

317 A Appendix

318 Denote

$$\mathcal{H}(t, x, s, u) = \langle f(t, x) + g(t, x)u, s \rangle - q(t, x) - u^T r(t, x)u. \quad (25)$$

Lemma 1. Let the value function $V_*(t, x)$ be continuously differentiable. Then, the following Hamilton-Jacobi-Bellman equation holds:

$$\frac{\partial V_*(t, x)}{\partial t} + \max_{u \in U} \mathcal{H}(t, x, \nabla_x V_*(t, x), u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n.$$

319 The proof can be found, for example, in J. Yong. Differential Games: A Concise Introduction. World
320 Scientific, University of Central Florida, 238 USA, 2015. doi: 10.1142/9121.

321 **Proof of Theorem 1.** Note that, due to (19), (21), and (25), we have

$$A(t, x, u)/\Delta t_k = \mathcal{H}(t, x, \nabla_x V_*(t, x), u) - \max_{u' \in U} \mathcal{H}(t, x, \nabla_x V_*(t, x), u') \quad (26)$$

322 for any $(t, x, u) \in [0, T] \times \mathbb{R}^n \times U$. Due to (9) and (11), we have

$$\|f(t, x) + g(t, x)u\| \leq c_{fg}(1 + \|z\|)e^{c_{fg}T} := \alpha_{fg}, \quad (t, x) \in S, \quad u \in U. \quad (27)$$

323 Let $\varepsilon > 0$. Since $V_*(t, x)$ is continuously differentiable, there exists $\delta > 0$ such that

$$\left| V_*(t', x') - V_*(t, x) - \frac{\partial V_*(t, x)}{\partial t}(t' - t) - \langle \nabla_x V_*(t, x), x' - x \rangle \right| \leq (|t - t'| + \|x - x'\|) \frac{\varepsilon}{1 + \alpha_{fg}} \quad (28)$$

324 for any $(t, x), (t', x') \in S$ satisfying $|t - t'| + \|x - x'\| \leq \delta$. Put $k_* = (1 + \alpha_{fg})T/\delta$.

325 Let $k \geq k_*$, $i \in \overline{0, k-1}$, $x \in S(t_i)$, $u \in U$, and $x' = x + (f(t_i, x) + g(t_i, x)u)\Delta t_k$. Due to (9) and
326 (11), we have $(t_{i+1}, x') \in S$. Then, from (27) and (28), we derive

$$\left| V_*(t_{i+1}, x') - V_*(t_i, x) - \frac{\partial V_*(t_i, x)}{\partial t} \Delta t_k - \langle \nabla_x V_*(t_i, x), f(t_i, x) + g(t_i, x)u \rangle \Delta t_k \right| \leq \varepsilon \Delta t_k, \quad (29)$$

Next, according to (18), (20), (25), (26), (29), and Lemma 1, we obtain

$$\begin{aligned} & \left| Q(t_i, x, u) + (q(t_i, x) + u^T r(t_i, x)u) \Delta t_k - \min_{u'} Q(t_{i+1}, x', u') \right| \\ &= \left| V_*(t_i, x) + A(t_i, x, u) + (q(t_i, x) + u^T r(t_i, x)u) \Delta t_k - V_*(t_{i+1}, x') \right| \\ &\leq \left| \frac{\partial V_*(t_i, x)}{\partial t} + \max_{u' \in U} \mathcal{H}(t_i, x, \nabla_x V_*(t_i, x), u') \right| \Delta t_k + \varepsilon \Delta t_k = \varepsilon \Delta t_k. \end{aligned}$$

327 The theorem is proved.

328 **Proof of Theorem 2.** Let $\varepsilon > 0$. Define k_* according to Theorem 1. Let $k \geq k_*$. Let $Q \in \mathbb{Q}$ satisfy
329 (22). Let x_i , $i \in \overline{0, k}$ be defined by (13), where $u_i \in \mu(t_i, x_i)$. Then, due to (12), (20), and (22), we
330 have

$$\begin{aligned} & V(0, z) - J_k(u_0, \dots, u_{k-1}) \\ &= \sum_{i=0}^{k-1} \left(Q(t_i, x_i, u_i) + (q(t_i, x_i) + u_i^T r(t_i, x_i)u_i) \Delta t_k - \max_{u' \in U} Q(t_{i+1}, x_{i+1}, u') \right) \leq T\varepsilon. \end{aligned} \quad (30)$$

331 Let us define $Q_* \in \mathbb{Q}$ according to (21). Let x_i^* , $i \in \overline{0, k}$ be defined by (13), where $u_i^* \in$
332 $\arg\max_{u \in U} Q_*(t_i, x_i, u)$. Then, due to the Theorem 1 and (20), we have

$$\left| V_*(t_i, x_i^*) + (q(t_i, x_i^*) + (u_i^*)^T r(t_i, x_i^*)u_i^*) \Delta t_k - V_*(t_{i+1}, x_{i+1}^*) \right| \leq \varepsilon \Delta t_k. \quad (31)$$

Note that, according to definition (10) of the value function V_* , we have $V_*(t_k, x) = \sigma(x)$, $x \in \mathbb{R}^n$. Then, from (20), (22), and (31), we derive

$$\begin{aligned} & 0 = \max_{u' \in U} Q(t_k, x_k^*, u') - \sigma(x_k^*) \\ &\leq \sum_{i=0}^{k-1} \left(\max_{u' \in U} Q(t_{i+1}, x_{i+1}^*, u') - Q(t_i, x_i^*, u_i^*) - (q(t_i, x_i^*) + (u_i^*)^T r(t_i, x_i^*)u_i^*) \Delta t_k \right) + V(0, z) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{k-1} \left(V_*(t_i, x_i^*) + (q(t_i, x_i^*) + (u_i^*)^T r(t_i, x_i^*) u_i^*) \Delta t_k - V_*(t_{i+1}, x_{i+1}^*) \right) - V_*(0, z) \\
& \leq 2T\varepsilon + V(0, z) - V_*(0, z).
\end{aligned}$$

333 Due to this inequality and inequality (30), we obtain $J_k(u_0, \dots, u_{k-1}) \geq V_*(0, z) - 3T\varepsilon$. Taking
334 into account (6) and (10) we have $\sup_{u(\cdot)} J(u(\cdot)) = V_*(0, z)$. Thus, the theorem is proved.

Proof of Theorem 3. Let $k \geq 4$ and $Q \in \mathbb{Q}_{NAF}$. Let us take $i = k - 1$ and $x = 2$. For the sake of brevity, denote $V = V(t_{k-1}, 2)$, $\mu = \mu(t_{k-1}, 2)$, and $P = P(t_{k-1}, 2)$. Then, in order to prove the theorem, we have to show that

$$\max_{u \in [-1, 1]} |V - (u - \mu)^2 P + u^2 \Delta t_k + (2 + u \Delta t_k)^2| > \Delta t_k / 8.$$

Arguing by contradiction assume that

$$\max_{u \in [-1, 1]} |V - (u - \mu)^2 P + u^2 \Delta t_k + (2 + u \Delta t_k)^2| \leq \Delta t_k / 8.$$

335 Then, for $u = -1$, $u = 1$, and $u = 0$, we have

$$-\Delta t_k / 8 \leq V - (1 + \mu)^2 P + \Delta t_k + (2 - \Delta t_k)^2 \leq \Delta t_k / 8, \quad (32)$$

$$336 \quad -\Delta t_k / 8 \leq V - (1 - \mu)^2 P + \Delta t_k + (2 + \Delta t_k)^2 \leq \Delta t_k / 8, \quad (33)$$

$$337 \quad -\Delta t_k / 8 \leq -V + \mu^2 P - 4 \leq \Delta t_k / 8. \quad (34)$$

338 Adding up inequalities (32), (33), and twice inequity (34), we derive

$$-\Delta t_k / 2 \leq 2P - 2\Delta t_k - 2\Delta t_k^2 \leq \Delta t_k / 2. \quad (35)$$

Adding up twice inequalities (32), (34), and inequity (35), we obtain

$$-\Delta t_k \leq -4\mu P - 8\Delta t_k \leq \Delta t_k.$$

From this estimate, taking into account (35) and the inequality $k \geq 4$, we conclude

$$\mu \leq -\frac{7\Delta t_k}{4P} \leq -\frac{7}{5 + 4\Delta t_k} \leq -\frac{7}{6} < -1.$$

339 This inequality contradicts the inclusion $\mu \in U = [-1, 1]$, which should be valid for $Q \in \mathbb{Q}_{NAF}$.