Beyond Independent Measurements: General Compressed Sensing with GNN Application

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NEURAL INFORMATION PROCESSING SYSTEMS

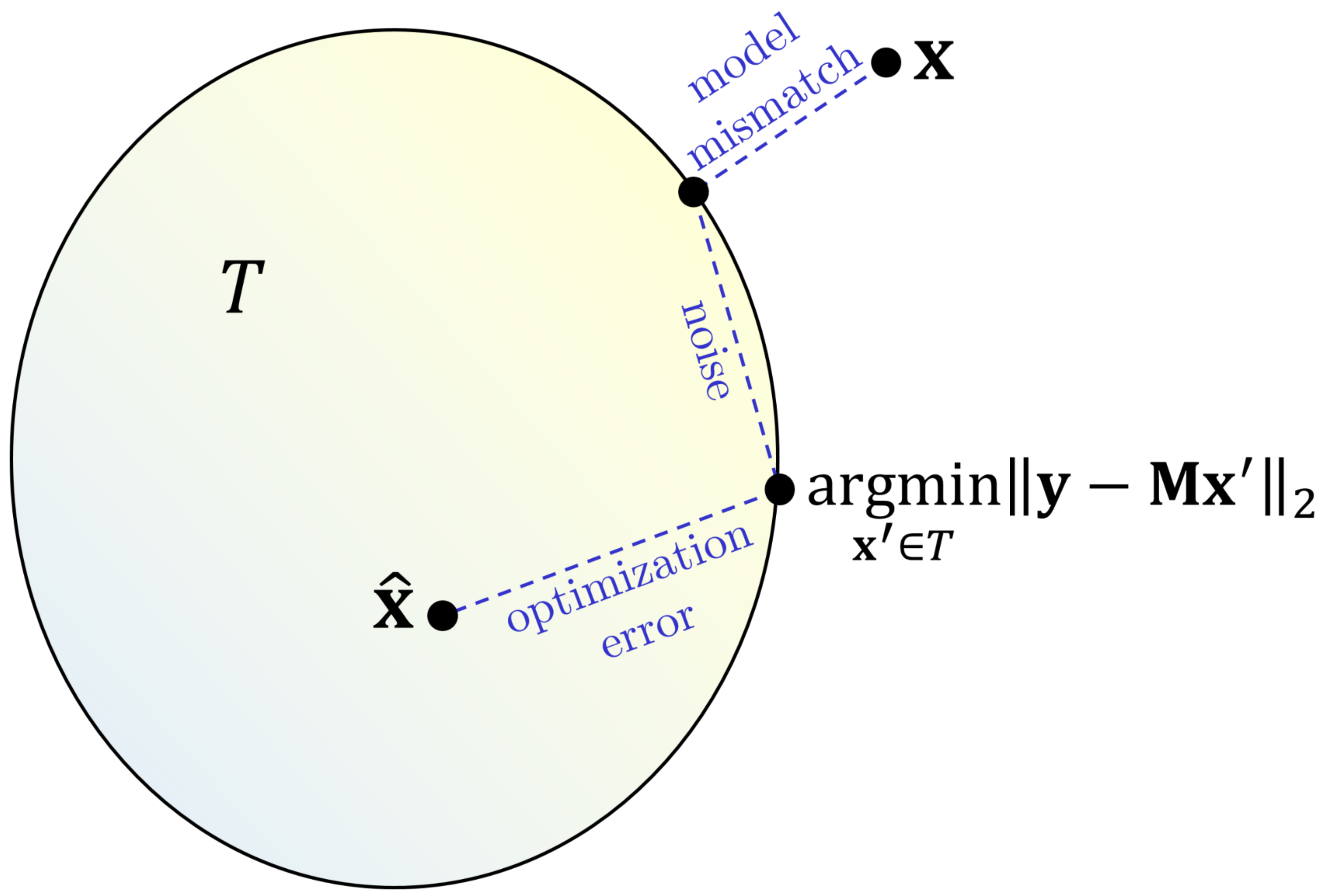
Overview

Recovery guarantee for a sparsity-free compressed sensing framework with great flexibility, e.g.

- Arbitrary cone T as the structure set
- Dependent measurements allowed
- Model mismatch ($\mathbf{x} \notin T$) allowed
- Inexact optimization allowed

When applied to the generative priors setting,

- Requires less measurements than the best known result
- Exhibits denoising with more measurements
- Highlights dependence on the model parameters



Theorem

$\mathbf{y} = \mathbf{B}\mathbf{A}\mathbf{x} + \mathbf{w}$ is given, where

- $\mathbf{B} \in \mathbb{R}^{l \times m}$ is arbitrary
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ has independent sub-g rows
- $\mathbf{x}$ is close to the structure set $T \subset \mathbb{R}^n$
- $\mathbf{w}$ is fixed or random (indep. of $\mathbf{A}$)

Let $w(\cdot)$ denote the Gaussian complexity of a set, and define

- $q = w^2\big((T - T) \cap \mathbb{S}^{n-1}\big)$
- $R(\mathbf{x}') = \|\mathbf{y} - \mathbf{B}\mathbf{A}\mathbf{x}'\|_2^2$ [empirical risk]
- $\mathrm{sr}(\mathbf{B}) = \|\mathbf{B}\|_F^2 / \|\mathbf{B}\|^2$ [stable rank]

Provided that

- $R(\hat{\mathbf{x}}) \le \min_{\mathbf{x}' \in T} R(\mathbf{x}') + \epsilon^2$, for $\hat{\mathbf{x}} \in T$
- $\mathrm{sr}(\mathbf{B}) \ge \mathcal{O}(q)$

Then, with high probability, $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 \lesssim$

$$\frac{\sqrt{q}}{\sqrt{\mathrm{sr}(\mathbf{B})}} \frac{\|\mathbf{w}\|_2}{\|\mathbf{B}\|_F} + \frac{\epsilon}{\|\mathbf{B}\|_F} + \frac{\sqrt{l}}{\sqrt{\mathrm{sr}(\mathbf{B})}} \mathrm{dist}(\mathbf{x}, T)$$

Complexity of $\mathrm{ran}(G)$

Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be a d-layer GNN with ReLU activation function, given by

$$G(\mathbf{z}) = \sigma\Big(\mathbf{A}_d \sigma\big(\cdots \sigma(\mathbf{A}_2 \sigma(\mathbf{A}_1 z)) \cdots\big)\Big).$$

Then, $T = \mathrm{ran}(G)$ is a cone and

$$w\big((T - T) \cap \mathbb{S}^{n-1}\big) \lesssim \sqrt{kd \log(p'/k)},$$

where $\mathbf{A}_i \in \mathbb{R}^{p_i \times p_{i-1}}$ and $p' = \left(\prod_{i=1}^d p_i\right)^{1/d}$.

CS with Generative Priors

Given $m = \mathcal{O}(kd \log(p'/k))$ “effective” sub-gaussian measurements, an approximate empirical risk minimizer robustly recovers signals close to $\mathrm{ran}(G)$, i.e. $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 \lesssim$

$$\frac{\sqrt{kd \log(p'/k)}}{m} \|\mathbf{w}\|_2 + \frac{\epsilon}{\sqrt{m}} + \mathrm{dist}\big(\mathbf{x}, \mathrm{ran}(G)\big)$$

w.h.p. Cf. [Bora et al, ‘17].

References

- Bora et al. "Compressed sensing using generative models." (2017).
- Jeong et al. "Sub-Gaussian matrices on sets: Optimal tail dependence and applications." (2020).