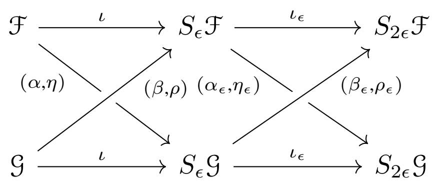
A. Proof of Theorem 4.10

Left inequality

As shown in (Bauer et al., 2015), $d_I\big(R,S\big) \leq d_{FD}\big(R_f,S_g\big)$ and, since the functional distortion distance on Reeb graphs corresponds to the first part of the functional distortion distance of barcode transforms (Definition 4.9), we obviously get $d_{FD}\big(R_f,S_g\big) \leq d_{FD}\big(BF,BG\big)$.

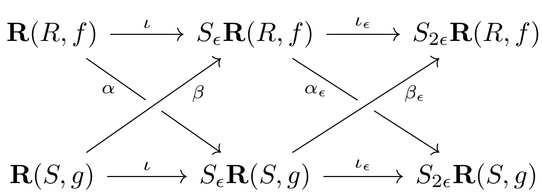
Right inequality

Let $S_\epsilon\colon \mathbf{Fun}(\mathcal{O}(\mathbb{R}),\mathbf{PVec}) \to \mathbf{Fun}(\mathcal{O}(\mathbb{R}),\mathbf{PVec})$ be the smoothing functor, defined by $S_\epsilon\mathcal{F}(U) := \mathcal{F}(U^\epsilon)$ and $\iota\colon \mathcal{F} \to S_\epsilon\mathcal{F}$ be defined by $\iota(U) := \mathcal{F}(U \subseteq U^\epsilon)$. In the following we denote an object of $\mathbf{PVec}$ by a tuple (I,D) representing a set I and a functor $D\colon I \to \mathbf{Vec}$. Suppose $\mathcal{F}$ and $\mathcal{G}$ are ϵ-interleaved, i.e. we have the following commutative diagram:



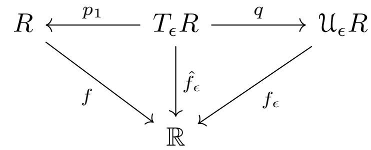
(4)

where α denotes the morphisms between the parameterizing sets and η denotes the morphisms between the parameterized vector spaces. Let $\mathbf{dom}\colon \mathbf{PVec} \to \mathbf{Set}$ be a forgetful functor defined on an object $(I,D) \in \mathbf{PVec}$ by $\mathbf{dom}\big((I,D)\big) := I$ and on a morphism $(\alpha,\eta)\colon (I,D) \to (I',D')$ by $\mathbf{dom}\big((\alpha,\eta)\big) := \alpha$. If we postcompose $\mathcal{F}$ with $\mathbf{dom}$ we obtain $\mathbf{dom} \circ \mathcal{F}(U) = \mathbf{dom}(\mathcal{F}(U)) = \pi_0(f^{-1}(U))$ and $\mathbf{dom} \circ \mathcal{F}(U \subseteq V) = \pi_0(f^{-1}(U) \subseteq f^{-1}(V))$. Hence, we get $\mathbf{dom} \circ \mathcal{F} = \mathcal{R}$ the categorical Reeb graph corresponding to (R,f). Denote by $\mathbf{R}\colon \mathbf{concreteReebgraphs} \to \mathbf{categoricalReebgraphs}$ the functor that sends a concrete Reeb graph (R,f) to the corresponding categorical Reeb graph $\mathcal{R}$ (see (de Silva et al., 2016)). We now apply $\mathbf{dom}$ on Equation (4) and obtain the following commutative diagram of $\mathbf{Set}$-valued functors:



(5)

By Proposition 4.29 in (de Silva et al., 2016), the smoothing of open sets S_ϵ is equivalent to the smoothing of the underlying geometric Reeb graphs. Let $T_\epsilon(R,f)$ be the ϵ-thickening of (R,f) defined by $T_\epsilon R := R \times [-\epsilon,\epsilon]$ and $\mathcal{U}_\epsilon(R,f)$ be the Reeb graph of $T_\epsilon(R,f)$ (the ϵ-smoothing of (R,f)). These spaces can be summarized by the following commutative diagram:



(6)

where p_1 is the projection to the first factor and q is the quotient map to the Reeb space. The map p_1 induces a natural isomorphism $\mathbf{R}T_\epsilon \implies S_\epsilon\mathbf{R}$ such that

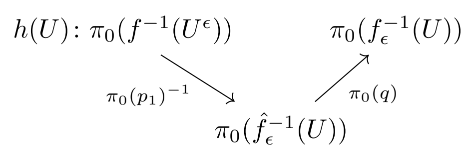
$$\Big(\mathbf{R}T_\epsilon(R,f)(U) \to S_\epsilon\mathbf{R}(R,f)(U)\Big)$$
$$=\Big(\pi_0(\hat{f}_\epsilon^{-1}(U)) \xrightarrow{\pi_0(p_1)} \pi_0(f^{-1}(U^\epsilon))\Big)$$

; (de Silva et al., 2016) Theorem 4.2. Moreover, the map q induces a natural isomorphism $\mathbf{R}T_\epsilon \implies \mathbf{R}\mathcal{U}_\epsilon$ such that

$$\Big(\mathbf{R}T_\epsilon(R,f)(U) \to \mathbf{R}\mathcal{U}_\epsilon(R,f)(U)\Big)$$
$$=\Big(\pi_0(\hat{f}_\epsilon^{-1}(U)) \xrightarrow{\pi_0(q)} \pi_0(f_\epsilon^{-1}(U))\Big)$$

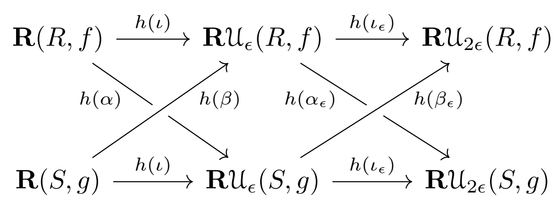
; (de Silva et al., 2016) Theorem 3.15. Let h denote the composition of the following natural isomorphisms:

$$h\colon S_\epsilon\mathbf{R} \Longrightarrow \mathbf{R}T_\epsilon \implies \mathbf{R}\mathcal{U}_\epsilon$$



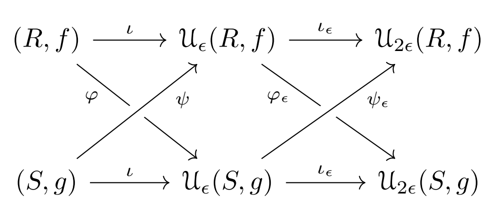
(7)

Applying h to Equation (5) yields



(8)

By Theorem 3.20 in (de Silva et al., 2016), the functor $\mathbf{R}$ is one part of an equivalence between the categories of concrete Reeb graphs and categorical Reeb graphs. If we apply the inverse functor $\mathbf{R}^{-1}$ (the display locale functor) to Equation (8) we obtain the following ϵ-interleaving of Reeb graphs:



(9)

Note that by the proof of Theorem 3.20 in (de Silva et al., 2016) and the following discussion the functors $\mathbf{R}$ and $\mathbf{R}^{-1}$ are actually inverse to each other, i.e. $\mathbf{R} \circ \mathbf{R}^{-1} = \mathrm{id}$ and $\mathbf{R}^{-1} \circ \mathbf{R} = \mathrm{id}$. In particular, we have that $\mathbf{R}(\varphi) = \mathbf{R} \circ \mathbf{R}^{-1}\big(h(\alpha)\big) = h(\alpha)$, i.e. , for all U, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\pi_0(f^{-1}(U)) & \xrightarrow{\pi_0(\varphi)} & \pi_0(g_\epsilon^{-1}(U)) \\
\big\downarrow = & & \big\downarrow = \\
\pi_0(f^{-1}(U)) & \xrightarrow{h(\alpha)} & \pi_0(g_\epsilon^{-1}(U)) \\
\big\downarrow = & & \big\downarrow h(U)^{-1} \\
\pi_0(f^{-1}(U)) & \xrightarrow{\alpha} & \pi_0(g^{-1}(U^\epsilon))
\end{array}
\tag{10}
$$

using the inverse of the isomorphism $h(U)$ in Equation (7). By the proof of Lemma 15 in (Bauer et al., 2015), there exist $\Phi\colon R \to S$ and $\Psi\colon S \to R$ sucht that

$$
\begin{aligned}
\sup_{\substack{(r,r'),(s,s')\\ \in C(\Phi,\Psi)}} \frac{1}{2}|d_f(r,r') - d_g(s,s')| &\leq 3(\epsilon+\delta) \\
||f - g \circ \Phi||_\infty &\leq \epsilon + \delta \\
||g - f \circ \Psi||_\infty &\leq \epsilon + \delta
\end{aligned}
\tag{11}
$$

for all sufficiently small $\delta > 0$. For $r \in R$, we now show that $BF(r)$ is close to $BG \circ \Phi(r)$ in the interleaving distance.

Let $\kappa > 0$, $t \in \mathbb{R}_{\geq 0}$ and $B(f(r),t) \subseteq \mathbb{R}$ be an open ball of radius t around $f(r)$. Since $|f(r) - g \circ \Phi(r)| \leq \epsilon + \delta$, if $\kappa > \epsilon + \delta$, we get:

$$
\begin{aligned}
B\big(f(r),t\big) &\subseteq B\big(g \circ \Phi(r), t+\kappa\big) \\
&\subseteq B\big(g \circ \Phi(r), t+\kappa+2\epsilon\big) \\
&\subseteq B\big(f(r), t+2(\kappa+\epsilon)\big)\,.
\end{aligned}
\tag{12}
$$

Therefore, by functoriality of $\mathcal{F}$ and the ϵ-interleaving between $\mathcal{F}$ and $\mathcal{G}$ in Equation (4) we obtain:

$$
\begin{array}{ccc}
\mathcal{F}\Big(B\big(f(r),t\big)\Big) & \xrightarrow{\mathcal{F}(\iota)} & \mathcal{F}\Big(B\big(f(r),t+2(\kappa+\epsilon)\big)\Big) \\
\mathcal{F}(\iota)\big\downarrow & & \big\uparrow \mathcal{F}(\iota) \\
\mathcal{F}\Big(B\big(g\circ\Phi(r),t+\kappa\big)\Big) & \xrightarrow{\mathcal{F}(\iota)} & \Big(B\big(g\circ\Phi(r),t+\kappa+2\epsilon\big)\Big) \\
(\alpha_{t+\kappa},\eta_{t+\kappa}) \searrow & & \nearrow (\beta_{t+\kappa+\epsilon},\rho_{t+\kappa+\epsilon}) \\
 & \mathcal{G}\Big(B\big(g\circ\Phi(r),t+\kappa+\epsilon\big)\Big) &
\end{array}
\tag{13}
$$

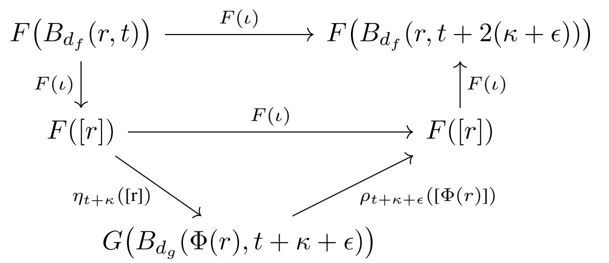
If we apply $\mathbf{dom}$ to Equation (13) we obtain:

$$
\begin{array}{ccc}
\pi_0\Big(f^{-1}\big(B(f(r),t)\big)\Big) & \xrightarrow{\pi_0(\iota)} & \pi_0\Big(f^{-1}\big(B(f(r),t+2(\kappa+\epsilon))\big)\Big) \\
\big\downarrow \pi_0(\iota) & & \big\uparrow \pi_0(\iota) \\
\pi_0\Big(f^{-1}\big(B(g\circ\Phi(r),t+\kappa)\big)\Big) & \xrightarrow{\pi_0(\iota)} & \pi_0\Big(f^{-1}\big(B(g\circ\Phi(r),t+\kappa+2\epsilon)\big)\Big) \\
\big\downarrow \alpha_{t+\kappa} & \nearrow \beta_{t+\kappa+\epsilon} & \\
\pi_0\Big(g^{-1}\big(B(g\circ\Phi(r),t+\kappa+\epsilon)\big)\Big) & &
\end{array}
\tag{14}
$$

Let $B_{d_f}(r,t)$ be the open ball of radius t around r in R. Since, $B_{d_f}(r,t) \subseteq f^{-1}(B(f(r),t))$ is by definition path-connected, $B_{d_f}(r,t) \in \pi_0\Big(f^{-1}\big(B(f(r),t)\big)\Big)$ and, since $r \in B_{d_f}(r,t)$, we have $B_{d_f}(r,t) = [r]$ the path-component of r in $f^{-1}(B(f(r),t))$. By the same argument, $B_{d_g}(\Phi(r), t+\kappa+\epsilon) = [\Phi(r)] \in \pi_0\Big(g^{-1}\big(B(g \circ \Phi(r), t+\kappa+\epsilon)\big)\Big)$. Moreover, $\pi_0(\iota)([r]) = [\iota(r)] = [r] \in \pi_0\Big(f^{-1}\big(B(g\circ\Phi(r),t+\kappa)\big)\Big)$. By using Equation (10) for $U = B(g \circ \Phi(r), t+\kappa)$ we obtain:

$$
\begin{array}{ccc}
\pi_0\big(f^{-1}(B(g\circ\Phi(r),t+\kappa))\big) & & \\
 & \searrow \pi_0(\varphi) & \\
\big\downarrow \alpha & & \pi_0\big(g_\epsilon^{-1}(B(g\circ\Phi(r),t+\kappa))\big) \\
 & \swarrow h^{-1} & \\
\pi_0\big(g^{-1}(B(g\circ\Phi(r),t+\kappa+\epsilon))\big) & &
\end{array}
\tag{15}
$$

By Equation (7), $h^{-1} := \pi_0(p_1) \circ \pi_0(q)^{-1}$ and, by (Bauer et al., 2015) Section 3.2, $\Phi := p_1 \circ \tilde{\varphi}_\delta$. Since $\tilde{\varphi}_\delta(r) \in \overline{\varphi}(B_{d_f}(r,\delta)) = q^{-1}\big(\varphi(B_{d_f}(r,\delta))\big)$, $\varphi(B_{d_f}(r,\delta))$ is path-connected and $\varphi(r) \in \varphi(B_{d_f}(r,\delta))$, we have that $[\varphi(r)] = [q(\tilde{\varphi}_\delta(r))] = \pi_0(q)([\tilde{\varphi}_\delta(r)])$. Hence, $\pi_0(q)^{-1}([\varphi(r)]) = [\tilde{\varphi}_\delta(r)]$. By definition of Φ, we have $[\Phi(r)] = [p_1 \circ \tilde{\varphi}_\delta(r)] = \pi_0(p_1)([\tilde{\varphi}_\delta(r)])$. Therefore, $h^{-1} \circ \pi_0(\varphi)([r]) = h^{-1}([\varphi(r)]) = \pi_0(p_1) \circ \pi_0(q)^{-1}([\varphi(r)]) = \pi_0(p_1)([\tilde{\varphi}_\delta(r)]) = [\Phi(r)] = \alpha_{t+\kappa}([r])$. By commutativity of Equation (14), $\beta_{t+\kappa+\epsilon} \circ \alpha_{t+\kappa}([r]) = \beta_{t+\kappa+\epsilon}([\Phi(r)]) = \pi_0(\iota)([r]) = [r]$. As a consequence, $\alpha_{t+\kappa} \circ \pi_0(\iota)(B_{d_f}(r,t)) = B_{d_g}(\Phi(r), t+\kappa+\epsilon)$ and $\pi_0(\iota) \circ \beta_{t+\kappa+\epsilon}(B_{d_g}(\Phi(r), t+\kappa+\epsilon)) = B_{d_f}(r, t+2(\kappa+\epsilon))$. Thus, the interleaving in Equation (13) yields the following commutative diagram in $\mathbf{Vec}$:



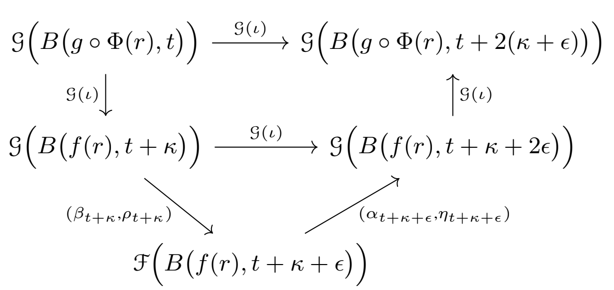
(16)

where $[r]$ and $[\Phi(r)]$ denote the topological path-components in the respective preimages.

We now start with $\Phi(r)$. Similar to Equation (12) we obtain the following inclusions of open intervals in $\mathbb{R}$:

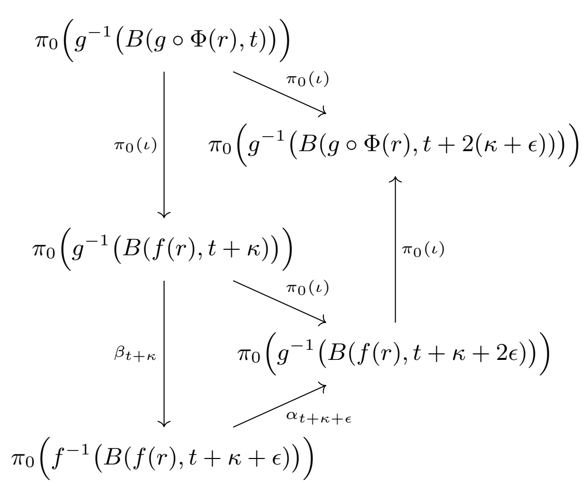
$$\begin{aligned} B\big(g\circ\Phi(r),t\big) &\subseteq B\big(f(r),t+\kappa\big) \\ &\subseteq B\big(f(r),t+\kappa+2\epsilon\big) \\ &\subseteq B\big(g\circ\Phi(r),t+2(\kappa+\epsilon)\big) \end{aligned} \tag{17}$$

for every $\kappa > \epsilon+\delta$. Therefore, by functoriality of $\mathcal{G}$ and the ϵ-interleaving between $\mathcal{F}$ and $\mathcal{G}$ in Equation (4) we obtain:



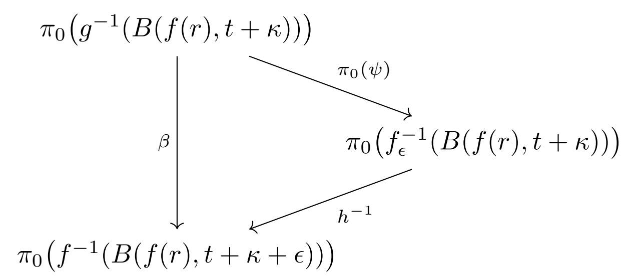
(18)

and, by applying $\mathbf{dom}$, we get:



(19)

As in the previous case, we have that $B_{d_g}(\Phi(r),t) \subseteq g^{-1}\big(B(g\circ\Phi(r),t)\big)$ is the path-component of $\Phi(r)$, i.e. $[\Phi(r)] = B_{d_g}(\Phi(r),t) \in \pi_0\Big(g^{-1}\big(B(g\circ\Phi(r),t)\big)\Big)$ and, analogously, $[r] = B_{d_f}(r,t+\kappa+\epsilon) \in \pi_0\Big(f^{-1}\big(B(f(r),t+\kappa+\epsilon)\big)\Big)$. We now use the analog of Equation (10) for $U = B(f(r),t+\kappa)$, ψ from the interleaving in Equation (9) and β to obtain:

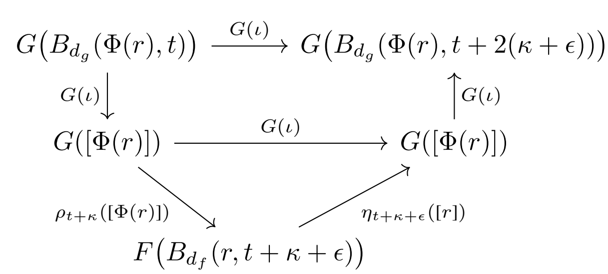


(20)

By Equation (7), $h^{-1} := \pi_0(p_1)\circ\pi_0(q)^{-1}$ and, by (Bauer et al., 2015) Section 3.2, $\Psi := p_1\circ\tilde{\psi}_\delta$. Since $\tilde{\psi}_\delta(\Phi(r)) \in \tilde{\psi}(B_{d_g}(\Phi(r),\delta)) = q^{-1}\big(\psi(B_{d_g}(\Phi(r),\delta))\big)$, $\psi(B_{d_g}(\Phi(r),\delta))$ is path-connected and $\psi(\Phi(r)) \in \psi(B_{d_g}(\Phi(r),\delta))$, we have that $[\psi(\Phi(r))] = [q(\tilde{\psi}_\delta(\Phi(r)))] = \pi_0(q)([\tilde{\psi}_\delta(\Phi(r))])$. Hence, $\pi_0(q)^{-1}([\psi(\Phi(r))]) = [\tilde{\psi}_\delta(\Phi(r))]$. By definition of Ψ, we have $[\Psi(\Phi(r))] = [p_1\circ\tilde{\psi}_\delta(\Phi(r))] = \pi_0(p_1)([\tilde{\psi}_\delta(\Phi(r))])$. Therefore, $h^{-1}\circ\pi_0(\psi)([\Phi(r)]) = h^{-1}([\psi(\Phi(r))]) = \pi_0(p_1)\circ\pi_0(q)^{-1}([\psi(\Phi(r))]) = \pi_0(p_1)([\tilde{\psi}_\delta(\Phi(r))]) = [\Psi(\Phi(r))] = \beta_{t+\kappa}([\Phi(r)])$.

From Equation (11) we get $\frac{1}{2}|d_f(r,\Psi\circ\Phi(r))| \leq 3(\epsilon+\delta)$. If $\kappa+\epsilon > 6(\epsilon+\delta)$, then $B_{d_f}(r,6(\epsilon+\delta)) \subseteq B_{d_f}(r,t+\kappa+\epsilon) \subseteq f^{-1}\big(B(f(r),t+\kappa+\epsilon)\big)$. Hence, since r and $\Psi\circ\Phi(r) \in B_{d_f}(r,t+\kappa+\epsilon)$ and $B_{d_f}(r,t+\kappa+\epsilon)$ is path-connected, $[r] = [\Psi\circ\Phi(r)] \in \pi_0\big(f^{-1}(B(f(r),t+\kappa+\epsilon))\big)$.

Therefore, starting with $B_{d_g}(\Phi(r),t) = [r]$, we obtain $\beta_{t+\kappa}\circ\pi_0(\iota)([\Phi(r)]) = \beta_{t+\kappa}([\Phi(r)]) = [\Psi\circ\Phi(r)] = [r] = B_{d_f}(r,t+\kappa+\epsilon)$. This implies that we can extract the following commutative diagram from Equation (18):



(21)

Now we define

$$\begin{aligned} &\mu_t\colon F\big(B_{d_f}(r,t)\big) \to G\big(B_{d_g}(\Phi(r),t+\kappa+\epsilon)\big) \\ &\mu_t := \eta_{t+\kappa}\circ F(\iota) \\ &\nu_t\colon G\big(B_{d_g}(\Phi(r),t)\big) \to F\big(B_{d_f}(r,t+\kappa+\epsilon)\big) \\ &\nu_t := \rho_{t+\kappa}\circ G(\iota) \end{aligned} \tag{22}$$

Since $\mathcal{F}$ and $\mathcal{G}$ are ϵ-interleaved we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}\big(B(f(r),t)\big) & \xrightarrow{\mathcal{F}(\iota)} & \mathcal{F}\big(B(g\circ\Phi(r),t+\kappa)\big) \\ {\scriptstyle(\alpha_t,\eta_t)}\downarrow & & \downarrow{\scriptstyle(\alpha_{t+\kappa},\eta_{t+\kappa})} \\ \mathcal{G}\big(B(f(r),t+\epsilon)\big) & \xrightarrow{\mathcal{G}(\iota)} & \mathcal{G}\big(B(g\circ\Phi(r),t+\kappa+\epsilon)\big) \end{array} \tag{23}$$

Following the component $B_{d_f}(r,t)$ we get

$$\begin{array}{ccc} F\big(B_{d_f}(r,t)\big) & \xrightarrow{F(\iota)} & F\big([r]\big) \\ {\scriptstyle\eta_t}\downarrow & & \downarrow{\scriptstyle\eta_{t+\kappa}} \\ G\big(\dots\big) & \xrightarrow{\mathcal{G}(\iota)} & G\big(B_{d_g}(\Phi(r),t+\kappa+\epsilon)\big) \end{array} \tag{24}$$

This implies that the map $\mu_t = \eta_{t+\kappa}\circ F(\iota) = G(\iota)\circ\eta_t$. Analogously we obtain that $\nu_t = \rho_{t+\kappa}\circ G(\iota) = F(\iota)\circ\rho_t$. Moreover, for $s < t \in \mathbb{R}_{\geq 0}$, the following diagram and its analog for ν obviously commute:

$$\begin{array}{ccc} F\big(B_{d_f}(r,s)\big) & \xrightarrow{F(\iota)} & F\big(B_{d_f}(r,t)\big) \\ {\scriptstyle\mu_s}\downarrow & & \downarrow{\scriptstyle\mu_t} \\ G\big(B_{d_g}(\Phi(r),s+\kappa+\epsilon)\big) & \xrightarrow{G(\iota)} & G\big(B_{d_g}(\Phi(r),t+\kappa+\epsilon)\big) \end{array} \tag{25}$$

Combining these results with Equation (16) and Equation (21), we obtain the following $(\kappa+\epsilon)$-interleaving:

$$\begin{array}{ccc} F\big(B_{d_f}(r,t)\big) & & G\big(B_{d_g}(\Phi(r),t)\big) \\ {\scriptstyle F(\iota)}\downarrow & {\scriptstyle\nu_t}\swarrow\;\searrow{\scriptstyle\mu_t} & \downarrow{\scriptstyle G(\iota)} \\ F\big(B_{d_f}(r,t+\kappa+\epsilon)\big) & & G\big(B_{d_g}(\Phi(r),t+\kappa+\epsilon)\big) \\ {\scriptstyle F(\iota)}\downarrow & {\scriptstyle\nu_{t+\kappa+\epsilon}}\swarrow\;\searrow{\scriptstyle\mu_{t+\kappa+\epsilon}} & \downarrow{\scriptstyle G(\iota)} \\ F\big(B_{d_f}(r,t+2(\kappa+\epsilon))\big) & & G\big(B_{d_g}(\Phi(r),t+2(\kappa+\epsilon))\big) \end{array} \tag{26}$$

Hence, $BF(r)$ and $BG(\Phi(r))$ are $(\kappa+\epsilon)$-interleaved for every $\kappa > 5\epsilon + 6\delta$. Since $\inf\{\kappa+\epsilon \mid \kappa > 5\epsilon+6\delta \text{ and } \delta > 0\} = 6\epsilon$, we finally obtain $d_I\big(BF(r), BG(\Phi(r))\big) \leq 6\epsilon$. By symmetry, we analogously obtain $d_I\big(BF(\Psi(s)), BG(s)\big) \leq 6\epsilon$. Together with Equation (11), these bounds imply the theorem.