A Topological Perspective on Causal Inference: Supplement

In this supplement we give proofs of all the main results in the text.

A Structural Causal Models (§2)

A.1 Background on Relations and Orders

Definition A.1.1. Let C be a set. Then a subset $R \subset C \times C$ is called a *binary relation* on C. We write cRc' if $(c, c') \in R$. The binary relation R is *well-founded* if every nonempty subset $D \subset C$ has a minimal element with respect to R, i.e., if for every nonempty $D \subset C$, there is some $d \in D$, such that there is no $d' \in D$ such that $d'Rd$. The binary relation $\prec\ \subset C \times C$ is a (strict) *total order* if it is irreflexive, transitive, and *connected*: either $c \prec c'$ or $c' \prec c$ for all $c \neq c' \in C$.

Example 1. The edges of a dag form a well-founded binary relation on its nodes. If $\mathbf{V} = \{V_n\}_{n \geq 0}$, then the binary relation $\rightarrow$ defined by $V_m \rightarrow V_n$ iff either $0 < m < n$ or $n = 0 < m$ is well-founded but not extendible to an ω-like total order (see Fact 2) and not locally finite: V_0 has infinitely many predecessors $V_1, V_2, \ldots$

A.2 Proofs

Proof of Proposition 1. We assume without loss that $\mathbf{U}(V) = \mathbf{U}$ for every $V \in \mathbf{V}$. For each $\mathbf{u} \in \chi_{\mathbf{U}}$, well-founded induction along $\rightarrow$ shows unique existence of a $m^{\mathcal{M}}(\mathbf{u}) \in \chi_{\mathbf{V}}$ solving $f_V\big(\pi_{\mathbf{Pa}(V)}(m^{\mathcal{M}}(\mathbf{u})), \mathbf{u}\big) = \pi_V(m^{\mathcal{M}}(\mathbf{u}))$ for each V. We claim the resulting function $m^{\mathcal{M}}$ is measurable. One has a clopen basis of cylinders, so it suffices to show each preimage $(m^{\mathcal{M}})^{-1}(v)$ is measurable. Recall that here v denotes the cylinder set $\pi_V^{-1}(\{v\}) \in \mathcal{B}(\chi_{\mathbf{V}})$, for $v \in \chi_V$. Once again this can be established inductively. Note that

$$(m^{\mathcal{M}})^{-1}(v) = \bigcup_{\mathbf{p} \in \chi_{\mathbf{Pa}(V)}} \Big[(m^{\mathcal{M}})^{-1}(\mathbf{p}) \cap \pi_{\mathbf{U}}\big(f_V^{-1}(\{v\}) \cap (\{\mathbf{p}\} \times \chi_{\mathbf{U}})\big)\Big].$$

which is a finite union (by local finiteness) of measurable sets (by the inductive hypothesis) and therefore measurable. Thus for any $\mathcal{M}$ the pushforward $p^{\mathcal{M}} = m_*^{\mathcal{M}}(P)$ is a measure on $\mathcal{B}(\chi_{\mathbf{V}})$ and gives the observational distribution (Definition 4). ☐

Remark on Definition 6. To see that $p_{\text{cf}}^{\mathcal{M}}$ thus defined is a measure, note that $p_{\text{cf}}^{\mathcal{M}} = p^{\mathcal{M}_A}$ and apply Proposition 1, where the model $\mathcal{M}_A$ is defined in Definition A.2.1. This is similar in spirit to the construction of "twinned networks" [2] or "single-world intervention graphs" [8]. ☐

Definition A.2.1. Given $\mathcal{M}$ as in Def. 3 and a collection of interventions A form the following *counterfactual model* $\mathcal{M}_A = \langle \mathbf{U}, A \times \mathbf{V}, \{f_{(\alpha,V)}\}_{(\alpha,V)}, P\rangle$, over endogenous variables $A \times \mathbf{V}$. The counterfactual model has the influence relation $\rightarrow'$, defined as follows. Where $\alpha', \alpha \in A$ let $(\alpha', V') \rightarrow' (\alpha, V)$ iff $\alpha' = \alpha$ and $V' \rightarrow V$. The exogenous space $\mathbf{U}$ and noise distribution P of $\mathcal{M}_A$ are the same as those of $\mathcal{M}$, the exogenous parents sets $\{\mathbf{U}(V)\}_V$ are also identical, and the functions are $\{f_{(\alpha,V)}\}_{(\alpha,V)}$ defined as follows. For any $\mathbf{W} := \mathbf{w} \in A$, $V \in \mathbf{V}$, $\mathbf{p} \in \chi_{\mathbf{Pa}(V)}$, and

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$\mathbf{u} \in \chi_{\mathbf{U}(V)}$ let

$$f_{(\mathbf{W}:=\mathbf{w},V)}\big((\mathbf{W}:=\mathbf{w},\mathbf{p}),\mathbf{u}\big) = \begin{cases}\pi_V(\mathbf{w}), & V \in \mathbf{W}\\ f_V(\mathbf{p},\mathbf{u}), & V \notin \mathbf{W}\end{cases}.$$

B Proofs from §3

Remark on exact characterizations of $\mathfrak{S}_3$, $\mathfrak{S}_2$. Rich probabilistic languages interpreted over $\mathfrak{S}_3$ and $\mathfrak{S}_2$ were axiomatized in [5]. This axiomatization, along with the atomless restriction, gives an exact characterization for the hierarchy sets. Standard form, defined below, gives an alternative characterization exhibiting each $\mathfrak{S}_3^{\prec}$ as a particular atomless probability space (Corollary B.1.1). For $\mathfrak{S}_2^{X\to Y}$ (or $\mathfrak{S}_2$ in the two-variable case) we need the characterization for the proof of the hierarchy separation result, so it is given explicitly as Lemma B.3.1 in the section below on 2VE-spaces. □

B.1 Standard Form

Fix $\prec$. Note that the map ϖ_3 restricted to $\mathfrak{M}_{\prec}$ does *not* inject into $\mathfrak{S}_3^{\prec}$, as any trivial reparametrizations of exogenous noise are distinguished in $\mathfrak{M}_{\prec}$. It is therefore useful to identify a "standard" subclass $\mathfrak{M}_{\prec}^{\mathrm{std}}$ on which ϖ_3 is injective with image $\mathfrak{S}_3^{\prec}$, and in which we lose no expressivity.

Notation. Let $\mathbf{Pred}(V) = \{V' : V' \prec V\}$ and denote a *deterministic* mechanism for V mapping a valuation of its predecessors to a value as $\mathtt{f}_V \in \chi_{\mathbf{Pred}(V)} \to \chi_V$. Write an entire collection of such mechanisms, one for each variable, as $\mathtt{f} = \{\mathtt{f}_V\}_V$. A set $\mathbf{B} \subset \mathbf{V}$ is *ancestrally closed* if $\mathbf{B} = \bigcup_{V\in\mathbf{B}} \mathbf{Pred}(V)$. For any ancestrally closed $\mathbf{B}$ let $\xi(\mathbf{B}) = \big\{(V,\mathbf{p}) : V \in \mathbf{B}, \mathbf{p} \in \chi_{\mathbf{Pred}(V)}\big\}$. Note that $\mathtt{F}(\mathbf{B}) = \bigtimes_{(V,\mathbf{p})\in\xi(\mathbf{B})} \chi_V$ encodes the set of all possible such collections of deterministic mechanisms, and we write, e.g., $\mathtt{f} \in \mathtt{F}(\mathbf{B})$. Abbreviate $\xi(\mathbf{V})$, $\mathtt{F}(\mathbf{V})$ for the entire endogenous variable set $\mathbf{V}$ as ξ, $\mathtt{F}$ respectively. We also use $\mathtt{f}$ to abbreviate the set

$$\bigcap_{\substack{V\in\mathbf{B}\\ \mathbf{p}\in\chi_{\mathbf{Pred}(V)}}} \pi^{-1}_{(\mathbf{Pred}(V):=\mathbf{p},V)}(\{\mathtt{f}(\mathbf{p})\}) \in \mathcal{B}(\chi_{A\times\mathbf{V}}) \tag{B.1}$$

so we can write, e.g., $p^{\mathcal{M}}_{\mathrm{cf}}(\mathtt{f})$ for the probability in $\mathcal{M}$ that the effective mechanisms $\mathtt{f}$ have been selected (by exogenous factors) for the variables $\mathbf{B}$.

Definition B.1.1. The SCM $\mathcal{M} = \langle \mathbf{U}, \mathbf{V}, \{f_V\}_V, P\rangle$ of Def. 3 is *standard form* over $\prec$, and we write $\mathcal{M} \in \mathfrak{M}_{\prec}^{\mathrm{std}}$, if we have that $\to\ =\ \prec$ for its influence relation, $\mathbf{U} = \{U\}$ for a single exogenous variable U with $\chi_U = \mathtt{F}$, $P \in \mathfrak{P}(\mathtt{F})$ for its exogenous noise space, and for every V, we have that $\mathbf{U}(V) = \mathbf{U} = \{U\}$ and the mechanism f_V takes $\mathbf{p}, (\{\mathtt{f}_V\}_V) \mapsto \mathtt{f}_V(\mathbf{p})$ for each $\mathbf{p} \in \chi_{\mathbf{Pred}(V)}$ and joint collection of deterministic functions $\{\mathtt{f}_V\}_V \in \mathtt{F} = \chi_U$.

Each unit $\mathbf{u}$ in a standard form model amounts to a collection $\{\mathtt{f}_V\}_V$ of deterministic mechanisms, and each variable is determined by a mechanism specified by the "selector" endogenous variable U.

Lemma B.1.1. Let $\mathcal{M} \in \mathfrak{M}_{\prec}$. Then there exists $\mathcal{M}^{\mathrm{std}} \in \mathfrak{M}_{\prec}^{\mathrm{std}}$ such that $\varpi_3(\mathcal{M}) = \varpi_3(\mathcal{M}^{\mathrm{std}})$.

Proof. To give $\mathcal{M}^{\mathrm{std}}$ define a measure $P \in \mathfrak{P}(\mathtt{F})$ as in Def. B.1.1 on a basis of cylinder sets by the counterfactual in $\mathcal{M}$

$$\begin{aligned}P\big(\pi^{-1}_{(V_1,\mathbf{p}_1)}(\{v_1\}) \cap \cdots \cap \pi^{-1}_{(V_n,\mathbf{p}_n)}(\{v_n\})\big)\\ = p^{\mathcal{M}}_{\mathrm{cf}}\big(\pi^{-1}_{(\mathbf{Pred}(V_1):=\mathbf{p}_1,V_1)}(\{v_1\}) \cap \cdots \cap \pi^{-1}_{(\mathbf{Pred}(V_n):=\mathbf{p}_n,V_n)}(\{v_n\})\big).\end{aligned} \tag{B.2}$$

To show that $\varpi_3(\mathcal{M}) = \varpi_3(\mathcal{M}^{\mathrm{std}})$ it suffices to show that any two models agreeing on all counterfactuals of the form (B.2) must agree on all counterfactuals in A. Suppose $\alpha_i \in A$, $V_i \in \mathbf{V}$, $v_i \in \chi_{V_i}$ for $i = 1,\dots,n$. Let $\mathbf{B} = \bigcup_i \mathbf{Pred}(V_i)$ and given $\mathtt{f} = \{\mathtt{f}_V\}_V$, define $\mathtt{f}_V^{\mathbf{W}:=\mathbf{w}}$ to be a constant function mapping to $\pi_V(\mathbf{w})$ if $V \in \mathbf{W}$ and $\mathtt{f}_V^{\mathbf{W}:=\mathbf{w}} = \mathtt{f}_V$ otherwise. Write $\mathtt{f} \vDash V = v$ if $\pi_V(\mathbf{v}) = v$ for that $\mathbf{v} \in \chi_{\mathbf{V}}$ such that $\mathtt{f}_V\big(\pi_{\mathbf{Pred}(V)}(\mathbf{v})\big) = \pi_V(\mathbf{v})$ for all V. Finally, note that

$$\bigcap_{i=1}^{n} \pi^{-1}_{(\alpha_i,V_i)}(\{v_i\}) = \bigsqcup_{\substack{\{\mathtt{f}_V\}_{V\in\mathbf{B}}\in\mathtt{F}(\mathbf{B})\\ \{\mathtt{f}_V^{\alpha_i}\}_{V\in\mathbf{B}}\vDash V_i=v_i\\ \text{for each } i}} \{\mathtt{f}_V\}_{V\in\mathbf{B}}$$

where each set in the finite disjoint union is of the form (B.1). Thus the measure of the left-hand side can be written as a sum of measures of such sets, which use only counterfactuals of the form (B.2), showing agreement of the measures (by Fact 1). ☐

Corollary B.1.1. $\mathfrak{S}_3^{\prec}$ bijects with the set of atomless measures in $\mathfrak{P}(\mathtt{F})$, which we denote $\mathfrak{S}^{\prec}_{\text{std}}$. We write the map as $\varpi^{\prec}_{\text{std}} : \mathfrak{S}_3^{\prec} \to \mathfrak{S}^{\prec}_{\text{std}}$. ☐

Where the order $\prec$ is clear, the above result permits us to abuse notation, using e.g. μ to denote either an element of $\mathfrak{S}_3^{\prec}$ or its associated point $\varpi^{\prec}_{\text{std}}(\mu)$ in $\mathfrak{S}^{\prec}_{\text{std}}$. We will henceforth indulge in such abuse.

Proof of Fact 4. The follows easily from Lem. B.1.2 below, adapted from Suppes and Zanotti [9, Thm. 1]. This shows that every atomless distribution is generated by some SCM; furthermore, it can chosen so as to exhibit no causal effects whatsoever. ☐

Definition B.1.2. Say that $\nu \in \mathfrak{P}\big(\mathtt{F}(\mathbf{V})\big)$ is *acausal* if $\nu(\pi^{-1}_{(V,\mathbf{p})}(\{v_1\}) \cap \pi^{-1}_{(V,\mathbf{p}')}(\{v_2\})\big) = 0$ for every $(V,\mathbf{p}), (V,\mathbf{p}') \in \xi$ and $v_1 \neq v_2 \in \chi_V$.

Lemma B.1.2. Let $\mu \in \mathfrak{P}(\chi_{\mathbf{V}})$ be atomless. Then there is a $\mathcal{M} \in \mathfrak{M}^{\text{std}}_{\prec}$ (see Def. B.1.1) with an acausal noise distribution such that $\mu = (\varpi_1 \circ \varpi_2 \circ \varpi_3)(\mathcal{M})$.

Proof. Consider $\nu \in \mathfrak{P}\big(\mathtt{F}(\mathbf{V})\big) = \mathfrak{P}\big(\bigtimes_{(V,\mathbf{p})} \chi_V\big)$ determined on a basis as follows: $\nu\big(\pi^{-1}_{(V_1,\mathbf{p}_1)}(\{v_1\}) \cap \dots \cap \pi^{-1}_{(V_n,\mathbf{p}_n)}(\{v_n\})\big) = \mu\big(\pi^{-1}_{V_1}(\{v_1\}) \cap \dots \cap \pi^{-1}_{V_n}(\{v_n\})\big)$. This is clearly acausal and atomless. ☐

B.2 Proofs from §3.2

Proof of Prop. 2 (Collapse set $\mathfrak{C}_1$ is empty). Let $\mu \in \mathfrak{S}_1$ and $\nu \in \mathfrak{S}^{\prec}_{\text{std}}$ with $(\varpi_1 \circ \varpi_2 \circ \varpi^{-1}_{\text{std}})(\nu) = \mu$. By Lemma B.1.2 we may assume ν is acausal. Let X be the first, and Y the second variable with respect to $\prec$. Note there are x^*, y^* such that $\mu(\pi_X^{-1}(\{x^*\}) \cap \pi_Y^{-1}(\{y^*\})) > 0$; let $x^\dagger \neq x^*$, $y^\dagger \neq y^*$. Consider ν' defined as follows where $\mathcal{F}_3$ stands for any set of the form $\pi^{-1}_{(V_1,\mathbf{p}_1)}(\{v_1\}) \cap \dots \cap \pi^{-1}_{(V_n,\mathbf{p}_n)}(\{v_n\}) \subset \mathtt{F}(\mathbf{V})$, for $V_i \in \mathbf{V}$, $\mathbf{p}_i \in \chi_{\mathbf{P}(V_i)}$, $v_i \in \chi_{V_i}$, and $\mathcal{F}_1$ is the corresponding $\pi^{-1}_{V_1}(\{v_1\}) \cap \dots \cap \pi^{-1}_{V_n}(\{v_n\}) \subset \chi_{\mathbf{V}}$.

$$\nu'\big(\pi^{-1}_{(X,())}(\{x\}) \cap \pi^{-1}_{(Y,(x^*))}(\{y_*\}) \cap \pi^{-1}_{(Y,(x^\dagger))}(\{y_\dagger\}) \cap \mathcal{F}_3\big) = \begin{cases} \mu\big(\pi_X^{-1}(\{x^*\}) \cap \pi_Y^{-1}(\{y^*\}) \cap \mathcal{F}_1\big), & x = x^*, y_* = y^* \neq y_\dagger \\ 0, & x = x^*, y_* = y^\dagger \neq y_\dagger \\ 0, & x = x^*, y_* = y_\dagger = y^* \\ \mu\big(\pi_X^{-1}(\{x^*\}) \cap \pi_Y^{-1}(\{y^\dagger\}) \cap \mathcal{F}_1\big), & x = x^*, y_* = y_\dagger = y^\dagger \\ \mu\big(\pi_X^{-1}(\{x^\dagger\}) \cap \pi_Y^{-1}(\{y\}) \cap \mathcal{F}_1\big), & x = x^\dagger \end{cases}$$

We claim that $\mu = \mu'$ where $\mu' = (\varpi_1 \circ \varpi_2)(\nu')$; it suffices to show agreement on sets of the form $\pi_X^{-1}(\{x\}) \cap \pi_Y^{-1}(\{y\}) \cap \mathcal{F}_1$. If $x = x^\dagger$ then the last case above occurs; if $x = x^*$ and $y = y^\dagger$ then we are in the fourth case; if $x = x^*$ and $y = y^*$ then exclusively the first case applies. In all cases the measures agree. Let $(\nu_\alpha)_\alpha = \varpi_2(\nu)$ and $(\nu'_\alpha)_\alpha = \varpi_2(\nu')$ be the Level 2 projections of ν, ν' respectively. Note that $\nu_{X:=x^\dagger}(y^\dagger) < \nu'_{X:=x^\dagger}(y^\dagger)$. This shows that the standard-form measures ν, ν' project down to different points in $\mathfrak{S}_2$ (in particular differing on the Y-marginal at the index corresponding to the intervention $X := x^\dagger$) while projecting to the same point in $\mathfrak{S}_1$. Thus $\mu \notin \mathfrak{C}_1$ and since μ was arbitrary, $\mathfrak{C}_1 = \varnothing$. ☐

Example 2 (Collapse set $\mathfrak{C}_2$ is nonempty)**.** We present a $\mu \in \mathfrak{S}^{\prec}_{\text{std}}$ for which $\varpi_2(\mu) \in \mathfrak{C}_2$. Let $\mathbf{S}_n \subset \mathbf{V}$ be the ancestrally closed (§B.1) set of the n least variables with respect to $\prec$ and X be the first variable with respect to $\prec$; thus, e.g., $\mathbf{S}_1 = \{X\}$. Where $\mathtt{f} = \{\mathtt{f}_V\}_{V \in \mathbf{S}_n} \in \mathtt{F}(\mathbf{S}_n)$, define $\mu(\mathtt{f}) = 0$ if there is any $V \in \mathbf{S}_n \setminus \{X\}$, $\mathbf{p} \neq (0, \dots, 0) \in \chi_{\mathbf{Pred}(V)}$ such that $\mathtt{f}_V(\mathbf{p}) = 0$, and otherwise define $\mu(\mathtt{f}) = 1/2^n$. Note that this example is *monotonic* in the sense of [1, 7].

We claim $\mu' = \mu$ for any $\mu' \in \mathfrak{S}^{\prec}_{\text{std}}$ projecting to the same Level 2, i.e., such that $\varpi_2(\mu') = \varpi_2(\mu)$; note that it suffices to consider only candidate counterexamples with order $\prec$ since $\varpi_2(\mu) \notin \mathfrak{S}_2^{\prec'}$

for any $\prec' \neq \prec$. It suffices to show that $\mu(\mathtt{f}) = \mu'(\mathtt{f})$ for any n and $\mathtt{f} = \{\mathtt{f}_V\}_{V \in \mathbf{S}_n}$; recall that in the measures, $\mathtt{f}$ denotes a set of the form (B.1). Let $(\mu_\alpha)_\alpha = \varpi_2(\mu) \in \mathfrak{S}_2^{\prec}$ and $(\mu'_\alpha)_\alpha = \varpi_2(\mu')$, with $(\mu_\alpha)_\alpha = (\mu'_\alpha)_\alpha$. Since $\mu'_{\mathbf{Pred}(V) := \mathbf{p}}(\pi_V^{-1}(\{1\})) = 1$ for any $V \in \mathbf{S}_n \setminus \{X\}$, $\mathbf{p} \neq (0, \dots, 0)$, probability bounds show $\mu'(\mathtt{f})$ vanishes unless $\mathtt{f}_V(\mathbf{p}) = 1$ for each such $\mathbf{p}$, in which case

$$\mu'(\mathtt{f}) = \mu'\Big(\bigcap_{i=1}^{n} \pi^{-1}_{(V_i, \{V_1, \dots, V_{i-1}\} := (0, \dots, 0))}(\{v_i\})\Big) \tag{B.3}$$

for some $v_i \in \chi_{V_i}$, where we have labeled the elements of $\mathbf{S}_n$ as $V_1, \dots, V_n$, with $V_1 \prec \dots \prec V_n$. We claim this is reducible—again using probabilistic reasoning alone—to a linear combination of quantities fixed by $(\mu'_\alpha)_\alpha$, the Level 2 projection of μ', which is the same as the projection $(\mu_\alpha)_\alpha$ of μ. This can be seen by an induction on the number $m = |M|$ where $M = \{i : v_i = 1\}$: note (B.3) becomes

$$\mu'\Big(\bigcap_{i \notin M} \pi^{-1}_{(V_i, \{V_1, \dots, V_{i-1}\} := (0, \dots, 0))}(\{0\})\Big)$$
$$- \sum_{M' \subsetneq M} \mu'\Big(\bigcap_{i \notin M'} \pi^{-1}_{(V_i, \{V_1, \dots, V_{i-1}\} := (0, \dots, 0))}(\{0\}) \cap \bigcap_{i \in M'} \pi^{-1}_{(V_i, \{V_1, \dots, V_{i-1}\} := (0, \dots, 0))}(\{1\})\Big)$$

and the inductive hypothesis implies each summand can be written in the sought form while the first term becomes $\mu'\big(\bigcap_{i \notin M} \pi^{-1}_{(V_i, ())}(\{0\})\big) = \mu'_{()}\big(\bigcap_{i \notin M} \pi^{-1}_{V_1}(\{0\})\big) = \mu_{()}\big(\bigcap_{i \notin M} \pi^{-1}_{V_1}(\{0\})\big)$. Here $()$ abbreviates the empty intervention $\varnothing := ()$. Thus any Level 3 quantity reduces to Level 2, on which the two measures agree by hypothesis.

B.3 Remarks on §3.3

Lemma B.3.1. Let $(\mu_\alpha)_\alpha \in \bigtimes_{\alpha \in A_2^{X \to Y}} \mathfrak{P}(\chi_{X,Y})$. Then $(\mu_\alpha)_\alpha \in \mathfrak{S}_2^{X \to Y}$ iff

$$\mu_{X := x}(x) = 1 \tag{B.4}$$

for every $x \in \chi_X$ and

$$\mu_{X := x}(y) \geq \mu_{()}(x, y) \tag{B.5}$$

for every $x \in \chi_X$, $y \in \chi_Y$. Here x, y abbreviates the basic set $\pi_X^{-1}(\{x\}) \cap \pi_Y^{-1}(\{y\})$.

Proof. It is easy to see that (B.4), (B.5) hold for any $(\mu_\alpha)_\alpha$. For the converse, consider the two-variable model over endogenous $\mathbf{Z} = \{X, Y\}$ with $X \prec Y$; note that $|\mathtt{F}(\mathbf{Z})| = 8$. A result of Tian et al. [10] gives that this model is characterized exactly by (B.4), (B.5) so for any such $(\mu_\alpha)_\alpha$ there is a distribution on $\mathtt{F}(\mathbf{Z})$ such that this model induces $(\mu_\alpha)_\alpha$. It is straightforward to extend this distribution to an atomless measure on $\mathtt{F}(\mathbf{V})$. □

C Proofs from §4

Proof of Prop. 4. This amounts to the continuity of projections in product spaces and marginalizations in weak convergence spaces. The latter follows easily from results in §3.1.3 of [4] or [3]. □

Proof of Thm. 2. We show how Theorem 3.2.1 of [4] can be applied to derive the result. Specifically, let $\Omega = \bigtimes_\alpha \chi_{\mathbf{V}}$. Let $\mathcal{I}$ be the usual clopen basis, and let W be the set of Borel measures $\mu \in \mathfrak{P}(\Omega)$ that factor as a product $\mu = \times_\alpha \mu_\alpha$ where each $\mu_\alpha \in \mathfrak{S}_1$ and $(\mu_\alpha)_\alpha \in \mathfrak{S}_2$. This choice of W corresponds exactly to our notion of experimental verifiability.

It remains to check that a set is open in W iff the associated set is open in $\mathfrak{S}_2$ (homeomorphism). It suffices to show their convergence notions agree. Suppose $(\nu_n)_n$ is a sequence, each $\nu_n \in W$, converging to $\nu = \times_\alpha \mu_\alpha \in W$. We have for each n that $\nu_n = \times_\alpha \mu_{n,\alpha}$ such that $(\mu_{n,\alpha})_\alpha \in \mathfrak{S}_2$. By Theorem 3.1.4 in [4], which is straightforwardly generalized to the infinite product, for each fixed α we have $(\mu_{n,\alpha})_n \Rightarrow \mu_\alpha$. This is exactly pointwise convergence in the product space $\mathfrak{S}_2$, and the same argument in reverse works for the converse. □

D Proofs from §5

We will use the following result to categorize sets in the weak topology.

Lemma D.0.1. If $X \subset \vartheta$ is a basic clopen, the map $p_X : (\mathfrak{S}, \tau^{\mathrm{w}}) \to ([0,1], \tau)$ sending $\mu \mapsto \mu(X)$ is continuous and open (in its image), where τ is as usual on $[0,1] \subset \mathbb{R}$.

Proof. Continuous: the preimage of the basic open $(r_1, r_2) \cap p_X(\mathfrak{S})$ where $r_1, r_2 \in \mathbb{Q}$ is $\{\mu : \mu(X) > r_1\} \cap \{\mu : \mu(X) < r_2\} = \{\mu : \mu(X) > r_1\} \cap \{\mu : \mu(\vartheta \setminus X) > 1 - r_2\}$, a finite intersection of the subbasic sets (1) from §4. See also Kechris [6, Corollary 17.21].

Open: if $X = \varnothing$ or ϑ, then $p_X(\mathfrak{S}) = \{0\}$ or $\{1\}$ resp., both open in themselves. Else $p_X(\mathfrak{S}) = [0,1]$; we show any $Z = p_X\big(\bigcap_{i=1}^n \{\mu : \mu(X_i) > r_i\}\big)$ is open. Consider a mutually disjoint, covering $\mathcal{D} = \big\{\bigcap_{i=0}^n Y_i : Y_0 \in \{X, \vartheta \setminus X\}, \text{ each } Y_i \in \{X_i, \vartheta \setminus X_i\}\big\}$ and space $\Delta = \{(\mu(D))_{D \in \mathcal{D}} : \mu \in \mathfrak{S}\} \subset \mathbb{R}^{2^{n+1}}$. Just as in the Lemma, we have $\mathsf{p}_S : \Delta \to [0,1]$, for each $S \subset \mathcal{D}$ taking $(\mu(D))_D \mapsto \sum_{D \in S} \mu(D)$. Note $Z = \mathsf{p}_{\{D : D \cap X \neq \varnothing\}}\big(\bigcap_{i=1}^n \mathsf{p}^{-1}_{\{D : D \cap X_i \neq \varnothing\}}((r_i, 1])\big)$ so it suffices to show p_S is continuous and open; this is straightforward. □

Full proof of Lem. 1. We show a stronger result, namely that the complement of the good set is nowhere dense. By rearrangement and laws of probability we find that the second inequality in (2) is equivalent to

$$\begin{aligned} \mu_x(y') &< \mu_{()}(x') + \mu_{()}(x, y') \\ 1 - \mu_x(y) &< \underbrace{\mu_{()}(x') + \mu_{()}(x)}_{1} - \mu_{()}(x, y) \\ \mu_x(y) &> \mu_{()}(x, y). \end{aligned}$$

Lemma B.3.1 then entails the non-strict analogues of all four inequalities in (2), (3) are met for any $(\mu_\alpha)_\alpha \in \mathfrak{S}_2^{X \to Y}$, so we show that converting each to an equality yields a nowhere dense set, whose finite union is also nowhere dense. Note that we have a continuous and surjective observational projection $\pi_{()} : \mathfrak{S}_2^{X \to Y} \to \mathfrak{P}(\chi_{\{X,Y\}})$, and the first inequality in (3) is met iff $(\mu_\alpha)_\alpha \in \big(p_{x,y} \circ \pi_{()}\big)^{-1}(\{0\})$ where $p_{x,y}$ is the map from Lemma D.0.1 and x, y denotes the set $\pi_X^{-1}(\{x\}) \cap \pi_Y^{-1}(\{y\}) \subset \chi_{\{X,Y\}}$. This is nowhere dense as it is the preimage of the nowhere dense set $\{0\} \subset [0,1]$ under a map which is continuous by Lemma D.0.1. The second inequality of (3) is wholly analogous after rearrangement.

As for (2), define a function $d : \mathfrak{S}_2^{X \to Y} \to [0,1]$ taking $(\mu_\alpha)_\alpha \mapsto \mu_{X:=x}(y') - \mu_{()}(x, y')$; this function d is continuous by Lemma D.0.1 and the continuity of addition and projection. Note that the first inequality of (2) holds iff $d((\mu_\alpha)_\alpha) = 0$. For any $\mu \in \mathfrak{S}_3^X$ such that $(\varpi_2^{X \to Y} \circ \varpi_2)(\mu) = (\mu_\alpha)_\alpha$, note that $d((\mu_\alpha)_\alpha) = \mu\big(x', y'_x\big)$ where x', y'_x abbreviates the basic set $\pi^{-1}_{((),X)}(\{x'\}) \cap \pi^{-1}_{(X:=x,Y)}(\{y'\}) \in \mathcal{B}(\chi_{A \times \mathbf{V}})$. Thus d is surjective, so that $d^{-1}(\{0\})$ is nowhere dense since $\{0\} \subset [0,1]$ is nowhere dense. The second inequality in (2) is again totally analogous. □

Proof of Lem. 2. Abbreviate μ_3 as μ, and without loss take $\mu \in \mathfrak{S}^{\prec}_{\mathrm{std}}$. Note that (2), (3) entail

$$0 < \mu(x', y'_x) < \mu(x'), \quad 0 < \mu(x', y'_{x'}) < \mu(x').$$

and therefore

$$0 < \mu\big(\pi^{-1}_{((),X)}(\{x'\}) \cap \pi^{-1}_{(x^*,Y)}(\{1\})\big) < \mu\big(\pi^{-1}_{((),X)}(\{x'\})\big)$$

for each $x^* \in \chi_X = \{0,1\}$. In turn this entails that there are some values $y_0, y_1 \in \{0,1\}$ such that $\mu(\Omega_1) > 0$, $\mu(\Omega_2) > 0$ where the disjoint sets $\{\Omega_i\}_i$ are defined as

$$\begin{aligned} \Omega_1 &= \pi^{-1}_{((),X)}(\{x'\}) \cap \pi^{-1}_{(X:=0,Y)}(\{y_0\}) \cap \pi^{-1}_{(X:=1,Y)}(\{y_1\}) \\ \Omega_2 &= \pi^{-1}_{((),X)}(\{x'\}) \cap \pi^{-1}_{(X:=0,Y)}(\{y_0^\dagger\}) \cap \pi^{-1}_{(X:=1,Y)}(\{y_1^\dagger\}) \end{aligned}$$

where in the second line, $y_0^\dagger = 1 - y_0$ and $y_1^\dagger = 1 - y_1$. Note that for $i = 1, 2$ we have conditional measures $\mu_i(S_i) = \frac{\mu(S_i)}{\mu(\Omega_i)}$ for $S_i \in \mathcal{B}(\Omega_i)$; further, Ω_i is Polish, since each is clopen. This implies

Ω_i is a standard atomless (since μ is) probability space under μ_i. By Kechris [6, Thm. 17.41], there are Borel isomorphisms $f_i : \Omega_i \hookrightarrow\!\!\!\!\!\rightarrow [0,1]$ pushing μ_i forward to Lebesgue measure λ, i.e., $\mu_i(f_i^{-1}(B)) = \lambda(B)$ for $B \in \mathcal{B}([0,1])$. Thus $g = f_2^{-1} \circ f_1 : \Omega_1 \hookrightarrow\!\!\!\!\!\rightarrow \Omega_2$ is μ_i-preserving: for $X_1 \in \mathcal{B}(\Omega_1)$,

$$\mu(g(X_1)) = \frac{\mu(\Omega_2)}{\mu(\Omega_1)}\mu(X_1). \tag{D.1}$$

Consider $\mu' = \varpi_3(\mathcal{M}')$ for a new $\mathcal{M}' \in \mathfrak{M}_{\prec}$, given as follows. Its exogenous valuation space is $\chi_{\mathbf{U}} = \Omega'$ where we define the sample space $\Omega' = \mathtt{F}(\mathbf{V}) \times \{\mathrm{T}, \mathrm{H}\}$; that is, a new exogenous variable representing a coin flip is added to some representation of the choice of deterministic standard form mechanisms. Fix constants $\varepsilon_1, \varepsilon_2 \in (0,1)$ with $\varepsilon_1 \cdot \mu(\Omega_1) = \varepsilon_2 \cdot \mu(\Omega_2)$ and define its exogenous noise distribution P by

$$P(X \times \{\mathrm{S}\}) = \begin{cases} (1-\varepsilon_1)\cdot\mu(X), & X \subset \Omega_1, \mathrm{S} = \mathrm{T} \\ \varepsilon_1 \cdot \mu(X), & X \subset \Omega_1, \mathrm{S} = \mathrm{H} \\ (1-\varepsilon_2)\cdot\mu(X), & X \subset \Omega_2, \mathrm{S} = \mathrm{T} \\ \varepsilon_2 \cdot \mu(X), & X \subset \Omega_2, \mathrm{S} = \mathrm{H} \\ \mu(X), & X \subset \mathtt{F}(\mathbf{V}) \setminus (\Omega_1 \cup \Omega_2), \mathrm{S} = \mathrm{T} \\ 0, & X \subset \mathtt{F}(\mathbf{V}) \setminus (\Omega_1 \cup \Omega_2), \mathrm{S} = \mathrm{H} \end{cases}. \tag{D.2}$$

Where $\mathtt{f} \in \mathtt{F}(\mathbf{V})$ and $V \in \mathbf{V}$ write $\mathtt{f}_V$ for the deterministic mechanism (of signature $\chi_{\mathbf{Pred}(V)} \to \chi_V$) for V in $\mathtt{f}$. (Note that each $\mathtt{f}$ is just an indexed collection of such mechanisms $\mathtt{f}_V$.) The function f'_V in $\mathcal{M}'$ is defined at the initial variable X as $f'_X(\mathtt{f}, \mathrm{S}) = \mathtt{f}_X$ for both values of S, and for $V \neq X$ is defined as follows, where $\mathbf{p} \in \mathbf{Pred}(V)$:

$$f'_V\big(\mathbf{p}, (\mathtt{f}, \mathrm{S})\big) = \begin{cases} (g(\mathtt{f}))_V(\mathbf{p}), & \mathtt{f} \in \Omega_1, \mathrm{S} = \mathrm{H},\ \pi_X(\mathbf{p}) = x \\ (g^{-1}(\mathtt{f}))_V(\mathbf{p}), & \mathtt{f} \in \Omega_2, \mathrm{S} = \mathrm{H},\ \pi_X(\mathbf{p}) = x \\ \mathtt{f}_V(\mathbf{p}), & \text{otherwise} \end{cases}. \tag{D.3}$$

We claim that $\varpi_2(\mu') = \varpi_2(\mu)$. It suffices to show for any $\mathbf{Z} \coloneqq \mathbf{z} \in A$ and $\mathbf{w} \in \chi_{\mathbf{W}}$, $\mathbf{W}$ finite, we have

$$\mu(\theta) = \mu'(\theta), \text{ where } \theta = \bigcap_{W \in \mathbf{W}} \pi^{-1}_{(\mathbf{Z}\coloneqq\mathbf{z}, W)}(\{\pi_W(\mathbf{w})\}). \tag{D.4}$$

Assume $\pi_Z(\mathbf{w}) = \pi_Z(\mathbf{z})$ for every $Z \in \mathbf{Z} \cap \mathbf{W}$, since both sides of (D.4) trivially vanish otherwise. Where $\mathtt{f} \in \mathtt{F}(\mathbf{V})$ write, e.g., $\mathtt{f} \vDash \theta$ if $m^{\mathcal{M}_A}(\mathtt{f}) \in \theta$, where $\mathcal{M}$ is a standard form model (Def. B.1.1); for $\omega' \in \Omega'$ write $\omega' \vDash' \theta$ if $m^{\mathcal{M}'_A}(\omega') \in \theta$. By the last two cases of (D.3) we have

$$\begin{aligned} \mu'(\theta) &= \sum_{\mathrm{S}=\mathrm{T},\mathrm{H}} P\big(\{(\mathtt{f}, \mathrm{S}) \in \Omega' : (\mathtt{f}, \mathrm{S}) \vDash' \theta\}\big) \\ &= \mu\big(\{\mathtt{f} \in \mathtt{F}(\mathbf{V}) \setminus (\Omega_1 \cup \Omega_2) : \mathtt{f} \vDash \theta\}\big) + \sum_{\substack{\mathrm{S}=\mathrm{T},\mathrm{H} \\ i=1,2}} P\big(\{(\mathtt{f}, \mathrm{S}) \in \Omega' : \mathtt{f} \in \Omega_i, (\mathtt{f}, \mathrm{S}) \vDash' \theta\}\big). \end{aligned} \tag{D.5}$$

Applying the first four cases of (D.2) and the third case of (D.3), the second term of (D.5) becomes

$$\sum_i \Big[\varepsilon_i \cdot \mu\big(\{\mathtt{f} \in \Omega_i : (\mathtt{f}, \mathrm{H}) \vDash' \theta\}\big) + (1-\varepsilon_i)\cdot\mu\big(\{\mathtt{f} \in \Omega_i : \mathtt{f} \vDash \theta\}\big)\Big]. \tag{D.6}$$

Either $X \in \mathbf{Z}$ and $\pi_X(\mathbf{z}) = x$, or not. In the former case: defining $X_i = \{\mathtt{f} \in \Omega_i : \mathtt{f} \vDash \theta\}$ for each $i = 1, 2$, the first two cases of (D.3) yield that

$$\begin{aligned} \{\mathtt{f} \in \Omega_1 : (\mathtt{f}, \mathrm{H}) \vDash' \theta\} &= \{\mathtt{f} \in \Omega_1 : g(\mathtt{f}) \vDash \theta\} = g^{-1}(X_2) \\ \{\mathtt{f} \in \Omega_2 : (\mathtt{f}, \mathrm{H}) \vDash' \theta\} &= \{\mathtt{f} \in \Omega_2 : g^{-1}(\mathtt{f}) \vDash \theta\} = g(X_1). \end{aligned} \tag{D.7}$$

Applying (D.7) and (D.1), (D.6) becomes

$$\begin{aligned} \varepsilon_1 \cdot \frac{\mu(\Omega_1)}{\mu(\Omega_2)} \cdot \mu(X_2) + (1-\varepsilon_1)\cdot\mu(X_1) + \varepsilon_2 \cdot \frac{\mu(\Omega_2)}{\mu(\Omega_1)} \cdot \mu(X_1) + (1-\varepsilon_2)\cdot\mu(X_2) \\ = \mu(X_1) + \mu(X_2), \end{aligned} \tag{D.8}$$

the final cancellation by choice of $\varepsilon_1, \varepsilon_2$. In the latter case: since $m^{\mathcal{M}}(\mathtt{f}) \in \pi_X^{-1}(\{x'\})$ for any $\mathtt{f} \in \Omega_1 \cup \Omega_2$, the third case of (D.3) gives $\{\mathtt{f} \in \Omega_i : (\mathtt{f}, \mathrm{H}) \vDash' \theta\} = X_i$. Thus (D.6) becomes (D.8) in either case. Putting in (D.8) as the second term in (D.5), we find $\mu(\theta) = \mu'(\theta)$.

Now we claim $\mu(\zeta) \neq \mu'(\zeta)$ for $\zeta = \zeta_0 \cap \zeta_1$ where $\zeta_1 = \pi^{-1}_{(X:=1,Y)}(\{y_1\})$ and $\zeta_0 = \pi^{-1}_{(X:=0,Y)}(\{y_0\})$. We have

$$\mu'(\zeta) = \mu\big(\{\mathtt{f} \in \Omega \setminus (\Omega_1 \cup \Omega_2) : \mathtt{f} \vDash \zeta\}\big) + \sum_{i=1,2} \Big[\varepsilon_i \cdot \mu\big(\{\mathtt{f} \in \Omega_i : (\mathtt{f}, \mathrm{H}) \vDash' \zeta\}\big) + (1-\varepsilon_i) \cdot \mu\big(\{\mathtt{f} \in \Omega_i : \mathtt{f} \vDash \zeta\}\big)\Big]. \tag{D.9}$$

First suppose that $x = 0$. If $\mathtt{f} \in \Omega_1$, then note that $(\mathtt{f}, \mathrm{H}) \vDash' \zeta_0$ iff $g(\mathtt{f}) \vDash \zeta_0$, but this is never so, since $g(\mathtt{f}) \in \Omega_2$. If $\mathtt{f} \in \Omega_2$, then $(\mathtt{f}, \mathrm{H}) \vDash' \zeta_1$ iff $\mathtt{f} \vDash \zeta_1$, which is never so again by choice of Ω_2. If $x = 1$ then we find that $(\mathtt{f}, \mathrm{H}) \nvDash \zeta_1$ (if $\mathtt{f} \in \Omega_1$) and $(\mathtt{f}, \mathrm{H}) \nvDash \zeta_0$ (if $\mathtt{f} \in \Omega_2$). Thus $(\mathtt{f}, \mathrm{H}) \nvDash' \zeta$ for any $\mathtt{f} \in \Omega_1 \cup \Omega_2$ and (D.9) becomes

$$\mu\big(\{\mathtt{f} \in \Omega : \mathtt{f} \vDash \zeta\}\big) - \sum_{i=1,2} \varepsilon_i \cdot \mu\big(\{\mathtt{f} \in \Omega_i : \mathtt{f} \vDash \zeta\}\big) = \mu\big(\{\mathtt{f} \in \Omega : \mathtt{f} \vDash \zeta\}\big) - \varepsilon_1 \cdot \mu(\Omega_1) < \mu(\zeta).$$

It is straightforward to check (via casework on the values y_0, y_1) that μ and μ' disagree also on the PNS: $\mu(y_x, y'_{x'}) \neq \mu'(y_x, y'_{x'})$ as well as its converse. As for the probability of sufficiency (Definition 10), note that

$$P(y_x \mid x', y') = \frac{P(y_x, x', y'_{x'}) + \overbrace{P(y_x, y'_x, x', x)}^{0}}{P(x', y')}$$

and it is again easily seen (given the definition of the Ω_i) that $\mu(y_x, x', y'_{x'}) \neq \mu'(y_x, x', y'_{x'})$ while the two measures agree on the denominator; similar reasoning shows disagreement on the probability of enablement, since

$$P(y_x \mid y') = \frac{P(y_x, y'_{x'}, x') + \overbrace{P(y_x, y'_x, x)}^{0}}{P(y')}. \qquad \square$$

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