A Auxiliary lemmas

The following lemma provides a useful upper bound on the optimal value of strongly convex function [28, 7, 25].

Lemma 4 *Let $\mathcal{X}$ be a convex set. Let $h : \mathcal{X} \to \mathcal{R}$ be α-strongly convex function on $\mathcal{X}$ and x_{opt} be an optimal solution of h, i.e., $x_{opt} = \arg\min_{x\in\mathcal{X}} h(x)$. Then, $h(x_{opt}) \leqslant h(x) - \frac{\alpha}{2}\|x - x_{opt}\|^2$ for any $x \in \mathcal{X}$.*

Proof *The proof of the lemma is based on the definition of α-strongly convex functions and the first-order optimality condition. Define the subgradient of $h(x)$ to be $\nabla h(x)$, According to the definition of strong convexity, we have*

$$h(x) \geqslant h(y) + \langle \nabla h(y), x - y\rangle + \frac{\alpha}{2}\|x - y\|^2. \tag{10}$$

Define $x_{opt} = \arg\min_{x\in\mathcal{X}} h(x)$. Let $y = x_{opt}$ in (10)*, we have*

$$h(x) \geqslant h(x_{opt}) + \langle \nabla h(x_{opt}), x - x_{opt}\rangle + \frac{\alpha}{2}\|x - x_{opt}\|^2.$$

We then conclude the proof based on the first-order optimality condition that for any $x \in \mathcal{X}$,

$$\langle \nabla h(x_{opt}), x - x_{opt}\rangle \geqslant 0.$$

The following lemma is the key to bridge the regret and the constraint violation.

Lemma 5 *Define*

$$h_t(x) := \langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x) + \alpha_t\|x - x_t\|^2.$$

Let x_{t+1} be the optimal solution returned by RECOO, i.e., $x_{t+1} = \arg\min_{x\in\mathcal{X}} h_t(x)$. We have for any $x \in \mathcal{X}$ that

$$\begin{aligned}&\langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant&\langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x) + \alpha_t\|x - x_t\|^2 - \alpha_t\|x - x_{t+1}\|^2.\end{aligned} \tag{11}$$

Proof *The proof is a direct application of Lemma 4. Note that $h_t(x)$ is $2\alpha_t$-strongly convex because $\langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x)$ is convex in x and $\alpha_t\|x - x_t\|^2$ is $2\alpha_t$-strongly convex.*

The following lemma is to provide the detailed calculations required for obtaining inequality (8).

Lemma 6 *Under Assumptions 1-3, we have*

$$\sum_{t=1}^{T}\frac{1}{t^{\frac{3}{2}+\varepsilon}} \leqslant 3,\quad \sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} \leqslant FD\left(1 + \frac{1}{\varepsilon}\right),\quad \sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}} \leqslant D^2.$$

Proof *Under Assumptions 1-3, we calculate these three terms as follows:*

$$\sum_{t=1}^{T}\frac{1}{t^{\frac{3}{2}+\varepsilon}} \leqslant 1 + \int_1^T \frac{1}{t^{\frac{3}{2}+\varepsilon}}dt \leqslant 3,$$

$$\sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} \leqslant \sum_{t=1}^{T}\frac{FD}{t^{1+\varepsilon}} \leqslant FD + \int_1^T \frac{FD}{t^{1+\varepsilon}}dt \leqslant \frac{FD(1+\varepsilon)}{\varepsilon},$$

$$\sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}} \leqslant D^2 + \sum_{t=2}^{T}\left(\frac{1}{t^{\frac{1}{2}+\varepsilon}} - \frac{1}{(t-1)^{\frac{1}{2}+\varepsilon}}\right)\|x_t - x^*\|^2 \leqslant D^2.$$

B Proof of lemmas in Theorem 1

B.1 Proof of Lemma 1

We prove the key self-bounding property in this section. Since (11) holds for any $x \in \mathcal{X}$ in Lemma 5, let $x = x^*$ such that

$$\begin{aligned}&\langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant&\langle \nabla f_t(x_t), x^* - x_t\rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2.\end{aligned} \tag{12}$$

We add $f_t(x_t)$ to both sides of (12) that

$$\begin{aligned}&f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant& f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t\rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\\ \leqslant& f_t(x^*) + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\\ \leqslant& f_t(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\end{aligned} \tag{13}$$

where the second inequality holds because $f_t(\cdot)$ is a convex function; the third inequality holds because x^* is an feasible point $g_t(x^*) \leqslant 0$ such that $\hat{g}_t^+(x^*) = 0$ holds.

By moving $f_t(x^*)$ to the left-hand side and $\alpha_t\|x_{t+1} - x_t\|^2$ to the right-hand side of (13), respectively, we have

$$\begin{aligned}&f_t(x_t) - f_t(x^*) + Q(t)\hat{g}_t^+(x_{t+1})\\ \leqslant&\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle - \alpha_t\|x_{t+1} - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\\ \leqslant&\frac{F^2}{4\alpha_t} + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\end{aligned}$$

where the last inequality holds by Assumption 2 that

$$\begin{aligned}&\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle - \alpha_t\|x_{t+1} - x_t\|^2\\ =&\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle - \alpha_t\|x_{t+1} - x_t\|^2 - \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} + \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t}\\ =&-\left\|\frac{\nabla f_t(x_t)}{2\sqrt{\alpha_t}} - \sqrt{\alpha_t}(x_{t+1} - x_t)\right\|^2 + \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t}\\ \leqslant&\frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \leqslant \frac{F^2}{4\alpha_t}.\end{aligned}$$

B.2 Proof of Lemma 2

Since $g_t(x)$ is convex function, we have $g_t^+(x)$ to be convex because $\max$ over convex functions is convex. Denoting by $\nabla g_t^+(x)$ the subgradient of $g_t^+(x)$, we have

$$g_t^+(x) \geqslant g_t^+(y) + \langle \nabla g_t^+(y), x - y\rangle.$$

Let $y = x_t, x = x_{t+1}$, we have

$$\begin{aligned}g_t^+(x_t) - g_t^+(x_{t+1}) &\leqslant \langle \nabla g_t^+(x_t), x_t - x_{t+1}\rangle\\ &\leqslant \|\nabla g_t^+(x_t)\|\|x_t - x_{t+1}\|\\ &\leqslant G\|x_t - x_{t+1}\| - \frac{G^2}{4\beta} - \beta\|x_t - x_{t+1}\|^2 + \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2\\ &\leqslant \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2\end{aligned}$$

where the second inequality holds because of Cauchy-Schwarz inequality; the third inequality holds because of Assumption 3. Take summation of the equality above from 1 to T, we have

$$\sum_{t=1}^{T}(g_t^+(x_t) - g_t^+(x_t + 1)) \leqslant \frac{TG^2}{4\beta} + \beta\sum_{t=1}^{T}\|x_t - x_{t+1}\|^2.$$

B.3 Proof of Lemma 3

According to Lemma 5, we have

$$\begin{aligned}&f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant& f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t\rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\\ \leqslant& f_t(x^*) + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2\end{aligned}$$

where the last inequality holds because of the convexity of $f_t(\cdot)$. By $g_t^+(x_{t+1}) \geqslant 0$ and $g_t^+(x^*) = 0$, we rearrange the inequality above and have

$$\|x_{t+1} - x_t\|^2 \leqslant \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t)) + \frac{1}{\alpha_t}\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle + \|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2$$

Take summation of the inequality above from $t = 1$ to T, we have

$$\begin{aligned}
&\sum_{t=1}^{T} \|x_{t+1} - x_t\|^2 \\
\leqslant &\sum_{t=1}^{T} \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t)) + \sum_{t=1}^{T} \frac{1}{\alpha_t}\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle + \sum_{t=1}^{T}(\|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2) \\
\leqslant &\sum_{t=1}^{T} \frac{2FD}{\alpha_t} + \|x^* - x_1\|^2 \\
\leqslant &4FD\sqrt{T} + D^2
\end{aligned}$$

where the second inequality holds because

$$f_t(x) - f_t(y) \leqslant \langle \nabla f_t(x), x - y\rangle \leqslant \|\nabla f_t(x)\|\|x - y\| \leqslant FD;$$

the last inequality holds because of $\alpha_t = \sqrt{t}$ and Assumption 1. The proof is completed.

C Corollary 1

Corollary 1 *Under Assumptions 1-3, let the learning rates be $\alpha_t = t^c, \eta_t = t^c, \gamma_t = t^{c+\varepsilon}, \forall t \in [T]$, where $c \in [1/2, 1)$ and $\varepsilon > 0$. RECOO algorithm achieves the following bounds on the regret and cumulative constraint violations:*

$$\begin{aligned}
\mathcal{R}(T) &\leqslant \left(\frac{F^2}{4(1-c)} + D^2\right) T^c, \text{ for both types of constraints,} \\
\mathcal{V}(T) &\leqslant F^2 + FD + \frac{FD}{\varepsilon} + D^2 \text{ for fixed constraints, and} \\
\mathcal{V}(T) &\leqslant \left(F^2 + \frac{G^2}{4} + FD\left(1 + \frac{2}{1-c} + \frac{1}{\varepsilon}\right) + 2D^2\right) T^{1-c/2} \text{ for adversarial constraints.}
\end{aligned}$$

The proof of corollary follows almost the same steps as in Theorem 1 where $c = 1/2$. Based on (5) in Lemma 1, we still have

$$f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{4\alpha_t} + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{14}$$

$$Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + |f_t(x_t) - f_t(x^*)| + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{15}$$

which is used to establish the bounds of regret and violation.

Regret Bound: we take summation of the inequality (14) from $t = 1, \cdots, T$ and have

$$\begin{aligned}
\sum_{t=1}^{T}(f_t(x_t) - f_t(x^*)) &\leqslant \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + \sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})\|x^* - x_t\|^2 \\
&\leqslant \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + D^2\sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})
\end{aligned}$$

where the last inequality holds by Assumption 1. Choose $\alpha_t = t^c$, we have

$$\sum_{t=1}^{T}\frac{1}{\alpha_t} = \sum_{t=1}^{T} t^{-c} \leqslant \frac{T^{1-c}}{1-c}, \quad \sum_{t=1}^{T}(\alpha_t - \alpha_{t-1}) = \sum_{t=1}^{T}(t^c - (t-1)^c) \leqslant T^c$$

It implies that

$$\sum_{t=1}^{T}(f_t(x_t) - f_t(x^*)) \leqslant \left(\frac{F^2}{4(1-c)} + D^2\right) T^{\max\{c,1-c\}},$$

which gives the regret bound with $O(T^{\max\{c,1-c\}})$.

Cumulative constraint violation bound: For constraint violation, we still have (15) and

$$g_t^+(x_{t+1}) \leqslant \frac{F^2}{4Q(t)\alpha_t\gamma_t} + \frac{|f_t(x_t) - f_t(x^*)|}{Q(t)\gamma_t} + \frac{\alpha_t}{Q(t)\gamma_t}\|x_t - x^*\|^2 - \frac{\alpha_t}{Q(t)\gamma_t}\|x_{t+1} - x^*\|^2,$$

Set $\gamma_t = t^c$ and $\eta_t = t^{c+\varepsilon}$, where $c \in [1/2, 1)$ and $\varepsilon > 0$, this implies that

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \sum_{t=1}^{T}\frac{F^2}{4t^{3c+\varepsilon}} + \sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{2c+\varepsilon}} + \sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{c+\varepsilon}}$$

By Lemma 6, we establish

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant F^2 + FD + \frac{FD}{\varepsilon} + D^2, \tag{16}$$

which proves $\mathcal{V}(T) := \sum_{t=1}^{T} g^+(x_t) \leqslant F^2 + FD + \frac{FD}{\varepsilon} + D^2$. Let's continue with (16) to prove the second part of Corollary 1 for the adversarial constraints. Recall Lemma 2 that under Assumptions 1-3, RECOO achieves for any $\beta > 0$

$$g_t^+(x_t) - g_t^+(x_{t+1}) \leqslant \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2.$$

From Lemma 2, it is required to quantify the stability term $\|x_t - x_{t+1}\|^2$ that is established in the following lemma.

Lemma 7 *Under Assumptions 1-3, let the learning rates be $\alpha_t = t^c, \eta_t = t^c, \gamma_t = t^{c+\varepsilon}, \forall t \in [T]$, where $c \in [1/2, 1)$ and $\varepsilon > 0$. RECOO achieves*

$$\sum_{t=1}^{T}\|x_{t+1} - x_t\|^2 \leqslant \frac{2FD}{1-c}T^{1-c} + D^2.$$

Proof *From Lemma 5,we still have:*

$$\begin{aligned}
&f_t(x_t) + \langle\nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2 \\
\leqslant& f_t(x_t) + \langle\nabla f_t(x_t), x^* - x_t\rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2 \\
\leqslant& f_t(x^*) + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2
\end{aligned}$$

According to $g_t^+(x_{t+1}) \geqslant 0$ and $g_t^+(x^*) = 0$ and rearrange the inequality, we have

$$\begin{aligned}
&\|x_{t+1} - x_t\|^2 \\
\leqslant& \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t)) + \frac{1}{\alpha_t}\langle\nabla f_t(x_t), x_t - x_{t+1}\rangle + \|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2.
\end{aligned}$$

Take summation of the inequality above, we have

$$\sum_{t=1}^{T}\|x_{t+1}-x_t\|^2 \leqslant \sum_{t=1}^{T}\frac{1}{\alpha_t}(f_t(x^*)-f_t(x_t))+\sum_{t=1}^{T}\frac{1}{\alpha_t}\langle\nabla f_t(x_t), x_t-x_{t+1}\rangle+\sum_{t=1}^{T}(\|x^*-x_t\|^2-\|x^*-x_{t+1}\|^2).$$

Since $\alpha_t = t^c$, we have:

$$\begin{aligned}
\sum_{t=1}^{T}\frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t) + \langle\nabla f_t(x_t), x_t - x_{t+1}\rangle) \leqslant& 2FD\sum_{t=1}^{T} t^{-c} \leqslant \frac{2FD}{1-c}T^{1-c}, \\
\sum_{t=1}^{T}(\|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2) \leqslant& D^2,
\end{aligned}$$

which proves Lemma 7.

Recall in Lemma 2 and we have

$$\sum_{t=1}^{T} g_t^+(x_t) = \sum_{t=1}^{T} g_{t+1}^+(x_t) + \sum_{t=1}^{T} g_t^+(x_t) - \sum_{t=1}^{T} g_{t+1}^+(x_t)$$
$$\leqslant \left(F^2 + FD + \frac{FD}{\varepsilon} + D^2\right) + \frac{G^2 T}{4\beta} + \beta\left(D^2 + \frac{2FD}{1-c}T^{1-c}\right).$$

Let $\beta = T^{c/2}$, we establish

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t) \leqslant \left(F^2 + \frac{G^2}{4} + FD\left(1 + \frac{2}{1-c} + \frac{1}{\varepsilon}\right) + 2D^2\right) T^{1-c/2},$$

which proves the corollary.

D Proof of Theorem 2

In this section, we prove Theorem 2 which considers strongly convex loss functions. According to the definition of μ-strongly convex, we have

$$f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle \leqslant f(x^*) - \frac{\mu}{2}\|x^* - x_t\|^2 \tag{17}$$

We start with inequality (12) by using Lemma 5:

$$\begin{aligned} &f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2 \\ \leqslant& f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2 \\ \leqslant& f_t(x^*) - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2 \end{aligned} \tag{18}$$

where the last inequality holds according to the strongly convex condition of $f_t(\cdot)$ in (17). By following the same steps that lead to inequality (5), we have

$$\begin{aligned} &f_t(x_t) - f_t(x^*) + Q(t)\hat{g}_t^+(x^*) \\ \leqslant& \frac{F^2}{4\alpha_t} - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2. \end{aligned} \tag{19}$$

This is the key "self-bounding" property when $f_t(\cdot)$ is strongly convex function, from which we obtain

$$f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{4\alpha_t} - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{20}$$

$$Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + |f_t(x_t) - f_t(x^*)| - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2. \tag{21}$$

Recall that $\alpha_t = \frac{\mu t}{2}$, $\eta_t = \sqrt{t}$, and $\gamma_t = t^{\frac{1}{2}+\varepsilon}, \varepsilon > 0$. We next establish the regret and violation bounds.

Regret bound: From (20), we immediately have

$$\begin{aligned} \sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leqslant& \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + \sum_{t=1}^{T}\left(\alpha_t - \alpha_{t-1} - \frac{\mu}{2}\right)\|x_t - x^*\|^2 \\ \leqslant& \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + D^2\sum_{t=1}^{T}\left(\alpha_t - \alpha_{t-1} - \frac{\mu}{2}\right) \end{aligned}$$

By the choice of $\alpha_t = \frac{\mu t}{2}$, we have

$$\sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{2\mu}(1 + \log T).$$

Following the same steps as in the proof of Theorem 1, we establish

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \frac{F^2}{2\mu}\sum_{t=1}^{T}\frac{1}{t^{2+\varepsilon}} + \sum_{t=1}^{T}\frac{|f_t(x_t)-f_t(x^*)|}{t^{1+\varepsilon}} \tag{22}$$

which implies that

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \frac{F^2}{\mu} + FD\left(1+\frac{1}{\epsilon}\right). \tag{23}$$

D.1 Violation bound: fixed constraints

For fixed constraints, inequality (23) implies $\mathcal{V}(T) := \sum_{t=1}^{T} g^+(x_t) \leqslant \frac{F^2}{\mu} + FD\left(1+\frac{1}{\epsilon}\right)$ because the constraint is fixed. We have proved the first part of Theorem 2 for the fixed constraints. Let's continue with (23) to prove the second part of Theorem 2 for the adversarial constraints.

D.2 Violation bound: adversarial constraints

By Lemma 2, we have

$$\sum_{t=1}^{T}(g_t^+(x_t) - g_t^+(x_{t+1})) \leqslant \frac{TG^2}{4\beta} + \beta\sum_{t=1}^{T}\|x_t - x_{t+1}\|^2. \tag{24}$$

From (18), we again establish the bound on $\sum_{t=1}^{T}\|x_t - x_{t+1}\|^2$ as follows

$$\begin{aligned}
&\sum_{t=1}^{T}\|x_t - x_{t+1}\|^2 \\
\leqslant &\sum_{t=1}^{T}\frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t) + \langle \nabla f_t(x_t), x_t - x_{t+1}\rangle) + \sum_{t=1}^{T}(\|x^*-x_t\|^2 - \|x^* - x_{t+1}\|^2) \\
\leqslant &\sum_{t=1}^{T}\frac{2FD}{\alpha_t} + \|x^* - x_1\|^2 \\
\leqslant &\frac{4FD}{\mu}(1+\log T) + D^2,
\end{aligned} \tag{25}$$

where the second inequality holds because

$$f_t(x) - f_t(y) \leqslant \langle \nabla f_t(x), x-y\rangle \leqslant \|\nabla f_t(x)\|\|x-y\| \leqslant FD, \forall x, y \in \mathcal{X};$$

the last inequality holds because of $\alpha_t = \frac{\mu t}{2}$ and Assumption 1. We combine (24) and (25) with $\beta = \sqrt{T/(1+\log T)}$ as follows

$$\begin{aligned}
\sum_{t=1}^{T}(g_t^+(x_t) - g_t^+(x_t+1)) &\leqslant \frac{TG^2}{4\beta} + \beta\sum_{t=1}^{T}\|x_t - x_{t+1}\|^2 \\
&\leqslant \frac{TG^2}{4\beta} + \beta\left(\frac{4FD}{\mu}(1+\log T) + D^2\right) \\
&= \left(\frac{G^2}{4} + \frac{4FD}{\mu}\right)\sqrt{T(1+\log T)} + D^2\sqrt{T/(1+\log T)}.
\end{aligned}$$

Combining the inequality above with inequality (23), we establish the second part of Theorem 2 for adversarial constraints.

E Proof of Theorem 3

We prove Theorem 3 with a dynamic baseline similar to that in [26], which is the solution to the following offline OCO with constraints:

$$\min_{x_t \in \mathcal{X}} \sum_{t=1}^{T} f_t(x_t) \tag{26}$$

$$\text{subject to: } g_t(x_t) \leqslant 0, \forall t \in [T], \tag{27}$$

$$\sum_{t=1}^{T} \|x_{t+1} - x_t\| \leqslant P_T. \tag{28}$$

where (28) imposes a path-length constraint on the baseline solution which limits change of $\{x_t\}$. Note that the solution to (26)-(28) with $P_T = 0$ reduces to the best fixed decision in hindsight. Let $\{x_t^*\}$ be the optimal solution to (26)-(28). We define the regret and cumulative constraint violation as follows

$$\mathcal{R}^{\text{dynamic}}(T) := \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_t^*), \tag{29}$$

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t). \tag{30}$$

Next, we state the key self-bounding property for establishing Theorem 3.

Lemma 8 *Let $\{x_t\}$ be the decision sequence generated by RECOO. Under Assumptions 1-3, the following inequality holds for any sequence $\{y_t\}$ with $y_t \in \mathcal{X}, \forall t$,*

$$\begin{aligned} & f_t(x_t) - f_t(y_t) + Q(t)\hat{g}_t^+(x_{t+1}) \\ \leqslant & \frac{F^2}{4\alpha_t} + Q(t)\hat{g}_t^+(y_t) + \alpha_t \|y_t - x_t\|^2 - \alpha_t \|y_t - x_{t+1}\|^2. \end{aligned} \tag{31}$$

Proof *According to Lemma 5, we have*

$$\begin{aligned} & \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t \|x_{t+1} - x_t\|^2 \\ \leqslant & \langle \nabla f_t(x_t), y_t - x_t \rangle + Q(t)\hat{g}_t^+(y_t) + \alpha_t \|y_t - x_t\|^2 - \alpha_t \|y_t - x_{t+1}\|^2. \end{aligned}$$

We add $f_t(x_t)$ to both sides of the inequality above

$$\begin{aligned} & f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t \|x_{t+1} - x_t\|^2 \\ \leqslant & f_t(x_t) + \langle \nabla f_t(x_t), y_t - x_t \rangle + Q(t)\hat{g}_t^+(y_t) + \alpha_t \|y_t - x_t\|^2 - \alpha_t \|y_t - x_{t+1}\|^2 \\ \leqslant & f_t(y_t) + Q(t)\hat{g}_t^+(y_t) + \alpha_t \|y_t - x_t\|^2 - \alpha_t \|y_t - x_{t+1}\|^2, \end{aligned}$$

which proves the lemma because

$$\langle \nabla f_t(x_t), x_t - x_{t+1} \rangle - \alpha_t \|x_{t+1} - x_t\|^2 \leqslant \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \leqslant \frac{F^2}{4\alpha_t}.$$

Regret bound with dynamic baseline: Let the optimal sequence be $\{x_t^*\}$. Substitute $\{y_t\} = \{x_t^*\}$ in (31). By $g_t(x_t^*) \leqslant 0$, we have

$$\begin{aligned} & f_t(x_t) - f_t(x_t^*) + Q(t)\hat{g}_t^+(x_{t+1}) \\ \leqslant & \frac{F^2}{4\alpha_t} + \alpha_t \|x_t^* - x_t\|^2 - \alpha_t \|x_t^* - x_{t+1}\|^2. \end{aligned}$$

Take the summation of the inequality above from $t = 1$ to T, we have

$$\mathcal{R}^{\text{dynamic}}(T) \leqslant \sum_{t=1}^{T} \frac{F^2}{4\alpha_t} + \sum_{t=1}^{T} \alpha_t \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2 \right) \tag{32}$$

Recall $\alpha_t = \sqrt{t}$, we have

$$\sum_{t=1}^{T} \frac{F^2}{4\alpha_t} \leqslant \frac{F^2\sqrt{T}}{2}.$$

For the second term, we have

$$\begin{aligned}
&\sum_{t=1}^{T} \alpha_t \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
=&\sum_{t=1}^{T} \sqrt{t} \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
=&\sum_{t=1}^{T} \sqrt{t}\|x_t^* - x_t\|^2 - \sqrt{t+1}\|x_{t+1}^* - x_{t+1}\|^2 + \sqrt{t+1}\|x_{t+1}^* - x_{t+1}\|^2 \\
&\quad - \sqrt{t}\|x_{t+1}^* - x_{t+1}\|^2 + \sqrt{t}\|x_{t+1}^* - x_{t+1}\|^2 - \sqrt{t}\|x_t^* - x_{t+1}\|^2 \\
\leqslant&\|x_1^* - x_1\|^2 + \sum_{t=1}^{T} \left(\sqrt{t+1} - \sqrt{t}\right) D^2 + \sum_{t=1}^{T} \sqrt{t} \left(\|x_{t+1}^* - x_{t+1}\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
\leqslant&D^2\sqrt{T+1} + 2DP_T\sqrt{T},
\end{aligned}$$

where the first inequality holds because of Assumption 1 and the last inequality holds because

$$\begin{aligned}
\sum_{t=1}^{T} |\|x_{t+1}^* - x_{t+1}\|^2 - \|x_t^* - x_{t+1}\|^2| =& \sum_{t=1}^{T} |\langle x_{t+1}^* - x_t^*, x_{t+1}^* - x_{t+1} + x_t^* - x_{t+1}\rangle| \\
\leqslant& \sum_{t=1}^{T} \|x_{t+1}^* - x_t^*\| \left(\|x_{t+1}^* - x_{t+1}\| + \|x_t^* - x_{t+1}\|\right) \\
\leqslant& 2D \sum_{t=1}^{T} \|x_{t+1}^* - x_t^*\|.
\end{aligned}$$

Therefore, we prove the regret in Theorem 3 as follows

$$\mathcal{R}(T) \leqslant \frac{F^2\sqrt{T}}{2} + (D^2 + 2DP_T)\sqrt{T+1} \leqslant \left(\frac{F^2}{2} + D^2 + 2DP_T\right)\sqrt{T+1}.$$

Cumulative hard constraint violation bound: When $\alpha_t = \eta_t = \sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}$ with $\varepsilon > 0$, the proof follows the same steps in the proof of Theorem 1 in Section 3.1 because the definition of constraint violation is the same.

F Refined results of RECOO with expert-tracking under the dynamic baseline

Motivated by [33], we combine RECOO with expert tracking techniques in [6] to improve the performance bounds w.r.t. P_T, similar in [26]. The intuition of the algorithm is to set up N parallel experts (N RECOO algorithms) and to track the best one. We state RECOO with expert tracking algorithm as follows.

A Rectified Online Optimization with Expert-Tracking Algorithm

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Initialization (N Experts): $f_0(x) = g_0(x) = 0, \forall x \in \mathcal{X}$, $x_{i,0} \in \mathcal{X}$, $Q_i(0) = 0$. The learning rates $\alpha_{i,t}, \eta_{i,t}, \gamma_t, \kappa$ and $w_{i,1} = \frac{N+1}{i(i+1)N}, \forall i \in [N]$.

For $t = 1, \cdots, T$,

- **Set:** $\hat{g}_{t-1}^{+}(x) = \gamma_{t-1} g_{t-1}^{+}(x)$.
- **Rectified decision:** find the optimal solution of $x_{i,t}$ for each expert and output x_t :

$$x_{i,t} = \arg\min_{x \in \mathcal{X}} \langle \nabla f_{t-1}(x_{i,t-1}), x - x_{i,t-1} \rangle + Q_i(t-1)\hat{g}_{t-1}^{+}(x) + \alpha_{i,t-1}\|x - x_{i,t-1}\|^2$$
$$x_t = \sum_{i=1}^{N} w_{i,t} x_{i,t}$$

- **Observe:** $\nabla f_t(\cdot)$ and $g_t(\cdot)$.
- **Rectified penalty update:** update $Q_i(t)$ and $w_{i,t}$ as follows:

$$Q_i(t) = \max\left(Q_i(t-1) + \hat{g}_t^{+}(x_t), \eta_{i,t}\right).$$
$$l_t(x) = \langle \nabla f_t(x_t), x - x_t \rangle$$
$$w_{i,t+1} = \frac{w_{i,t} e^{-\kappa l_t(x_{i,t})}}{\sum_{i=1}^{N} w_{i,t} e^{-\kappa l_t(x_{i,t})}}$$

Before presenting the main result of RECOO with expert-tracking algorithm, we impose an additional assumption on the loss functions as in [33].

Assumption 5 *The loss functions are bounded by a constant C such that $|f_t(x)| \leqslant C, \forall x \in \mathcal{X}, \forall t$.*

We are ready to show RECOO with expert-tracking algorithm improves Theorem 3 in the following.

Corollary 2 *Let $N = \left\lfloor \frac{1}{2}\log_2(1+T) \right\rfloor + 1$, $\kappa = 1/\sqrt{T}$. Let the learning rates be $\alpha_{i,t} = \sqrt{t}/2^{i-1}, \eta_{i,t} = 2^{i-1}\sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}$, $\forall i \in [N]$, where $\varepsilon > 0$. Let $\{x_t^*\}$ be the optimal solution to* (2) *with $P_T = \sum_{t=1}^{T-1} \|x_{t+1}^* - x_t^*\|$. Under Assumptions 1-3, RECOO with expert-tracking algorithm achieves the following bounds on the regret and cumulative constraint violations:*

$$\mathcal{R}^{dynamic}(T) \leqslant \left[(2F^2 + 4D^2)\sqrt{1 + \frac{P_T}{D}} + \frac{C^2}{2} + 2\ln\left(\left\lfloor \frac{1}{2}\log_2\left(1 + \frac{P_T}{D}\right)\right\rfloor + 2\right)\right]\sqrt{T+1},$$
$$\mathcal{V}(T) \leqslant 2FD\left(1 + \frac{1}{\varepsilon}\right) + F^2(1 + \log(1+T)) + 2D^2.$$

We first introduce Lemma 1 in [33], which quantifies the difference between the weighted output of all experts with the best expert.

Lemma 9 (Lemma 1 in [33]) *Let $\{x_{i,t}\}$ and $\{x_t\}$ be the sequence generated by RECOO with expert-tracking algorithm, we have*

$$\sum_{t=1}^{T} l_t(x_t) - \min_{i \in [N]} \left\{ \sum_{t=1}^{T} l_t(x_{i,t}) + \frac{1}{\kappa} \ln \frac{1}{w_{i,1}} \right\} \leqslant \frac{\kappa C^2 T}{2}.$$

Substitute the learning rate of $\alpha_{i,t}$ in (32) in the proof of Theorem 3, we have

$$\sum_{t=1}^{T} (f_t(x_{i,t}) - f_t(x_t^*)) + \sum_{t=1}^{T} Q_i(t)\hat{g}_t^{+}(x_{i,t+1}) \leqslant 2^i F^2 \sqrt{T} + \frac{D^2\sqrt{T+1}}{2^{i-1}} + \frac{2DP_T\sqrt{T}}{2^{i-1}}. \quad (33)$$

Recall $N = \left\lfloor \frac{1}{2}\log_2(1+T) \right\rfloor + 1$, there exists $i_0 = \left\lfloor \frac{1}{2}\log_2(1 + \frac{P_T}{D}) \right\rfloor + 1 \in [N]$ such that

$$2^{i_0 - 1} \leqslant \sqrt{1 + \frac{P_T}{D}} \leqslant 2^{i_0}. \quad (34)$$

Regret bound with dynamic baseline: Let $i = i_0$ in (33) and by Lemma 9, we have:

$$\begin{aligned}\sum_{t=1}^{T}(f_t(x_{i_0,t}) - f_t(x_t^*)) &\leqslant 2^{i_0}F^2\sqrt{T} + \frac{D^2\sqrt{T+1}}{2^{i_0-1}} + \frac{2DP_T\sqrt{T}}{2^{i_0-1}} \\ &\leqslant 2F^2\sqrt{T\left(1+\frac{P_T}{D}\right)} + \frac{D^2\sqrt{T+1}+2DP_T\sqrt{T}}{2^{i_0-1}} \\ &\leqslant 2F^2\sqrt{T\left(1+\frac{P_T}{D}\right)} + \frac{4}{2^{i_0}}\left(D^2\sqrt{T+1}\left(1+\frac{P_T}{D}\right)\right) \\ &\leqslant (2F^2+4D^2)\sqrt{(T+1)\left(1+\frac{P_T}{D}\right)}\end{aligned}$$

where the second and the last inequalities hold by using (34). Moreover, we have

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{i_0,t}) \leqslant \sum_{t=1}^{T} l_t(x_t) - \sum_{t=1}^{T} l_t(x_{i_0,t}) \leqslant \frac{\kappa C^2 T}{2} + \frac{1}{\kappa}\ln\frac{1}{w_{i_0,1}}$$

where the first inequality holds by the convexity of $f_t(\cdot)$ and the second inequality holds by Lemma 9. Recall $\kappa = \frac{1}{\sqrt{T}}$ and $w_{i,1} = \frac{N+1}{i(i+1)N}$, we have

$$\ln\frac{1}{w_{i_0,1}} \leqslant \ln(i_0(i_0+1)) \leqslant 2\ln(i_0+1) \leqslant 2\ln\left(\left\lfloor\frac{1}{2}\log_2\left(1+\frac{P_T}{D}\right)\right\rfloor + 2\right).$$

Combine all these inequalities, we have:

$$\begin{aligned}\mathcal{R}(T) =& \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{i_0,t}) + \sum_{t=1}^{T} f_t(x_{i_0,t}) - \sum_{t=1}^{T} f_t(x_t^*) \\ \leqslant& (2F^2+4D^2)\sqrt{(T+1)\left(1+\frac{P_T}{D}\right)} + \left(\frac{C^2}{2} + 2\ln\left(\left\lfloor\frac{1}{2}\log_2\left(1+\frac{P_T}{D}\right)\right\rfloor+2\right)\right)\sqrt{T}\end{aligned}$$

which proves the regret in Corollary 2 and establish $O(\sqrt{P_T T})$ regret bound.

Violation bound: Since $g_t^+(x)$ is convex, we have:

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) = \sum_{t=1}^{T} g_t^+\left(\sum_{i=1}^{N} w_{i,t+1}x_{i,t+1}\right) \leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} w_{i,t+1}g_t^+(x_{i,t+1}) \leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} g_t^+(x_{i,t+1}).$$

By the inequality (31), we have

$$Q_i(t)\hat{g}_t^+(x_{i,t+1}) \leqslant f_t(x_t^*) - f_t(x_{i,t}) + \frac{F^2}{4\alpha_t} + \alpha_{i,t}\|x_t^* - x_{i,t}\|^2 - \alpha_{i,t}\|x_t^* - x_{i,t+1}\|^2.$$

Let $\alpha_{i,t} = \sqrt{t}/2^{i-1}$, $\eta_{i,t} = 2^{i-1}\sqrt{t}$, $\gamma_t = t^{\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$, we have

$$\begin{aligned}g_t^+(x_{i,t+1}) &\leqslant \frac{|f_t(x_t^*) - f_t(x_{i,t})|}{\eta_{i,t}\gamma_t} + \frac{F^2}{4\alpha_{i,t}\eta_{i,t}\gamma_t} + \frac{\alpha_{i,t}}{\eta_{i,t}\gamma_t}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2) \\ &\leqslant \frac{1}{2^{i-1}}\frac{FD}{t^{1+\varepsilon}} + \frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \frac{1}{4^{i-1}}\frac{1}{t^{\frac{1}{2}+\varepsilon}}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2),\end{aligned}$$

which implies

$$\begin{aligned}\sum_{t=1}^{T} g_t^+(x_{i,t+1}) &\leqslant \frac{1}{2^{i-1}}\sum_{t=1}^{T}\frac{FD}{t^{1+\varepsilon}} + \sum_{t=1}^{T}\frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \frac{1}{4^{i-1}}\sum_{t=1}^{T}\frac{1}{t^{\frac{1}{2}+\varepsilon}}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2) \\ &\leqslant \frac{FD}{2^{i-1}}\left(1+\frac{1}{\varepsilon}\right) + F^2 + \frac{D^2}{4^{i-1}}\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{t=1}^{T} g_t^+(x_{t+1}) &\leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} g_t^+(x_{i,t+1}) \\ &\leqslant \sum_{i=1}^{N}\left(\frac{FD}{2^{i-1}} + F^2 + \frac{D^2}{4^{i-1}}\right) \\ &\leqslant 2FD\left(1+\frac{1}{\varepsilon}\right) + F^2(1+\log(1+T)) + 2D^2\end{aligned}$$

which proves the violation in Corollary 2.