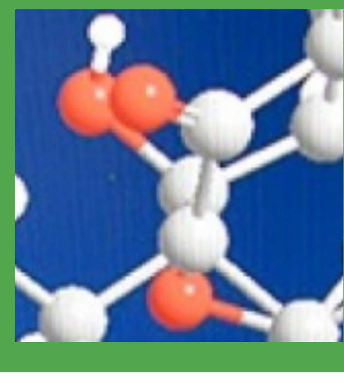
K-theoretic Persistent Cohomology



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Overview: We propose K-theoretic persistent cohomology (KPCH), which equips the Grothendieck group of persistence modules with lambda-operations arising from exterior powers. This yields new persistence layers that measure concurrency of structures, thus allowing to distinguish graph structures indistinguishable by usual persistent homology.

Objective: Beyond additive summaries of H^p (e.g. total persistence), quantify simultaneity of classes. KPCH does this by placing $[H^p]$ in a K-theoretic λ-ring and extracting exterior-power layers $\Lambda^i H^p$ that are computable, stable, and interpretable as concurrency.

Setup: Fix a field k and a totally ordered $I \subset \mathbb{R}$. A persistence module is a functor

$$M:\ I \to \mathbf{Vect}_k, \qquad t \mapsto M_t, \quad (s \le t) \mapsto \phi^M_{s\le t} : M_s \to M_t.$$

Let $\mathsf{PMod}_{\mathrm{pfd}}$ denote pointwise fin-dim modules. The interval decomposition thm yields a barcode $\mathcal{B}(M)$ (a finite multisite of intervals when M is tame) and $M \cong \bigoplus_{J\in\mathcal{B}(M)} I_J$.
Define Betti curve $\beta_M(t) := \dim_k M_t$ and total persistence $\mathrm{TP}(\mathcal{B}(M)) := \sum_{[b,d)\in\mathcal{B}(M)} (d-b)$.

K-theory and λ-structure:
Let $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ be the Grothendieck group generated by iso. classes [X] with relations [B] = [A] + [C] for every short exact 0 → A → B → C → 0 in $\mathsf{PMod}_{\mathrm{pfd}}$.
Define multiplication by $[M]\cdot[N] := [M\otimes N]$
Define $\lambda^n([M]) := [\Lambda^n M]$ where $(\Lambda^n M)_t = \Lambda^n(M_t)$ with structure maps $\Lambda^n(\phi^M_{s\le t})$.

Theorem (special λ-structure): $K_0(\mathsf{PMod}_{\mathrm{pfd}})$ with · and λ^n satisfies

$$\lambda^0(x) = 1, \quad \lambda^1(x) = x, \quad \lambda_t(x+y) = \lambda_t(x)\lambda_t(y), \quad \lambda_t = \sum_{n\ge 0} \lambda^n t^n.$$

For a filtered space $(X_\alpha)_{\alpha\in I}$, set $\kappa_p := [H^p(X_\alpha;k)] \in K_0$
and interpret $\lambda^i(\kappa_p)$ as the K-class of the new persistence layer $\Lambda^i H^p$.

Interval calculus: If $M \cong \bigoplus_{r\in R} I_{J_r}$, then the following hold:

$$\Lambda^i M \ \cong \bigoplus_{r_1<\cdots<r_i} I_{J_{r_1}\cap\cdots\cap J_{r_i}}, \qquad \mathcal{B}(\Lambda^i M) = \left\{[\max b_{\ell_j},\, \min d_{\ell_j})\right\}_{\ell_1<\cdots<\ell_i}$$

keeping only positive-length intersections. Thus Λ^i records i-fold concurrency windows.

Stability: Let d_I be the interleaving distance and d_B the bottleneck distance,

$$d_I(\Lambda^i M, \Lambda^i N) \ \le\ d_I(M,N) \qquad \Rightarrow \qquad d_B(\mathrm{Dgm}(\Lambda^i M), \mathrm{Dgm}(\Lambda^i N)) \ \le\ d_B(\mathrm{Dgm}(M), \mathrm{Dgm}(N)).$$

Integral formula: $\mathrm{TP}(\mathcal{B}(\Lambda^i M)) \ = \ \int_I \binom{\beta_M(t)}{i}\, dt, \qquad \dim_k(\Lambda^i M_t) = \binom{\beta_M(t)}{i}.$

This enables a single-pass computation of the total i-fold concurrency form $\mathcal{B}(M)$.

Experiment Results:

Feature	Sequential	Overlap	AUROC	Acc.
$\mathrm{TP}(\mathcal{B}(\Lambda^2 H^1))$	0.00	0.62	0.99	0.97
$\mathrm{TP}(\mathcal{B}(H^1))$	1.58	1.60	0.52	0.55
Max lifetime in H^1	0.86	0.88	0.54	0.53