Missing Derivation from Section C.3

Anonymous Author(s)
Affiliation
Address
email

Let $\tilde{\pi}'$ be as in Section C.3. Choose any $c \in \mathcal{C}$ and $v \in \mathcal{V}$. For all $t \in [T]$ define $\xi_t := \tilde{\pi}'(c, (v, t), \circ)$. For all $t \in [T]$ let $\mathcal{H}_t := \mathcal{A}^{[t]}$ and for all $e \in \mathcal{H}_t$ let:

$$\Lambda(e) := \sum_{s \in [t-1]} [\![e(s) \neq e(s+1)]\!]$$

We take the inductive hypothesis over $t \in [T]$ that there exists a distribution $\varphi_t \in \Delta_{\mathcal{H}_t}$ such that for all $s \in [t]$ and $a \in \mathcal{A}$ we have:

$$\xi_s(a) = \sum_{e \in \mathcal{H}_t} [\![e(s) = a]\!] \varphi_t(e)$$

and we have:

$$\sum_{e \in \mathcal{H}_t} \varphi_t(e) \Lambda(e) = \frac{1}{2} \sum_{s \in [t-1]} \sum_{a \in \mathcal{A}} |\xi_s(a) - \xi_{s+1}(a)|$$

The inductive hypothesis clearly holds for $t = 1$ as then both sides of the above equation are equal to zero. Now suppose it holds for t. Let $\mathcal{S}$ be the set of all $a \in \mathcal{A}$ with $\xi_{t+1}(a) \leq \xi_t(a)$. For all $a \in \mathcal{S}$ let $r(a) := \xi_{t+1}(a)/\xi_t(a)$. For all $a \in \mathcal{A} \setminus \mathcal{S}$ let:

$$q(a) := \frac{\xi_{t+1}(a) - \xi_t(a)}{\sum_{a' \in \mathcal{A} \setminus \mathcal{S}} (\xi_{t+1}(a') - \xi_t(a'))}$$

For all $e \in \mathcal{H}_t$ and $a \in \mathcal{A}$ let $\gamma(e, a)$ be the function $e' \in \mathcal{H}_{t+1}$ with $e'(t+1) = a$ and whose restriction onto $[t]$ is equal to e. For all $a \in \mathcal{A} \setminus \mathcal{S}$ and $e \in \mathcal{H}_t$ with $e(t) = a$ define:

$$\varphi_{t+1}(\gamma(e, a)) := \varphi_t(e)$$

For all $a \in \mathcal{S}$, $a' \in \mathcal{A} \setminus \mathcal{S}$ and $e \in \mathcal{H}_t$ with $e(t) = a$ define:

$$\varphi_{t+1}(\gamma(e, a)) := r(a) \varphi_t(e)$$

and define:

$$\varphi_{t+1}(\gamma(e, a')) := q(a')(1 - r(a)) \varphi_t(e)$$

For all other $e' \in \mathcal{H}_{t+1}$ we define $\varphi_{t+1}(e') := 0$.

Note that for all $s \in [t]$ and $a \in \mathcal{A}$ we have:

$$\xi_s(a) = \sum_{e \in \mathcal{H}_t} [\![e(s) = a]\!] \varphi_t(e) = \sum_{e \in \mathcal{H}_t} [\![e(s) = a]\!] \sum_{a' \in \mathcal{A}} \varphi_{t+1}(\gamma(e, a')) = \sum_{e' \in \mathcal{H}_{t+1}} [\![e'(s) = a]\!] \varphi_{t+1}(e')$$

and for all $a \in \mathcal{S}$ we have:

$$\xi_{t+1}(a) = r(a)\xi_t(a) = r(a) \sum_{e \in \mathcal{H}_t} [\![e(t) = a]\!] \varphi_t(e) = \sum_{e \in \mathcal{H}_t} [\![e(t) = a]\!] \varphi_{t+1}(\gamma(e, a)) = \sum_{e' \in \mathcal{H}_{t+1}} [\![e'(t+1) = a]\!] \varphi_{t+1}(e')$$

Submitted to 39th Conference on Neural Information Processing Systems (NeurIPS 2025). Do not distribute.

and for all $a \in \mathcal{A} \setminus \mathcal{S}$ we have:

$$\begin{aligned}
\xi_{t+1}(a) &= \xi_t(a) + (\xi_{t+1}(a) - \xi_t(a)) \\
&= \xi_t(a) + q(a) \sum_{a' \in \mathcal{A} \setminus \mathcal{S}} (\xi_{t+1}(a') - \xi_t(a')) \\
&= \xi_t(a) + q(a) \sum_{a' \in \mathcal{S}} (\xi_t(a') - \xi_{t+1}(a')) \\
&= \xi_t(a) + q(a) \sum_{a' \in \mathcal{S}} (1 - r(a'))\xi_t(a') \\
&= \sum_{e \in \mathcal{H}_t} [\![e(t) = a]\!] \varphi_t(e) + q(a) \sum_{a' \in \mathcal{S}} (1 - r(a')) \sum_{e \in \mathcal{H}_t} [\![e(t) = a']\!] \varphi_t(e) \\
&= \sum_{e \in \mathcal{H}_t} [\![e(t) = a]\!] \varphi_t(e) + \sum_{a' \in \mathcal{S}} \sum_{e \in \mathcal{H}_t} [\![e(t) = a']\!] q(a)(1 - r(a'))\varphi_t(e) \\
&= \sum_{e' \in \mathcal{H}_{t+1}} [\![e'(t+1) = a]\!] \varphi_{t+1}(e')
\end{aligned}$$

We have now shown that for all $s \in [t+1]$ and $a \in \mathcal{A}$ we have:

$$\xi_s(a) = \sum_{e \in \mathcal{H}_{t+1}} [\![e(s) = a]\!] \varphi_{t+1}(e)$$

as required.

For all $a \in \mathcal{A} \setminus \mathcal{S}$ and $e \in \mathcal{H}_t$ with $e(t) = a$ we have:

$$\sum_{a' \in \mathcal{A}} \varphi_{t+1}(\gamma(e, a'))\Lambda(\gamma(e, a')) = \varphi_{t+1}(\gamma(e, a))\Lambda(\gamma(e, a)) = \varphi_t(e)\Lambda(e)$$

and for all $a \in \mathcal{S}$ and $e \in \mathcal{H}_t$ with $e(t) = a$ we have:

$$\begin{aligned}
\sum_{a' \in \mathcal{A}} \varphi_{t+1}(\gamma(e, a'))\Lambda(\gamma(e, a')) &= \varphi_{t+1}(\gamma(e, a))\Lambda(\gamma(e, a)) + \sum_{a' \in \mathcal{A} \setminus \mathcal{S}} \varphi_{t+1}(\gamma(e, a'))\Lambda(\gamma(e, a')) \\
&= \varphi_{t+1}(\gamma(e, a))\Lambda(e) + (\Lambda(e) + 1) \sum_{a' \in \mathcal{A} \setminus \mathcal{S}} \varphi_{t+1}(\gamma(e, a')) \\
&= r(a)\varphi_t(e)\Lambda(e) + (\Lambda(e) + 1)(1 - r(a))\varphi_t(e) \\
&= \varphi_t(e)\Lambda(e) + (1 - r(a))\varphi_t(e)
\end{aligned}$$

Hence, we have:

$$\begin{aligned}
\sum_{e' \in \mathcal{H}_{t+1}} \varphi_{t+1}(e')\Lambda(e') &= \sum_{e \in \mathcal{H}_t} \varphi_t(e)\Lambda(e) + \sum_{a \in \mathcal{S}} (1 - r(a)) \sum_{e \in \mathcal{H}_t} [\![e(t) = a]\!] \varphi_t(e) \\
&= \sum_{e \in \mathcal{H}_t} \varphi_t(e)\Lambda(e) + \sum_{a \in \mathcal{S}} (1 - r(a))\xi_t(a) \\
&= \sum_{e \in \mathcal{H}_t} \varphi_t(e)\Lambda(e) + \sum_{a \in \mathcal{S}} (\xi_t(a) - \xi_{t+1}(a)) \\
&= \frac{1}{2} \sum_{s \in [t-1]} \sum_{a \in \mathcal{A}} |\xi_s(a) - \xi_{s+1}(a)| + \frac{1}{2} \sum_{a \in \mathcal{A}} |\xi_t(a) - \xi_{t+1}(a)| \\
&= \frac{1}{2} \sum_{s \in [t]} \sum_{a \in \mathcal{A}} |\xi_s(a) - \xi_{s+1}(a)|
\end{aligned}$$

as required.

So the inductive hypothesis holds for all $t \in [T]$. In particular it holds for $t = T$ which completes the derivation.