Appendix

The appendix is organized as follows.

- In Appendices A and B, we provide all the missing proofs from the main body of the work;
- In Appendix C, we design a polynomial-time algorithm to compute an EF1 allocation with at least $1/(4n^2)$ fraction of the maximum social welfare for n heterogeneous agents;
- In Appendix D, we present some interesting results connecting EF1 and Nash social welfare;
- In Appendix E, we show a new algorithm to compute a $2/3$-MMS allocation for two agents.

A Missing Proofs of Section 3

A.1 Proof of Lemma 3.2

Proof. Without loss of generality, assume all edges have weight 1. In the greedy partition $(M_1, \cdots, M_n)$ of M^*, for any $i \in N$,

$$|M_i| \geq |M_n| = \lfloor \frac{|M^*|}{n} \rfloor.$$

Let $(O_1, \cdots, O_n)$ be an optimal max-min allocation. If $\mathsf{opt} = |O_n| > |M_n|$, then for all $i \in N$,

$$|O_i| \geq \lfloor \frac{|M^*|}{n} \rfloor + 1.$$

Thus

$$\sum_{i \in N} |O_i| \geq n \cdot \lfloor \frac{|M^*|}{n} \rfloor + n > |M^*|,$$

which is a contradiction with M^* being a maximum matching. $\square$

A.2 Proof of Lemma 3.3

Proof. Denote by $O = (O_1, O_2, \cdots, O_n)$ the optimal solution before eliminating any edge, where $u(O_1) \geq u(O_2) \geq \cdots \geq u(O_n)$ and $\mathsf{opt}(\mathcal{I}) = u(O_n)$. Under the maximum matching M, consider the greedy partition $(M_1, M_2, \cdots, M_n)$, where $u(M_1) \geq u(M_2) \geq \cdots \geq u(M_n)$. In greedy partition procedure, all edges are sorted in descending order of their weights and each time we select the edge with the largest weight in the remaining edge set and allocate it to the bundle with the least total utility. If $|M_1| \geq 2$, consider the last edge e added to M_1, we have $w(M_n) \geq w(e)$, since there exists at least one edge added to M_n before edge e is added to M_1. Since in the greedy procedure, edges are added to the bundle with least utility, we have $w(M_n) \geq w(M_1/e)$. Furthermore, we have

$$\begin{aligned} w(M_n) &\geq \frac{1}{2}(w(e) + w(M_1/e)) \geq \frac{1}{2} w(M_1) \\ &\geq \frac{1}{2n} \sum_{i=1}^{n} w(M_i) \geq \frac{1}{2n} \sum_{i=1}^{n} u(O_i) \\ &\geq \frac{1}{2} u(O_n), \end{aligned}$$

and the lemma holds accordingly. $\square$

A.3 Proof of Lemma 3.4

Proof. Let $\mathcal{I}'' = (G, N, w'')$ be the instance obtained from $\mathcal{I}$ by halving all its edge weights. Let opt, opt' and opt'' be the optimal values of instance $\mathcal{I}$, $\mathcal{I}'$ and $\mathcal{I}''$, respectively. It is easy to see that

$$\mathsf{opt}'' = \frac{1}{2} \cdot \mathsf{opt}.$$

Moreover, the weight of all edges in instance $\mathcal{I}'$ is at least as large as that in instance $\mathcal{I}''$, and thus

$$\mathsf{opt}' \geq \mathsf{opt}'' = \frac{1}{2} \cdot \mathsf{opt}.$$

Finally, since $u_i(X_i) \geq \alpha \cdot \mathsf{opt}'$ for all $i \in N$, then

$$u_i(X_i) \geq \frac{\alpha}{2} \cdot \mathsf{opt},$$

and thus the lemma holds. □

A.4 Proof of Claim 3.5

Proof. We first consider Case 1. For any O_i, if $w(e) < w(e_1)$ for all $e \in E(O_i)$, then $u(O_i)$ does not decrease. If $w(e) \geq w(e_1)$ for some $e \in E(O_i)$, $u(O_i) \geq w(e_1) \geq 2 \cdot \mathsf{opt}(\mathcal{I})$ and after decreasing the weights to $w(e_1)/2$, $u'(O_i) \geq \mathsf{opt}(\mathcal{I})$, implying the existence of an allocation with the minimum utility no smaller than $\mathsf{opt}(\mathcal{I})$, which means $\mathsf{opt}(\mathcal{I}') = \mathsf{opt}(\mathcal{I})$.

Second, we consider Case 2 when $w(e_1) < 2 \cdot \mathsf{opt}(\mathcal{I})$. It is straightforward that $2 \cdot \mathsf{opt}(\mathcal{I}') > \mathsf{opt}(\mathcal{I})$ since $w(e_1) > \mathsf{opt}(\mathcal{I})$ and after decreasing the weights of some edges to $w(e_1)/2$, $u'(O_i) \geq w(e_1)/2 > \mathsf{opt}(\mathcal{I})/2$ (If $e_1 \in O_i$, $u'(O_i) \geq w(e_1)/2$. Otherwise, $u'(O_i) = u(O_i) \geq \mathsf{opt}(\mathcal{I}) > \mathsf{opt}(\mathcal{I})/2$). Next we show $|M'_1| \geq 2$ which implies $w'(M'_1) \leq 2 \cdot w'(M'_n)$. For the sake of contradiction, assume $M'_1 = \{e'_1\}$. Note that at this moment, e'_1 must be an edge with the largest weight in the graph, which means $w'(e'_1) = w'(M'_1) \geq \cdots \geq w'(M'_n)$, and thus

$$w'(M') \leq n \cdot w'(e'_1) < n \cdot w'(O'_n) \leq \sum_{i \in N} w'(O'_i).$$

This is a contradiction with M' being a maximum matching in G, which completes the proof. □

A.5 Proof of Theorem 3.6

Before proving Theorem 3.6, we first show several technical lemmas. In the following, denote by $Y = (Y_1, \cdots, Y_n)$ the partial allocation after the **for** loop in Step 10 of Algorithm 2.

Lemma A.1. *Y is EF1.*

Proof. If $Q = \emptyset$, by definition, the allocation is already EF1. In the following, assume $Q \neq \emptyset$. Note that only the agents in $i \in Q$ has one vertex removed from $V(M_i)$ and for any $i \notin Q$, $Y_i = V(M_i)$. Particularly, $Y_n = V(M_n)$.

Fix any $i \in N \setminus \{n\}$. Let (v_{i1}, v_{i2}) be the edge selected in Step 11, i.e., the edge with the smallest weight in M_i. By the definition of greedy partition,

$$u(V(M_n)) \geq u(V(M_i) \setminus \{v_{i1}, v_{i2}\}). \tag{1}$$

We have the following claims.

Claim A.2. *Agent n does not envy agent i for more than one item in the partial allocation Y.*

Proof. The claim is straightforward if $i \notin Q$ since there is no edge between n and i. If $i \in Q$, then $Y_i = V(M_i) \setminus \{v_{i1}\}$ and by Inequality (1),

$$u(V(M_n)) \geq u(V(M_i) \setminus \{v_{i1}, v_{i2}\}) = u(Y_i \setminus \{v_{i2}\}),$$

implying n does not envy Y_i for more than one item. □

Claim A.3. *Agent i does not envy agent n in Y.*

Proof. If $i \notin Q$, the bundles of agent i and n do not change in the **for** loop in Step 10. Since M_n has the smallest weight in the greedy partition of M^*, we have

$$u(Y_i) = u(V(M_i)) \geq u(V(M_n)) = u(Y_n).$$

If $i \in Q$, since there is an edge from n to i, we have

$$u(Y_i) \geq \min_{v \in V(M_i)} u(V(M_i) \setminus \{v\}) > u(V(M_n)),$$

which means i does not envy n. □

Combining Claims A.2 and A.3, we have that for any two agents i and j,

$$u(Y_i) \geq u(Y_n) \geq u(Y_j \setminus \{v\}) \text{ for some } v \in Y_j,$$

which means i does not envy j for more than one item. This completes the proof of Lemma A.1. □

To prove the approximation ratio of Algorithm 2, we need the following lemma.

Lemma A.4. $|M_i| \geq 3$ *for all* $i \in Q$.

Proof. If the in-degree of agent i is non-zero, then agent n must envy i for more than one item, and

$$u(Y_n) < u(V(M_i) \setminus \{v\}) \text{ for any } v \in V(M_i). \tag{2}$$

First, it is easy to see that $|M_i| \neq 1$ since the removal of any node v makes the remaining utility be 0 and thus Equation (2) does not hold.

Next we show $|M_i| \neq 2$. For the sake of contradiction, assume $M_i = \{e, e'\}$ with $e = (v_1, v_2)$ and $e' = (v'_1, v'_2)$. Without loss of generality, we further assume $w(e) \geq w(e')$. Then it must be that $w(M_n) \geq w(e)$, otherwise e' cannot be added to M_i. Note that since M^* is a maximum weighted matching in G, $\{e, e'\}$ must be a maximum weighted matching in $G[M_i]$. If there exist edges in $G[M_i]$ whose weights are greater than $w(e)$, these edges must be adjacent to the same node, denoted by $\bar{v}$; otherwise they can form another matching with weight greater than $w(M_i)$. Thus by removing $\bar{v}$ from $G[M_i]$, the maximum matching in the remaining graph contains at most one edge, and all the remaining edges have weight at most $w(e)$, which means the maximum matching in $G[V(M_i) \setminus \{\bar{v}\}]$ brings utility no larger than $w(e)$. Therefore,

$$u(V(M_i) \setminus \{\bar{v}\}) \leq w(e) \leq w(M_n),$$

which is a contradiction with Equation (2). Combining the above two cases, we have $|M_i| \geq 3$. □

Based on the claims and lemmas presented above, we present the proof of Theorem 3.6 below.

Proof of Theorem 3.6. Let $(X_1, \cdots, X_n)$ be the allocation returned by Algorithm 2. If the allocation is from Step 6, then it must be EF1. This is because X_n has the smallest value and thus nobody envies n and each of X_i with $1 \leq i \leq n-1$ contains only two nodes which means the removal of one of them brings utility 0 to any agent. It also achieves the optimal social welfare since all edges in M^* are allocated to some agents.

Next we consider the case when the allocation is obtained from Step 18. By Lemma A.1, after the **for** loop in Step 10, the partial allocation is EF1. To show the final allocation to be EF1, it suffices to show that the **for** loop in Step 14 preserves EF1. This is true as in each round, only the bundle with the smallest value can be allocated one more item whose removal makes it smallest again.

Finally, we consider the social welfare loss. For each agent $i \in Q$, we observe that at most one node will be removed from $V(M_i)$ in the **for** loop in Step 10 and the **for** loop in Step 14 can only increase i's utility. Since the removed node v_{i1} is from the edge with the smallest weight in M_i, by Lemma A.4, we have

$$u(X_i) \geq \frac{2}{3} \cdot u(V(M_i)) \text{ for all } i \in N \setminus \{n\},$$

Moreover, for agent n and any $i \neq n$,

$$u(X_n) \geq u(M_i \setminus \{(v_{i1}, v_{i2})\}) \geq \frac{2}{3} \cdot u(V(M_i)). \tag{3}$$

Therefore

$$\begin{aligned}
\frac{\sum_{i \in N} u(X_i)}{\mathsf{sw}^*} &\geq \frac{\sum_{i \in N \setminus \{n\}} \frac{2}{3} \cdot u(V(M_i)) + u(V(M_n))}{\sum_{i \in N} u(V(M_i))} \\
&= \frac{2}{3} + \frac{1}{3} \cdot \frac{u(V(M_n))}{\sum_{i \in N} u(V(M_i))} \\
&\geq \frac{2}{3} + \frac{1}{3} \cdot \frac{u(V(M_n))}{(\frac{3}{2}(n-1)+1) \cdot u(V(M_n))} \\
&= \frac{2}{3} + \frac{2}{9n-3},
\end{aligned}$$

where the second inequality is because of Inequality (3) and we complete the proof of Theorem 3.6. □

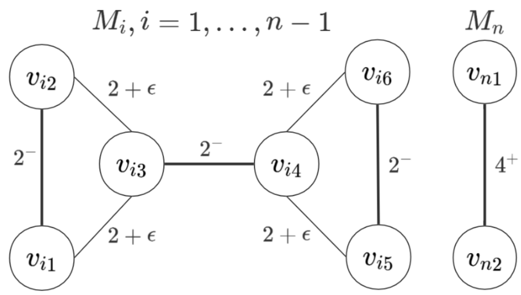


Figure 2: A graph contains n connected components where the first $n-1$ components are identical as shown by $M_i, i = 1, \cdots, n-1$, and the last component is a single edge as shown by M_n.

Tight Example. We show that the analysis in Theorem 3.6 is asymptotically tight. Consider the example in Figure 2, where 2^- means $2-\epsilon^2$ and 4^+ means $4+3(n-1)\epsilon^2$. Let $\epsilon > 0$ be a sufficiently small number, say $1/n^2$. The maximum matching M^* contains all the bold edges and $\mathsf{sw}^* = w(M^*) = 6(n-1)+4$. By Algorithm 2, the greedy-partition of M^* is $(M_1, \cdots, M_n)$ as shown in Figure 2. However, it is not EF1: for $1 \le i \le n-1$, by removing any vertex from M_i, the maximum matching in the remaining graph has weight at least $4+\epsilon > 4+3(n-1)\epsilon^2 = w(M_n)$. After the **for** loop in Step 10 in Algorithm 2, for $1 \le i \le n-1$, one vertex in each M_i is removed and is reallocated to M_n in the **for** loop in Step 14. Thus the remaining social welfare is at most

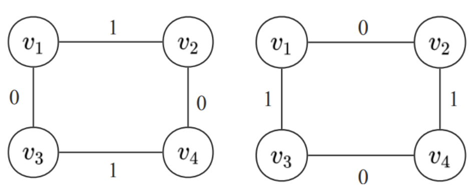
$$2 \cdot (2+\epsilon) \cdot (n-1) + 4 \to \frac{2}{3} \cdot \mathsf{sw}^*.$$

Remark. By Theorem 3.6, if $n = 2$, the approximation ratio is 4/5 and when $n \to \infty$ the approximation ratio is 2/3. Unfortunately, we were not able to prove an upper bound where the optimal social welfare cannot be achieved by any EF1 allocation. We conjecture that there is always an EF1 allocation that achieves the optimal social welfare sw^*.

B Missing Proofs of Section 4

B.1 Proof of Theorem 4.1

Proof. Consider the example as shown in Figure 3. The graph containing four nodes $\{v_1, v_2, v_3, v_4\}$ is allocated to two agents whose valuations (i.e., edge weights) are shown in Figure 3(a) and 3(b) respectively. It can be verified that $\mathsf{MMS}_i = 1$ for both $i = 1, 2$. However, no matter how we allocate the vertices to the agents, one of them receives utility of 0. □



(a) Agent 1's Metric (b) Agent 2's Metric

Figure 3: A bad example for which no allocation has bounded approximation of MMS fairness.

B.2 Proof of Theorem 4.2

Similar to Theorem 3.6, we want to find an EF1 allocation which also has high social welfare. Unfortunately, with heterogeneous agents, the fraction of efficiency loss can be as large as $1-1/n$.

Note that the optimal social welfare sw^* is no longer the maximum matching under a single metric, which can be computed by

$$\mathsf{sw}^* = \max_{X \in \Pi_n(V)} \sum_{i \in N} u_i(X_i).$$

Proof. Now, we give an instance where, for any $\epsilon > 0$, every EF1 allocation has social welfare at most $(1/n + \epsilon) \cdot \mathsf{sw}^*$. If $\epsilon \geq 1 - 1/n$, it holds trivially since no allocation can have social welfare more than sw^*. In the following, we assume $\epsilon < 1 - 1/n$.

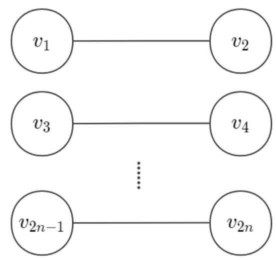


Figure 4: A graph with n disjoint edges is allocated to n agents.

Consider a graph with n disjoint edges, as shown in Figure 4, which is to be allocated to n agents. For each edge, agent 1 has value 1, and the other agents have value ϵ. The maximum social welfare $\mathsf{sw}^* = n$ is achieved by allocating all edges to agent 1. However, to guarantee EF1, at most one edge can be given to agent 1. The maximum welfare of an EF1 allocation is therefore at most $1 + (n-1) \cdot \epsilon$ (each agent receives exactly one edge). The largest ratio is

$$\frac{1 + (n-1) \cdot \epsilon}{n} < \frac{1}{n} + \epsilon,$$

which completes the proof of the Theorem, since ϵ can be arbitrarily small constant. $\square$

B.3 Proof of Theorem 4.3

We first prove Theorem 4.3 in Appendix B.3.1 and then show our analysis is tight in Appendix B.3.2.

B.3.1 The Proof

Before proving Theorem 4.3, we first give several useful lemmas.

Lemma B.1. *During the execution of Algorithm 3, the partial allocation maintains EF1.*

Proof. During the execution of Algorithm 3, two main cases within the **while** loop in Step 5 change the partial allocation, namely

- Case 1. Directly Allocate;
- Case 2. Exchange and Allocate.

Consider an arbitrary round $t \geq 1$. In Case 1, a single edge is allocated to one agent if and only if such allocation still guarantees EF1. Now, we consider Case 2. If $i \in A_t$ is the agent who is able to pick a subset $V^* \subseteq P$ to maintain his own utility, i.e., $u_i(V^*) = u_i(X_i)$, we show that any other agent does not envy i for more than one item after agent i receives bundle V^*. Let M_i^* be the maximum matching of $G[V^*]$ for agent i. We first consider agent $j^* \in A_l, l \in [t, \tau]$. Note that replacing X_i by V^* does not change the number of edges in the maximum matching M_i as well as the size of i's bundle X_i. Thus, we have

$$u_{j^*}(X_{j^*}) = |M_{j^*}| \geq |M_i| = |M_i^*| = \frac{|V^*|}{2} \geq u_{j^*}(V^*),$$

where the last inequality holds because for binary valuations, the valuation of a bundle for one agent is at most half the size of the bundle. Therefore, agent $j^* \in A_l, l \in [t, \tau]$ does not envy agent i up to more than one item after i replaces its bundle with V^*. Next, we consider agent $j^* \in A_l, l \in [t-1]$. For the sake of contradiction, assume $u_{j^*}(X_{j^*}) < u_{j^*}(V^*)$, which means that there exists at least one edge e such that $w_{j^*}(e) = 1$ as well as a bundle $V_{j^*} \subseteq P$ such that $u_{j^*}(V_{j^*}) = u_{j^*}(X_{j^*})$. Therefore, in the lth round of the **while** loop, a single edge e with weight 1 is added to agent j^* if it does not break EF1. Otherwise, there exists an agent j' who envies agent j^* before adding edge e. In such case, Algorithm 3 will execute the bundle-exchanging procedure in Step 13-17 in $l < t$th round of the **while** loop, which is a contradiction with the tth round of the **while** loop being executed. We complete the proof of Lemma B.1. □

For any graph $G = (V, E)$ and n different binary valuations $v_i(\cdot)$ on G, we call a matching M social welfare maximizing if M is a maximum matching on the graph $G' = (V, E')$ where for any $e \in E$ if and only there exists i such that $v_i(e) = 1$. Let M^* denote the social welfare maximizing matching on the input graph G of Algorithm 3. Let V_R be the set of unallocated items after we move out of the **while** loop in Step 25, and M_R be the social welfare maximizing matching on the induced subgraph of V_R. Let V_L be the set of allocated items after Step 25 and M_L be the welfare maximizing matching on V_L. Actually, $|M_L|$ is the social welfare that Algorithm 3 produces after Step 25. Then we have the following.

Lemma B.2. $|M_L| \geq |M_R|$.

Proof. We note that when Algorithm 3 moves out of the **while** loop in Step 5, any agent i values the unallocated items no more than its own bundle, i.e., $u_i(X_i) \geq u_i(V_R)$. Then we have $|M_L| = \sum_{i=1}^{n} u_i(X_i) \geq \sum_{i=1}^{n} u_i(V_R) \geq |M_R|$, which completes the proof. □

Based on the claims and lemmas presented above, we are ready to prove Theorem 4.3

Proof of Theorem 4.3. Let M_m be a welfare maximizing matching on the bipartite graph induced by V_L and V_R. i.e., finding as many disjoint edges e_{ij} as possible such that $v_i \in V_R$ and $v_j \in V_L$. Observe that the maximum number of vertices within V_L equals half the number of the edges in the maximum matching M_L, i.e., $|V_L| = 2|M_L|$. Therefore, the size of M_m is at most $2|M_L|$ (each vertex $v_1 \in V_L$ combined with another vertex $v_2 \in V_R$ to form a matching). Therefore, we have $|M^*| \leq 2|M_L| + |M_R|$. Furthermore, we have

$$\begin{aligned}\frac{u(M_L)}{\mathsf{sw}^*} = \frac{|M_L|}{|M^*|} &\geq \frac{|M_L|}{2|M_L| + |M_R|} \\ &\geq \frac{|M_L|}{3|M_L|} \geq \frac{1}{3},\end{aligned}$$

where the second inequality holds because $|M_L| \geq |M_R|$ proved in Lemma B.2. Since Step 26 can only increase the social welfare, we have proved the social welfare guarantee.

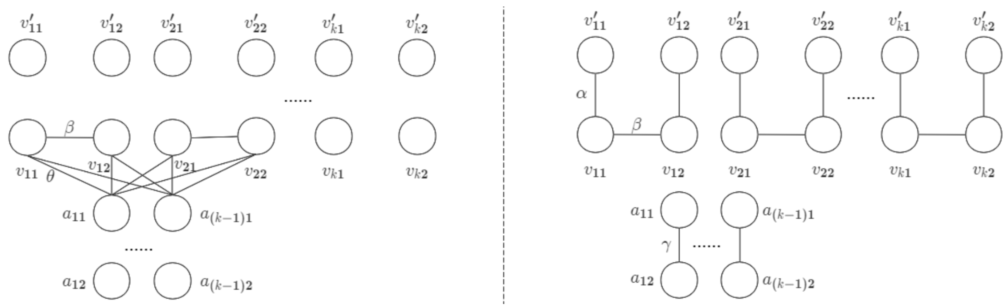
It remains to see the running time of the algorithm. In each iteration of the **while** loop in Step 5, the utility of exact one agent increases by 1. Since the maximum possible welfare is bounded by $O(|V|^2)$, the **while** loop will execute for at most $O(|V|^2)$ times. The envy-cycle elimination procedure in Step 26 will execute at most $O(|V|)$ times. Thus, Algorithm 3 runs in $O(|V|^2 + |V|) = O(|V|^2)$ time.

The proof of Theorem 4.3 is completed. □

B.3.2 Tight Example

We show that the analysis in Theorem 4.3 is asymptotically tight. Consider the example as shown in Figure 5. Let $k > 4$ be a constant. Denote by $\alpha_{ij}, i \in [k], j \in [2]$ the edge between node v_{ij} and node v'_{ij}, $\beta_i, i \in [k]$ the edge between node v_{i1} and v_{i2}, $\gamma_i, i \in [k-1]$ the edge between node a_{i1} and a_{i2}. Let $\theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4}, i \in [k]$ be the edge between node a_{i1} and node v_{11}, a_{i1} and node v_{12}, a_{i1} and node v_{21}, a_{i1} and node v_{22}, respectively. Obviously, allocating all the nodes to agent 2 and allocating nothing to agent 1 result in the optimal social welfare, i.e.,

$$\mathsf{sw}^* = u_2(V) = \sum_{i \in [k], j \in [2]} w_2(\alpha_{ij}) + \sum_{i \in [k-1]} w_2(\gamma_i) = 3k - 1. \tag{4}$$



(a) Agent 1's Metric (b) Agent 2's Metric

Figure 5: The graph is partitioned among two agents with binary valuations.

The corresponding maximum matching M^* contains $2k$ edges $\alpha_{ij}, i \in [k], j \in [2]$ and $k-1$ edges $\gamma_i, i \in [k-1]$. Now, we consider the worst case achieved by Algorithm 3 running on this example, which results in a total utility of $k+3$. In the first two rounds of the **while** loop in Step 5, each agent picks exactly one of the two edges β_1 and β_2 (w.l.o.g. agent 1 picks β_1 and agent 2 picks β_2). Following that agent 2 picks all the remaining edges $\beta_i, i \in [3, k]$ and arbitrary two edges $\gamma_i, i \in [k-1]$ (w.l.o.g. γ_1 and γ_2). We then move out of the **while** loop since (1) for agent 1, $u_1(P) = 0$; (2) for agent 2, $u_2(P) < u_2(X_2)$ and allocating any other edge $\gamma_i, i \in [3, k-1]$ to it will break EF1. Thus, we execute the envy-cycle elimination procedure on the remaining items, i.e., allocating all the remaining vertices in P to agent 1 with the EF1 allocation being completed. For agent 2, the maximum matching in $G[X_2]$ containing edges $\beta_i, i \in [2, k], \gamma_i, i \in [2]$. We thus have $u_2(X_2) = k - 1 + 2 = k + 1$. For agent 1, the maximum matching in $G[X_1]$ containing edges a_{11} and a_{12}. Therefore, $u_1(X_1) = 2$. The total social welfare is $u_1(X_1) + u_2(X_2) = k + 1 + 2 = k + 3$. Thus

$$\lim_{k \to +\infty} \frac{k+3}{3k-1} = \frac{1}{3},$$

which completes the proof of the theorem.

B.4 Proof of Theorem 4.4

We first prove Theorem 4.4 in Appendix B.4.1 and then show that the approximation ratio guarantee is tight, i.e., no algorithm is better than $1/3$-approximate in Appendix B.4.2.

B.4.1 The Proof

Proof. Denote (M_1, M_2) as a social welfare maximizing allocation. Consider the following two cases:

- Case 1: $\exists e \in E$ such that $w_i(e) \geq \frac{1}{3} \cdot \mathsf{sw}^*, i \in \{1, 2\}$;
- Case 2: $\forall e \in E$, $w_i(e) < \frac{1}{3} \cdot \mathsf{sw}^*, i \in \{1, 2\}$.

For Case 1, giving edge e to agent i and running the envy-cycle elimination procedure on remaining vertices can find an EF1 allocation, which, at the same time, guarantees the total utility no less than $1/3$ of the maximum possible social welfare.

Next, we consider Case 2. There are two subcases.

- Subcase 1: $u_i(M_i) \geq \frac{1}{3} \cdot \mathsf{sw}^*$ for all $i \in \{1, 2\}$;
- Subcase 2: $\exists i \in \{1, 2\}$ such that $u_i(M_i) < \frac{1}{3} \cdot \mathsf{sw}^*$.

For Subcase 1, if such allocation guarantees EF1, the theorem holds. Otherwise, agent 1 envies agent 2 since we assume $u_1(M_1) \leq u_2(M_2)$. We then reallocate the item $v \in X_2$ to agent 1 one by one

(a) Agent 1's weight for the graph (b) Agent 2's weight for the graph

Figure 6: An example where any EF1 allocation guarantees at most $(1/3 + 2\epsilon)$ of the maximum social welfare.

until such allocation guarantees EF1. The total utility is $u_1(X_1) + u_2(X_2) \geq u_1(M_1) \geq (1/3)\mathsf{sw}^*$. Therefore, we complete the proof for this subcase.

Consider Subcase 2. Without loss of generality, assume $u_1(M_1) < (1/3)\mathsf{sw}^*$. If allocation (M_1, M_2) guarantees EF1, the theorem is proved. Otherwise, by the assumption that $u_1(M_1) \leq u_2(M_2)$, agent 1 envies agent 2 more than one item. By $u_1(M_1) < (1/3)\mathsf{sw}^*$, we have $u_2(M_2) > (2/3)\mathsf{sw}^*$. Now, we consider to remove items from agent 2's bundle to agent 1's bundle. First, sorting the edges within M_2 by decreasing order according to their valuation to agent 2. In each iteration, we pick an edge within agent 2's bundle with largest weight and give one endpoint to agent 1. If the allocation still admits EF1, we give another endpoint to agent 1 and pick another edge with largest weight in agent 2's remaining bundle. Repeat above procedure until agent 1 envies agent 2 up to exact one item. When Algorithm 4 completed, at most one edge e within M_2 is destroyed, i.e., one endpoint of e is allocated to agent 1 and another endpoint still remains in X_2. If e is the edge with largest weight in M_2, we have $u_2(X_2) \geq u_2(M_2 \setminus \{e\}) > (1/3)\mathsf{sw}^*$, where the last inequality holds because $u_2(M_2) > (2/3)\mathsf{sw}^*$ and $w_2(e) < (1/3)\mathsf{sw}^*$. We thus complete the proof of the theorem. Otherwise, we next show that $u_2(X_2) \geq (1/3)u_2(M_2)$. Denote by X_2' be the set of items given to agent 1. We have

$$w_2(e) \leq u_2(X_2') \leq u_2(X_2), \tag{5}$$

where the first inequality holds because at least one edge within M_2 with larger weight is allocated to agent 1 before and the second inequality holds since otherwise agent 2 will envy agent 1. We thus derive

$$u_2(X_2) \geq \frac{1}{3}(u_2(X_2) + u_2(X_2') + w_2(e)) \geq \frac{1}{3}u_2(M_2). \tag{6}$$

Furthermore

$$\begin{aligned} u_1(X_1) + u_2(X_2) &\geq u_1(M_1) + \frac{1}{3}u_2(M_2) \\ &\geq \mathsf{sw}^* - u_2(M_2) + \frac{1}{3}u_2(M_2) \\ &= \mathsf{sw}^* - \frac{2}{3}u_2(M_2) \geq \frac{1}{3}\mathsf{sw}^*, \end{aligned} \tag{7}$$

where the last inequality holds because $u_2(M_2) \leq \mathsf{sw}^*$. Since in each iteration, at most one item is removed from agent 2 to agent 1, Algorithm 4 runs in $poly(|V|)$ time. We complete the proof of the theorem.

B.4.2 Tight Example

We next show the approximation of $1/3$ is optimal. Consider the example in Fig. 6(a) and Fig. 6(b). It is not hard to verify that the maximum social welfare without fairness constraint is $\mathsf{sw}^* = 3$ by allocating all the items to agent 1. However, for any allocation where agent 1 has utility no smaller than 2, the allocation is not EF1 to agent 2 since agent 2 always has utility 0 in such allocations. Therefore, the maximum social welfare generated by EF1 allocations is no greater than $1 + 2\epsilon$. Thus

$$\lim_{x \to 0} \frac{1 + 2\epsilon}{3} = \frac{1}{3}, \tag{8}$$

which means the approximation ratio of $1/3$ is optimal. $\square$

C EF1 Allocation with Bounded Social Welfare Guarantee

Now we are ready to present an algorithm to find an EF1 allocation with bounded social welfare guarantee in polynomial time.

Theorem C.1. *For any instance $\mathcal{I} = (G, N)$, Algorithm 5 returns an EF1 allocation with social welfare at least $1/(4n^2) \cdot \mathsf{sw}^*(\mathcal{I})$ in polynomial time.*

Algorithm 5: Computing EF1 Allocations for n Heterogeneous Agents with Distinct Weights

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Input: Instance $\mathcal{I} = (G, N, w)$ with $G = (V, E)$.
Output: Allocation $\mathbf{X} = (X_1, \cdots, X_n)$.

1: Initialize $X_i \leftarrow \emptyset, i \in N$. Let M_i be the maximum matching in $G[X_i]$ for agent i. Denote by $\mathcal{G}' = (N, \mathcal{E})$ the envy-graph on $\mathbf{X}$.
2: Let $P = V \setminus (X_1 \cup \cdots \cup X_n)$ be the set of unallocated items (called *pool*).
3: Denote H as the set of agents who are not envied by any other agents. Initialize $H \leftarrow N$.
4: Let agent i^* determine a maximum matching M_{i^*} in graph G. Denote by R the set of the remaining edges within the maximum matching. Initialize $R \leftarrow M_{i^*}$.
5: Sort the edges $e \in M_{i^*}$ by non-increasing order according to their weight to agent i^*.
6: Let agent i^* pick one edge with largest weight $w_{i^*}(e)$ (with ties broken arbitrarily).
7: **while** $\{R \neq \emptyset\}$ **do**
8:     Select one agent $i \in H$.
9:     **if** $\{P_i = \emptyset\}$ **then**
10:         Select one edge $e \in R$ with largest weight to agent i^*. Give one endpoint v_1 of e to agent i and put another endpoint v_2 in the corresponding pool P_i, i.e., $R \leftarrow R \setminus \{e\}$, $X_i \leftarrow X_i \cup \{v_1\}$, $P_i \leftarrow P_i \cup \{v_2\}$, $P \leftarrow P \setminus \{v_1, v_2\}$.
11:         Update the envy-graph $\mathcal{G}'$ and set H.
12:     **else**
13:         Give the node $v \in P_i$ to agent i, i.e., $P_i \leftarrow \emptyset$, $X_i \leftarrow X_i \cup \{v\}$.
14:         Update the envy-graph $\mathcal{G}'$ and set H.
15:     **end if**
16: **end while**
17: Return all the vertices within P_i to the pool P, i.e., $P \leftarrow P \bigcup_{i \in N} P_i$.
18: Execute the envy-cycle elimination procedure running on the remaining items P.
19: Return the allocation $(X_1, \cdots, X_n)$.

Without loss of generality, we assume i^* to be the agent who has the maximum value of $u_i(V), i \in N$. Denote by $S = (S_1, \cdots, S_n)$ the partial allocation when we first move out of the **while** loop in Step 7. Before presenting the proof of Theorem C.1, we first present a useful lemma.

Lemma C.2. $\sum\limits_{i \in N} u_{i^*}(X_i) \geq \frac{1}{2} u_{i^*}(V)$.

Proof. During the execution of Algorithm 5, there is at most one node v_i in pool $P_i, i \in [n]$. Consider P_i when we first move out of the **while** loop in Step 7. If $P_i \neq \emptyset$, w.l.o.g. suppose $v_i \in P_i$ is one endpoint of edge $e_i \in M_{i^*}$. Let N_1 be the set of agents such that $P_i \neq \emptyset$. Since the edges are picked by non-increasing order of their weight to agent i^*, we have $u_{i^*}(S_i) \geq w_{i^*}(e_i), i \in N_1$. Furthermore, we have

$$2u_{i^*}(S_i) \geq u_{i^*}(S_i) + w_{i^*}(e_i) \geq u_{i^*}(S_i \cup \{v_i\}).$$

Thus, $u_{i^*}(S_i) \geq \frac{1}{2}u_{i^*}(S_i \cup \{v_i\}), i \in N_1$. We have

$$\begin{aligned}
\sum_{i\in N} u_{i^*}(X_i) \geq& \sum_{i\in N} u_{i^*}(S_i) \\
&\geq \frac{1}{2}\sum_{i\in N_1} u_{i^*}(S_i \cup \{v_i\}) + \sum_{i\in N\setminus N_1} u_{i^*}(S_i) \\
&\geq \frac{1}{2}\sum_{i\in N_1} u_{i^*}(S_i \cup \{v_i\}) + \frac{1}{2}\sum_{i\in N\setminus N_1} u_{i^*}(S_i) \\
&= \frac{1}{2}\sum_{i\in N} u_{i^*}(V),
\end{aligned}$$

where the last equality holds because the nodes within the remaining pool P after we move out of the **while** loop in Step 7 do not have any effect on the maximum matching M_{i^*}. We complete the proof of Lemma C.2. □

Now we are ready to prove Theorem C.1.

Proof of Theorem C.1. Let N_e be the set of agents that agent i^* envies. Since Algorithm 5 admits EF1, there exists one node $v \in X_i, i \in N_e$ such that $u_{i^*}(X_{i^*}) \geq u_{i^*}(X_i \setminus \{v\}), i \in N_e$. For any agent $i \in N_e$, w.l.o.g. assume v_1 is the node such that $u_{i^*}(X_{i^*}) \geq u_{i^*}(X_i \setminus \{v_1\})$ and $v_1, v_2 \in S_i$ are two endpoints of the edge $e \in M_{i^*}$. Since agent i^* first picks the edge with the largest weight to itself, we have $u_{i^*}(X_{i^*}) \geq w_{i^*}(e) = u_{i^*}(\{v_1, v_2\})$. By the definition of EF1, $u_{i^*}(X_{i^*}) \geq u_{i^*}(X_i \setminus \{v_1\}) \geq u_{i^*}(X_i \setminus \{v_1, v_2\})$ holds. Thus, we have $u_{i^*}(X_{i^*}) \geq \frac{1}{2}(u_{i^*}(X_i \setminus \{v_1, v_2\}) + u_{i^*}(\{v_1, v_2\})) = \frac{1}{2}u_{i^*}(X_i)$. Let $(X_1^*, \cdots, X_n^*)$ be the welfare maximization allocation, where $X_i^*, i \in N$ is the set of vertices allocated to agent i. Therefore

$$\begin{aligned}
n \cdot u_{i^*}(X_{i^*}) &\geq \frac{1}{2}\sum_{i\in N} u_{i^*}(X_i) \geq \frac{1}{2}\cdot\frac{1}{2}u_{i^*}(V) \\
&\geq \frac{1}{4}\cdot\frac{1}{n}\sum_{i\in N} u_i(V) \geq \frac{1}{4n}\sum_{i\in N} u_i(X_i^*) \\
&= \frac{1}{4n}\mathsf{sw}^*(\mathcal{I}),
\end{aligned}$$

where the second inequality follows by Lemma C.2 and the third inequality holds because of the assumption that i^* is the agent with the largest value of $u_i(V), i \in N$. Since in each iteration one node is allocated to an agent, the time complexity of Algorithm 5 is at most $O(|V|^2)$, completing the proof of Theorem C.1. □

D Nash Social Welfare and EF1 Allocations

Before introducing Nash social welfare, We first generalize the definition of EF1 to envy-free up to k items and its approximations. For $\alpha \geq 0$, allocation $(X_1, \cdots, X_n)$ is α-approximate envy-free up to k items (α-EFk) if for any i and j, there exists $S = \{g_1, \cdots, g_k\} \subseteq X_j$ such that

$$u_i(X_i) \geq \alpha \cdot u_i(X_j \setminus S).$$

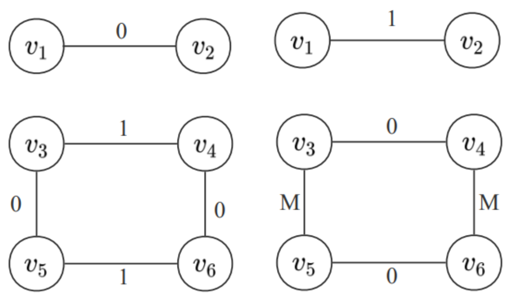
Now, we are ready to give several results concerning nash social welfare, which is (informally) the product of all agents' utilities. It is proved in [Caragiannis *et al.*, 2019] that under additive valuations, EF1 and Pareto Optimality (PO) are compatible and an allocation that maximizes Nash social welfare is always simultaneously EF1 and PO. Here, an allocation is PO if there is no alternative allocation that makes an agent better off without making anyone worse off. Recently, Wu *et al.* [2021] showed that with subadditive valuations, a Nash social welfare maximizer is still PO but only 1/4-EF1. In our results, we observe that when the valuations are measured by maximum matchings, Nash social welfare maximizer is PO and EF2 for any matching valuations. However, the Nash social welfare maximizer does not have any bounded approximation guarantee on EF1.

Proposition D.1. *An allocation that maximizes Nash social welfare does not have bounded approximation ratio on EF1.*

Proof. Consider the example shown in Figure 7, where the two agents have different metrics as shown in Figure 7(a) and Figure 7(b) respectively. Let M be a sufficiently large number. The unique allocation that maximizes the Nash social welfare is to assign $X_1 = \{v_3, v_4, v_5, v_6\}$ to agent 1 and $X_2 = \{v_1, v_2\}$ to agent 2. However, for any $v \in X_1$,

$$u_2(X_1 \setminus \{v\}) = M \gg 1 = u_2(X_2).$$

When M goes to infinity, the allocation does not have any bounded approximation ratio. $\square$



(a) Agent 1's Metric (b) Agent 2's Metric

Figure 7: The graph containing a single edge and a square is allocated to two agents.

Note that even with identical valuations, the Nash social welfare maximizing allocation does not guarantee EF1. Consider the example shown in Figure 8, where ϵ is arbitrarily small. To maximize the Nash social welfare, one possible allocation is that $X_1 = \{v_1, v_2\}$ and $X_2 = \{v_3, v_4, v_5, v_6, v_7, v_8\}$. However, this allocation does not guarantee EF1 for agent 1 since the removal of any vertex in X_2 still admits a matching with weight at least 8.5, which is greater than $u_1(X_1) = 8 + \epsilon$.

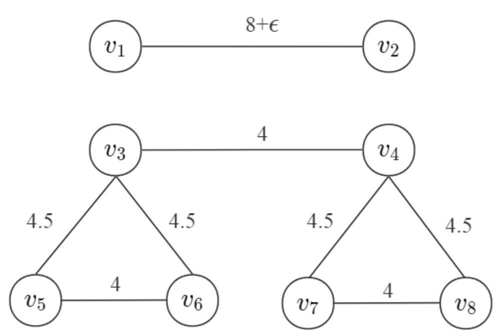


Figure 8: The graph is allocated to two agents with identical valuations. The allocation maximizing Nash social welfare fails to guarantee EF1.

Although Proposition D.1 is disappointing in general, an allocation that maximizes Nash social welfare is PO and EF2, and when the valuations are identical, it ensures 2/3-EF1.

Proposition D.2. *A Nash social welfare maximizing allocation is PO and EF2 for any matching-induced valuations, and is 2/3-EF1 if the valuations are identical.*

Proof. We only prove for the case of two agents, and the proof for arbitrary number of agents is the same. Given any graph $G = (V, E)$, suppose (V_1, V_2) is an allocation that maximizes the Nash social welfare. Denote by $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ the induced subgraphs by V_1 and V_2 respectively. Denote by M_1 and M_2 the maximum matchings in G_1 and G_2. To show (V_1, V_2) is EF2, we can regard the edges in M_1 and M_2 as items and by the result from [Caragiannis *et al.*, 2019], the allocation must be envy-free up to one edge, which guarantees EF2, since deleting one edge can be achieved by deleting two nodes.

Next we show the allocation is 2/3-EF1 for identical valuations. For the sake of contradiction, suppose that for any $v \in V_2$,

$$u(V_1) = w(M_1) < \frac{2}{3} \cdot u(V_2 \setminus \{v\}) \leq \frac{2}{3} \cdot w(M_2).$$

By Lemma A.4, $|M_2| \geq 3$ and thus the smallest edge in M_2 denoted by e has weight at most $1/3 \cdot w(M_2)$. Thus

$$w(M_1) + w(e) < \frac{2}{3} \cdot w(M_2) + \frac{2}{3} \cdot w(M_2) = w(M_2). \tag{9}$$

Consider a new allocation by assigning edge e to agent 1. The resulting maximum matchings in each subgraph becomes $M_1 \cup \{e\}$ and $M_2 \setminus \{e\}$. The Nash social welfare of the new allocation is

$$\begin{aligned}
&w(M_1 \cup \{e\}) \cdot w(M_2 \setminus \{e\}) \\
&= (w(M_1) + w(e)) \cdot (w(M_2) - w(e)) \\
&= w(M_1) \cdot w(M_2) + (w(M_2) - w(M_1) - w(e)) \cdot w(e) \\
&> w(M_1) \cdot w(M_2),
\end{aligned}$$

where the inequality is by (9). However, this is a contradiction with the fact that (V_1, V_2) maximizes the Nash social welfare. □

E An Improved MMS Allocation Algorithm for Two Homogeneous Agents

When there are only two agents, Algorithm 1 can be refined as shown in Algorithm 6, and the approximation ratio can be improved to $2/3$. Algorithm 6 is similar with Algorithm 1; we first compute a maximum matching M^* and a max-min partition (M_1, M_2) with $w(M_1) \geq w(M_2)$. If $|M_1| \geq 2$, we output the corresponding allocation. Otherwise, in graph G, we directly delete the edge that M_1 contains. We repeat the above procedure until all edges are removed.

Algorithm 6: Max-Min Allocation for 2 Agents

Input: Instance $\mathcal{I} = (G, N, u)$ with $G = (V, E; w)$.
Output: Allocation $\mathbf{X} = (X_1, X_2)$.

Find a maximum matching M^* in G. Denote by V' the set of unmatched vertices by M^*.
Find the greedy partition (M_1, M_2) of edges in M^* such that $w(M_1) \geq w(M_2)$.
Let $Max = w(M_2)$.
Set $X_1 = V(M_1)$.
Set $X_2 = V(M_2) \cup V'$.
while $w(M_1) > 2w(M_2)$ **do**
  By Lemma 3.3. M_1 must contain only one edge. Suppose $M_1 = \{e^*\}$.
  Delete edge e^*.
  Re-compute a maximum matching M^*.
  Re-set V' to be unmatched vertices by M^*.
  Re-compute the greedy partition (M_1, M_2) of M^* such that $w(M_1) \geq w(M_2)$.
  if $Max < w(M_2)$ **then**
    $Max = w(M_2)$.
    Set $X_1 = V(M_1)$.
    Set $X_2 = V(M_2) \cup V'$.
  end if
end while
Output allocation (X_1, X_2).

Theorem E.1. *Algorithm 6 outputs an allocation that is $2/3$-approximate max-min fair in polynomial time .*

Proof. Given an Instance $\mathcal{I} = (G, N, u)$ with $G = (V, E; w)$. Denote by $O = (O_1, O_2)$ the optimal solution, where $u(O_1) \geq u(O_2)$ and $\mathsf{opt}(\mathcal{I}) = u(O_2)$. The first time when we reach the **while** loop, if $w(M_1) \leq 2 \cdot w(M_2)$, allocation $(X_1, X_2) = (V(M_1), V(M_2) \cup V')$ has been output. By Algorithm 6, we have

$$w(M_1) \geq u(O_1) \geq u(O_2) \geq w(M_2).$$

Moreover,

$$\begin{aligned} w(M_2) &\geq \frac{1}{3} \cdot (w(M_1) + w(M_2)) \\ &\geq \frac{1}{3} \cdot (u(O_1) + u(O_2)) \\ &\geq \frac{1}{3} \cdot 2 \cdot u(O_2) = \frac{2}{3} \cdot u(O_2). \end{aligned}$$

We move into the **while** loop if $w(M_1) > 2 \cdot w(M_2)$. In such case, M_1 contains only one edge, i.e., $|M_1| = 1$. Suppose $M_1 = \{e^*\}$. There are two subcases:

- Case 1: $e^* \in O_1 \cup O_2$
- Case 2: $e^* \notin O_1 \cup O_2$

First. consider $|M_1| = 1$ and $e^* \in O_1 \cup O_2$. We have: $e^* \in O_1$ and $w(M_2) = u(O_2) = \mathsf{opt}(\mathcal{I})$. The optimal solution has been found and recorded. Therefore, the approximation ratio of the max-min partition is 1. Then the **while** loop is executed for the next round. When $e^* \notin O_1 \cup O_2$, edge e^* is deleted. Let (M_1', M_2') be the greedy partition after deleting edge e^*. Then there are two subcases:

- Subcase 1: $|M_1'| \geq 2$
- Subcase 2: $|M_1'| = 1$

For Subcase 1, we will get out of the **while** loop and a $2/3$-approximate max-min allocation has been determined. For Subcase 2, the **while** loop is executed for the next round. Since we are not sure whether $e^* \in O_1 \cup O_2$, the **while** loop is executed for at most $O(m^2)$ rounds (m is the number of nodes in graph $G(V, E)$). The output (X_1, X_2) is at least $2/3$-approximate max-min fair allocation. Thus, the theorem holds. ☐

Lemma E.2. *Algorithm 6 outputs an allocation that is* $1/2$*-approximate max-min fair by eliminating at most two edges.*

Proof. Given an Instance $\mathcal{I} = (G, N, u)$ with $G = (V, E; w)$. Denote by $O = (O_1, O_2)$ the optimal solution before eliminating any edge, where $u(O_1) \geq u(O_2)$ and $\mathsf{opt}(\mathcal{I}) = u(O_2)$. Initially, under the maximum matching M', we find the greedy max-min partition (M_1, M_2) such that $w(M_1) \geq w(M_2)$. If $|M_1| \geq 2$, by Lemma 3.3, (M_1, M_2) is a $1/2$-approximation max-min partition. Next, consider that M_1 contains only one edge. Suppose $M_1 = \{e_1\}$ and $e_1 \notin O_1 \cup O_2$ (otherwise, by Case 1 in Theorem E.1, $w(M_2) = u(O_2) = \mathsf{opt}(\mathcal{I})$. The optimal solution has been found). If we eliminate edge e_1, under the re-computed maximum matching, we find the greedy max-min partition $(M_1^{'}, M_2^{'})$. Let $O' = (O_1', O_2')$ denote the optimal solution after eliminating edge e_1, where $u(O_1') \geq u(O_2')$ and $\mathsf{opt}'(\mathcal{I}') = u(O_2')$. $M_1^{'} = \{e_1^{'}\}$ and $e_1^{'} \notin O_1' \cup O_2'$. We first show that the two edges e_1 and $e_1^{'}$ have one common endpoint. By Algorithm 6,

$$w(e_1) > u(O_1) \geq u(O_2) > w(M_2),$$

and

$$w(e_1^{'}) > u(O_1') \geq u(O_2') > w(M_2^{'}).$$

Hence, $w(e_1^{'}) > w(M_2)$. Furthermore,

$$w(e_1) + w(e_1^{'}) > w(e_1) + w(M_2).$$

Therefore, edges e_1 and $e_1^{'}$ make a maximum matching, which implies that there exists an allocation to improve the max-min value from $u(O_2)$ to $w(e_1')$. It results in a contradiction. Hence, the two edges e_1 and $e_1^{'}$ have one common endpoint. Furthermore, $e_1 \notin O_1 \cup O_2$ (otherwise, suppose

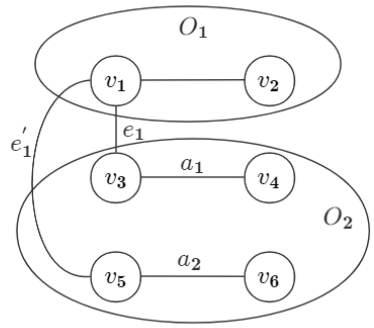


Figure 9: The graph is allocated to two agents with identical valuations. Two large edges e_1 and e_1' have one common endpoint and their other endpoints connect two distinct edges.

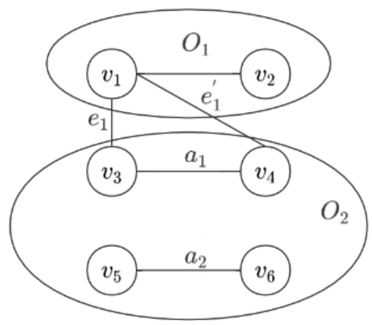


Figure 10: The graph is allocated to two agents with identical valuations. Two large edges e_1 and e_1' have one common endpoint and their other endpoints are the two endpoints of another same edge.

$e_1 \in O_1$. Then O_1 can be replaced by edge e_1 to get larger welfare, which makes a contradiction). Therefore, there are two cases. First, we consider Case 1 (as shown in Figure 9): suppose a_1 and a_2 are two edges in O_2, and $w(a_1) \geq w(a_2)$. Denote $O_2^* = O_2/\{a_1, a_2\}$. Thus

$$\max(w(a_1), w(a_2)) \geq \frac{1}{2}(w(a_1) + w(a_2)),$$

and then

$$w(a_1) + u(O_2^*) \geq \frac{1}{2}u(O_2).$$

Accordingly, either

$$w(M_2) \geq w(a_1) + u(O_2^*)$$

or

$$w(M_2^{'}) \geq w(a_1) + u(O_2^*)$$

holds; otherwise, replacing M_2 or $M_2^{'}$ with $a_1 \cup O_2^*$ can make a matching with larger welfare. Therefore, $w(M_2) \geq 1/2 \cdot u(O_2)$ or $W(M_2^{'}) \geq 1/2 \cdot u(O_2)$. Next, we consider Case 2 (as shown in Figure 10). Suppose after the two edges e_1 and $e_1^{'}$ have been deleted, under the new maximum matching M'', we find the greedy partition $(M_1^{''}, M_2^{''})$. If $|M_1^{''}| \geq 2$, by Theorem E.1, the 2/3-approximate max-min partition can be found. If $|M_1^{''}| = 1$, then

$$w(M_2^{''}) \geq w(a_1) + O(B^{'}) \geq \frac{1}{2}u(B_2).$$

Thus a $1/2$-approximate max-min partition has been found, and the lemma holds. $\square$

F Examples

F.1 An Example where Envy-cycle Elimination Algorithm does not Work

Consider a path of four nodes $v_1 \to v_2 \to v_3 \to v_4$, and two agents have the same weight 1 on all three edges (v_1, v_2), v_2, v_3 and v_3, v_4. By ency-cycle elimination algorithm, we may first allocate the items in the following order: v_1 to agent 1, then v_2 to agent 2, then v_3 to agent 1 and finally v_4 to agent 2. Note that $u_1(\{v_1, v_3\}) = v_2(\{v_2, v_4\}) = 0$, however, the optimal social welfare is 2 by allocating $\{v_1, v_2\}$ to agent 1 and $\{v_3, v_4\}$ to agent 2. Thus the approximation ratio of the social welfare is unbounded.

F.2 e_1 May not Have the Largest Weight

In the execution of Algorithm 1, the edge e_1, which is the only edge in M_1, may not have the largest weight. Consider an instance with two agents and the graph is shown in Figure 11. By Algorithm 1, in the first round of the *while* loop, we find a maximum matching, say, $M^* = \{e_{12}, e_{34}, e_{56}, e_{78}\}$. The greedy partition of M^* is $M_1 = \{e_{34}\}$, $M_2 = \{e_{56}\}$, $M_3 = \{e_{12}, e_{78}\}$. Since $w(M_3) = 4 < 1/2 \cdot w(M_1) = 32$, $H = \{e_{34}, e_{45}, e_{56}\}$, which contains all the edges with at least $w(e_{34}) = 32$. Noe that in this case, e_1 does not have the highest 64.

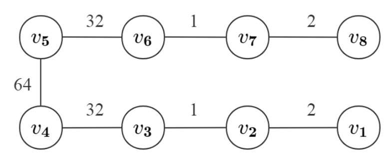


Figure 11: The graph is allocated to three agents with identical valuations.