infer

$$\begin{aligned}
\mathbb{E}_{\mathbf{P}}\left[T^{\mathcal{A}}\right] &\geq \frac{(m-1)\ln((2.4\gamma)^{-1})}{\mathrm{kl}((1+h)/2, (1-h)/2)} \\
&\geq (m-1)\ln((2.4\gamma)^{-1})\left(\frac{2h^2}{1-h} + \frac{2h^2}{1+h}\right)^{-1} \\
&= \frac{(m-1)(1-h^2)\ln((2.4\gamma)^{-1})}{4h^2}.
\end{aligned}$$

□

Remark F.5. *Suppose $\mathcal{A}$ solves $\mathcal{P}_k^{k,\gamma}(\Delta^h)$, let $\mathbf{p} \in \Delta_k^h$ and write $i := \mathrm{mode}(\mathbf{p})$. According to Prop. C.1 we have*

$$\mathbb{E}_{\mathbf{p}}\left[T^{\mathcal{A}}\right] \geq \max_{l \in [m]\setminus\{i\}} \frac{1-2\gamma}{2\phi_{l,i}(\mathbf{p})(p_l + p_i)} \left\lceil \frac{\ln((1-\gamma)/\gamma)}{\ln\left((1/2+\phi_{l,i}(\mathbf{p}))/(1/2-\phi_{l,i}(\mathbf{p}))\right)} \right\rceil =: \mathrm{LB}_1(\mathbf{p}, \gamma)$$

with $\phi_{l,i}(\mathbf{p}) := \frac{p_i - p_l}{2(p_l + p_i)}$, and Thm. 5.2 guarantees

$$\mathbb{E}_{\mathbf{p}}\left[T^{\mathcal{A}}\right] \geq \frac{\ln((2.4\gamma)^{-1})}{k-1} \sum_{l \in [k]\setminus\{i\}} \left(p_l \ln\left(\frac{p_l}{p_i}\right) + p_i \ln\left(\frac{p_i}{p_l}\right)\right)^{-1} =: \mathrm{LB}_2(\mathbf{p}, \gamma).$$

In an empirical study we observed $\mathrm{LB}_1(\mathbf{p}, \gamma) > \mathrm{LB}_2(\mathbf{p}, \gamma)$ for all of 1000 parameters $\mathbf{p}$ sampled iid and uniformly at random from Δ_k^0, for any $(k, \gamma) \in \{5, 10, 15\} \times \{0.01, 0.05, 0.1\}$. For example, we have $\mathrm{LB}_1((0.2, 0.2, 0.15, 0.2, 0.25), 0.05) \approx 252 > 152.9 \approx \mathrm{LB}_2((0.2, 0.2, 0.15, 0.2, 0.25), 0.05)$. This indicates that the instance-wise lower bound of Prop. C.1 is larger than that from Thm. 5.2.

G Additional Experiments

G.1 Comparison of DKWT with PAC-WRAPPER

In this section, we provide further experimental results. First, we repeat the experiment regarding the comparison of DKWT and PW from Section 7 for $\boldsymbol{\theta} = (1, 2^{-1}, 2^{-2}, \ldots, 2^{-9})$, with $\gamma = 0.1$ and for different values of k. Table 5 shows the results obtained with 10 repetitions. Similar to the results in the main paper, both algorithms apparently keep the desired confidence of 90%, but PW requires far more samples for this. The fact that the observed sample complexities are not throughout decreasing in k is supposedly due to the large standard errors and the little number of repetitions. However, they strongly indicate that DKWT outperforms PW in terms of sample complexity.

Table 5: Comparison of DKWT with PAC-WRAPPER (PW) on $\boldsymbol{\theta} = (1, 2^{-1}, 2^{-2}, \ldots, 2^{-9})$

	$T^{\mathcal{A}}$		Accuracy	
k	DKWT	PW	DKWT	PW
2	**8310** (0.0)	2509460 (226634.0)	1.00	1.00
3	**4078** (348.9)	46277676 (30635546.4)	1.00	1.00
4	**3925** (1014.3)	775101 (108535.7)	1.00	1.00
5	**3397** (529.2)	6450264 (1363336.3)	1.00	1.00
6	**2213** (465.0)	130069344 (77405795.5)	1.00	1.00
7	**2856** (507.4)	253206333 (125199242.0)	1.00	1.00
8	**3817** (608.9)	27159632 (12458792.0)	1.00	1.00
9	**2855** (680.7)	146229360 (79427860.6)	1.00	1.00

Next, we compare DKWT with PW on synthetic data considered in [35], where PW has first been introduced. We restrict ourselves to $\boldsymbol{\theta}^{\text{arith}}, \boldsymbol{\theta}^{\text{geo}} \in [0,1]^{16}$ defined via

$$\theta_1^{\text{arith}} := 1, \quad \forall i \in [15] : \theta_{i+1}^{\text{arith}} := \theta_i^{\text{arith}} - 0.06,$$
$$\theta_1^{\text{geo}} := 1, \quad \forall i \in [15] : \theta_{i+1}^{\text{geo}} := \frac{4}{5} \cdot \theta_i^{\text{geo}},$$

because the other synthetic datasets considered in Fig. 2 of [35] (i.e., **g1** and **b1**) are not in $PM_k^m(\exists\text{GCW} \wedge \Delta^0)$, which is formally required for DKWT. For $\boldsymbol{\theta} \in \{\boldsymbol{\theta}^{\text{arith}}, \boldsymbol{\theta}^{\text{geo}}\}$ we execute DKWT with $\gamma = 0.01$ for 1000 repetitions on feedback generated by $\mathbf{P}(\boldsymbol{\theta})$ and report the mean termination time (and standard error in brackets) as well as the observed accuracy in Table 6. A look at Fig. 2 of [35] reveals that DKWT indeed outperforms PW on both datasets while still keeping its theoretical guarantees.

Table 6: Results of DKWT on $\boldsymbol{\theta}^{\text{arith}}$ and $\boldsymbol{\theta}^{\text{geo}}$

	$T^{\mathcal{A}}$ of $\mathcal{A}$ =DKWT	Accuracy
$\boldsymbol{\theta}^{\text{arith}}$	1277781 (22284.0)	1.00
$\boldsymbol{\theta}^{\text{geo}}$	55132 (910.5)	1.00

G.2 Comparison of DKWT with SELECT, SEEBS and EXPLORE-THEN-VERIFY

The authors of [29] restrict themselves in the analysis of their algorithm SEEBS to probability models $\mathbf{P} \in PM_2^m(\exists\text{GCW} \wedge \Delta^0)$, which fulfill both of the following conditions:

- *Strong stochastic transitivity (SST):* For all distinct distinct $i, j, k \in [m]$ with $\mathbf{P}(i|\{i,j\}) \geq 1/2$ and $\mathbf{P}(j|\{j,k\}) \geq 1/2$ we have
$$\mathbf{P}(i|\{i,k\}) \geq \max\{\mathbf{P}(i|\{i,j\}), \mathbf{P}(j|\{j,k\})\}$$
- *Stochastic triangle inequality (STI):* For all distinct $i, j, k \in [m]$ we have
$$|\mathbf{P}(i|\{i,k\}) - 1/2| \leq |\mathbf{P}(i|\{i,j\}) - 1/2| + |\mathbf{P}(j|\{j,k\}) - 1/2|.$$

In particular, SEEBS is only proven to identify the correct GCW with confidence $\geq 1-\gamma$ for any $\mathbf{P}$ in a set $PM_2^m(\exists \text{GCW} \wedge \Delta^0 \wedge \text{SST} \wedge \text{STI}) \subsetneq PM_2^m(\exists \text{GCW} \wedge \Delta^0)$.

Table 7 shows the observed termination times (and standard errors theoreof in brackets) of DKWT, SELECT, SEEBS and EtV compared on $PM_2^m(\exists \text{GCW} \wedge \Delta^h)$ in the same manner as done in Section 7, where the true value of h is revealed to SELECT but not to DKWT, SEEBS or EtV. The results for $m \in \{5, 10\}$ are averaged over 100 repetitions and are partly the same as shown in Table 4, the results for $m \in \{15, 20\}$ are averaged over 10 repetitions. Table 8 shows the corresponding accuracies observed during the experiment underlying Table 7. Almost any algorithm in any case achieves an accuracy of $\geq 95\%$, the only exception is SELECT for $m = 20$ and $h = 0.20$ and this is supposedly due to the little number of repetitions considered. The results indicate again that DKWT outperforms SEEBS but not SELECT. Since SELECT obtains as further information the true value of h, this is not at all surprising.

Table 7: Comparison of DKWT, SELECT, SEEBS and EXPLORE-THEN-VERIFY (EtV)

		$T^{\mathcal{A}}$			
m	h	DKWT	SELECT	SEEBS	EtV
5	0.20	6010 (293.2)	**252** (4.2)	7305 (432.1)	8601 (589.2)
5	0.15	8874 (460.0)	**460** (7.3)	13393 (904.5)	11899 (986.9)
5	0.10	15769 (1457.1)	**989** (17.0)	19802 (1543.2)	260171 (210678.1)
5	0.05	31454 (4127.4)	**3924** (68.6)	36855 (3533.2)	156534 (115903.1)
10	0.20	14334 (492.8)	**565** (2.5)	16956 (617.9)	26115 (969.2)
10	0.15	18563 (734.5)	**1009** (4.2)	27527 (1126.7)	32548 (2514.6)
10	0.10	33040 (1625.1)	**2245** (9.7)	47330 (2138.2)	68858 (11304.5)
10	0.05	78660 (6517.2)	**8971** (39.2)	83877 (5842.6)	220098 (92484.9)
15	0.20	21932 (1618.1)	**803** (13.9)	28605 (2161.5)	54197 (5307.3)
15	0.15	27446 (2500.0)	**1436** (12.3)	38084 (4985.3)	78753 (27741.4)
15	0.10	45737 (6709.6)	**3248** (20.7)	67383 (8117.1)	116014 (24282.2)
15	0.05	114152 (18704.0)	**12993** (82.7)	108738 (19780.4)	2804238 (2560594.1)
20	0.20	32038 (1209.2)	**1154** (8.7)	40910 (2893.1)	78286 (3451.5)
20	0.15	39792 (3923.6)	**2080** (12.6)	58793 (4828.0)	122582 (24065.7)
20	0.10	87667 (13380.8)	**4616** (32.3)	105249 (13231.8)	631195 (281883.6)
20	0.05	134628 (21743.3)	**18375** (138.2)	164439 (30175.4)	2094505 (1694236.4)

Table 8: Accuracies of DKWT, SELECT, SEEBS and EXPLORE-THEN-VERIFY (EtV) corresponding to the experiment of Table 7

		Accuracy			
m	h	DKWT	SELECT	SEEBS	EtV
5	0.20	1.00	0.97	1.00	1.00
5	0.15	1.00	1.00	1.00	1.00
5	0.10	1.00	0.99	1.00	1.00
5	0.05	1.00	1.00	1.00	1.00
10	0.20	1.00	0.95	1.00	1.00
10	0.15	1.00	0.98	1.00	1.00
10	0.10	1.00	0.99	1.00	1.00
10	0.05	1.00	1.00	1.00	1.00
15	0.20	1.00	1.00	1.00	1.00
15	0.15	1.00	1.00	1.00	1.00
15	0.10	1.00	1.00	1.00	1.00
15	0.05	1.00	1.00	1.00	1.00
20	0.20	1.00	0.90	1.00	1.00
20	0.15	1.00	1.00	1.00	1.00
20	0.10	1.00	1.00	1.00	1.00
20	0.05	1.00	1.00	1.00	1.00

G.3 Comparison of DKWT with Alg. 5

Finally, we compare DKWT and Alg. 5 by means of their average sample complexity and accuracy when executed on 1000 instances $\mathbf{P}$, which were drawn independently and uniformly at random from (a) $PM_k^5(\exists \text{GCW} \wedge \Delta^h)$ and (b) $PM_k^5(\exists h\text{GCW} \wedge \Delta^{0.01})$. We choose $\gamma = 0.05$ and restrict ourselves due to $PM_5^5(\exists \text{GCW} \wedge \Delta^h) = PM_5^5(\exists h\text{GCW} \wedge \Delta^{0.01})$ to $k \in \{2, 3, 4\}$. Similarly as in our comparison to SELECT, Alg. 5 is revealed the true value of h and started with this as parameter. The results are collected in (a) Table 9 and (b) Table 10. In any of the cases (a) and (b), DKWT apparently outperforms Alg. 5 if h is smaller than some threshold h_0, and the value of h_0 appears to be significantly larger for (a) than for (b). This indicates that Alg. 5 may be preferable over DKWT if $h(\mathbf{P})$ is small and $\mathbf{P} \in PM_k^m(\exists h'\text{GCW} \wedge \Delta^0)$ holds for some a priori known $h' \in (0, 1/2)$.

Table 9: Comparison of DKWT with Alg. 5 on $PM_k^5(\exists \text{GCW} \wedge \Delta^h)$

		$T^{\mathcal{A}}$		Accuracy	
k	h	DKWT	Alg. 5	DKWT	Alg. 5
2	0.9	4155 (0.0)	**2664** (0.0)	1.00	1.00
2	0.7	**4155** (0.0)	4405 (0.0)	1.00	1.00
2	0.5	**4155** (0.0)	8630 (0.0)	1.00	1.00
2	0.3	**4195** (12.7)	23970 (0.0)	1.00	1.00
2	0.1	**14729** (423.4)	215695 (0.0)	1.00	1.00
3	0.9	2298 (0.0)	**1464** (0.0)	1.00	1.00
3	0.7	**2298** (0.0)	2418 (0.0)	1.00	1.00
3	0.5	**2298** (0.0)	4737 (0.0)	1.00	1.00
3	0.3	**2381** (17.5)	13155 (0.0)	1.00	1.00
3	0.1	**14933** (436.1)	118383 (0.0)	1.00	1.00
4	0.9	1428 (0.0)	**1356** (0.0)	1.00	1.00
4	0.7	**1428** (0.0)	2238 (0.0)	1.00	1.00
4	0.5	**1428** (0.0)	4386 (0.0)	1.00	1.00
4	0.3	**1492** (15.0)	12183 (0.0)	1.00	1.00
4	0.1	**13449** (306.4)	109626 (0.0)	1.00	1.00

Table 10: Comparison of DKWT with Alg. 5 on $PM_k^5(\exists h\text{GCW} \wedge \Delta^{0.01})$

		$T^{\mathcal{A}}$		Accuracy	
k	h	DKWT	Alg. 5	DKWT	Alg. 5
2	0.9	53913 (7092.4)	**2477** (8.0)	1.00	1.00
2	0.7	63647 (8322.8)	**4124** (13.0)	1.00	1.00
2	0.5	54370 (6753.8)	**8167** (24.2)	1.00	1.00
2	0.3	59488 (7738.0)	**23275** (53.4)	1.00	1.00
2	0.1	**60682** (7256.5)	214358 (236.4)	1.00	1.00
3	0.9	40359 (6188.7)	**1464** (0.0)	1.00	1.00
3	0.7	27069 (3621.2)	**2418** (0.0)	1.00	1.00
3	0.5	37362 (5774.2)	**4737** (0.0)	1.00	1.00
3	0.3	31553 (4551.6)	**13155** (0.0)	1.00	1.00
3	0.1	**45929** (5277.3)	118383 (0.0)	1.00	1.00
4	0.9	24164 (4446.0)	**1356** (0.0)	1.00	1.00
4	0.7	39088 (6293.2)	**2238** (0.0)	1.00	1.00
4	0.5	31835 (5462.0)	**4386** (0.0)	1.00	1.00
4	0.3	31796 (5131.8)	**12183** (0.0)	1.00	1.00
4	0.1	**48202** (5765.3)	109626 (0.0)	1.00	1.00