Beyond Independent Measurements: General Compressed Sensing with GNN Application
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Overview
Recovery guarantee for a sparsity-free compressed sensing framework with great flexibility, e.g.
• Arbitrary cone $T$ as the structure set
• Dependent measurements allowed
• Model mismatch ($x \notin T$) allowed
• Inexact optimization allowed

When applied to the generative priors setting,
• Requires less measurements than the best known result
• Exhibits denoising with more measurements
• Highlights dependence on the model parameters

Theorem
$y = BAx + w$ is given, where
• $B \in \mathbb{R}^{l \times m}$ is arbitrary
• $A \in \mathbb{R}^{m \times n}$ has independent sub-g rows
• $x$ is close to the structure set $T \subset \mathbb{R}^n$
• $w$ is fixed or random (indep. of $A$)

Let $w(\cdot)$ denote the Gaussian complexity of a set, and define
• $q = w^2(T - T) \cap S^{n-1}$
• $R(x') = ||y - BAx'||_2$ [empirical risk]
• $sr(B) = ||B||_F^2 / ||B||^2$ [stable rank]

Provided that
• $R(\hat{x}) \leq \min_{x' \in T} R(x') + \epsilon^2$, for $\hat{x} \in T$
• $sr(B) \geq \mathcal{O}(q)$

Then, with high probability, $||\hat{x} - x||_2 \leq$
$$\frac{\sqrt{q}}{\sqrt{sr(B)}} ||w||_2 + \frac{\epsilon}{\sqrt{m}} + \frac{\sqrt{I}}{\sqrt{sr(B)}} \text{dist}(x, T)$$

Complexity of $\text{ran}(G)$
Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be a $d$-layer GNN with ReLU activation function, given by
$G(z) = \sigma(A_d \sigma(\cdots \sigma(A_2 \sigma(A_1 z) \cdots))$.

Then, $T = \text{ran}(G)$ is a cone and
$w((T - T) \cap S^{n-1}) \leq \sqrt{kd} \log(p'/k)$,
where $A_i \in \mathbb{R}^{p'_i \times p_{i-1}}$ and $p' = (\prod_{i=1}^d p_i)^{1/d}$.

CS with Generative Priors
Given $m = \mathcal{O}(kd \log(p'/k))$ “effective” sub-gaussian measurements, an approximate empirical risk minimizer robustly recovers signals close to $\text{ran}(G)$, i.e. $||\hat{x} - x||_2 \leq \sqrt{kd} \log(p'/k) ||w||_2 + \frac{\epsilon}{\sqrt{m}} + \text{dist}(x, \text{ran}(G))$

References